Discrete Comput Geom 19:19–31 (1998)

Discrete & Computational
Geometry

Counting d-Step Paths in Extremal Dantzig Figures*

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Abstract. The d-step conjecture asserts that every d-polytope P with $2d$ facets has an edge-path of at most d steps between any two of its vertices. Klee and Walkup showed that to prove the d-step conjecture, it suffices to verify it for all Dantzig figures $(P, \mathbf{w}_1, \mathbf{w}_2)$, which are simple d-polytopes with $2d$ facets together with distinguished vertices $\mathbf{w}_1$ and $\mathbf{w}_2$ which have no common facet, and to consider only paths between $\mathbf{w}_1$ and $\mathbf{w}_2$. This paper counts the number of d-step paths between $\mathbf{w}_1$ and $\mathbf{w}_2$ for certain Dantzig figures $(P, \mathbf{w}_1, \mathbf{w}_2)$ which are extremal in the sense that P has the minimal and maximal vertices possible among such d-polytopes with $2d$ facets, which are $d^2 - d + 2$ vertices (lower bound theorem) and $2\binom{\lfloor \frac{3}{2}d - \frac{1}{2} \rfloor}{\lfloor \frac{d}{2} \rfloor}$ vertices (upper bound theorem), respectively. These Dantzig figures have exactly 2^{d-1} d-step paths.

1. Introduction

The *(bounded) Hirsch conjecture* asserts that the diameter of the graph of any d-polytope having n facets is at most $n - d$. Let $\Delta(d, n)$ denote the maximal diameter of any (d, n)-polytope; the Hirsch conjecture asserts that

$$\Delta(d, n) \leq n - d \qquad \text{for all} \quad n > d > 0.$$

The *d-step conjecture* is a special case of the Hirsch conjecture and asserts that

$$\Delta(d, 2d) = d \qquad \text{for all} \quad d \geq 1.$$

Although seemingly less general, the d-step conjecture is known to be equivalent to the Hirsch conjecture. Klee and Walkup [9, Theorem 2.8] showed that $\Delta(d, 2d)$ is attained

* The second author was supported in part by NSF Research Initiation Award and ONR Grant N-00014-96-1-0281.

by specifying which vertices are contained in each face. Two polytopes have the same *combinatorial type* (or are *combinatorially equivalent*) if there is a one-to-one onto, incidence-preserving mapping of faces of one to faces of the other. The *combinatorial type of a Dantzig figure* $(P, \mathbf{w}_1, \mathbf{w}_2)$ is defined similarly, except that incidence-preserving mappings between Dantzig figures are required to take distinguished vertices of one to distinguished vertices of the other. The truth of the strong d-step conjecture for a particular Dantzig figure $(P, \mathbf{w}_1, \mathbf{w}_2)$ depends only on its combinatorial type. Henceforth in this paper the word "polytope" or "Dantzig figure" refers only to its combinatorial type.

We describe the incidence structure of Dantzig figures using facets. Let $H_1, H_2, \ldots, H_{2d}$ denote the hyperplanes determined by the $2d$ facets of a Dantzig figure $(P, \mathbf{w}_1, \mathbf{w}_2)$. We address each vertex $\mathbf{v}$ of P by the indices of the facets determining it. The *vertex address*

$$\mathbf{v} := [i_1, i_2, \ldots, i_d], \qquad i_1 < i_2 < \cdots < i_d. \tag{2.1}$$

means that $\mathbf{v} = H_{i_1} \cap H_{i_2} \cap \cdots \cap H_{i_d}$, with $i_1 < i_2 < \cdots < i_d$. A *normalized Dantzig figure* is one in which $\mathbf{w}_1 = H_1 \cap H_2 \cap \cdots \cap H_d$ and $\mathbf{w}_2 = H_{d+1} \cap H_{d+2} \cap \cdots \cap H_{2d}$, so that

$$\mathbf{w}_1 = [1, 2, \ldots, d] \qquad \text{and} \qquad \mathbf{w}_2 = [d+1,\ d+2, \ldots, 2d]. \tag{2.2}$$

A *distinguished path* in a Dantzig figure $(P, \mathbf{w}_1, \mathbf{w}_2)$ is a d-step path between $\mathbf{w}_1$ and $\mathbf{w}_2$ in the graph of P. We use the following simple characterization of distinguished paths.

Lemma 2.1. *Let $(P, \mathbf{w}_1, \mathbf{w}_2)$ be a normalized Dantzig figure. The following are equivalent:*

(i) *The set of vertices $\mathbf{v}_0 = \mathbf{w}_1,\ \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_d = \mathbf{w}_2$ is a d-step path in the graph of P, i.e., a distinguished path.*

(ii) *The set of vertices $\mathbf{v}_0 = \mathbf{w}_1, \mathbf{v}_1, \ldots, \mathbf{v}_d = \mathbf{w}_2$ of P has the property that each vertex address $\mathbf{v}_{i+1}$ is obtained from $\mathbf{v}_i$ by removing one facet index j with $1 \le j \le d$ and adding one facet index j' with $d+1 \le j' \le 2d$.*

Here (ii) implies that the vertex labels on any such distinguished path are lexicographically increasing. Based on (ii), we sometimes call the change of addresses from $\mathbf{v}_i$ to $\mathbf{v}_{i+1}$ a *move* of j to j'.

Proof. (i) $\Rightarrow$ (ii). An edge deletes one facet, and a new vertex adds exactly one facet. In going from $\mathbf{w}_1$ to $\mathbf{w}_2$ we must add the facets $H_{d+1}, \ldots, H_{2d}$ and we must remove the facets $H_1, \ldots, H_d$. Since there are only d-steps, (ii) follows.

(ii) $\Rightarrow$ (i). We need only check that $[\mathbf{v}_i, \mathbf{v}_{i+1}]$ is an edge of P, i.e., is not interior to any k-face of P for $k \ge 2$. However, (ii) shows that $\operatorname{conv}[\mathbf{v}_i, \mathbf{v}_{i+1}]$ lies in the intersection of $d-1$ facet hyperplanes given by the $d-1$ common facet indices of $\mathbf{v}_i$ and $\mathbf{v}_{i+1}$. Since P is a simple polytope, any k-face for $k \ge 1$ is contained in exactly $d-k$ facets, hence $[\mathbf{v}_i, \mathbf{v}_{i+1}]$ is an edge of P. □

3. Lower Bound Dantzig Figures: Truncation Polytopes

Barnette [1] proved that any simple d-polytope having n facets has at least $n(d-1) - (d+1)(d-2)$ vertices. He also proved that for dimension $d \geq 4$ equality is attained only for *truncation polytopes*, which are polytopes recursively constructed from a d-simplex by adding facets, one at a time, each of which cuts off a single vertex of the previous polytope. More precisely, we have a sequence of polytopes $P_0, P_1, \dots, P_{n-d-1}$, in which P_0 is a d-simplex, and P_i is obtained from P_{i-1} by "truncating" one vertex by a facet that replaces it by d vertices, see Fig. 3.1. (This construction is sometimes given in its dual form for simplicial polytopes, and the resulting polytopes are called *stacked polytopes*.)

For simple d-polytopes with $2d$ facets, the lower bound is $d^2 - d + 2$ vertices. There are many combinatorially distinct types of truncation polytopes having $2d$ facets, and most of them contain no antipodal vertices, so do not give rise to any Dantzig figures. We construct below a particular family of truncation polytopes with n facets for $n \geq d+1$, among which the case $n = 2d$ gives a Dantzig figure.

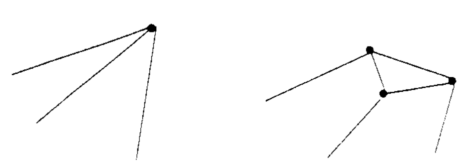
Let $H_1, H_2, \dots, H_n$ denote the hyperplanes giving the facets of a general truncation polytope P. We label vertex $\mathbf{v}$ of P by the labels of the d hyperplanes determining $\mathbf{v}$, numbered in increasing order. To construct P, we start with the d-simplex P_0 determined by hyperplanes $H_1, H_2, \dots, H_{d+1}$, so that P_0 has the vertices with addresses $\{[1, 2, \dots, \hat{i}, \dots, d+1]\colon 1 \leq i \leq d+1\}$, where the symbol $\hat{i}$ means that i is omitted from the address. Among these vertices, the lexicographically minimal vertex is $[1, 2, \dots, (d-1), d]$, and the lexicographically maximal vertex is $[2, 3, \dots, d, (d+1)]$. At step k, for $1 \leq k \leq n-d-1$, we truncate polytope P_{k-1} with a hyperplane H_{d+k+1}, to obtain a new truncation polytope P_k. Then $P = P_{n-d-1}$.

The *maximal truncation polytope* $T_d(n)$ is the truncation polytope with n facets obtained as above, by choosing the lexiographically maximal vertex of P_{k-1} to truncate at step k. Thus at the kth step, the vertex truncated has address $[k+1, k+2, \dots, k+d]$ and the d new vertices of P_{k+1} have addresses $\{[k+1, \dots, \widehat{k+i}, \dots, k+d, k+d+1]\colon 1 \leq i \leq d\}$.

For the case $n = 2d$, the $d^2 - d + 2$ vertices of $T_d(2d)$ consist of the two distinguished "end vertices" $\mathbf{w}_1 = [1, 2, \dots, d]$ and $\mathbf{w}_2 = [d+1, d+2, \dots, 2d]$, plus the $d^2 - d$ "non-end" vertices which have addresses

$$[i, i+1, \dots, \widehat{i+k}, \dots, i+d] \qquad \text{for} \quad 1 \leq k \leq d-1 \quad \text{and} \quad 1 \leq i \leq d. \qquad (3.1)$$

In particular, $(T_d(2d), \mathbf{w}_1, \mathbf{w}_2)$ is a normalized Dantzig figure.



(i) (ii)

Fig. 3.1. Truncating a vertex. (i) Before truncation and (ii) after truncation.

Theorem 3.1. *For each $d \geq 2$, the normalized Dantzig figure $(T_d(2d), [1, 2, \ldots, d], [d+1, \ldots, 2d])$ with $d^2 - d + 1$ vertices has exactly 2^{d-1} d-step paths in its graph between $[1, 2, \ldots, d]$ and $[d+1, \ldots, 2d]$.*

We obtain this from the following more detailed result.

Theorem 3.2. *Let $\mathbf{v}_i$ denote the vertex of $T_d(2d)$ that is reached from $[1, 2, \ldots, d]$ along the edge that deletes H_i, which has the address*

$$\mathbf{v}_i = [1, 2, \ldots, \hat{i}, \ldots, d, d+1], \qquad 1 \leq i \leq d. \tag{3.2}$$

The number of distinguished paths in $(T_d(2d), [1, 2, \ldots, d], [d+1, \ldots, 2d])$ that pass through $\mathbf{v}_i$ is 2^{d-1-i} for $1 \leq i \leq d-1$ and is 1 for $i = d$.

Theorem 3.1 follows directly, since (3.2) runs over all d vertices adjacent to $\mathbf{w}_1$, and $1 + \sum_{j=1}^{d-1} 2^{d-1-j} = 2^{d-1}$.

Proof of Theorem 3.2. We first introduce some additional notation concerning vertex addresses. We add a vertical bar to separate facet numbers at most d from those at least $d+1$. For example, when $d = 4$, the vertex label $[1, 2, 5, 7]$ becomes $[1, 2|5, 7]$.

The "non-end" vertices of $T_d(2d)$ have labels consisting of $d+1$ consecutive facet indices, from i to $d+i$, from which exactly one facet index is omitted, which is neither of the endpoints i or $d+i$. We call the omitted facet index a "bubble," since it interrupts the consecutive sequence of indices of the vertex address.

We call any move between two vertices $\mathbf{v}$ and $\mathbf{v}'$ of $T_d(2d)$ a *legal move* if the vertex address of $\mathbf{v}'$ is obtained from that of $\mathbf{v}$ by removing a facet number $j \leq d$ to the left of the vertical bar and adding a facet number $j' \geq d+1$ to the right of the vertical bar. Lemma 2.1 asserts that every step in a d-step distinguished path is a legal move.

The following claim is verified by inspection, using (3.1).

Claim 1. *The legal moves from a vertex of $T_d(2d)$ to a new vertex depend on the location of the "bubble," and are as follows:*

(i) *If the "bubble" is to the left of the vertical bar, then the only legal move is to replace the index i by $d+i+1$ if the "bubble" is not at $i+1$, and by $d+i+2$ otherwise. In the first case the "bubble" stays where it is, while in the second case the "bubble" moves to the right of the vertical bar, to the position $d+i+1$.*

(ii) *If the "bubble" is to the right of the vertical bar, we can remove any element j to the left of the bar, except i, and use it to fill the "bubble." If so, the "bubble" moves to the left of the vertical bar, to position $\hat{j}$. Alternatively, the label i can move to the right of the vertical bar, to position $d+i+1$.*

We now prove Theorem 3.1 by induction on the dimension d. The base case $d = 2$ is easily checked directly. Now suppose that it is true for all dimensions up to $d-1$ and consider dimension d.

First, we treat paths which pass through $\mathbf{v}_d = [1, 2, \ldots, d-1, \hat{d}|d+1]$. The "bubble" is to the left of the vertical bar, hence rule (i) of the claim forces a unique legal move for $d-2$ steps, ending at the vertex with address $[d-1, \hat{d}|d+1, \ldots, 2d-1]$. There is now a unique legal final step possible to $\mathbf{w}_2 = [d+1, \ldots, 2d-1, 2d]$, hence there is exactly one d-step distinguished path through $\mathbf{v}_d$.

Next, consider paths which pass through $\mathbf{v}_{d-1} = [1, 2, \ldots, \widehat{d-1}, d|d+1]$. The first $d-2$ legal steps are uniquely forced by rule (i), and end at the vertex with address $[d|d+1, d+2, \ldots, 2d-2, \widehat{2d-1}, 2d]$. Now there is a unique final legal move to $\mathbf{w}_2$, hence there is exactly one d-step distinguished path through $\mathbf{v}_{d-1}$.

For the cases of paths which pass through $\mathbf{v}_i$ for $1 \leq i \leq d-2$, the first $i-1$ legal steps are uniquely forced by rule (i), with the path reaching the vertex

$$\mathbf{v}^* = [i+1, i+2, \ldots, d|d+1, \ldots, \widehat{d+i}, d+i+1].$$

Here $\mathbf{v}^*$ has a "bubble" to the right of the vertical bar.

Claim 2. *The $(d-i)$-step legal paths from $\mathbf{v}^*$ to $\mathbf{w}_2$ in $T_d(2d)$ are in one-to-one correspondence with distinguished $(d-i)$-step paths for the normalized Dantzig figure $(T_{d-i}(2d-2i), \mathbf{w}_1', \mathbf{w}_2')$ in $\mathbf{R}^{d-i}$.*

The one-to-one correspondence arises by deleting the $2i$ facets labeled $1, 2, \ldots, i$ and $d+1, \ldots, \widehat{d+i}, d+i+1$, and then renumbering the remaining facets in increasing order, denoting them $\bar{1}, \bar{2}, \ldots, \overline{2(d-i)}$, in which $d+i$ is mapped to $\overline{d-i+1}$, and $d+i+2$ to $\overline{d-i+2}$. Then $\mathbf{v}^*$ corresponds to the initial vertex

$$\mathbf{w}_1' = [\bar{1}, \bar{2}, \ldots, \overline{d-1}]$$

of $T_{d-i}(2(d-i))$. We now observe that any sequence of legal moves that results from applying rules (i) and (ii) starting at the vertex $\mathbf{v}^*$ of $T_d(2d)$ can be exactly matched by the same sequence of legal moves by applying rules (i) and (ii) to $\mathbf{w}_1'$ of $T_{d-1}(2d-2i)$, under the renumbering above, and vice versa. This holds because deleted vertices can never be moved again in any legal d-step path, by Lemma 2.1. For example, a rule (ii) move of type j to fill the "bubble" of $\mathbf{v}_1$ gives the next vertex

$$\mathbf{v} = [i+1, i+2, \ldots, \hat{j}, \ldots, d|d+1, \ldots, d+i, d+i+1],$$

which corresponds to the vertex

$$\mathbf{v}' = [\bar{1}, \bar{2}, \ldots, \widehat{\overline{j-i}}, \ldots, \overline{d-i}|\overline{d-i+1}]$$

of $T_{d-1}(2d-2i)$. A case (ii) "end move" to

$$\mathbf{v} = [i+2, i+3, \ldots, d|d+1, \ldots, \widehat{d+i}, d+i+1, d+i+2]$$

corresponds to the "end move"

$$\mathbf{v}' = [\bar{2}, \bar{3}, \ldots, \overline{d-i}|\widehat{\overline{d-i+1}}, \overline{d-i+2}],$$

of $T_{d-i}(2d-2i)$. This proves Claim 2.

Table 3.1. d-Step paths for $T_4(8)$.

1234	1234	1234	1234
2346	2346	2346	2346
3467	3467	2456	2356
4678	3567	4568	3567
5678	5678	5678	5678
1234	1234	1234	1234
1345	1345	1245	1235
3457	3457	2456	2356
4578	3567	4568	3567
5678	5678	5678	5678

The induction hypothesis for $d-i$ gives that the total number of distinguished $(d-i)$-step paths in $T_{d-i}(2d-2i)$ is

$$1+\sum_{i=1}^{d-i-1} 2^{d-i-1-j} = 2^{d-i-1}.$$

Thus Claim 2 shows that there are 2^{d-i-1} d-step paths through $\mathbf{v}_i$ for $1 \le i \le d-2$, which completes the induction step. □

Table 3.1 gives the eight 4-step paths for $(T_4(8), [1234], [5678])$, grouped by the value of $\mathbf{v}_i$.

We conclude this section with a uniqueness result.

Theorem 3.3. *For $d \ge 4$, there is a unique combinatorial type of a d-dimensional Dantzig figure having d^2-d+2 vertices, which is given by $(T_d(2d), [1,2,3,\dots,d], [d+1, d+2,\dots,2d])$.*

Proof. By Barnette's result, any such Dantzig figure is a truncation polytope P. Let P be constructed from the d-simplex S_d by truncating $d-1$ successive vertices. By suitably renumbering the facets of S_d and the added facets, we may suppose that the distinguished vertices are $\mathbf{w}_1 = [1,2,\dots,i,d+1,d+2,\dots,2d-i]$ and $\mathbf{w}_2 = [i+1,i+2,\dots,d,2d-i+1,2d-i+2,\dots,2d]$. In order to create these vertices using $d-1$ truncations, the sequence of truncations leading to $\mathbf{w}_1$ must add facets $H_{d+1},\dots,H_{2d-i}$ and delete facets $H_{i+1},\dots,H_d$ in some order, while for $\mathbf{w}_2$ it adds facets $H_{2d-i+1},\dots,H_{2d}$ and deletes $H_1,\dots,H_i$, in some order. In particular, no truncation operation for $H_{d+1},\dots,H_{2d+i}$ ever involves a vertex containing $H_{2d-i+1},\dots,H_{2d}$, hence these truncation operations mutually commute. Thus we may first make all the truncations $H_{d+1},\dots,H_{2d-i}$ (in some order), followed by H_{2d-i+1} through H_{2d} (in some order). Next we may relabel these vertices so that they enter in consecutive order $H_{d+1},\dots,H_{2d-i}$, followed by $H_{2d-i+1},\dots,H_{2d}$. Finally, we may relabel the facets of S_d so that when the facet H_{d+j} enters, then the facet H_j leaves. This relabeling is a permutation of $\{1,2,\dots,i\}$ and $\{i+1,i+2,\dots,d\}$ separately. The resulting sequence of vertex addresses of the vertices

that are truncated are

$$[1,2,\dots,i,\ i+j+1,\ i+j+2,\dots,d,d+1,d+2,\dots,d+j], \qquad 1 \le j \le d-i,$$

and

$$[1,2,\dots,j,\ i+1,\ i+2,\dots,d,\ 2d-i-1,\dots,2d-j], \qquad 1 \le j \le i.$$

Now we can explicitly write down all the vertices of the resulting truncation polytope $(P,\mathbf{w}_1,\mathbf{w}_2)$ and check that there is a further permutation σ of the facet labels that sends it to $(T_d(2d), [1,2,3,\dots,d], [d+1, d+2,\dots,2d].)$ The permutation σ is given by

$$\sigma(k) = \begin{cases} d+1-k, & 1 \le k \le i, \\ d-i+k, & i+1 \le k \le d, \\ i+1-k, & d+1 \le k \le 2d-i+1, \\ k-2d+i, & 2d-i \le k \le 2d. \end{cases}$$

This completes the proof. □

The unique four-dimensional Dantzig figure with 14 vertices appears in Table 4 of [4] as $(P_2^8, [1367], [2458])$.

4. Upper Bound Dantzig Figures: Cyclic Polytopes

We recall the construction of a cyclic polytope $C_d(n)$ on n vertices. The *moment curve* in $\mathbf{R}^d$ is the curve $\{\mathbf{x}(t) = (t, t^2, \dots, t^d) : t \in \mathbf{R}\}$. A cyclic polytope $C_d(n)$ is the convex hull of n points on the moment curve, i.e., $C_d(n) = \operatorname{conv}\{\mathbf{x}(t_1),\dots,\mathbf{x}(t_n) : t_1 < t_2 < \dots < t_n\}$. Such polytopes have a well-defined combinatorial type independent of the choice of vertices. The Upper Bound Theorem states that $C_d(n)$ has the maximum number of facets among all d-polytopes having n vertices. The *dual cyclic polytope* $C_d^*(n)$ denotes the combinatorial type of any polytope that is polar to some cyclic polytope $C_d(n)$.

Proposition 4.1 (Gale's Evenness Condition). *For $n \ge d \ge 2$ the polytope $C_d^*(n)$ is a simple d-polytope. Given a d-subset $S \subseteq \{1,2,\dots,n\}$, the point*

$$\mathbf{v} = \bigcap_{i \in S} H_i$$

is a vertex of $C_d^*(n)$ if and only if for any two elements j_1 and j_2 not in S, with $j_1 < j_2$, the number of elements of S between j_1 and j_2 is even, i.e.,

$$\#\{k : k \in S,\ j_1 < k < j_2\} \equiv 0 \pmod 2 \qquad \text{for} \quad j_1, j_2 \in S. \tag{4.1}$$

Equivalently, all maximal blocks of consecutive elements of S which do not contain either 1 or n must contain an even number of elements.

Proof. Gale's evenness condition for facets of the cyclic polytope appears in [12, p. 84] or in [15, p. 14]. Proposition 4.1 states the dual form, which follows from the fact that polarity reverses the incidence structure of all faces, see Section 2.3 of [15]. □

We apply Gale's evenness condition to derive the following result.

Theorem 4.1. *For each $d \ge 2$, the normalized Dantzig figure $(C_d^*(2d), \{1, 2, \ldots, d\}, \{d+1, d+2, \ldots, 2d\})$ with $2\binom{\lfloor \frac{1}{2}d - \frac{1}{2} \rfloor}{\lfloor \frac{1}{2}d \rfloor}$ vertices has exactly 2^{d-1} d-step paths in its graph between $\{1, 2, \ldots, d\}$ and $\{d + 1, \ldots, 2d\}$.*

We obtain this from the following more detailed result.

Theorem 4.2. *Let $\mathbf{v}_i$ denote the vertex of $C_d^*(2d)$ reached from $\{1, 2, \ldots, d\}$ along the edge deleting H_i, so that*

$$\mathbf{v}_i = \{1, 2, \ldots, \hat{i}, \ldots, m_i\}, \qquad 1 \le i \le d + 1, \tag{4.2}$$

in which $m_i = d + 1$ or $2d$ according as $d - i$ is odd or even. The number of distinguished paths of $(C_d^*(2d), \{1, 2, \ldots, d\}, \{d + 1, \ldots, 2d\})$ that pass through $\mathbf{v}_i$ is $\binom{d-1}{i-1}$, for $1 \le i \le d$.

Theorem 4.1 follows immediately from Theorem 4.2, since $\sum_{i=1}^{d} \binom{d-1}{i-1} = 2^{d-1}$.

Proof of Theorem 4.2. Although $C_d^*(2d)$ has an exponentially large number of vertices, only a polynomial size subset of them appear in the totality of all distinguished paths. We will subsequently show that these are a subset of all vertices having addresses which consist of at most three maximal blocks of consecutive integers; the number of such vertices is $O(d^3)$.

To describe these vertex addresses we introduce a suggestive terminology. Call a consecutive sequence of vertices $Y_1 = \{1, 2, \ldots, i\}$ a (*departure*) *dock* and call a consecutive sequence $Y_2 = \{k + 1, k + 2, \ldots, 2d\}$ an (*arrival*) *dock*. We call a maximal consecutive sequence of vertices $X = \{r, r + 1, \ldots, s\}$ with $1 < r < s < 2d$ a *ship*. A general vertex address has indices that group into either 0, 1, or 2 docks and some number of ships, i.e., $\mathbf{v} = [Y_1, X_1, X_2, \ldots, X_j, Y_2]$, in which we permit Y_1 or Y_2 to be the empty set, and $j = 0$ may occur.

Claim 1. *A vertex $\mathbf{v}$ occurs in some distinguished path if and only if its address consists of two docks Y_1, Y_2 and a single ship X, in which up to two of X, Y_1, Y_2 may be the empty set, and the ship, if present, has an even number of vertices, and contains at least one of the indices d and $d + 1$.*

Call a vertex of the above type a *voyage vertex*. To prove Claim 1 we will need to characterize the legal moves possible from a voyage vertex. Recall that Lemma 2.1 states that on any distinguished path the address of $\mathbf{v}_{i+1}$ is obtained from that of $\mathbf{v}_i$ by replacing one index $j \le d$ with an index $j' \ge d + 1$.

Claim 2. *The legal moves possible from a voyage vertex $\mathbf{v} = [Y_1, X, Y_2]$ which remove an index $j \le d$ and add an index $j' \ge d + 1$ are of the following forms:*

(i) *An "initial move" from $\{1, 2, \ldots, d\}$ replaces an index i $(1 \le i \le d)$ by $2d$ if $d - i$ is even, and by $d + 1$ if $d - i$ is odd.*

(ii) *If $Y_1 = \{1, 2, \ldots, i\} \ne \emptyset$ and $Y_2 = \{k + 1, \ldots, 2d\}$ or $\emptyset$, then a "docking move" removes i and either adds k if $Y_2 \ne \emptyset$ or adds $2d$ if $Y_2 = \emptyset$.*

(iii) *If the ship $X = \{r, r + 1, \ldots, s\} \ne \emptyset$, with $r \le d$ and $s \ge d + 1$, a "sailing move" removes r and adds $s + 1$, yielding $X' = \{r + 1, r + 2, \ldots, s + 1\}$.*

(iv) *If $Y_1 = \{1, 2, \ldots, i\} \ne \emptyset$ with $i < d$ and $Y_2 = \{k + 1, \ldots, 2d\}$ or $\emptyset$ and $1 \le i - 2m < i$, a "submarine move" removes $i - 2m$ and either adds k if $Y_2 \ne \emptyset$ or adds $2d$ if $Y_2 = \emptyset$. This move produces a ship $X' = \{i - 2m + 1, \ldots, i\}$.*

(v) *If the ship $X = \{r, r + 1, \ldots, s\} \ne \emptyset$, with $r \le d$, a "shipwreck move" removes some index $r + 2m$ with $r < r + 2m \le d$ and adds $s + 1$. It produces two ships $X_1' = \{r, \ldots, r + 2m - 1\}$ and $X_2' = \{r + 2m + 1, \ldots, s + 1\}$.*

In cases (i)–(iii) a legal move from a voyage vertex leads to another voyage vertex. In cases (iv) and (v) the new vertex is not a voyage vertex. In proving Claim 1 we must rule out moves of either of these types on a distinguished path.

The terminology "ship" and "dock" is motivated by Claim 2. Assuming that cases (iv) and (v) are ruled out, Claim 2 implies that on a distinguished path a ship can be created only at an initial move, and it then "sails" from $X = \{i + 1, i + 2, \ldots, d + 1\}$ to $X' = \{d + 1, d + 2, \ldots, 2d - i + 1\}$ by a sequence of $d - i$ "sailing" moves. A sequence of $i - 1$ "docking moves" removes the departure dock $Y_1 = \{1, 2, \ldots, i - 1\}$ and reconstructs it as the arrival dock $Y_2 = \{2d - i + 2, \ldots, 2d\}$. Claim 2 allows this sequence of "sailing" and "docking" moves to occur in any order, and the number of distinguished paths this accounts for is equal to the number of different ordered sequences of $d - i$ red balls and $i - 1$ green balls; this number is $\binom{d-1}{i-1}$. Thus Theorem 4.2 follows from Claim 1 and Claim 2.

We prove Claim 2 first. Gale's evenness condition (Proposition 4.1) asserts that a sequence of the form $[Y_1, X, Y_2]$ is the address of a vertex of $C_d^*(2d)$, if and only if the ship X contains an even number of elements. All of the permitted moves in cases (i)–(v) produce a new address that satisfies Gale's evenness condition, hence they are legal moves. We must show that they are exhaustive. Since Y_2 only contains elements in $\{d + 1, \ldots, 2d\}$, any element in it is never moved. We show that cases (i)–(v) cover all possible elements that could be moved from Y_1 and X.

To verify case (i), first note that the index i can move only to $d + 1$ or $2d$, because any move to $d + 1 < j' < 2d$ creates a ship of length 1, violating Gale's evenness condition. For $1 \le i \le d - 1$, moving i creates a ship starting in position $i + 1$ which extends at least to d, hence i must move to $d + 1$ or $2d$ as necessary to create a ship of even length. Finally, for $i = d$ a move to $d + 1$ creates a ship of length 1, which is ruled out, so case (i) follows.

To verify case (ii), we argue by contradiction. If the last index i of Y_1 is removed, and is not replaced by an index j' which is just before the smallest index in Y_2 (which is $2d$ if $Y_2 = \emptyset$), then j' either falls adjacent to a ship and changes its length from even to odd, or else it forms a new ship of length 1, both of which contradict Gale's evenness condition.

To verify case (iii), we note that moving the smallest index r in a ship must always be to the other end $s+1$ of the ship, for if not, the length of the ship changes from even to odd.

To verify case (iv), we note that removing an interior index $1 \le j < i$ of the dock $Y_1 = \{1, \ldots, i\}$ must have $j = i - 2m$. For if $j = i - 2m - 1$, then the removed index must be replaced by index $j' = i + 1$ to create a ship of even length, and then $j' \le d$, contradicting the hypothesis that $j' \geq d+1$. Thus $j = i - 2m$, and if it is replaced by j', then $j' = k - 1$, otherwise, j' is either a ship of length 1 or j' is adjacent to the ship X and changes its length from even to odd.

The final case (v) concerns removing some index j of a ship X which is not its smallest vertex. This vertex cannot be the largest vertex of X because it changes the length of the ship from even to odd. If j is any other index of X, its removal splits the ship into (at least) two ships. The removed vertex $j = r + 2m$, in order that the ship $[r, r+1, \ldots, r+2m-1]$ has even length, and the added index must be $s+1$ in order that the second ship $[r+2m+1, \ldots, s, s+1]$ have even length. This establishes Claim 2.

It remains to prove Claim 1. We first show that only voyage vertices occur on any distinguished path. Such a path starts at $[1, 2, \ldots, d]$. It suffices to show that of the legal moves in Claim 2, those of type (i), (ii), and (iii) move a voyage vertex to another voyage vertex, and that moves in cases (iv) and (v) never occur on any distinguished path. To handle cases (i), (ii), and (iii) it suffices to observe that, if the move takes $[Y_1, X, Y_2]$ to $[Y_1', X', Y_2']$, then either $X' = \emptyset$ or X' contains one of d and $d+1$.

To show that a case (iv) move cannot occur, we argue by contradiction. If it did occur, it would produce a vertex $\mathbf{v} = [Y_1', X', Y_2']$ containing a ship $X' = [i-2m+1, \ldots, i]$ whose largest index $i \leq d-1$. Now Lemma 2.1 says that all moves on a distinguished path remove a vertex $j \leq d$ and add a vertex $j' \geq d+1$. In particular, no subsequent vertex on the distinguished path can ever contain either index $i+1$ or $i-2m$, so this ship can never "sail." On a distinguished path all vertices eventually are moved to be larger than d, hence there is a first time on the path after $\mathbf{v}$ that some index j in X' is removed and replaced with $j' \geq d+1$. The resulting address contains the set of indices $X - \{j\}$, and does not contain $i+1$, j, and $i-2m$. Since $|X - \{j\}|$ is odd, this address contains a ship with an odd number of elements, which contradicts Gale's evenness condition.

To show that a case (v) move can never occur, we again argue by contradiction. If it did occur, the vertex reached is $\mathbf{v} = [Y_1', X_1', X_2', Y_2']$, in which $X_1' = \{r, r+1, \ldots, r+2m-1\}$ is a ship with $r + 2m \leq d$. Now the ship X' can never "sail," and we obtain the same contradiction as in case (iv).

We conclude that every distinguished path consists only of voyage vertices, by induction on the number steps in a partial path, using Claim 2.

To show that all voyage vertices actually occur in some distinguished path, it suffices to observe that each move of type (i)–(iii) switches one index $j \leq d$ to an index $j' \geq d+1$, and that each voyage vertex has at least one predecessor and one successor under a move of one of these types. Thus, given such a vertex, we can extend it by predecessors back to $[1, 2, \ldots, d]$ and by successors to $[d+1, d+2, \ldots, 2d]$, and the resulting path takes exactly d steps by the index switching property. This establishes Claim 1. □

Table 4.1 below gives the eight 4-step paths for $(C_4^*(8), [1234], [5678])$, grouped by the value of $\mathbf{v}_i$.

There seems to be no obvious pattern to the number of d-step paths for Dantzig figures having the maximal number of vertices, in four-dimensional examples. Let P_{36}^8 denote the noncyclic polytope with 20 vertices taken from Table 4 of [4]. Then we have

$$\#(C_4^*(8), [1256], [3478]) = 12,$$
$$\#(P_{36}^8, [2345], [1678]) = 8,$$
$$\#(P_{36}^8, [1256], [3478]) = 12.$$

Table 4.1. d-Step paths for $C_4^*(8)$.

1234	1234	1234	1234
2345	1348	1348	1348
3456	3478	1458	1458
4567	4578	4578	1568
5678	5678	5678	5678
1234	1234	1234	1234
1245	1245	1245	1238
1458	1458	1256	1278
4578	1568	1568	1678
5678	5678	5678	5678

Acknowledgment

In 1994 Steven T. Ikeler had mentioned that, in his low-dimensional numerical calculations on a family of Dantzig figures (constructed by the second author), he had observed 2^{d-1} d-step paths in dual cyclic $(d, 2d)$-polytopes. We thank him for sharing his numerical observation with us.

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Received September 24, 1995, *and in revised form December* 12, 1995, *and April* 8, 1996.