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BOUNDS FOR LATTICE POLYTOPES CONTAINING A FIXED NUMBER OF INTERIOR POINTS IN A SUBLATTICE

JEFFREY C. LAGARIAS AND GÜNTER M. ZIEGLER

ABSTRACT. A lattice polytope is a polytope in $\mathbf{R}^n$ whose vertices are all in $\mathbf{Z}^n$. The volume of a lattice polytope $\mathbf{P}$ containing exactly $k \geq 1$ points in $d\mathbf{Z}^n$ in its interior is bounded above by $kd^n(7(kd+1))^{n2^{n+1}}$. Any lattice polytope in $\mathbf{R}^n$ of volume V can after an integral unimodular transformation be contained in a lattice cube having side length at most $n \cdot n!\, V$. Thus the number of equivalence classes under integer unimodular transformations of lattice polytopes of bounded volume is finite. If $\mathbf{S}$ is any simplex of maximum volume inside a closed bounded convex body $\mathbf{K}$ in $\mathbf{R}^n$ having nonempty interior, then $\mathbf{K} \subseteq (n+2)\mathbf{S} - (n+1)\mathbf{s}$ where $m\mathbf{S}$ denotes a homothetic copy of $\mathbf{S}$ with scale factor m, and $\mathbf{s}$ is the centroid of $\mathbf{S}$.

1. Introduction. A *lattice polytope* in $\mathbf{R}^n$ is a convex polytope all of whose vertices are lattice points, i.e. points in $\mathbf{Z}^n$. A *rational polytope* $\mathbf{P}$ is a convex polytope with all vertices in $\mathbf{Q}^n$. The *denominator* of a rational polytope $\mathbf{P}$ is the smallest integer $d \geq 1$ such that $d\mathbf{P}$ is a lattice polytope.

For each $n \geq 2$ there are lattice polytopes in $\mathbf{R}^n$ of arbitrarily large volume containing no interior lattice points, and for $n \geq 3$ there are lattice simplices of arbitrarily large volume whose vertices are their only lattice points. However D. Hensley [5] proved that any lattice polytope $\mathbf{P}$ in $\mathbf{R}^n$ containing *exactly* $k \geq 1$ interior lattice points has volume bounded by a finite bound $V(n,k)$, and furthermore the total number of lattice points in the interior and on the boundary of such $\mathbf{P}$ is bounded by a finite bound $J(n,k)$.

The main purpose of this paper is to sharpen Hensley's upper bounds for $V(n,k)$ and $J(n,k)$, and to extend his results to apply to lattice polytopes containing a fixed number $k \geq 1$ of interior points in a given sublattice Λ of $\mathbf{Z}^n$. We also prove finiteness of the number of equivalence classes of such polytopes under lattice-point preserving affine maps. Finally, we prove that any closed convex body $\mathbf{K}$ in $\mathbf{R}^n$ contains a simplex $\mathbf{S}$ such that $\mathbf{K} \subseteq (-n)\mathbf{S} + (n+1)\mathbf{s}$ and $\mathbf{K} \subseteq (n+2)\mathbf{S} - (n+1)\mathbf{s}$, where $\mathbf{s}$ is the centroid of $\mathbf{S}$, and if $\mathbf{K}$ is a lattice polytope then one can choose $\mathbf{S}$, $(-n)\mathbf{S} + (n+1)\mathbf{s}$, and $(n+2)\mathbf{S} - (n+1)\mathbf{s}$ to all be lattice simplices.

In extending Hensley's bounds, we treat first the special case $\Lambda = d\mathbf{Z}^n$. This case arises in considering rational polytopes of denominator d containing k interior lattice points in $\mathbf{Z}^n$, after rescaling to clear the denominator.

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THEOREM 1. *Let $V(n,k,d)$ denote the maximal volume of a lattice polytope in $\mathbf{R}^n$ that contains exactly $k \geq 1$ points in $d\mathbf{Z}^n$ in its interior, and let $J(n,k,d)$ denote the maximum number of lattice points $J(n,k,d)$ inside or on the boundary of such a polytope. Then $V(n,k,d)$ and $J(n,k,d)$ are finite, with*

$$V(n,k,d) \leq kd^n\big(7(kd+1)\big)^{n2^{n+1}}, \tag{1.1}$$

and

$$J(n,k,d) \leq n + n!\,kd^n\big(7(kd+1)\big)^{n2^{n+1}}. \tag{1.2}$$

The proof follows the general approach of Hensley's proof, obtaining an improvement by sharpening his basic Diophantine approximation lemma. (Hensley's bound for $V(n,k,1)$ is roughly $k(4k)^{n!+1}$.)

Any bound on $V(n,k,d)$ must have double exponential dependence on n. In §2 we generalize examples of Zaks, Perles and Wills [10] to show that for $n \geq 2$,

$$V(n,k,d) \geq \frac{k+1}{n!}(d+1)^{2^{n-1}-1},$$
$$J(n,k,d) \geq k(d+1)^{2^{n-2}}.$$

The bound (1.1) is probably far from the truth in its dependence on k, however, and conjectured extremal examples (see Proposition 2.6) suggest that $V(n,k,d)$ grows linearly in k as $k \to \infty$ with n and d fixed.

Exact formulae for $V(n,k,d)$ are known in a few cases. One has

$$V(1,k,d) = (k+1)d,$$

and a result of Scott [9] gives

$$V(2,k,1) = \begin{cases} 9/2 & \text{for } k = 1, \\ 2(k+1) & \text{for } k \geq 2. \end{cases}$$

The bounds of Theorem 1 immediately yield bounds applicable to a general (full rank) sublattice Λ of $\mathbf{Z}^n$. Let d be the smallest positive integer such that $d\mathbf{Z}^n \subset \Lambda$. If $\lambda_i = \min\{\lambda \in \mathbf{N} : \lambda \mathbf{e}_i \in \Lambda\}$, then $\Lambda_0 = \langle \lambda_1\mathbf{e}_1, \ldots, \lambda_n\mathbf{e}_n\rangle$ is a sublattice of Λ, and $d\mathbf{Z}^n \subseteq \Lambda$ requires $d\mathbf{Z}^n \subseteq \Lambda_0$ so that $d = l.c.m.(\lambda_1, \ldots, \lambda_n)$. Since for each i there is a basis of Λ whose first vector is $\lambda_i\mathbf{e}_i$, one has $\lambda_i \mid \det(\Lambda)$, so that $d \mid \det(\Lambda)$. If the columns of the integer matrix M are a basis of Λ then $\det(\Lambda) = |\det(M)|$ and $\operatorname{adj}(M) = |\det(M)|M^{-1}$ is an integer matrix. Furthermore $\tilde{M} = \frac{d}{\det(\Lambda)}\operatorname{adj}(M)$ is also an integer matrix, because $\tilde{M}M = dI$, and the columns of $\tilde{M}$ express a basis of the sublattice $d\mathbf{Z}^n$ of Λ in terms of the basis M of Λ, hence are integral. The linear map $\Phi: \mathbf{R}^n \to \mathbf{R}^n$ given by $\Phi(x) = \tilde{M}\mathbf{x}$ has $\Phi(\mathbf{Z}^n) \subseteq \mathbf{Z}^n$ and $\Phi(\Lambda) = d\mathbf{Z}^n$, and its determinant is $d^n\big(\det(\Lambda)\big)^{-1}$. If a lattice polytope

$\mathbf{P}$ contains exactly $k \geq 1$ interior lattice points in Λ, then $\Phi(\mathbf{P})$ is a lattice polytope containing exactly k interior lattice points in $d\mathbf{Z}^n$, hence

$$\mathrm{Vol}\big(\Phi(\mathbf{P})\big) \leq V(n,k,d),$$

so that

$$\text{(1.3)} \qquad \mathrm{Vol}(\mathbf{P}) \leq \big(\det(\Lambda)\big) d^{-n} V(n,k,d),$$

and one also obtains

$$\text{(1.4)} \qquad \#(\mathbf{P} \cap \mathbf{Z}^n) \leq J(n,k,d).$$

The second question we study concerns the finiteness of the number of integral equivalence classes of such polytopes. The group of *lattice point preserving maps* $\mathcal{L}_n(\mathbf{Z})$ consists of those affine maps L with $L(\mathbf{Z}^n) = \mathbf{Z}^n$. They are exactly the maps $L(\mathbf{x}) = G\mathbf{x} + \mathbf{m}$ with $G \in GL(n,\mathbf{Z})$ and $\mathbf{m} \in \mathbf{Z}^n$. The subgroup $\mathcal{L}_{n,d}(\mathbf{Z})$ contains all such maps which also have $L(d\mathbf{Z}^n) = d\mathbf{Z}^n$; they consist of those maps $L \in \mathcal{L}_n(\mathbf{Z})$ having $\mathbf{m} \in d\mathbf{Z}^n$. Two polytopes $\mathbf{P}_1$ and $\mathbf{P}_2$ are *integrally equivalent* if $L(\mathbf{P}_1) = \mathbf{P}_2$ for $L \in \mathcal{L}_n(\mathbf{Z})$. Integrally equivalent polytopes have the same number of lattice points in each corresponding k-dimensional face. Two polytopes are *d-integrally equivalent* if $L(\mathbf{P}_1) = \mathbf{P}_2$ for $L \in \mathcal{L}_{n,d}(\mathbf{Z})$; such polytopes have the same number of lattice points in both $\mathbf{Z}^n$ and $d\mathbf{Z}^n$ on corresponding faces.

We establish the finiteness of the number of integral equivalence classes of lattice polytopes of bounded volume, as a consequence of the following result. A *lattice cube* is a cube with sides parallel to the coordinate axes whose vertices are lattice points.

THEOREM 2. *Any lattice polytope in $\mathbf{R}^n$ of volume $\leq V$ is integrally equivalent under a map $\mathbf{x} \to U\mathbf{x}$ with $U \in GL(n,\mathbf{Z})$ to a lattice polytope contained in a lattice cube of side length at most $n \cdot n!\, V$.*

The bound of Theorem 2 is reasonably tight since the lattice simplex $\mathbf{S}_n$ with vertices $\mathbf{v}_0 = \mathbf{0}$ and $\mathbf{v}_i = \mathbf{e}_i$ for $1 \leq i \leq n-1$ and $\mathbf{v}_n = \lfloor n!\, V \rfloor \mathbf{e}_n$ has volume $\mathrm{Vol}(\mathbf{S}_n) \leq V$ and for any $L \in \mathcal{L}_n(\mathbf{Z})$ the simplex $L(\mathbf{S}_n)$ is not contained in any lattice cube of side length $\frac{1}{\sqrt{n}}(n!)V$.

The finiteness of the number of integral equivalence classes of lattice polytopes of volume $\leq V$ follows immediately from Theorem 2. By a translation in $\mathbf{Z}^n$ we may move the cube inside $\{(x_1,\ldots,x_n) : 0 \leq x_i \leq n \cdot n!\, V\}$. Since there are only finitely many lattice points in this cube, there are at most finitely many integral equivalence types of such polytopes. If we wish to preserve membership in $d\mathbf{Z}^n$ as well, this translation must be in $d\mathbf{Z}^n$ and we can move the cube into $\{(x_1,\ldots,x_n) : 0 \leq x_i \leq n \cdot n!\, V + d\}$. The finiteness of integral equivalence classes for lattice simplices for $n = 3$ was previously established by Reznick [8, Section 3].

We also prove several properties of maximal volume simplices contained in a convex body $\mathbf{K}$, some of which are used in the proof of Theorem 2.

THEOREM 3. *(a) Suppose $\mathbf{K}$ is a closed bounded convex body in $\mathbf{R}^n$ with nonempty interior. Let $\mathbf{S}$ be any simplex of maximal volume contained in $\mathbf{K}$, and let $\mathbf{s}$ be its centriod. Then*

$$\text{(1.5)} \qquad \mathbf{K} \subseteq (-n)\mathbf{S} + (n+1)\mathbf{s},$$

and

$$\text{(1.6)} \qquad \mathbf{K} \subseteq (n+2)\mathbf{S} - (n+1)\mathbf{s}.$$

(b) Any convex polytope $\mathbf{K}$ contains a maximal volume simplex $\mathbf{S}$ whose vertices are vertices of $\mathbf{K}$. In particular if $\mathbf{K}$ is a lattice polytope then this $\mathbf{S}$ is a lattice simplex, and both $(-n)\mathbf{S} + (n+1)\mathbf{s}$ and $(n+2)\mathbf{S} - (n+1)\mathbf{s}$ are lattice simplices.

The study of maximal volume simplices in a convex body goes back at least to Rado [7, pp. 242–244], who showed that the centroid $\mathbf{s}$ of a maximal volume simplex in a convex body $\mathbf{K}$ as in part (a) has the property that any chord in $\mathbf{K}$ through $\mathbf{s}$ is divided into two segments of ratio $k : l$ satisfying $\frac{1}{n} \leq \frac{k}{l} \leq n$. The inclusion $\mathbf{K} \subseteq (-n)\mathbf{S} + (n+1)\mathbf{s}$ is a well-known result traceable back to Mahler [6, pp. 111–116], and appears in Andrews [1, Lemma 2]. The observation that $\mathbf{K} \subseteq (n+2)\mathbf{S} - (n+1)\mathbf{s}$ is apparently new.

These two inclusions in part (a) are both sharp for all $n \geq 2$, in the sense that the minimal $c_n > 0$ such that $\mathbf{S} \subseteq \mathbf{K} \subseteq c_n\mathbf{S} + (c_n - 1)\mathbf{s}$ is $c_n = n+2$, and the minimal $|c_n|$ with $c_n < 0$ is $c_n = -n$, see the end of §4.

2. **Proof of Theorem 1.** We first consider a lattice simplex $\mathbf{S}$ in $\mathbf{R}^n$ and let $(\alpha_0, \alpha_1, \ldots, \alpha_n)$ denote the barycentric coordinates of an interior point $\mathbf{w} \in d\mathbf{Z}^n$ in $\mathbf{S}$. The basic idea (due to Hensley [5]) is to show that $\mathbf{w}$ cannot be too close to a face of $\mathbf{S}$, i.e. that its barycentric coordinates are bounded away from 0 and 1. This bounds the coefficient of asymmetry of $\mathbf{S}$ around the lattice point $\mathbf{w}$, which leads to a bound on its volume by a generalization of Minkowski's convex body theorem due to Mahler.

The lower bound in the following one-sided Diophantine approximation lemma provides the basic ingredient in the proof. This result sharpens Lemma 3.1 in Hensley [5]. (Hensley's lemma yields roughly the bound $\delta(n,d) \geq (4d)^{-n!-1}$.)

LEMMA 2.1. *For $d \geq 1$ let $\delta(n,d)$ be the largest constant such that for all positive real numbers $\alpha_1, \ldots, \alpha_n > 0$ satisfying*

$$1 \geq \sum_{i=1}^{n} \alpha_i > 1 - \delta(n,d)$$

there exist integers $Q, P_1, \ldots, P_n$ with $Q > 0$, all $P_i \geq 0$, such that

(1) $\displaystyle\sum_{i=1}^{n} \frac{P_i}{Q} = 1$,

(2) $\displaystyle\alpha_i > \frac{dP_i}{dQ+1}$ *for* $1 \leq i \leq n$,

(3) $1 \le dQ + 1 \le \delta(n,d)^{-1}$.

Then

$$\frac{d}{t_{n+1,d}-1} \ge \delta(n,d) \ge \left(7(d+1)\right)^{-2^{n+1}}, \tag{2.1}$$

where $t_{n,d}$ is determined by $t_{1,d} = d+1$ and the recursion $t_{n,d} = t_{n-1,d}^2 - t_{n-1,d} + 1$.

One can easily prove by induction on n that

$$(d+1)^{2^{n-1}} \ge t_{n,d} \ge (d+1)^{2^{n-2}},$$

where the lower bound is derived using $u_{n,d} = t_{n,d} - 1$, which satisfies $u_{n,d} = u_{n-1,d}^2 + u_{n-1,d}$. These inequalities show that the lower bound in (2.1) is qualitatively similar in order of magnitude to the upper bound.

PROOF. The upper bound in (2.1) is obtained on choosing $\alpha_i = \frac{d}{t_{i,d}}$ for $1 \le i \le n$. One can easily prove by induction on n that $t_{n+1,d} - 1 = d\prod_{i=1}^n t_{i,d}$ and

$$\sum_{i=1}^n \alpha_i = 1 - \frac{d}{t_{n+1,d}-1}.$$

Now there is no approximation satisfying (1)–(3), for if there were then (2) would give $dQ + 1 > P_i t_{i,d}$ for all i. This implies that $dQ \ge P_i t_{i,d}$ since $t_{i,d} \in \mathbf{Z}$, hence

$$\frac{d}{t_{i,d}} \ge \frac{P_i}{Q}, \quad 1 \le i \le n.$$

Consequently

$$1 - \frac{d}{t_{n-1,d}-1} = \sum_{i=1}^n \alpha_i \ge \sum_{i=1}^n \frac{P_i}{Q} = 1,$$

a contradiction.

The main content of the lemma is the lower bound in (2.1). The proof is by induction on n, holding d fixed. It's true for all d in the base case $n = 1$, on taking $\delta(1,d) = \frac{1}{d+1}$ with $Q = P_1 = 1$. The upper bound in (2.1) holds with equality for this case.

Now suppose $n \ge 2$ and that the lower bound in (2.1) is true for all values smaller than n. Reorder the α_i so that $\alpha_1 \ge \alpha_2 \ge \cdots \ge \alpha_n > 0$, and since $\sum_{i=1}^n \alpha_i \ge \frac{1}{2}$ (using the upper bound in (2.1)) we have $\alpha_1 \ge \frac{1}{2n}$. Let $\frac{1}{\Delta_{n,d}}$ denote a lower bound for $\delta(n,d)$, which will be determined in the proof (by (2.11) below), and choose $\Delta_{1,d} = d+1$. We set $\sum_{i=1}^n \alpha_i = 1 - \mu$ with $0 < \mu < \frac{1}{\Delta_{n,d}}$.

If there is some $j < n$ such that

$$\alpha_1 + \cdots + \alpha_j > 1 - \frac{1}{\Delta_{j,d}},$$

then by the induction hypothesis there exists $(Q, P_1, \dots, P_j)$ satisfying (1)–(3) for $(\alpha_1, \dots, \alpha_j)$, and on setting $P_{j+1} = \cdots = P_n = 0$ we obtain a solution to (1)–(3) for $(\alpha_1, \dots, \alpha_n)$. Thus we need only consider the case that

$$\alpha_{j+1} + \cdots + \alpha_n \ge \frac{1}{\Delta_{j,d}}, \quad 1 \le j \le n-1, \tag{2.2}$$

holds. Now the ordering of the α_i's gives

$$(n-j)\alpha_{j+1} \ge \alpha_{j+1} + \alpha_{j+2} + \cdots + \alpha_n,$$

which with (2.2) yields

$$\alpha_{j+1} \ge \frac{1}{n\Delta_{j,d}}, \quad 1 \le j \le n-1. \tag{2.3}$$

By Minkowski's convex body theorem ([3, p. 71]) there exists a nonzero lattice point in the open symmetric convex body $\mathbf{K} = \mathbf{K}(Q, P_2, \dots, P_n)$ in $\mathbf{R}^n$ defined by

$$|Q| < R, \tag{2.4a}$$

$$|Q\alpha_i - P_i| < \min\left(\frac{1}{d}\alpha_i, \frac{1}{2n^2(d+1)}\right), \quad i \ge 2, \tag{2.4b}$$

provided that $\mathrm{Vol}(\mathbf{K}) > 2^n$, that is provided

$$R\prod_{i=2}^n \min\left(\frac{1}{d}\alpha_i, \frac{1}{2n^2(d+1)}\right) > 1. \tag{2.5}$$

Using the facts that $\alpha_j < 1/2$ for $i \ge 2$ and (2.3) we obtain, for $i \ge 2$,

$$\min\left(\frac{1}{d}\alpha_i, \frac{1}{2n^2(d+1)}\right) > \frac{\alpha_i}{n^2(d+1)} \ge \frac{1}{n^3(d+1)\Delta_{i-1,d}}.$$

Thus (2.5) is certainly satisfied whenever

$$R \ge n^{3n-3}(d+1)^{n-1}\prod_{i=1}^{n-1}\Delta_{i,d}. \tag{2.6}$$

Take a nonzero solution $(Q, P_2, \dots, P_n)$ in $\mathbf{K}$, and observe that $Q \ne 0$ because $Q = 0$ implies by (2.4b) that all $P_i = 0$, a contradiction. We may suppose that $Q > 0$ since $(-Q_1 - P_1, \dots, -P_n)$ is also in $\mathbf{K}$, and (2.4b) then shows that all $P_i \ge 0$ for $i \ge 2$.

Now define P_1 by

$$P_1 = Q - \sum_{j=2}^n P_j,$$

which makes (1) hold. We also have by (2.4b) that

$$(dQ+1)\alpha_i = dP_i + \alpha_i + d(Q\alpha_i - P_i) > dP_i \tag{2.7}$$

for $2 \le i \le n$, which verifies (2) except for $i = 1$. Next we show that $P_1 \ge 0$. If $\tilde{\alpha}_1 = \alpha_1 + \mu = 1 - \sum_{i=2}^n \alpha_i$, then

$$Q\tilde{\alpha}_1 - P_1 = Q\left(1 - \sum_{i=2}^n \alpha_i\right) - \left(Q - \sum_{i=2}^n P_i\right)$$
$$= -\sum_{i=2}^n (Q\alpha_i - P_i).$$

Hence using $\tilde{\alpha}_1 \geq \alpha_1 \geq \frac{1}{2n}$,

$$|Q\tilde{\alpha}_1 - P_1| \leq \sum_{i=2}^{n} |Q\alpha_i - P_i| \leq \sum_{i=2}^{n} \frac{1}{2n^2(d+1)} < \frac{1}{d+1}\tilde{\alpha}_1. \tag{2.8}$$

Thus P_1 is the nearest integer to $Q\tilde{\alpha}_1$, hence $P_1 \geq 0$.

We claim that (2) and (3) will hold provided $\Delta_{n,d}$ and R are suitably chosen. To check (2) we need only treat the case $i = 1$, by (2.7). We have, using (2.8) and (2.4a),

$$\begin{aligned}(dQ+1)\alpha_1 &= (dQ+1)\tilde{\alpha}_1 - (dQ+1)\mu \\ &= dP_1 + \tilde{\alpha}_1 + d(Q\tilde{\alpha}_1 - P_1) - (dQ+1)\mu \\ &\geq dP_1 + \tilde{\alpha}_1 - \frac{d}{d+1}\tilde{\alpha}_1 - (dR+1)\mu \\ &> dP_1 + \frac{1}{d+1}\tilde{\alpha}_1 - (dR+1)\frac{1}{\Delta_{n,d}}.\end{aligned}$$

This shows that (2) holds provided that

$$dR + 1 \leq \frac{1}{2n(d+1)}\Delta_{n,d}, \tag{2.9}$$

since $\tilde{\alpha}_1 \geq \frac{1}{2n}$. Also the inequality (2.9) guarantees that (3) holds, since $1 \leq Q \leq R$.

Thus to prove existence it suffices to choose $\Delta_{n,d}$ large enough that an R exists satisfying (2.6) and (2.9). Now (2.9) holds if

$$R \leq \frac{1}{2n(d+1)^2}\Delta_{n,d}.$$

This condition will allow an R for which (2.6) holds to exist provided that

$$\frac{1}{2n(d+1)^2}\Delta_{n,d} \geq n^{3n-3}(d+1)^{n-1}\prod_{i=1}^{n-1}\Delta_{i,d}. \tag{2.10}$$

It suffices to choose

$$\Delta_{n,d} = n^{3n}(d+1)^{n+1}\prod_{i=1}^{n-1}\Delta_{i,d}, \tag{2.11}$$

for $\Delta_{n,d}$ to make (2.10) hold for $n \geq 2$ and this completes the induction step.

To complete the proof, we show that

$$\Delta_{n,d} \leq \left(7(d+1)\right)^{2^{n+1}}.$$

Indeed (2.11) for $n \geq 2$ gives the recursion

$$\log \Delta_{n,d} = 3n\log n + (n+1)\log(d+1) + \sum_{i=1}^{n-1}\log(\Delta_{i,d})$$

with $\Delta_{1,d} = d+1$. This recursion can be solved explicitly, yielding the following inequalities (in which the logarithms are to base 2):

$$\begin{aligned}\log\Delta_{n,d} &= 3n\log n + 3\sum_{i=2}^{n-1}2^{n-i-1}i\log i + (5\cdot 2^{n-2} - 1)\log(d+1) \\ &< 3\cdot 2^{n-1}\sum_{i\geq 2}2^{-i}(i\log i) + 5\cdot 2^{n-2}\log(d+1) \\ &< 3\cdot 2^{n-1}\sum_{i\geq 2}2^{-i}i(i-1) + 5\cdot 2^{n-2}\log(d+1) \\ &= 3\cdot 2^{n+1} + 5\cdot 2^{n-2}\log(d+1) < 2^{n+1}\log\left(7(d+1)\right).\end{aligned}$$

■

Hensley conjectured that the upper bound in (2.1) holds with equality for $d = 1$ and all n, and we extend this to conjecture that it holds for all n and d. The proof showed the conjecture is true for $n = 1$ and all d, and we have also verified it in the cases $(n, d) = (2,1), (3,1), (2,2)$ and $(2,3)$.

LEMMA 2.2. *If* $\mathbf{S}$ *is a lattice simplex in* $\mathbf{R}^n$ *with* $k = \#\left(d\mathbf{Z}^n \cap \mathrm{Int}(\mathbf{S})\right) \geq 1$, *and if* $(\alpha_0, \ldots, \alpha_n)$ *are the barycentric coordinates of an interior point* $\mathbf{w}$ *in* $d\mathbf{Z}^n$ *then*

$$\delta(n, dk) \leq \alpha_i \leq 1 - n\delta(n, dk).$$

PROOF. Suppose not, so that some $\alpha_i < \delta(n, dk)$, which we may take to be α_0. Lemma 2.1 applies to $(\alpha_1, \ldots, \alpha_n)$ and the $(Q, P_1, \ldots, P_n)$ it produces satisfies

$$(jQ+1)\alpha_i > jP_i, \quad 1 \leq i \leq n$$

for $1 \leq j \leq kd$, If $\mathbf{v}_i$ are the vertices of $\mathbf{S}$ then

$$\mathbf{x}_m = (mdQ+1)\mathbf{w} + m\sum_{i=1}^{n} dP_i\mathbf{v}_i$$

for $0 \leq m \leq k$ are distinct points in $d\mathbf{Z}^n \cap \mathrm{Int}(\mathbf{S})$, a contradiction. ■

Theorem 1.1 for a lattice simplex $\mathbf{S}$ follows from Lemma 2.1 and the following bound.

LEMMA 2.3. *Suppose that* $\mathbf{S}$ *is a lattice simplex in* $\mathbf{R}^n$ *such that* $k = \#\left(d\mathbf{Z}^n \cap \mathrm{Int}(\mathbf{S})\right) \geq 1$. *Then*

$$\mathrm{Vol}(\mathbf{S}) \leq \frac{1}{n!}(k+1)d^n\delta(n, dk)^{-n}.$$

PROOF. We adapt the proof of Theorem 3.4 in [5]. Let Φ be an affine map that takes $\mathbf{S}$ to the "standard simplex" $\mathbf{S}_0$ having vertices $\mathbf{0}, \mathbf{e}_1, \ldots, \mathbf{e}_n$ in $\mathbf{R}^n$. Let $\Lambda = \Phi(\mathbf{Z}^n)$, so that Λ is a (possibly noninteger) lattice of determinant $|\det(\Phi)|$ and $\mathbf{S}$ has volume $\mathrm{Vol}(\mathbf{S}) = \frac{1}{n!}|\det(\Phi)|^{-1}$.

Suppose that $\mathbf{y} \in d\mathbf{Z}^n \cap \mathrm{Int}(\mathbf{S}_0)$ and set $\mathbf{v} = \Phi(\mathbf{y}) = \sum_{i=1}^{n}\alpha_i\mathbf{e}_i$, where α_i are barycentric coordinates. The region $\mathbf{R} = \{\mathbf{v} + \mathbf{u} : |u_i| < \alpha_i \text{ for } 1 \leq i \leq n\}$ is centrally symmetric about $\mathbf{v}$, and $\Phi(d\mathbf{Z}^n) = \mathbf{v} + d\Lambda$ is a coset of the lattice $d\Lambda$. By van der Corput's theorem ([4, p. 51]) **R** contains at least the greatest integer strictly less than $\left(\prod_{i=1}^{n}\alpha_i\right)\frac{1}{d^n}|\det(\Phi)|^{-1}$ distinct pairs of points $\mathbf{v}\pm\mathbf{u}$ where each $\mathbf{u}\in d\Lambda$ is nonzero. Now let $\mathbf{u}=\sum_{i=1}^{n}u_i\mathbf{e}_i$ with $|u_i|<\alpha_i$ for all i. Then at least one of $\mathbf{v}+\mathbf{u}$ and $\mathbf{v}-\mathbf{u}$ is in $\text{Int}(\mathbf{S}_0)$ if some $\alpha_i>1/2$ and both $\mathbf{v}\pm\mathbf{u}$ are in $\text{Int}(\mathbf{S}_0)$ otherwise. Thus Lemma 2.2 yields

$$k=\#\big(d\mathbf{Z}^n\cap\text{Int}(\mathbf{S})\big)=\#\big((\mathbf{v}+d\Lambda)\cap\text{Int}(\mathbf{S}_0)\big)\geq\frac{1}{d^n}\left(\prod_{i=1}^{n}\alpha_i\right)|\det(\Phi)|^{-1}-1,$$
$$\geq d^{-n}\delta(n,kd)^n n!\ \text{Vol}(\mathbf{S})-1.$$

∎

To prove Theorem 1 for a general lattice polytope **P** we follow Hensley's arguments exactly. As a consequence of Lemma 2.2 one has:

LEMMA 2.4. *Let* **F** *be a lattice polytope in* $\mathbf{R}^n$ *of dimension* $n-1$. *Let* $\mathbf{x}_0$ *be a lattice point not in the* $(n-1)$*-dimensional hyperplane containing* **F** *and let* **P** *be the conical lattice polytope which is the convex hull of* **F** *and* $\mathbf{x}_0$. *Suppose* $k=\#\big(d\mathbf{Z}^n\cap\text{Int}(\mathbf{P})\big)\geq 1$. *If* $\mathbf{x}_1,\dots,\mathbf{x}_m$ *are the lattice vertices of* **F** *then for any barycentric representation of* **y** *contained in* $d\mathbf{Z}^n\cap\text{Int}(\mathbf{P})$ *as* $\mathbf{y}=\sum_{i=0}^{m}\alpha_i\mathbf{x}_i$ *with all* $\alpha_i\geq 0$, $\sum_{i=0}^{m}\alpha_i=1$, *one has*

$$\delta(n,dk)\leq\alpha_0\leq 1-\delta(n,dk).$$

PROOF. See Hensley, [5, Corollary 3.2]. ∎

The *coefficient of asymmetry* $\sigma(\mathbf{K},\mathbf{x})$ of a convex body **K** about a point **x** is

$$\sigma(\mathbf{K},\mathbf{x})=\sup_{\|\mathbf{y}\|=1}\frac{\max\{\lambda:\mathbf{x}+\lambda\mathbf{y}\in\mathbf{K}\}}{\max\{\lambda:\mathbf{x}-\lambda\mathbf{y}\in\mathbf{K}\}}.$$

Using Lemma 2.4 one finds that the coefficient of asymmetry $\sigma(\mathbf{P},\mathbf{y})$ of a lattice polytope **P** having $\#\big(d\mathbf{Z}^n\cap\text{Int}(\mathbf{P})\big)=k\geq 1$ about any $\mathbf{y}\in\big(d\mathbf{Z}^n\cap\text{Int}(\mathbf{P})\big)$ satisfies

$$\sigma(\mathbf{P},\mathbf{y})\leq\frac{1-\delta(n,kd)}{\delta(n,kd)}. \tag{2.12}$$

Now we use the following extension of a theorem of Mahler (see [4, p. 52]).

THEOREM 2.5. *If* **K** *is any convex body having* $k=\#\big(d\mathbf{Z}^n\cap\text{Int}(\mathbf{K})\big)\geq 1$, *such that the coefficient of assymmetry* $\sigma(\mathbf{P},\mathbf{y})$ *about some* $\mathbf{y}\in d\mathbf{Z}^n\cap\text{Int}(\mathbf{K})$ *satisfies* $\sigma(\mathbf{P},\mathbf{y})\leq\frac{1-\delta}{\delta}$ *then*

$$\text{Vol}(\mathbf{K})\leq k\left(\frac{d}{\delta}\right)^n.$$

PROOF. By rescaling coordinates by a factor of d we may suppose without loss of generality that $d=1$, and by a further translation we may suppose that $\mathbf{y}=\mathbf{0}$. We argue by contradiction. If $\text{Vol}(\mathbf{K})>k\delta^{-n}$, then one can choose $\varepsilon>0$ small enough that $\mathbf{K}'=(1-\varepsilon)\mathbf{K}$ has $\text{Vol}(\mathbf{K}')>k\delta^{-n}$. Then put $\mathbf{K}''=(1+\sigma)^{-1}\mathbf{K}'=\delta^{-1}\mathbf{K}'$, and $\text{Vol}(\mathbf{K}'')>k$. By van der Corput's theorem ([4, p. 51]) $\mathbf{K}''$ contains points $\mathbf{x},\mathbf{y}_1,\dots,\mathbf{y}_{k+1}$ such that all $\mathbf{y}_i-\mathbf{x}\in\mathbf{Z}^n$. Now $-\frac{1}{\sigma}\mathbf{x}\in\mathbf{K}''$ by definition of $\sigma=\sigma(\mathbf{K},\mathbf{0})=\sigma(\mathbf{K}'',\mathbf{0})$. By convexity

$$\frac{1}{1+\sigma}(\mathbf{y}_i-\mathbf{x})=\frac{1}{1+\sigma}\mathbf{y}_i+\frac{\sigma}{1+\sigma}\left(-\frac{1}{\sigma}\mathbf{x}\right)\in\mathbf{K}'',$$

hence all $\mathbf{y}_i-\mathbf{x}\in\mathbf{K}'$. Since $\mathbf{K}'\subseteq\text{Int}(\mathbf{K})$, there are $k+1$ interior lattice points in **K**, a contradiction. ∎

We have now completed all the work for Theorem 1. In fact, applying Theorem 2.5 to (2.12) yields

$$\text{Vol}(\mathbf{P})\leq kd^n\delta(n,kd)^{-n},$$

and (1.1) follows using Lemma 2.1. If **P** is a lattice simplex Lemma 2.3 gives a slightly stronger bound for $n\geq 2$.

A theorem of Blichfeldt ([2],[3, p. 69]) asserts that any body **P** containing J lattice points spanning $\mathbf{R}^n$ has $\text{Vol}(\mathbf{P})\geq\frac{J-n}{n!}$, which yields $J\leq n+n!\ \text{Vol}(\mathbf{P})$, and (1.2) follows. ∎

We give lower bounds for $V(n,k,d)$ and $J(n,k,d)$ by extending examples of Zaks, Perles and Wills [10]. These involve the sequences $t_{n,d}$ defined in Lemma 2.1.

PROPOSITION 2.6. *The lattice simplex* $\mathbf{S}_{n,k,d}$ *having vertices* $\mathbf{v}_0=\mathbf{0}$, $\mathbf{v}_i=t_{i,d}\mathbf{e}_i$ *for* $1\leq i\leq n-1$, *and* $\mathbf{v}_n=(k+1)(t_{n,d}-1)\mathbf{e}_n$ *contains exactly* k *interior lattice points in* $d\mathbf{Z}^n$. *Hence*

$$V(n,k,d)\geq\frac{k+1}{n!}\left(\prod_{i=1}^{n-1}t_{i,d}\right)(t_{n,d}-1)=\frac{k+1}{n!}\frac{1}{d}(t_{n,d}-1)^2. \tag{2.13}$$

and

$$J(n,k,d)\geq(k+1)(t_{n,d}-1).$$

This proposition gives the lower bounds stated in § 1 using $t_{n,d}>(d+1)^{2^{n-2}}$ for $n\geq 2$.

PROOF. We show that

$$\text{Int}(\mathbf{S}_{n,k,d})\cap d\mathbf{Z}^n=\{(d,d,\dots,d,id):1\leq i\leq k\}.$$

Let $(\alpha_0,\alpha_1,\dots,\alpha_n)$ denote the barycentric coordinates of a lattice point $\mathbf{w}=\sum_{i=0}^{n}\alpha_i\mathbf{v}_i\in d\mathbf{Z}^n$ in $\text{Int}(\mathbf{S}_{n,k,d})$. By induction on i for $1\leq i\leq n-1$ starting from $i=1$ one shows that $\alpha_i=\frac{d}{t_{i,d}}$ using the relation

$$\sum_{j=1}^{i}\frac{d}{t_{j,d}}=1-\frac{d}{t_{i+1,d}-1}, \tag{2.14}$$

because necessarily $\alpha_j=\frac{md}{t_{i,d}}$ for some $m\geq 1$, and choosing $m\geq 2$ gives $\sum_{j=1}^{i}\alpha_j>1$, a contradiction. Next (2.14) allows only $\alpha_n=\frac{md}{(k+1)(t_{n,d}-1)}$ with $1\leq m\leq k$. Since $\alpha_0=1-\sum_{j=1}^{n}\alpha_i$ one checks that these barycentric coordinates actually yield the k lattice points in $d\mathbf{Z}^n$ above. ∎

It is possible that equality holds in (2.13) for all $(n,k,d)\neq(2,1,1)$. This is however an open problem even for $n=2$. Furthermore it is possible that the only lattice polytopes attaining equality in (2.13) are lattice simplices unless $(n,d)=(2,1)$.

3. **Proof of Theorem 2.** First consider the case that the polytope is a simplex $\mathbf{S}$ having vertices $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_n \in \mathbf{Z}^n$. Consider the lattice Λ spanned by the basis vectors $\mathbf{w}_i = \mathbf{v}_i - \mathbf{v}_0$ for $1 \le i \le n$. Then Λ is a sublattice of $\mathbf{Z}^n$ and

$$\det(\Lambda) = [\mathbf{Z}^n : \Lambda] = n!\ \mathrm{Vol}(\mathbf{S}) \le n!\ V.$$

Let B be the integer matrix whose i^{th} row is $\mathbf{w}_i$, so that $|\det(B)| = \det(\Lambda)$. If $\mathbf{P}_0$ is the parallelepiped $\left\{\mathbf{y} : \mathbf{y} = \sum_{i=1}^n y_i \mathbf{w}_i,\ 0 \le y_i \le 1\right\}$ then $\mathbf{S}$ is contained in the translated parallelepiped $\mathbf{v}_0 + \mathbf{P}_0$. Now there is a matrix $U \in GL(n, \mathbf{Z})$ taking the basis matrix to the lower-triangular form (Hermite normal form):

$$UB = \begin{bmatrix} a_{11} & & & \\ a_{21} & a_{22} & & \\ \vdots & & \ddots & \\ a_{n1} & \cdots & & a_{nn} \end{bmatrix}, \tag{3.1}$$

with $0 \le a_{ji} < a_{ii}$ for $j > i$ and all $a_{ii} > 0$ ([3, p. 13]). Now $|\det(B)| = \prod_{i=1}^n a_{ii} \le n!\,V$, hence $1 \le a_{ii} \le n!\,V$ and the parallelepiped generated by the row vectors of UB is contained in the cube $\{\mathbf{x} : 0 \le x_i \le n!\,V \text{ for } 1 \le i \le n\}$. The map $\mathbf{x} \longrightarrow U\mathbf{x} \in \mathcal{L}_n$ takes $\mathbf{S}$ to $U\mathbf{S}$, which is contained in this parallelepiped, and thus lies in a lattice cube of side at most $n!\,V$.

Now suppose that $\mathbf{P}$ is an arbitrary lattice polytope. We assume that Theroem 3 is proved. By Theorem 3(b) it contains a maximal volume simplex $\mathbf{S}$ which is a lattice simplex. The argument above shows that there exists a transformation $U \in GL(n, \mathbf{Z})$ such that $\mathbf{x} \longrightarrow U\mathbf{x}$ maps $\mathbf{S}$ to a lattice simplex $\mathbf{S}_1$ contained in a lattice cube $\mathbf{C}$ of side $n!\,V$, and maps $\mathbf{P}$ to a lattice polytope $\mathbf{P}_1$. Then $\mathbf{S}_1$ is a maximal volume simplex in $\mathbf{P}_1$, so by Theorem 3(a) $\mathbf{P}_1$ is contained in the lattice simplex $(-n)\mathbf{S}_1 + (n+1)\mathbf{s}$, where $\mathbf{s}$ is the centroid of $\mathbf{S}_1$, and $(n+1)\mathbf{s} \in \mathbf{Z}^n$. Consequently $\mathbf{P}_1$ is contained in the lattice cube $(-n)\mathbf{C} + (n+1)\mathbf{s}$ of side $n \cdot n!\,V$. ∎

4. **Proof of Theorem 3.** Let $\mathbf{S}$ be any maximal volume simplex in the bounded convex body $\mathbf{K}$, and let $\mathbf{v}_0, \ldots, \mathbf{v}_n$ be the vertices of $\mathbf{S}$. By making a translation if necessary we may assume that the centroid of $\mathbf{S}$ is $\mathbf{0}$, i.e. $\sum_{i=0}^n \mathbf{v}_i = \mathbf{0}$. Our object is then to show that $\mathbf{K} \subseteq (-n)\mathbf{S}$ and $\mathbf{K} \subseteq (n+2)\mathbf{S}$. Let H_i be the hyperplane spanned by all the vertices except $\mathbf{v}_i$, and let $d_i = \mathrm{dist}(\mathbf{v}_i, H_i)$. Define H_i^+, H_i^- to be the two hyperplanes parallel to H_i such that H_i^+ contains $\mathbf{v}_i$ while H_i^- is at distance d_i from H_i with H_i separating H_i^- from $\mathbf{v}_i$. We claim that $\mathbf{K}$ is contained in the closed region $\mathbf{R}_i$ between H_i^+ and H_i^-. For if $\mathbf{y} \in \mathbf{K}$ were outside this region, then the simplex spanned by $\mathbf{y}$ and all $\mathbf{v}_j$ for $j \ne i$ would have volume bigger than $\mathrm{Vol}(\mathbf{S})$, a contradiction. Hence $\mathbf{K} \subset \bigcap_{i=0}^n \mathbf{R}_i$.

We will show that

$$\bigcap_{i=0}^n \mathbf{R}_i = (n+2)\mathbf{S} \cap (-n)\mathbf{S}, \tag{4.1}$$

which implies part (a) of the theorem. Since $\mathbf{S}$ has nonzero volume, all points in $\mathbb{R}^n$ have unique barycentric coordinates $\mathbf{y} = \sum_{i=0}^n \beta_i \mathbf{v}_i$ with $\sum_{i=0}^n \beta_i = 1$. The region $\mathbf{R}_i$ is given by the barycentric coordinates:

$$\mathbf{R}_i = \left\{\mathbf{y} = \sum_{j=0}^n \beta_j \mathbf{v}_j : \sum_{j=0}^n \beta_j = 1 \text{ and } |\beta_i| \le 1\right\}.$$

This is clear since if $\mathbf{y} = \sum_{j=0}^n \beta_j \mathbf{v}_j$ then $\mathrm{dist}(\mathbf{y}, H_i) = |\beta_i| d_i$. Hence

$$\bigcap_{i=1}^n \mathbf{R}_i = \left\{\mathbf{y} = \sum_{j=0}^n \beta_j \mathbf{v}_j : \sum_{j=0}^n \beta_j = 1 \text{ and all } |\beta_j| \le 1\right\}. \tag{4.2}$$

Since $\sum_{i=0}^n \mathbf{v}_i = \mathbf{0}$ by hypothesis,

$$\begin{aligned}(-n)\mathbf{S} &= \left\{\mathbf{y} = \sum_{j=0}^n \alpha_j(-n\mathbf{v}_j) : \sum_{j=0}^n \alpha_j = 1 \text{ and all } \alpha_j \ge 0\right\} \\ &= \left\{\mathbf{y} = \sum_{j=0}^n \beta_j \mathbf{v}_j : \sum_{j=0}^n \beta_j = 1 \text{ and all } \beta_j \le 1\right\},\end{aligned} \tag{4.3}$$

where $\beta_j = -n\alpha_j + 1$. Similarly

$$\begin{aligned}(n+2)\mathbf{S} &= \left\{\mathbf{y} = \sum_{j=0}^n \alpha_j(n+2)\mathbf{v}_j : \sum_{j=0}^n \alpha_j = 1 \text{ and all } \alpha_j \ge 0\right\} \\ &= \left\{\mathbf{y} = \sum_{j=0}^n \beta_j \mathbf{v}_j : \sum_{j=0}^n \beta_j = 1 \text{ and all } \beta_j \ge -1\right\}\end{aligned} \tag{4.4}$$

where $\beta_j = (n+2)\alpha_j - 1$. The equality (4.1) follows on comparing (4.2)–(4.4).

To prove part (b), let $\mathbf{P}$ be a convex polytope having nonzero volume. We wish to show that $\mathbf{P}$ contains a maximal volume simplex whose vertices are all vertices of $\mathbf{P}$. Let $\mathbf{S}'$ be a maximal volume simplex contained in $\mathbf{P}$. If it has a vertex $\mathbf{w}'$ not a vertex of $\mathbf{P}$, consider the linear program of maximizing the (oriented) distance of a point in $\mathbf{P}$ from the hyperplane spanned by the other n vertices of $\mathbf{S}'$. Some vertex $\mathbf{w}''$ of $\mathbf{P}$ is an optimal point for this linear program, so we can replace $\mathbf{w}'$ by $\mathbf{w}''$ to obtain a new maximal volume simplex for $\mathbf{P}$ which has one fewer vertex not a vertex of $\mathbf{P}$. Continuing in this way, we eventually obtain a maximal volume simplex $\mathbf{S}$ all of whose vertices are vertices of $\mathbf{P}$.

If $\mathbf{P}$ is a lattice polytope this $\mathbf{S}$ is a lattice simplex. If its vertices are $\mathbf{v}_0, \ldots, \mathbf{v}_n$ then $(n+1)\mathbf{s} = \sum_{i=0}^n \mathbf{v}_i \in \mathbf{Z}^n$. Hence $(-n)\mathbf{S} + (n+1)\mathbf{s}$ and $(n+2)\mathbf{S} - (n+1)\mathbf{s}$ are lattice simplices. ∎

REMARKS. (1) If $\mathbf{P}$ is a lattice polytope having the maximum volume simplex $\mathbf{S}$ which is a lattice simplex, then

$$\bigcap_{i=0}^n \mathbf{R}_i = (n+2)\mathbf{S} \cap (-n)\mathbf{S}$$

is a lattice polytope. For (4.2) implies that its vertices are contained in the set of lattice points $\left\{\sum_{i=0}^n \beta_i \mathbf{v}_i : \sum_{i=0}^n \beta_i = 1 \text{ and all } \beta_i \in \{1, 0, -1\}\right\}$.

(2) The inclusion $\mathbf{K} \subset (-n)\mathbf{S}+(n+1)\mathbf{s}$ is sharp in the sense that if $\mathbf{K} \subset c_n\mathbf{S}+(1-c_n)\mathbf{s}$ for all $\mathbf{K}$ and $c_n < 0$ then $c_n \leq -n$. Take $\mathbf{K}$ to be a simplex

$$\mathbf{S} = \mathrm{conv}(\mathbf{0}, \mathbf{e}_1, \dots, \mathbf{e}_n).$$
$$= \left\{\mathbf{x} \in \mathbf{R}^n : \text{ all } x_i \geq 0 \text{ and } \sum_{i=1}^{n} x_i \leq 1\right\}.$$

Then $\mathbf{s} = \left(\frac{1}{n+1}, \dots, \frac{1}{n+1}\right)$ and for $c_n < 0$ one has

$$c_n\mathbf{S} = \left\{\mathbf{x} \in \mathbf{R}^n : \text{ all } x_i \leq 0 \text{ and } \sum_{i=1}^{n} x_i \geq c_n\right\}.$$

Hence

$$c_n\mathbf{S} + (1-c_n)\mathbf{s} = \left\{\mathbf{x} \in \mathbf{R}^n : \text{ all } x_i \leq \frac{1-c_n}{n+1} \text{ and } \sum_{i=1}^{n} x_i \geq \frac{1}{n+1}(n+c_n)\right\}.$$

To obtain $\mathbf{e}_1$ in this region requires $c_n \leq -n$.

(3) The inclusion $\mathbf{K} \subset (n+2)\mathbf{S}-(n+1)\mathbf{s}$ is sharp in the sense that if $\mathbf{K} \subset c_n\mathbf{S}+(1-c_n)\mathbf{s}$ for all $\mathbf{K}$ and $c_n > 0$ then $c_n \geq n+2$. Let

$$\mathbf{K} = \mathrm{conv}\{\pm\mathbf{e}_i : 1 \leq i \leq n\}$$

be the n-dimensional cross-polytope. A maximum volume simplex $\mathbf{S}$ in $\mathbf{K}$ is given by

$$\mathbf{S} = \mathrm{conv}\{-\mathbf{e}_1, \mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n\}$$
$$= \left\{\mathbf{x} \in \mathbf{R}^n : x_2 \geq 0, \dots, x_n \geq 0, \pm 1 + \sum_{i=2}^{n} x_i \leq 1\right\}.$$

of volume $\frac{2}{n!}$, with centroid $\mathbf{s} = \left(0, \frac{1}{n+1}, \dots, \frac{1}{n+1}\right)$. This holds because every lattice simplex in $\mathbf{K}$ has this form after a suitable permutation of the coordinate axes, and after sending certain $x_i \longrightarrow -x_i$. Now suppose $c_n > 0$ is such that $\mathbf{K} \subseteq c_n\mathbf{S} - (c_n - 1)\mathbf{s}$. Computation yields

$$c_n\mathbf{S} = \left\{x \in \mathbf{R}^n : x_2 \geq 0, \dots, x_n \geq 0, \pm x_1 + \sum_{i=2}^{n} x_i \leq c_n\right\},$$

hence

$$c_n\mathbf{S} - (c_n - 1)\mathbf{s}$$
$$= \left\{x \in \mathbf{R}^n : x_2 \geq \frac{1-c_n}{n+1}, \dots, x_n \geq \frac{1-c_n}{n+1}, \pm x_1 + \sum_{i=2}^{n} x_i \leq \frac{c_n}{n+1} + \frac{n+1}{n-1}\right\}.$$

For $n \geq 2$ the condition $-\mathbf{e}_2 \in c_n\mathbf{S} - (c_n - 1)\mathbf{s}$ requires $-1 \geq \frac{1-c_n}{n+1}$, which is $c_n \geq n+2$.

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AT&T Bell Laboratories
Room 2C-373
Murray Hill, New Jersey
U.S.A. 07974

Universität Augsburg
Augsburg, Germany