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AN ALTERNATIVE ALGORITHM FOR COUNTING LATTICE POINTS IN A CONVEX POLYTOPE

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Abstract. We provide an alternative algorithm for counting lattice points in the convex polytope $\{x \in \mathbb{R}^n | Ax = b, x \geq 0\}$. It is based on an exact (tractable) formula for the case $A \in \mathbb{Z}^{m\times(m+1)}$ that we repeatedly use for the general case $A \in \mathbb{Z}^{m\times n}$.

1. Introduction

Consider the (not necessarily compact) polyhedron

$$\Omega(y) = \{x \in \mathbb{R}^n \mid Ax = y; \quad x_k \geq 0\}, \tag{1.1}$$

with $y \in \mathbb{Z}^m$ and $A \in \mathbb{Z}^{m\times n}$ for $n > m$; and let $f : \mathbb{Z}^m \to \mathbb{R}$ be the function

$$y \mapsto f(y) := \sum_{x\in\Omega(y)\cap\mathbb{N}^n} \mathrm{e}^{c'x}, \tag{1.2}$$

where the vector $c \in \mathbb{R}^n$ is chosen small enough (even negative) to ensure that $f(y)$ is well defined even when $\Omega(y)$ is not compact. The notation c' stands for the transpose of the vector c, so that: $c'x = c_1x_1 + \cdots + c_nx_n$. If $\Omega(y)$ is compact, then $f(y)$ provides us with the exact number of points in the set $\Omega(y) \cap \mathbb{N}^n$ by either choosing $c := 0$, or taking $\lim_{c\to 0} f(y)$ (or even rounding up $f(y)$ to the nearest integer for c sufficiently close to zero).

In recent works, Barvinok [3], Barvinok and Pommersheim [4], Brion and Vergne [8], Pukhlikov and Khovanskii [11] have provided nice exact formulas for $f(y)$. For instance, with y fixed, Barvinok [3] considers $f(y)$ as the generating function (evaluated at $z := \mathrm{e}^c \in \mathbb{C}^n$) of the indicator function $x \mapsto I_{\Omega(y)\cap\mathbb{N}^n}(x)$, for the set $\Omega(y) \cap \mathbb{N}^n$; and provides a decomposition into a sum of simpler generating functions associated with supporting cones (those decomposed into unimodular cones as well). De Loera et al [10] have implemented Barvinok's counting algorithm, in the software `LattE`, which runs in time polynomial in the problem size when the dimension is fixed. Let us also mention the software developed by Verdoolaege [16], which extends the

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`LattE` software to handle *parametric polytopes*. In a dual approach, Brion and Vergne [8] consider the generating function $F : \mathbb{C}^m \to \mathbb{C}$ of f, that is,

$$z \mapsto F(z) := \sum_{y \in \mathbb{Z}^m} f(y) z^y, \tag{1.3}$$

for which they provide a generalized residue formula to next obtain $f(y)$ in closed form. As a result of both approaches, $f(y)$ is finally expressed as a weighted sum over the vertices of $\Omega(y)$. Similarly, Beck [5], and Beck, Diaz and Robins [6] have provided a complete analysis based on residue techniques for the case of a tetrahedron ($m = 1$); and mentioned the possibility of evaluating $f(b)$ for general polytopes by means of residues as well. Despite of its theoretical interest, Brion and Vergne's formula may not be directly *tractable* because it contains many products with complex coefficients (roots of unity) which makes the formula difficult to evaluate numerically. However, in some cases, this formula can be exploited to yield an efficient algorithm as e.g. in Baldoni-Silva, De Loera and Vergne [2] for flow polytopes; in Beck and Pixton [7] for transportation polytopes; and more generally when the matrix A is totally unimodular as in the work of Cochet [9]. Finally, in Lasserre and Zeron [12], we have provided two algorithms based on Cauchy residue techniques to invert the generating function F in (1.3), and an alternative algebraic technique based on partial fraction expansions of the generating function (using the Hilbert NullstellenSatz). A nice feature of the latter technique is to avoid computing residues.

Contribution. The goal of this paper, as a sequel to [12], is to provide a recursive algorithm to compute $f(y)$ in the spirit of the algebraic technique briefly outlined in [12, §7]; but now in a more constructive and explicit way. Like in Brion and Vergne, we use the generating function F in (1.3), and we provide a decomposition into simpler rational fractions whose *inversion* is easily obtained. To avoid handling complex roots of unity, we do not use residues *explicitly*, but we build up the required decomposition in a recursive manner. Properly speaking, we inductively calculate constants $Q_{\sigma,\beta}$ and a fixed positive integer M, all of them completely independent of y (and of the magnitude $\|c\|$ as well), such that the *counting* function f is given by the finite sum :

$$f(y) \;=\sum_{A_\sigma} \sum_{\beta\in\mathbb{Z}^m,\,\|\beta\|\le M} Q_{\sigma,\beta} \times \left\{ \begin{array}{cl} \mathrm{e}^{c'_\sigma x} & \text{if } x := A_\sigma^{-1}[y-\beta] \in \mathbb{N}^m, \\ 0 & \text{otherwise;} \end{array} \right.$$

where the first finite sum is computed over all invertible $[m \times m]$-square sub-matrices A_σ of A. This formula is presented in Theorem 2.6, after the necessary notation is introduced in Section §2.

Crucial in our algorithm is an explicit decomposition in a *closed form* (and so, an explicit formula for $f(y)$) for the case $n = m+1$; that we next repeatedly use for the general case $n > m+1$. Our closed form expression for the case $n = m+1$ is immediately computable and *tractable*, for it does *not* contain complex coefficients such as the roots of unity in Brion and Vergne's formula.

The computational complexity is $\mathrm{O}[(m+1)^{n-m}\Lambda]$, where the coefficient Λ depends only on the entries of A and the vector c, but *not* on the magnitude of y (cf. (4.12)). Actually, Λ depends on the ratio between the entries of the vector c, but *not* on the magnitude of $\|c\|$. However, Λ is exponential in the input size of A.

Thus, the formulas presented in section §3 give us a simple procedure for calculating $f(y)$ in the case $n = m+1$. Moreover, the recursive algorithm presented in section §4 is particularly attractive for calculating $f(y)$ in all cases where $n-m$ is relatively small, *no matter* the magnitude of y. However, this algorithm becomes less efficient whenever $n = m+k$, with relatively large values of k.

Analyzing the algorithm presented in §4 against the algorithm (via integration) that we introduced in [12], we can conclude that they are complementary in the sense that the algorithm presented in [12] is attractive when m is small, whereas the algorithm presented in this paper is attractive when $n-m$ is small, no matter how large m and n could be.

The paper is organized as follows: In §2 we present our main result, an exact expression of $f(y)$ provided that its generating function $F(z)$ is decomposed into a sum of some rational fractions. In §3 we obtain this explicit decomposition for the case $n = m+1$, as well as the corresponding expression for $f(y)$. In §4 we present a recursive algorithm whose output is the required decomposition for the general case $n \ge m+1$.

2. Main result

2.1. **Notation and definitions.** The notation $\mathbb{R}$, $\mathbb{Q}$ and $\mathbb{Z}$ stand for the usual sets of real, rational and integer numbers respectively; moreover, the set of natural numbers $\{0, 1, 2, \ldots\}$ is denoted by $\mathbb{Z}_+$ or $\mathbb{N}$. The notation c' and A' stand for the respective transposes of the vector $c \in \mathbb{R}^n$ and the matrix $A \in \mathbb{Z}^{m \times n}$. Moreover, the k-th column of the matrix $A \in \mathbb{Z}^{m \times n}$ is denoted by

$$A_k := (A_{1,k}, \ldots, A_{m,k})'.$$

When $y = 0$, $\Omega(0)$ in (1.1) is a convex cone with *dual* cone

$$\Omega(0)^* := \{b \in \mathbb{R}^n \,|\, b'x \geq 0 \quad \text{for every} \quad x \in \Omega(0)\}. \tag{2.1}$$

We may now define the following open set

$$\Gamma := \{c \in \mathbb{R}^n \,|\, -c > b \quad \text{for some} \quad b \in \Omega(0)^*\}. \tag{2.2}$$

Notice that Γ and $\Omega(0)^*$ are both equal to $\mathbb{R}^n$ whenever $\Omega(0)$ is the singleton $\{0\}$, which is the case if $\Omega(y)$ is compact.

On the other hand, we will suppose from now on that the matrix $A \in \mathbb{Z}^{m \times n}$ has maximal rank (see the comment before §2.2).

Definition 2.1. Let $p \in \mathbb{N}$ satisfy $m \leq p \leq n$, and let $\eta = \{\eta_1, \eta_2, \ldots \eta_p\} \subset \mathbb{N}$ be an ordered set with cardinality $|\eta| = p$ and $1 \leq \eta_1 < \eta_2 < \ldots < \eta_p \leq n$. Then

(i) η is said to be a *basis* of order p if the $[m \times p]$ sub-matrix

$$A_\eta := \left[A_{\eta_1} | A_{\eta_2} | \cdots | A_{\eta_p}\right]$$

has maximal rank, that is, $\operatorname{rank}(A_\eta) = m$.

(ii) For $m \leq p \leq n$, let

$$\mathbb{J}_p := \{\eta \subset \{1, \ldots, n\} \,| \quad \eta \text{ is a basis of order } p\} \tag{2.3}$$

be the set of bases of order p.

Notice that $\mathbb{J}_n = \{\{1, 2, \ldots n\}\}$ because A has maximal rank.

Lemma 2.2. *Let η be any subset of $\{1, 2, \ldots, n\}$.*

(i) *If $|\eta| = m$ then $\eta \in \mathbb{J}_m$ if and only if A_η is invertible.*

(ii) *If $|\eta| = q$ with $m < q \leq n$, then $\eta \in \mathbb{J}_q$ if and only if there exists a basis $\sigma \in \mathbb{J}_m$ such that $\sigma \subset \eta$.*

Proof. (i) is immediate because A_η is a square matrix; and A_η is invertible if and only if A_η has maximal rank.

Next, (ii) also follows from the fact that A_η has maximal rank if and only if A_η contains a square invertible sub-matrix. $\square$

Lemma 2.2 automatically implies $\mathbb{J}_m \neq \emptyset$ because the matrix A must contain at least one square invertible sub-matrix (we are supposing that A has maximal rank). Besides, $\mathbb{J}_p \neq \emptyset$ for $m < p \leq n$, because $\mathbb{J}_m \neq \emptyset$.

Finally, given a basis $\eta \in \mathbb{J}_p$ for $m \leq p \leq n$, and three vectors $z \in \mathbb{C}^m$, $c \in \mathbb{R}^n$ and $w \in \mathbb{Z}^m$, we introduce the following notation

$$
\begin{array}{rcl}
z^w & := & z_1^{w_1} z_2^{w_2} \cdots z_m^{w_m}, \\
c_\eta & := & (c_{\eta_1}, c_{\eta_2}, \ldots c_{\eta_p})', \\
\|w\| & := & \max\{|w_1|, |w_2|, \ldots |w_m|\}.
\end{array} \tag{2.4}
$$

Definition 2.3. The vector $c \in \mathbb{R}^n$ is said to be *regular* if for every basis $\sigma \in \mathbb{J}_{m+1}$, there exist a non-zero vector $v(\sigma) \in \mathbb{Z}^{m+1}$ such that :

$$
A_\sigma v(\sigma) = 0 \quad \text{and} \quad c'_\sigma v(\sigma) \neq 0. \tag{2.5}
$$

Notice that $c \neq 0$ whenever c is regular. Moreover, there are infinitely many vectors $v \in \mathbb{Z}^{m+1}$ such that $A_\sigma v = 0$, because $\operatorname{rank}(A_\sigma) = m < n$. Thus, the vector $c \in \mathbb{R}^n$ is regular if and only if

$$
c_j - c'_\pi A_\pi^{-1} A_j \neq 0, \quad \forall \pi \in \mathbb{J}_m, \quad \forall j \notin \pi;
$$

which is the regularity condition used in Brion and Vergne [8], except we do not require $c_j \neq 0$ for all $j = 1, \ldots, n$.

As already mentioned, we will suppose that the matrix $A \in \mathbb{Z}^{m \times n}$ in (1.1)–(1.2) has maximal rank. That is, the m rows of A, $v(j) = (A_{j,1}, \ldots, A_{j,n})$, $j = 1, \ldots, m$, are linearly independent. For suppose that A has not maximal rank. Then we can find $0 \neq \beta \in \mathbb{Z}^m$ such that $0 = \beta_1 v(1) + \cdots + \beta_m v(m)$ and $\beta \neq 0$. Assume that $\beta_1 \neq 0$. The equation $y = Ax$ has a solution $x \in \mathbb{N}^n$ if and only if x is a solution of the system of equations

$$
\begin{array}{rcl}
y_j & = & v(j)x \quad \text{for} \quad 2 \leq j \leq m, \quad \text{and} \\
y_1 & = & v(1)x \; = \; -\displaystyle\sum_{j=2}^m \beta_j v(j)x/\beta_1 \; = \; \displaystyle\sum_{j=2}^m y_j \beta_j/\beta_1.
\end{array}
$$

So, if $y_1 \neq \sum_{j=2}^m y_j\beta_j/\beta_1$ then $f(y) = 0$; otherwise we can eliminate the equation $y_1 = v(1)x$ from $y = Ax$ (because it does not depend on the free variable x) and use instead the trivial relationship $\beta_1 y(1)+\cdots+\beta_m y(m) = 0$.

2.2. **Generating function.** An appropriate tool for computing the exact value of $f(y)$ is the *generating function* $F : \mathbb{C}^m \to \mathbb{C}$,

$$z \mapsto F(z) := \sum_{y\in\mathbb{Z}^m} f(y) z^y, \tag{2.6}$$

with z^y defined in (2.4). This generating function was already considered in Brion and Vergne [8], with $\lambda := (\ln z_1, \ldots \ln z_m)$.

Proposition 2.4. *Let f and $\mathcal{F}$ be like in (1.2) and (2.6) respectively, and let $c \in \Gamma$. Then :*

$$F(z) = \prod_{k=1}^n \frac{1}{(1 - e^{c_k} z_1^{A_{1,k}} z_2^{A_{2,k}} \cdots z_m^{A_{m,k}})}, \tag{2.7}$$

on the domain

$$\begin{array}{l} (\,|z_1|, \ldots |z_m|\,) \in D, \quad \textit{with} \\ D := \{\rho \in \mathbb{R}^m \mid \rho > 0; \quad e^{c_k} \rho^{A_k} < 1,\ k = 1, \ldots n\}. \end{array} \tag{2.8}$$

Proof. Apply the definition (2.6) of F to obtain :

$$F(z) = \sum_{y\in\mathbb{Z}^m} z^y \left[\sum_{x\in\mathbb{N}^n,\, Ax=y} \mathrm{e}^{c'x} \right] = \sum_{x\in\mathbb{N}^n} \mathrm{e}^{c'x}\, z^{Ax}.$$

On the other hand,

$$\mathrm{e}^{c'x}\, z^{Ax} = \prod_{k=1}^n \left(\mathrm{e}^{c_k} z_1^{A_{1,k}} \cdots z_m^{A_{m,k}} \right)^{x_k}.$$

The domain D in (2.8) is not empty because $c \in \Gamma$. Indeed, a variant of Farkas' Lemma (see Corollary 7.1e in Schrijver [13, p. 89]) states that the system $A'u \leq b$ has a solution if and only if $b'x \geq 0$ for every vector $x \geq 0$ with $Ax = 0$. Whence, the system $A'u \leq b$ will have a solution whenever b is in the dual cone $\Omega(0)^*$. Moreover, recalling the definition (2.2) of Γ, we can deduce that $A'u < -c$ has indeed a solution $\breve{u} \in \mathbb{R}^m$ because $c \in \Gamma$. Thus, we also have that $(\mathrm{e}^{\breve{u}_1}, \mathrm{e}^{\breve{u}_2}, \ldots \mathrm{e}^{\breve{u}_m})^{A_k} < \mathrm{e}^{-c_k}$ for every $1 \leq k \leq n$, and so $\rho := (\mathrm{e}^{\breve{u}_1}, \ldots \mathrm{e}^{\breve{u}_m}) \in D$.

Thus, the condition $\left| e^{c_k} z_1^{A_{1,k}} \dots z_m^{A_{m,k}} \right| < 1$ holds whenever $1 \le k \le n$ and $(|z_1|, \dots |z_m|) \in D$, so

$$\begin{aligned} F(z) &= \prod_{k=1}^{n} \sum_{x_k=0}^{\infty} \left(e^{c_k} z_1^{A_{1,k}} \cdots z_m^{A_{m,k}} \right)^{x_k} \\ &= \prod_{k=1}^{n} \frac{1}{(1 - e^{c_k} z_1^{A_{1,k}} \cdots z_m^{A_{m,k}})}. \end{aligned}$$

□

2.3. **Inverting the generating function.** We will compute the exact value of $f(y)$ by first determining an appropriate expansion of the generating function in the form

$$F(z) = \sum_{\sigma \in \mathbb{J}_m} \frac{Q_\sigma(z)}{\prod_{k \in \sigma} (1 - e^{c_k} z^{A_k})}, \tag{2.9}$$

where the coefficients $Q_\sigma : \mathbb{C}^m \to \mathbb{C}$ are *rational* functions with a finite Laurent series

$$z \mapsto Q_\sigma(z) = \sum_{\beta \in \mathbb{Z}^m, \|\beta\| \le M} Q_{\sigma,\beta} z^\beta. \tag{2.10}$$

In (2.10), the strictly positive integer M is fixed and each $Q_{\sigma,\beta}$ is a real number which can be computed with algebraic operations when the numbers $\{e^{c_j}\}_{j=1}^n$ are given. An important observation is that the integer M does *not* depend on the right-hand-side y. It only depends on A and c, but not on the magnitude of c.

Remark 2.5. The decomposition (2.9) is *not* unique (at all) and there are several ways to obtain such a decomposition. For instance, Brion and Vergne [8, §2.3, p. 815] provide an explicit decomposition of $F(z)$ into elementary rational fractions of the form

$$F(z) = \sum_{\sigma \in \mathbb{J}_m} \sum_{g \in G(\sigma)} \frac{1}{\prod_{j \in \sigma} \left(1 - \gamma_j(g)(e^{c_j} z^{A_j})^{1/q}\right)} \frac{1}{\prod_{k \notin \sigma} \delta_k(g)}, \tag{2.11}$$

where $G(\sigma)$ is a certain set of cardinality q, and the coefficients $\{\gamma_j(g), \delta_k(g)\}$ involve certain roots of unity. The fact that c is *regular* ensures that (2.11) is well-defined. Thus, in principle, we could obtain (2.9) from (2.11), but this would require a highly nontrivial analysis and manipulation of the coefficients $\{\gamma_j(g), \delta_k(g)\}$. In the sequel, we provide an alternative algebraic approach that avoids manipulating these complex coefficients.

If F satisfies (2.9) then we get the following result.

Theorem 2.6. *Let $A \in \mathbb{Z}^{m\times n}$ be of maximal rank, f be as in (1.2) with $c \in \Gamma$, and assume that the generating function F in (2.6) satisfies (2.9)–(2.10). Then :*

$$f(y) \;=\; \sum_{\sigma\in\mathbb{J}_m} \quad \sum_{\beta\in\mathbb{Z}^m,\, \|\beta\|\leq M} Q_{\sigma,\beta}\, E_\sigma(y-\beta) \tag{2.12}$$

with

$$E_\sigma(y-\beta) \;=\; \left\{ \begin{array}{ll} \mathrm{e}^{c'_\sigma x} & \textit{if} \quad x := A_\sigma^{-1}[y-\beta] \in \mathbb{N}^m, \\ 0 & \textit{otherwise;} \end{array} \right. \tag{2.13}$$

where $c_\sigma \in \mathbb{R}^m$ was defined in (2.4).

Proof. Recall that $z^{A_k} = z_1^{A_{1,k}} \cdots z_m^{A_{m,k}}$, according to (2.4). On the other hand, in view of (2.8), the inequality $|\mathrm{e}^{c_k}\, z^{A_k}| < 1$ holds for every $1 \leq k \leq n$; and so the following expansion holds as well for each $\sigma \in \mathbb{J}_m$:

$$\prod_{k\in\sigma} \frac{1}{1-\mathrm{e}^{c_k}\, z^{A_k}} \;=\; \prod_{k\in\sigma} \left[\sum_{x_k\in\mathbb{N}} \mathrm{e}^{c_k x_k}\, z^{A_k x_k} \right] \;=\; \sum_{x\in\mathbb{N}^m} \mathrm{e}^{c'_\sigma x}\, z^{A_\sigma x}.$$

Next, suppose that a decomposition (2.9)–(2.10) exists. Then the following relationship is easy to establish.

$$\begin{aligned} F(z) \;&=\; \sum_{\sigma\in\mathbb{J}_m} \quad \sum_{x\in\mathbb{N}^m} Q_\sigma(z)\, \mathrm{e}^{c'_\sigma x}\, z^{A_\sigma x} \\ &=\; \sum_{\sigma\in\mathbb{J}_m} \quad \sum_{\beta\in\mathbb{Z}^m,\, \|\beta\|\leq M} \quad \sum_{x\in\mathbb{N}^m} Q_{\sigma,\beta}\, \mathrm{e}^{c'_\sigma x}\, z^{\beta+A_\sigma x}. \end{aligned} \tag{2.14}$$

Notice that both equations in (2.6) and (2.14) are equal. Hence, if we want to obtain the exact value of $f(y)$ from (2.14), we only have to sum up all terms with exponent $\beta + A_\sigma x$ equal to y. That is, recalling that A_σ is invertible for every $\sigma \in \mathbb{J}_m$ (see Lemma 2.2),

$$f(y) \;=\; \sum_{\sigma\in\mathbb{J}_m} \quad \sum_{\beta\in\mathbb{Z}^m,\, \|\beta\|\leq M} Q_{\sigma,\beta} \times \left\{ \begin{array}{ll} \mathrm{e}^{c'_\sigma x} & \text{if } x := A_\sigma^{-1}[y-\beta] \in \mathbb{N}^m; \\ 0 & \text{otherwise;} \end{array} \right.$$

which is exactly(2.12). □

Remark 2.7. Observe that function $f(y)$ in Theorem 2.6 can be rewritten as a weighted sum of $\mathrm{e}^{c'x}$ at some integral points $x \in \mathbb{N}^n$, namely

$$f(y) \;=\; \sum_{\sigma\in\mathbb{J}_m} \left(\sum_{\beta} Q_{\sigma,\beta}\, \mathrm{e}^{c'\breve{x}(\sigma,\beta)} \right), \tag{2.15}$$

where the second finite sum is calculated over all $\beta \in \mathbb{Z}^m$ such that $\|\beta\| \leq M$ and $A_\sigma^{-1}[y-\beta] \in \mathbb{N}^m$. Moreover, each vector $\breve{x}(\sigma,\beta) \in \mathbb{N}^n$ is an *integral* point. Indeed, given $x := A_\sigma^{-1}(y-\beta)$ inside $\mathbb{N}^m$ like in (2.13), we define the integral vector $\breve{x}(\sigma,\beta) \in \mathbb{N}^n$ by setting the entries:

$$[\breve{x}(\sigma,\beta)]_j = \begin{cases} x_k & \text{if } j = \sigma_k \text{ for some } 1 \leq k \leq m, \\ 0 & \text{if } j \notin \sigma; \end{cases}$$

for $j = 1, \ldots n$. Clearly, we have that $\mathrm{e}^{c'_\sigma x} = \mathrm{e}^{c'\breve{x}(\sigma,\beta)}$ from which equation (2.15) follows. In addition, these integral points $\breve{x}(\sigma,\beta) \in \mathbb{N}^n$ have at most m nontrivial coordinates and their convex hull defines an integral polyhedron (that is, a polyhedron with integral vertices).

In view of Theorem 2.6, $f(y)$ is easily obtained once the rational functions $Q_\sigma(z)$ in the decomposition (2.9) are available. As already pointed out, the decomposition (2.9)–(2.10) is not unique and the purpose of the next section (§3) is to provide :

- a simple decomposition (2.9) for which the expression of the coefficients Q_σ are easily calculated in the case $n = m+1$;

whereas in §4 we present :

- a recursive algorithm to provide the Q_σ in the general case $n > m+1$.

3. The case $n = m + 1$

In this section we completely solve the case $n = m+1$, that is, we provide an explicit expression of $f(y)$. We first need some essential intermediate algebraic calculations, in order to deduce the decomposition (2.9)–(2.10) of $F(z)$ when $n = m+1$.

3.1. **Some auxiliary rational functions.** Let sgn : $\mathbb{R} \to \mathbb{Z}$ be the *sign* function defined by

$$t \mapsto \operatorname{sgn}(t) := \begin{cases} 1 & \text{if } \; t > 0, \\ -1 & \text{if } \; t < 0, \\ 0 & \text{otherwise.} \end{cases}$$

Besides, adopt the convention that any sum with negative superindex :

$$\sum_{r=0}^{-1} (\cdots) \quad \text{is identically equal to zero.}$$

Now, given a fixed integer $n > 0$, for every $k = 1, \ldots n$, we are going to construct auxiliary functions $P_k : \mathbb{Z}^n \times \mathbb{C}^n \to \mathbb{C}$, such that each $w \mapsto P_k(v,w)$

is a rational function of the variable $w \in \mathbb{C}^n$. Given a vector $v \in \mathbb{Z}^n$, we define :

$$
\begin{aligned}
(3.1) \qquad\qquad P_1(v,w) &:= \sum_{r=0}^{|v_1|-1} w_1^{\operatorname{sgn}(v_1)\,r}. \\
P_2(v,w) &:= (w_1^{v_1}) \sum_{r=0}^{|v_2|-1} w_2^{\operatorname{sgn}(v_2)\,r}, \\
P_3(v,w) &:= (w_1^{v_1} w_2^{v_2}) \sum_{r=0}^{|v_3|-1} w_3^{\operatorname{sgn}(v_3)\,r}, \\
\vdots &:= \qquad \vdots \\
P_n(v,w) &:= \left(\prod_{j=1}^{n-1} w_j^{v_j} \right) \sum_{r=0}^{|v_n|-1} w_n^{\operatorname{sgn}(v_n)\,r}.
\end{aligned}
$$

Obviously, we have that $P_k(v,w) = 0$ whenever $v_k = 0$. Moreover, we claim that :

Lemma 3.1. *Let $v \in \mathbb{Z}^n$ and $w \in \mathbb{C}^n$. The functions P_k defined in (3.1) satisfy*

$$
(3.2) \qquad \sum_{k=1}^{n} \left(1 - w_k^{\operatorname{sgn}(v_k)}\right) P_k(v,w) \;=\; 1 \;-\; w^v.
$$

Proof. First, notice that

$$
\left(1 - w_1^{\operatorname{sgn}(v_1)}\right) P_1(v,w) \;=\; \left(1 - w_1^{\operatorname{sgn}(v_1)}\right) \sum_{r=0}^{|v_1|-1} w_1^{\operatorname{sgn}(v_1)\,r} \;=\; 1 - w_1^{v_1}.
$$

Previous equalities are obvious when $v_1 = 0$. We have similar formulas for $2 \leq k \leq n$,

$$
\left(1 - w_k^{\operatorname{sgn}(v_k)}\right) P_k(v,w) \;=\; (1 - w_k^{v_k}) \prod_{j=1}^{k-1} w_j^{v_j} \;=\; \prod_{j=1}^{k-1} w_j^{v_j} \;-\; \prod_{j=1}^{k} w_j^{v_j}.
$$

Therefore, adding together all the terms in equation (3.2) yields

$$
\sum_{k=1}^{n} \left(1 - w_k^{\operatorname{sgn}(v_k)}\right) P_k(v,w) \;=\; 1 \;-\; \prod_{j=1}^{n} w_j^{v_j}.
$$

$\square$

3.2. **Solving the case** $n = m+1$. We now use the algebraic expansions of §3.1 to calculate the function $f(y)$ in (1.2) where $\Omega(y)$ is given in (1.1) and $A \in \mathbb{Z}^{m\times(m+1)}$ is a maximal rank matrix.

Theorem 3.2. *Let $n = m+1$ be fixed, $A \in \mathbb{Z}^{m\times n}$ a maximal rank matrix and let $c \in \Gamma$ be regular. Let $v \in \mathbb{Z}^n$ be a non-zero vector such that $Av = 0$ and $c'v \neq 0$ (cf. Definition 2.3). Define the vector*

$$w := (e^{c_1} z^{A_1}, e^{c_2} z^{A_2}, \dots e^{c_n} z^{A_n}). \tag{3.3}$$

Then :

(i) *The generating function $F(z)$ in (2.6) has the expansion*

$$F(z) = \sum_{k=1}^{n} \frac{Q_k(z)}{\prod_{j\neq k}(1 - e^{c_j} z^{A_j})} = \sum_{\sigma\in\mathbb{J}_m} \frac{Q_\sigma(z)}{\prod_{j\in\sigma}(1 - e^{c_j} z^{A_j})}, \tag{3.4}$$

where the rational functions $z \mapsto Q_k(z)$ are defined by :

$$Q_k(z) := \begin{cases} P_k(v,w)/(1 - e^{c'v}) & \text{if } v_k > 0, \\ -w_k^{-1} P_k(v,w)/(1 - e^{c'v}) & \text{if } v_k < 0, \\ 0 & \text{otherwise;} \end{cases} \tag{3.5}$$

for $1 \leq k \leq n$. Each function P_k in (3.5) is defined as in (3.1). Notice that the first sum in equation (3.4) is done only over the indexes k for which $v_k \neq 0$, because $Q_k(z) = 0$ whenever $v_k = 0$.

(ii) *Given $y \in \mathbb{Z}^m$, the function $f(y)$ in (1.2) is directly obtained by applying Theorem 2.6.*

Proof. (i) Since c is regular, let $v \in \mathbb{Z}^n$ be a vector such that $Av = 0$ and $c'v \neq 0$, see (2.5) in Definition 2.3. Let $w \in \mathbb{C}^n$ be the vectors defined in (3.3). We can easily deduce that

$$w^v = \prod_{j=1}^{n} \left(\mathrm{e}^{c_j} z^{A_j}\right)^{v_j} = \mathrm{e}^{c'v} z^{Av} = \mathrm{e}^{c'v} \neq 1. \tag{3.6}$$

Next, let $z \mapsto Q_k(z)$ be the rational function defined in (3.5). Then, from Lemma 3.1,

$$\begin{aligned} \sum_{k=1}^{n} \left(1 - \mathrm{e}^{c_k} z^{A_k}\right) Q_k(z) &= \sum_{k=1}^{n} \left(1 - w_k^{\operatorname{sgn}(v_k)}\right) \frac{P_k(v,w)}{1 - \mathrm{e}^{c'v}} \\ &= \frac{1 - w^v}{1 - \mathrm{e}^{c'v}} = 1. \end{aligned} \tag{3.7}$$

Multiplying together the generating function (2.7) with the left side of (3.7), yields the expansion

$$F(z) \;=\; \sum_{k=1}^{n} \frac{Q_k(z)}{\prod_{j\neq k}(1-\mathrm{e}^{c_j}\, z^{A_j})}; \tag{3.8}$$

which gives us the first equality in (3.4).

(ii) Since $c \in \Gamma$, the function $F(z)$ is indeed the generating function of $f(y)$. Next, consider the ordered sets

$$\sigma(k) \;=\; \{1, 2, \cdots, k-1, k+1, \cdots, n\} \quad \text{for} \quad k = 1, \ldots, n. \tag{3.9}$$

In order to apply Theorem 2.6, we only need to prove that each square sub-matrix $A_{\sigma(k)}$ is indeed invertible for every $k = 1, \ldots, n$ with $v_k \neq 0$. Recall that $Q_k(z) = 0$ whenever $v_k = 0$, and that $\sigma(k)$ is an element of $\mathbb{J}_m$ precisely when $A_{\sigma(k)}$ is invertible. We know that $A \in \mathbb{R}^{m\times n}$ has maximal rank, so A has m linearly independent columns. With no loss of generality, we may assume that the first m columns A_k are linearly independent, for $k = 1, \ldots, m$. Thus, the square matrix

$$A_{\sigma(n)} \;=\; [A_1|A_2|\cdots|A_m] \quad \text{is invertible,}$$

recall that $n = m + 1$; and so $\sigma(n) = \{1, \ldots, m\}$ defined in (3.9) is an element of $\mathbb{J}_m$. On the other hand, since $A_{\sigma(n)}$ is invertible and the vector $v \in \mathbb{Z}^n$ satisfies

$$0 \;=\; Av \;=\; A_{\sigma(n)}(v_1, v_2, \ldots v_m)' \;+\; A_n v_n$$

with $v \neq 0$, we automatically have that the n-entry $v_n \neq 0$; and so the n-column of A is equal to $A_n = \sum_{j=1}^{m} \frac{-v_j}{v_n} A_j$. Whence, for every $1 \leq k \leq m$ with $v_k \neq 0$, the square matrix

$$A_{\sigma(k)} \;=\; [A_1|\ldots|A_{k-1}|A_{k+1}|\ldots|A_m|A_n]$$

is clearly invertible because the column A_k of $A_{\sigma(n)}$ has been substituted with the linear combination $A_n = \sum_{j=1}^{m} \frac{-v_j}{v_n} A_j$ whose coefficient $-v_k/v_n$ is different from zero. That is, the set $\sigma(k)$ in (3.9) is an element of $\mathbb{J}_m$ for every $1 \leq k \leq m$ with $v_k \neq 0$.

Therefore, the expansion (3.8) can be re-written

$$F(z) \;=\; \sum_{v_k\neq 0} \frac{Q_k(z)}{\prod_{j\in\sigma(k)}(1-\mathrm{e}^{c_j}\, z^{A_j})} \;=\; \sum_{\sigma\in\mathbb{J}_m} \frac{Q_\sigma(z)}{\prod_{j\in\sigma}(1-\mathrm{e}^{c_j}\, z^{A_j})},$$

with

$$Q_\sigma(z) \;\equiv\; \left\{ \begin{array}{cl} Q_k(z) & \text{if } \sigma = \sigma(k) \text{ for some index } k \text{ with } v_k \neq 0, \\ 0 & \text{otherwise.} \end{array} \right.$$

And so, a closed form of $f(y)$ is obtained by applying Theorem 2.6. $\square$

Remark 3.3. In the case where $n = m+1$ and $\Omega(y)$ is compact, a naive way to evaluate $f(y)$ is as follows. Suppose that $B := [A_1|\ldots|A_m]$ is invertible. One may then calculate $\rho := \max\{x_{m+1} \,|\, Ax = y,\, x \geq 0\}$. Thus, the evaluation of $f(y)$ reduces to summing up $\sum_x \mathrm{e}^{c'x}$ over all vectors $x = (\hat{x}, x_{m+1}) \in \mathbb{N}^{m+1}$ such that $x_{m+1} \in [0,\rho] \cap \mathbb{N}$ and $\hat{x} := B^{-1}[y - A_{m+1}x_{m+1}]$. This may work very well for reasonable values of ρ; but clearly, ρ depends on the magnitude of y. On the other hand, the computational complexity of the algorithm presented in §3 does *not* depend on y. Indeed, the bound M in (2.12) of Theorem 2.6, does not depend at all on y. Moreover, the algorithm also applies to the case where $\Omega(y)$ is not compact.

To illustrate the difference, consider the following trivial example where $n = 2$, $m = 1$, $A = [1,1]$ and $c = [0,a]$ with $a \neq 0$. The generating function $F(z)$ in (2.6) and (2.7) is the rational function

$$F(z) \;=\; \frac{1}{(1-z)(1-\mathrm{e}^a z)}.$$

Setting $v = (-1,1)$ and $w = (z, \mathrm{e}^a z)$, one obtains

$$\begin{aligned} 1 \;&=\; (1-z)\,Q_1(z) \;+\; (1-\mathrm{e}^a z)\,Q_2(z) \\ &=\; (1-z)\frac{-z^{-1}P_1(v,w)}{1-\mathrm{e}^a} \;+\; (1-\mathrm{e}^a z)\frac{P_2(v,w)}{1-\mathrm{e}^a} \\ &=\; (1-z)\frac{-z^{-1}}{1-\mathrm{e}^a} \;+\; (1-\mathrm{e}^a z)\frac{z^{-1}}{1-\mathrm{e}^a}, \end{aligned}$$

an illustration of the Hilbert Nullstellensatz applied to the two polynomials $z \mapsto (1-z)$ and $z \mapsto (1-\mathrm{e}^a z)$, which have no common zero in $\mathbb{C}$.

And so, the generating function $F(z)$ gets expanded to

$$F(z) \;=\; \frac{-z^{-1}}{(1-\mathrm{e}^a)(1-\mathrm{e}^a z)} \;+\; \frac{z^{-1}}{(1-\mathrm{e}^a)(1-z)}. \tag{3.10}$$

Finally, using Theorem 2.6, we obtain $f(y)$ in closed form by

$$f(y) \;=\; \frac{-\mathrm{e}^{a(y+1)}}{1-\mathrm{e}^a} + \frac{\mathrm{e}^{0(y+1)}}{1-\mathrm{e}^a} \;==\; \frac{1-\mathrm{e}^{(y+1)a}}{1-\mathrm{e}^a}. \tag{3.11}$$

Looking back at (2.10) we may see that $M = 1$ (and obviously does not depend on y) and so the evaluation of $f(y)$ via (2.12) in Theorem 2.6 (as described in (3.11) is done in 2 elementary steps, no matter the magnitude of y. On the other hand, the naive procedure would require y elementary steps.

Remark 3.4. We have already mentioned that the expansion of the generating function $F(z)$ is not unique. In the trivial example of Remark 3.3, we may also expand $F(z)$ as the following sum of linear fractions

$$F(z) = \frac{\mathrm{e}^a}{(\mathrm{e}^a - 1)(1 - \mathrm{e}^a z)} - \frac{1}{(\mathrm{e}^a - 1)(1 - z)},$$

which is not the same as the expansion in (3.10). However, applying Theorem 2.6 again yields the same formula (3.11) for $f(y)$.

3.3. **An algorithm for the case $n = m + 1$.** Let $\mathbb{R}[z, z^{-1}]$ be the set of polynomials with real coefficients and entries in z_t and z_t^{-1}, for $1 \leq t \leq m$; so negative exponents are allowed. That is, a rational function Q is an element of $\mathbb{R}[z, z^{-1}]$ if and only if Q has a *finite* Laurent series, like in 2.10. Considering Theorems 2.6 and 3.2, the algorithm for the case $n = m+1$ can be written as follows :

Procedure `Solve`(A, c, y).
Input: $A \in \mathbb{Z}^{m \times n}$ full rank and $n = m + 1$; $c \in \mathbb{R}^n$ regular; $y \in \mathbb{Z}^m$.
Output: $\{Q_k\}_{k=1}^n \subset \mathbb{R}[z, z^{-1}]$ in (3.4) and $f(y)$ in (2.12).

- Step 0: Compute a vector $v \in \mathbb{Z}^n$ such that $Av = 0$ and $c'v \neq 0$.
- Step 1: Compute polynomials $\{w \mapsto P_k(v, w)\}$ from (3.1).
- Step 2: Compute polynomials $\{z \mapsto Q_k(z)\}$ from (3.5); and let $L_k := \{Q_{k,\beta} \neq 0\}_{\beta \in \mathbb{N}^m}$ be the list of nonzero coefficients of $Q_k(z)$, for all nonzero $Q_k(z)$. Set $f(y) := 0$. Set $k := 1$.
- Step 3: While $k \leq n$ do:
  - If $v_k = 0$ go to Label{skip}.
  - If $v_k \neq 0$, let $A_\sigma \in \mathbb{Z}^{m \times m}$ be the nonsingular matrix obtained from A by deleting k-column A_k; and let $c_\sigma \in \mathbb{R}^m$ be the vector obtained from c by deleting k-entry c_k.
  - While $L_k \neq \emptyset$ do: Pick $Q_{k,\beta} \in L_k$ and solve $A_\sigma x = y - \beta$. Set $L_k := L_k \setminus \{Q_{k,\beta}\}$. If $x \in \mathbb{N}^m$ then $f(y) = f(y) + \mathrm{e}^{c'_\sigma x} Q_{k,\beta}$.
  - Label{skip}. Set $k := k + 1$.

4. The general case $n > m+1$

We now consider the case $n > m+1$ and obtain the decomposition (2.9) that permits to compute $f(y)$ by invoking Theorem 2.6. The idea is to use the results of §3 recursively, and we exhibit a decomposition (2.9) in the general case $n > m+1$, by induction.

The following result is proved with the same arguments as in the proof of Theorem 3.2.

Proposition 4.1. *Let $A \in \mathbb{Z}^{m\times n}$ be a maximal rank matrix and $c \in \Gamma$ be regular. Suppose that the generating function F in (2.6)–(2.7) has the expansion*

$$F(z) \;=\; \sum_{\pi\in\mathbb{J}_p} \frac{Q_\pi(z)}{\prod_{j\in\pi}(1-e^{c_j}z^{A_j})}, \tag{4.1}$$

for some integer p with $m < p \le n$, and for some rational functions $z \mapsto Q_\pi(z)$, explicitly known and with a finite Laurent's series expansion (2.10).

Then, F also has the expansion

$$F(z) \;=\; \sum_{\breve{\pi}\in\mathbb{J}_{p-1}} \frac{Q^*_{\breve{\pi}}(z)}{\prod_{j\in\breve{\pi}}(1-e^{c_j}z^{A_j})}, \tag{4.2}$$

where the rational functions $z \mapsto Q^*_{\breve{\pi}}(z)$ are constructed explicitly, and have a finite Laurent's series expansion (2.10).

Proof. Let $\pi \in \mathbb{J}_p$ be any given basis with $m < p \le n$ and such that $Q_\pi(z) \not\equiv 0$ in (4.1). We are going to build up simple rational functions $z \mapsto R^\pi_\eta(z)$, where $\eta \in \mathbb{J}_{p-1}$, such that the expansion

$$\frac{1}{\prod_{j\in\pi}(1-\mathrm{e}^{c_j}z^{A_j})} \;=\; \sum_{\eta\in\mathbb{J}_{p-1}} \frac{R^\pi_\eta(z)}{\prod_{j\in\eta}(1-\mathrm{e}^{c_j}z^{A_j})} \quad \text{holds.} \tag{4.3}$$

Invoking Lemma 2.2, there exists a basis $\breve{\sigma} \in \mathbb{J}_m$ such that $\breve{\sigma} \subset \pi$. Pick any index $g \in \pi \setminus \breve{\sigma}$, so the basis $\sigma := \breve{\sigma} \cup \{g\}$ is indeed an element of $\mathbb{J}_{m+1}$, because of Lemma 2.2 again. Next, since c is regular, pick a vector $v \in \mathbb{Z}^{m+1}$ such that $A_\sigma v = 0$ and $c'_\sigma v \neq 0$, like in (2.5). The statements below follow from the same arguments as in the proof of Theorem 3.2(i), so we briefly outline the proof. Define the vector

$$w \;:=\; (\mathrm{e}^{c_1}z^{A_1}, \mathrm{e}^{c_2}z^{A_2}, \cdots, \mathrm{e}^{c_n}z^{A_n}) \in \mathbb{C}^n; \tag{4.4}$$

so that, with same notation as in (2.4), $w_\sigma \in \mathbb{C}^{m+1}$. Like in (3.6), we may deduce that $(w_\sigma)^v = \mathrm{e}^{c'_\sigma v} \neq 1$. Moreover, define the rational functions

$$(4.5) \qquad R^\pi_k(z) \,:=\, \begin{cases} P_k(v, w_\sigma)/(1-\mathrm{e}^{c'_\sigma v}) & \text{if } v_k > 0, \\ -[w_\sigma]_k^{-1} P_k(v, w_\sigma)/(1-\mathrm{e}^{c'_\sigma v}) & \text{if } v_k < 0, \\ 0 & \text{otherwise;} \end{cases}$$

where the functions P_k are defined as in (3.1), for $1 \leq k \leq m+1$. Thus,

$$(4.6) \qquad 1 \,=\, \sum_{k=1}^{m+1} (1-[w_\sigma]_k)\, R^\pi_k(z) \,=\, \sum_{k=1}^{m+1} \left(1-\mathrm{e}^{c_{\sigma_k}} z^{A_{\sigma_k}}\right) R^\pi_k(z),$$

like in (3.7). From (4.6) one easily obtains

$$(4.7) \qquad \frac{1}{\prod_{j\in\pi}(1-\mathrm{e}^{c_j} z^{A_j})} \,=\, \sum_{k=1}^{m+1} R^\pi_k(z) \left[\prod_{j\in\pi,\, j\neq\sigma_k} \frac{1}{1-\mathrm{e}^{c_j} z^{A_j}}\right].$$

Notice that the sum in (4.7) is done only over the integers k for which $v_k \neq 0$, because $R^\pi_k(z) = 0$ whenever $v_k = 0$. Next, we use the same arguments as in the proof of Theorem 3.2(ii). With no loss of generality, suppose that the ordered sets $\breve\sigma \subset \sigma \subset \pi$ are given by :

$$(4.8) \qquad \pi \,:=\, \{1, 2, \ldots, p\}, \quad \sigma \,:=\, \breve\sigma \cup \{p\} \quad \text{and} \quad p \notin \breve\sigma.$$

Notice that $\sigma_k \in \breve\sigma$, for $1 \leq k \leq m$, and $\sigma_{m+1} = p$. Besides, consider the ordered sets

$$(4.9) \qquad \eta(k) \,=\, \{j \in \pi \,|\quad j \neq \sigma_k\} \quad \text{for} \quad k = 1, \ldots, m+1.$$

We next show that each sub-matrix $A_{\eta(k)}$ has maximal rank for every $k = 1, \ldots, m+1$ with $v_k \neq 0$. Notice that $|\eta(k)| = p-1$ because $|\pi| = p$; whence, the set $\eta(k)$ is indeed an element of $\mathbb{J}_{p-1}$ precisely when $A_{\eta(k)}$ has maximal rank. Now, we have that $\breve\sigma \subset \eta(m+1)$, for the index $\sigma_{m+1} = p$ is contained in $\pi \backslash \breve\sigma$. Therefore, since $\breve\sigma \in \mathbb{J}_m$, Lemma 2.2 implies that $\eta(m+1)$ in (4.9) is an element of $\mathbb{J}_{p-1}$, the square sub-matrix $A_{\breve\sigma}$ is invertible, and $A_{\eta(m+1)}$ has maximal rank. On the other hand, the vector $v \in \mathbb{Z}^{m+1}$ satisfies

$$0 \,=\, A_\sigma v \,=\, A_{\breve\sigma}(v_1, v_2, \ldots v_m)' \,+\, A_p v_{m+1}$$

with $v \neq 0$, so we may conclude that the last entry $v_{m+1} \neq 0$. We can now express the p-column of A as the finite sum $A_p = \sum_{j=1}^m \frac{-v_j}{v_{m+1}} A_{\sigma_j}$. Whence, for every $1 \leq k \leq m$ with $v_k \neq 0$, the matrix

$$A_{\eta(k)} \,=\, [A_1|\cdots|A_{\sigma_k - 1}|A_{\sigma_k+1}|\cdots|A_p]$$

has maximal rank, because the column A_{σ_k} of $A_{\sigma(m+1)}$ has been substituted with the linear combination $A_p = \sum_{j=1}^m \frac{-v_j}{v_{m+1}} A_{\sigma_j}$ whose coefficient $-v_k/v_{m+1}$ is different from zero. Thus, the set $\eta(k)$ defined in (4.9) is an element of $\mathbb{J}_{p-1}$ for every $1 \leq k \leq m$ with $v_k \neq 0$. Therefore, (4.7) can be re-written

$$\begin{aligned} \frac{1}{\prod_{j\in\pi}(1-\mathrm{e}^{c_j} z^{A_j})} &= \sum_{v_k\neq 0} \frac{R_k^\pi(z)}{\prod_{j\in\eta(k)}(1-\mathrm{e}^{c_j}\, z^{A_j})} \\ &= \sum_{\eta\in\mathbb{J}_{p-1}} \frac{R_\eta^\pi(z)}{\prod_{j\in\eta}(1-\mathrm{e}^{c_j}\, z^{A_j})}, \end{aligned}$$

which is the desired identity (4.3) with

$$R_\eta^\pi(z) \equiv \left\{ \begin{array}{cl} R_k^\pi(z) & \text{if } \eta = \eta(k) \text{ for some index } k \text{ with } v_k \neq 0, \\ 0 & \text{otherwise.} \end{array} \right.$$

On the other hand, it is easy to see that all rational functions R_k^π and R_η^π have finite Laurent series (2.10), because each R_k^π is defined in terms of P_k in (4.5), and each rational function P_k in (3.1) also has a finite Laurent series. Finally, (4.2) follows easily. Compounding (4.1) and (4.3) together, yields

$$F(z) = \sum_{\eta\in\mathbb{J}_{p-1}} \sum_{\pi\in\mathbb{J}_p} \frac{R_\eta^\pi(z)\, Q_\pi(z)}{\prod_{k\in\eta}(1-\mathrm{e}^{c_k} z^{A_k})}, \tag{4.10}$$

so that the decomposition (4.2) holds by setting Q_η^* identically equal to the finite sum $\sum_{\pi\in\mathbb{J}_p} R_\eta^\pi Q_\pi$ for every $\eta \in \mathbb{J}_{p-1}$. □

Notice that the sum in (4.1) runs over the bases of order p, whereas the sum in (4.2) runs over the bases of order $p-1$. Hence, repeated applications of Proposition 4.1 yields a decomposition of the generating function F into a sum over the bases of order m, which is the decomposition described in (2.9)–(2.10). Namely,

Corollary 4.2. *Let $A \in \mathbb{Z}^{m\times n}$ be a maximal rank matrix, and let $c \in \Gamma$ be regular. Let f be as in (1.2) and F be its generating function (2.6)–(2.7). Then :*

(i) *$F(z)$ has the expansion*

$$F(z) = \sum_{\sigma\in\mathbb{J}_m} \frac{Q_\sigma(z)}{\prod_{k\in\sigma}(1-e^{c_k} z^{A_k})}, \tag{4.11}$$

for some rational functions $z \mapsto Q_\sigma(z)$ which can be built up explicitly, and with finite Laurent series (2.10).

(ii) *For every $y \in \mathbb{Z}^m$, the function $f(y)$ is obtained from Theorem 2.6.*

Proof. The point (i) is proved by induction. Notice that (2.7) can be rewritten

$$F(z) \;=\; \sum_{\pi \in \mathbb{J}_n} \frac{1}{\prod_{k\in\pi}(1-\mathrm{e}^{c_k} z^{A_k})},$$

because $\mathbb{J}_n = \{\{1,2,\dots n\}\}$ and A has maximal rank (see (2.3)). Thus, from Proposition 4.1, (4.2) holds for $p = n-1$ as well. And more generally, repeated applications of Proposition 4.1 show that (4.2) holds for all $m \leq p < n$. However, (4.11) is precisely (4.2) with $p = m$.

On the other hand, (ii) follows because as $c \in \Gamma$, $F(z)$ is the generating function of $f(y)$, and has the decomposition (4.11) required to apply Theorem 2.6. □

4.1. **An algorithm for the general case $n > m+1$.** Let $\mathbb{R}[z, z^{-1}]$ be the set of polynomials with real coefficients and entries in z_t and z_t^{-1}, so negative exponents are allowed. In view of Theorems 2.4 and 2.6, the recursive algorithm for the general case reads as follows:

Procedure `Solve2`(A, c, y).
Input: $A \in \mathbb{Z}^{m\times n}$ full rank and $n > m+1$; $c \in \mathbb{R}^n$ regular; $y \in \mathbb{Z}^m$;
Output: $f(y)$ as in (2.12).

Step 0, initialization :

- Calculate the set of bases $\mathbb{J}_m$, so that $\sigma \in \mathbb{J}_m$ if and only if the square matrix $A_\sigma = [A_{\sigma_1}|\cdots|A_{\sigma_m}]$ is invertible.
- Set $p := n$, and $z \mapsto Q_{\breve{\pi}}(z) \equiv 1$ with $\breve{\pi} := \{1, 2, \dots, n\}$. Thus, $Q_{\breve{\pi}}(z) \in \mathbb{R}[z, z^{-1}]$ and :

$$F(z) = \sum_{\pi\in\mathbb{J}_p} \frac{Q_\pi(z)}{\prod_{j\in\pi}(1-\mathrm{e}^{c_j} z^{A_j})} \quad \text{with} \quad \mathbb{J}_p = \{\{1,2,\dots,n\}\}.$$

Step 1: While $p \geq m+1$ do:

- For every $\pi \in \mathbb{J}_p$ with $Q_\pi(z) \not\equiv 0$ do:
  - Pick a basis $\breve{\sigma} \in \mathbb{J}_m$ such that $\breve{\sigma} \subset \pi$.
  - Pick a point $g \in \pi \setminus \breve{\sigma}$; and let $\sigma := \breve{\sigma} \cup \{g\} \in \mathbb{J}_{m+1}$.

– Let $A_\sigma := [A_j]_{j\in\sigma} \in \mathbb{Z}^{m\times(m+1)}$ and $c_\sigma := [c_j]_{j\in\sigma} \in \mathbb{R}^{m+1}$.
– Apply steps $0, 1, 2$ of `Solve`(A_σ, c_σ, y). That is :
– Compute $v \in \mathbb{Z}^{m+1}$ such that $A_\sigma v = 0$ and $c'_\sigma v \neq 0$.
– Compute polynomials $\{w_\sigma \mapsto P_k(v, w_\sigma)\}$ from (3.1).
– Compute polynomials $\{z \mapsto R^\pi_k(z)\}$ from (4.5); so that :

$$\frac{1}{\prod_{j\in\pi}(1-\mathrm{e}^{c_j}z^{A_j})} = \sum_{k=1}^{m+1} R^\pi_k(z)\left[\prod_{j\in\pi,\,j\neq\sigma_k}\frac{1}{1-\mathrm{e}^{c_j}z^{A_j}}\right].$$

– For every $\eta \in \mathbb{J}_{p-1}$ do: Set $R^\pi_\eta := 0$.
– For every $1 \leq k \leq m+1$ with $v_k \neq 0$ do: Set $\eta[k] := \pi \setminus \{\sigma_k\} \in \mathbb{J}_{p-1}$ and $R^\pi_{\eta[k]}(z) := R^\pi_k(z) \in \mathbb{R}[z, z^{-1}]$.
– We finally have the identity :

$$\frac{Q_\pi(z)}{\prod_{j\in\pi}(1-\mathrm{e}^{c_j}z^{A_j})} = \sum_{\eta\in\mathbb{J}_{p-1}}\frac{R^\pi_\eta(z)Q_\pi(z)}{\prod_{j\in\eta}(1-\mathrm{e}^{c_j}z^{A_j})}.$$

- For every $\eta \in \mathbb{J}_{p-1}$ do: Set $Q_\eta(z) := \sum_{\pi\in\mathbb{J}_p} R^\pi_\eta(z)Q_\pi(z)$.
- Hence, each $Q_\eta(z) \in \mathbb{R}[z, z^{-1}]$ and :

$$F(z) = \sum_{\eta\in\mathbb{J}_{p-1}}\frac{Q_\eta(z)}{\prod_{j\in\eta}(1-\mathrm{e}^{c_j}z^{A_j})}.$$

- Set $p := p - 1$.

Step 2: We have obtained the decomposition :

$$F(z) = \sum_{\pi\in\mathbb{J}_m}\frac{Q_\pi(z)}{\prod_{j\in\pi}(1-\mathrm{e}^{c_j}z^{A_j})}, \quad \text{where each} \quad Q_\pi(z) \in \mathbb{R}[z, z^{-1}].$$

Since $F(z)$ is now in the form (2.9) required to apply Theorem 2.6, we thus obtain $f(y)$ from (2.12) in Theorem 2.6.

4.2. **Computational complexity.** First, observe that the procedures `Solve` and `Solve2`, defined in §3.3 and §4.1, compute the coefficients of the polynomials $Q_\sigma(z)$ in the decomposition (2.12) of $F(z)$. This computation involves only algebraic operations, provided the matrix $A \in \mathbb{Z}^{m\times n}$ and the vector $c \in \mathbb{R}^n$ are given. Recall that a main step in these procedures is to calculate a vector $v \in \mathbb{Z}^{m+1}$ such that $A_\sigma v = 0$ and $c'_\sigma v \neq 0$. Thus, for practical implementation, one should directly consider working with a rational vector $c \in \mathbb{Q}^n$. Next, concerning the real numbers $\{\mathrm{e}^{c_k}\}_k$ used in the procedures `Solve` and `Solve2`, according to §3.3 and §4.1, one may easily see that the entries e^{c_k} can be treated symbolically. Indeed, we need the numerical value of each e^{c_k} only at the very final step, i.e., for evaluating $f(y)$ via (2.12) in Theorem 2.6; see the illustrative example in the next section. Therefore, only at the very final stage, one needs a good rational approximation of e^{c_j} in $\mathbb{Q}^n$ to evaluate $f(y)$.

Having said these, the computational complexity is essentially determined by the number of coefficients $\{Q_{\sigma,\beta}\}$ in equation (2.12); or equivalently, by the number of nonzero coefficients of the polynomials $\{Q_\sigma(z)\}$ in the decomposition (2.9)–(2.10). Define

$$\Lambda := \max_{\sigma\in\mathbb{J}_{m+1}} \left\{ \min\{ \|v\| \quad | \quad A_\sigma v = 0,\ c'_\sigma v \neq 0,\ v \in \mathbb{Z}^{m+1}\} \right\}. \tag{4.12}$$

In the case $n = m+1$ (see §3.1), each polynomial $Q_\sigma(z)$ has at most Λ terms. This follows directly from (3.1) and (3.5).

For $n = m+2$, we have at most $(m+1)^2$ polynomials $Q_\sigma(z)$ in (2.9); and again, each one of them has at most Λ non-zero coefficients. Therefore, in the general case $n > m$, we end up with at most $(m+1)^{n-m}\Lambda$ terms in (2.12). Thus, the computational complexity is equal to $\mathrm{O}[(m+1)^{n-m}\Lambda]$. As a nice feature of the algorithm, notice that the computational complexity does *not* depends on the right-hand-side $y \in \mathbb{Z}^m$. Moreover, notice that the constant Λ does not change (at all) if we multiply the vector $c \in \mathbb{Q}^n$ for any real $r \neq 0$, because $c'_\sigma v \neq 0$ if and only if $rc'_\sigma v \neq 0$. Hence, we can also conclude that the computational complexity does *not* depends on the magnitude of $\|c\|$, it only depends on the ratio between the entries of c. However, as shown in the following simple example kindly suggested by an anonymous referee, Λ is exponential in the input size of A. Indeed, if

$$A = \begin{bmatrix} 1 & a & a^2 \\ 1 & a+1 & (a+1)^2 \end{bmatrix},$$

then necessarily, every solution $v \in \mathbb{Z}^3$ of $Av = 0$, is an integer multiple of the vector $(a^2+a, -2a-1, 1)$, and so $\Lambda = O(a^2)$. Finally, the constant $M > 0$ in (2.10) and (2.12), depends polynomially on Λ.

5. Illustrative Example

Consider the following example with $n = 6, m = 3$ and data

$$A := \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 2 & 1 & 0 & 0 & 1 & 0 \\ 0 & 2 & 1 & 0 & 0 & 1 \end{bmatrix}, \quad c := (c_1, \ldots, c_6),$$

so that $F(z)$ is equal to the rational fraction

$$\frac{1}{(1 - \mathrm{e}^{c_1} z_1 z_2^2)(1 - \mathrm{e}^{c_2} z_1 z_2 z_3^2)(1 - \mathrm{e}^{c_3} z_1 z_3)(1 - \mathrm{e}^{c_4} z_1)(1 - \mathrm{e}^{c_5} z_2)(1 - \mathrm{e}^{c_6} z_3)}.$$

Let us apply the algorithm developed in §4.1.

Step 0: The set $\mathbb{J}_3$ is composed by all bases $\{i, j, k\}$, with $1 \leq i \leq j \leq k \leq 6$, but the exceptions: $\{1, 4, 5\}$ and $\{3, 4, 6\}$.

Set $z \mapsto Q_{\breve{\pi}}(z) \equiv 1$ when $\breve{\pi} = \{1, 2, \ldots, 6\}$.

Step 1: $p = 6$. With $\pi = \{1, 2, \ldots, 6\} \in \mathbb{J}_6$, define the vector

$$w = (\, \mathrm{e}^{c_1} z_1 z_2^2,\ \mathrm{e}^{c_2} z_1 z_2 z_3^2,\ \mathrm{e}^{c_3} z_1 z_3,\ \mathrm{e}^{c_4} z_1,\ \mathrm{e}^{c_5} z_2,\ \mathrm{e}^{c_6} z_3 \,).$$

Notice that the element k in the base π indeed represents the k-th column of A, the k-th entry of w and the k-th factor in the denominator of $F(z)$. Now, choose $\breve{\sigma} := \{4, 5, 6\}$ and $\sigma := \{3, 4, 5, 6\}$. Let $v := (-1, 1, 0, 1) \in \mathbb{Z}^4$ solve $A_\sigma v = 0$. We obviously have that $w_\sigma = (\mathrm{e}^{c_3} z_1 z_3, \mathrm{e}^{c_4} z_1, \mathrm{e}^{c_5} z_5, \mathrm{e}^{c_6} z_3)$ and $(w_\sigma)^v = \mathrm{e}^{c_4 + c_6 - c_3}$. Therefore, applying equations (3.1) and (4.5), we get

$$\begin{aligned} R_1^\pi(z) &= \frac{-(\mathrm{e}^{c_3} z_1 z_3)^{-1}}{1 - \mathrm{e}^{c_4 + c_6 - c_3}}, \\ R_2^\pi(z) &= \frac{(\mathrm{e}^{c_3} z_1 z_3)^{-1}}{1 - \mathrm{e}^{c_4 + c_6 - c_3}}, \\ R_3^\pi(z) &= 0, \\ R_4^\pi(z) &= \frac{\mathrm{e}^{(c_4 - c_3)} z_3^{-1}}{1 - \mathrm{e}^{c_4 + c_6 - c_3}}. \end{aligned}$$

Hence, we can easily calculate that :

$$1 = (1 - \mathrm{e}^{c_3} z_1 z_3) R_1^\pi(z) + (1 - \mathrm{e}^{c_4} z_1) R_2^\pi(z) + (1 - \mathrm{e}^{c_6} z_3) R_4^\pi(z).$$

Notice that the term $(1 - \mathrm{e}^{c_3} z_1 z_3) R_1^\pi(z)$ will *kill* the element 3 in the base π. Moreover, the terms $(\cdots) R_2^\pi(z)$ and $(\cdots) R_3^\pi(z)$ will also kill the respective entries 4 and 6 in the base π, so

$$
\begin{aligned}
F(z) &= \frac{R_1^\pi(z)}{(1-\mathrm{e}^{c_1}z_1z_2^2)(1-\mathrm{e}^{c_2}z_1z_2z_3^2)(1-\mathrm{e}^{c_4}z_1)(1-\mathrm{e}^{c_5}z_2)(1-\mathrm{e}^{c_6}z_3)} \\
&+ \frac{R_2^\pi(z)}{(1-\mathrm{e}^{c_1}z_1z_2^2)(1-\mathrm{e}^{c_2}z_1z_2z_3^2)(1-\mathrm{e}^{c_3}z_1z_3)(1-\mathrm{e}^{c_5}z_2)(1-\mathrm{e}^{c_6}z_3)} \\
&+ \frac{R_3^\pi(z)}{(1-\mathrm{e}^{c_1}z_1z_2^2)(1-\mathrm{e}^{c_2}z_1z_2z_3^2)(1-\mathrm{e}^{c_3}z_1z_3)(1-\mathrm{e}^{c_4}z_1)(1-\mathrm{e}^{c_5}z_2)} \\
&= \sum_{j=1}^{3} \frac{Q_{\eta_5[j]}(z)}{\prod_{k\in\eta_5[j]}(1-\mathrm{e}^{c_k}z^{A_k})}, \quad \text{with } \eta_5[j]\in\mathbb{J}_5,\ j=1,2,3.
\end{aligned}
$$

One has $\eta_5[1]=\{1,2,4,5,6\}$, $\eta_5[2]=\{1,2,3,5,6\}$, $\eta_5[3]=\{1,2,3,4,5\}$ and $Q_{\eta_5[j]}(z)=R_j^\pi(z)$ for $j=1,2,3$.

Step 1: $p=5$.

- Analyzing $\eta_5[1]=\{1,2,4,5,6\}\in\mathbb{J}_5$, choose $\breve{\sigma}=\{4,5,6\}$ and $\sigma:=\{1,4,5,6\}$. Let $v:=(-1,1,2,0)\in\mathbb{Z}^4$ solve $A_\sigma v=0$. We have that $w_\sigma=(\mathrm{e}^{c_1}z_1z_2^2,\mathrm{e}^{c_4}z_1,\mathrm{e}^{c_5}z_2,\mathrm{e}^{c_6}z_3)$, so we get

$$
\begin{aligned}
R_1^{\eta_5[1]}(z) &= \frac{-(\mathrm{e}^{c_1}z_1z_2^2)^{-1}}{1-\mathrm{e}^{-c_1+c_4+2c_5}}, \\
R_2^{\eta_5[1]}(z) &= \frac{(\mathrm{e}^{c_1}z_1z_2^2)^{-1}}{1-\mathrm{e}^{-c_1+c_4+2c_5}}, \\
R_3^{\eta_5[1]}(z) &= \frac{(\mathrm{e}^{c_4-c_1}z_2^{-2})(1+\mathrm{e}^{c_5}z_2)}{1-\mathrm{e}^{-c_1+c_4+2c_5}}, \\
R_4^{\eta_5[1]}(z) &= 0.
\end{aligned}
$$

One may easily verify that

$$
1=(1-\mathrm{e}^{c_1}z_1z_2^2)R_1^{\eta_5[1]}(z)+(1-\mathrm{e}^{c_4}z_1)R_2^{\eta_5[1]}(z)+(1-\mathrm{e}^{c_5}z_2)R_3^{\eta_5[1]}(z).
$$

Notice that the terms associated to $R_1^{\eta_5[1]}(z)$, $R_2^{\eta_5[1]}(z)$ and $R_3^{\eta_5[1]}(z)$ kill the respective entries 1, 4 and 5 in the base $\eta_5[1]$.

- Analyzing $\eta_5[2]=\{1,2,3,5,6\}\in\mathbb{J}_5$, choose $\breve{\sigma}=\{3,5,6\}$ and $\sigma:=\{2,3,5,6\}$. Let $v:=(-1,1,1,1)\in\mathbb{Z}^4$ solve $A_\sigma v=0$. We have that $w_\sigma=(\mathrm{e}^{c_2}z_1z_2z_3^2,\mathrm{e}^{c_3}z_1z_3,\mathrm{e}^{c_5}z_2,\mathrm{e}^{c_6}z_3)$, so we get

$$\begin{aligned}
R_1^{\eta_5[2]}(z) &= \frac{-(\mathrm{e}^{c_2} z_1 z_2 z_3^2)^{-1}}{1-\mathrm{e}^{-c_2+c_3+c_5+c_6}}, \\
R_2^{\eta_5[2]}(z) &= \frac{(\mathrm{e}^{c_2} z_1 z_2 z_3^2)^{-1}}{1-\mathrm{e}^{-c_2+c_3+c_5+c_6}}, \\
R_3^{\eta_5[2]}(z) &= \frac{\mathrm{e}^{c_3-c_2} z_2^{-1} z_3^{-1}}{1-\mathrm{e}^{-c_2+c_3+c_5+c_6}}, \\
R_4^{\eta_5[2]}(z) &= \frac{\mathrm{e}^{c_5+c_3-c_2} z_3^{-1}}{1-\mathrm{e}^{-c_2+c_3+c_5+c_6}}.
\end{aligned}$$

We may easily verify that

$$\begin{aligned}
1 = (1-\mathrm{e}^{c_2} z_1 z_2 z_3^2) R_1^{\eta_5[2]}(z) + (1-\mathrm{e}^{c_3} z_1 z_3) R_2^{\eta_5[2]}(z) + \\
+(1-\mathrm{e}^{c_5} z_2) R_3^{\eta_5[2]}(z) + (1-\mathrm{e}^{c_6} z_3) R_4^{\eta_5[2]}(z).
\end{aligned}$$

Notice that the terms associated to $R_1^{\eta_5[2]}(z)$, $R_2^{\eta_5[2]}(z)$, $R_3^{\eta_5[2]}(z)$ and $R_4^{\eta_5[2]}(z)$ kill the respective entries 2, 3, 5 and 6 in the base $\eta_5[2]$.

- Analyzing $\eta_5[3] = \{1,2,3,4,5\} \in \mathbb{J}_5$, choose $\breve{\sigma} = \{3,4,5\}$ and $\sigma := \{2,3,4,5\}$. Let $v := (-1,2,-1,1) \in \mathbb{Z}^4$ solve $A_\sigma v = 0$. We have that $w_\sigma = (\mathrm{e}^{c_2} z_1 z_2 z_3^2, \mathrm{e}^{c_3} z_1 z_3, \mathrm{e}^{c_4} z_1, \mathrm{e}^{c_5} z_2)$, so we get

$$\begin{aligned}
R_1^{\eta_5[3]}(z) &= \frac{-(\mathrm{e}^{c_2} z_1 z_2 z_3^2)^{-1}}{1-\mathrm{e}^{-c_2+2c_3-c_4+c_5}}, \\
R_2^{\eta_5[3]}(z) &= \frac{(\mathrm{e}^{c_2} z_1 z_2 z_3^2)^{-1}(1+\mathrm{e}^{c_3} z_1 z_3)}{1-\mathrm{e}^{-c_2+2c_3-c_4+c_5}}, \\
R_3^{\eta_5[3]}(z) &= \frac{-(\mathrm{e}^{c_4} z_1)^{-1}(\mathrm{e}^{2c_3-c_2} z_1 z_2^{-1})}{1-\mathrm{e}^{-c_2+2c_3-c_4+c_5}}, \\
R_4^{\eta_5[3]}(z) &= \frac{\mathrm{e}^{2c_3-c_2-c_4} z_2^{-1}}{1-\mathrm{e}^{-c_2+2c_3-c_4+c_5}}.
\end{aligned}$$

We may easily verify that

$$\begin{aligned}
1 = (1-\mathrm{e}^{c_2} z_1 z_2 z_3^2) R_1^{\eta_5[3]}(z) + (1-\mathrm{e}^{c_3} z_1 z_3) R_2^{\eta_5[3]}(z) + \\
+(1-\mathrm{e}^{c_4} z_1) R_3^{\eta_5[3]}(z) + (1-\mathrm{e}^{c_5} z_2) R_4^{\eta_5[3]}(z).
\end{aligned}$$

Notice that the terms associated to $R_1^{\eta_5[3]}(z)$, $R_2^{\eta_5[3]}(z)$, $R_3^{\eta_5[3]}(z)$ and $R_4^{\eta_5[3]}(z)$ kill the respective entries 2, 3, 4 and 5 in the base $\eta_5[3]$.

Therefore, we have the following expansion of $F(z)$.

$$
\begin{aligned}
F(z) &= \frac{Q_{\eta_5[1]}(z) R_1^{\eta_5[1]}(z)}{(1-\mathrm{e}^{c_2} z_1 z_2 z_3^2)(1-\mathrm{e}^{c_4} z_1)(1-\mathrm{e}^{c_5} z_2)(1-\mathrm{e}^{c_6} z_3)} \\
&+ \frac{Q_{\eta_5[1]}(z) R_2^{\eta_5[1]}(z) + Q_{\eta_5[2]}(z) R_2^{\eta_5[2]}(z)}{(1-\mathrm{e}^{c_1} z_1 z_2^2)(1-\mathrm{e}^{c_2} z_1 z_2 z_3^2)(1-\mathrm{e}^{c_5} z_2)(1-\mathrm{e}^{c_6} z_3)} \\
&+ \frac{Q_{\eta_5[1]}(z) R_3^{\eta_5[1]}(z)}{(1-\mathrm{e}^{c_1} z_1 z_2^2)(1-\mathrm{e}^{c_2} z_1 z_2 z_3^2)(1-\mathrm{e}^{c_4} z_1)(1-\mathrm{e}^{c_6} z_3)} \\
&+ \frac{Q_{\eta_5[2]}(z) R_1^{\eta_5[2]}(z)}{(1-\mathrm{e}^{c_1} z_1 z_2^2)(1-\mathrm{e}^{c_3} z_1 z_3)(1-\mathrm{e}^{c_5} z_2)(1-\mathrm{e}^{c_6} z_3)} \\
&+ \frac{Q_{\eta_5[2]}(z) R_3^{\eta_5[2]}(z)}{(1-\mathrm{e}^{c_1} z_1 z_2^2)(1-\mathrm{e}^{c_2} z_1 z_2 z_3^2)(1-\mathrm{e}^{c_3} z_1 z_3)(1-\mathrm{e}^{c_6} z_3)} \\
&+ \frac{Q_{\eta_5[2]}(z) R_4^{\eta_5[2]}(z) + Q_{\eta_5[3]}(z) R_3^{\eta_5[3]}(z)}{(1-\mathrm{e}^{c_1} z_1 z_2^2)(1-\mathrm{e}^{c_2} z_1 z_2 z_3^2)(1-\mathrm{e}^{c_3} z_1 z_3)(1-\mathrm{e}^{c_5} z_2)} \\
&+ \frac{Q_{\eta_5[3]}(z) R_1^{\eta_5[3]}(z)}{(1-\mathrm{e}^{c_1} z_1 z_2^2)(1-\mathrm{e}^{c_3} z_1 z_3)(1-\mathrm{e}^{c_4} z_1)(1-\mathrm{e}^{c_5} z_2)} \\
&+ \frac{Q_{\eta_5[3]}(z) R_2^{\eta_5[3]}(z)}{(1-\mathrm{e}^{c_1} z_1 z_2^2)(1-\mathrm{e}^{c_2} z_1 z_2 z_3^2)(1-\mathrm{e}^{c_4} z_1)(1-\mathrm{e}^{c_5} z_2)} \\
&+ \frac{Q_{\eta_5[3]}(z) R_4^{\eta_5[3]}(z)}{(1-\mathrm{e}^{c_1} z_1 z_2^2)(1-\mathrm{e}^{c_2} z_1 z_2 z_3^2)(1-\mathrm{e}^{c_3} z_1 z_3)(1-\mathrm{e}^{c_4} z_1)} \\
&= \sum_{j=1}^{9} \frac{Q_{\eta_4[j]}(z)}{\prod_{k \in \eta_4[j]} (1-\mathrm{e}^{c_k} z^{A_k})}, \quad \text{with } \eta_4[j] \in \mathbb{J}_4,\ j = 1, 2, \ldots, 9.
\end{aligned}
$$

Step 1: $p = 4 = m+1$. At this step we obtain the required decomposition (4.11), that is, we express $F(z)$ as the sum

$$
(5.1) \qquad F(z) = \sum_j \frac{Q_{\eta_3[j]}(z)}{\prod_{k \in \eta_3[j]} (1-\mathrm{e}^{c_k} z^{A_k})}, \quad \text{with } \eta_3[j] \in \mathbb{J}_3 = \mathbb{J}_m,\ \forall j.
$$

The above sum contains only 16 terms (not detailed here) out of the potentially $\binom{6}{3} = 20$ terms. For illustration, we only provide the term $Q_{\eta_3[j]}(z)$ relative to the basis $\eta_3[j] = \{2, 5, 6\} \in \mathbb{J}_3$.

- With $\eta_4[1] = \{2, 4, 5, 6\} \in \mathbb{J}_4$, choose $\breve{\sigma} := \{4, 5, 6\}$ and $\sigma := \{2, 4, 5, 6\}$. Let $v := (-1, 1, 1, 2) \in \mathbb{Z}^4$ solve $A_\sigma v = 0$. We have that
$w_\sigma = (\mathrm{e}^{c_2} z_1 z_2 z_3^2, \mathrm{e}^{c_4} z_1, \mathrm{e}^{c_5} z_2, \mathrm{e}^{c_6} z_3)$, so we get

$$\begin{aligned}
R_1^{\eta_4[1]}(z) &= \frac{-(\mathrm{e}^{c_2} z_1 z_2 z_3^2)^{-1}}{1-\mathrm{e}^{2c_6+c_5+c_4-c_2}},\\
R_2^{\eta_4[1]}(z) &= \frac{(\mathrm{e}^{c_2} z_1 z_2 z_3^2)^{-1}}{1-\mathrm{e}^{2c_6+c_5+c_4-c_2}},\\
R_3^{\eta_4[1]}(z) &= \frac{\mathrm{e}^{c_4-c_2} (z_2 z_3^2)^{-1}}{1-\mathrm{e}^{2c_6+c_5+c_4-c_2}},\\
R_4^{\eta_4[1]}(z) &= \frac{(\mathrm{e}^{c_4+c_5-c_2} z_3^{-2})(1+\mathrm{e}^{c_6} z_3)}{1-\mathrm{e}^{2c_6+c_5+c_4-c_2}}.
\end{aligned}$$

$$\begin{aligned}
1 = (1-\mathrm{e}^{c_2} z_1 z_2 z_3^2) R_1^{\eta_4[1]}(z) + (1-\mathrm{e}^{c_4} z_1) R_2^{\eta_4[1]}(z) + \\
+(1-\mathrm{e}^{c_5} z_2) R_3^{\eta_4[1]}(z) + (1-\mathrm{e}^{c_6} z_3) R_4^{\eta_4[1]}(z).
\end{aligned}$$

Notice that the term associated to $R_2^{\eta_4[1]}$ kills the entry 4 in the base $\eta_4[1] = \{2,4,5,6\}$, so we are getting the desired base $\eta_3[1] = \{2,5,6\}$.

- With $\eta_4[2] = \{1,2,5,6\} \in \mathbb{J}_4$, choose $\breve{\sigma} := \{2,5,6\}$ and $\sigma := \{1,2,5,6\}$. Let $v := (-1,1,1,-2) \in \mathbb{Z}^4$ solve $A_\sigma v = 0$. We have that
$w_\sigma = (\mathrm{e}^{c_1} z_1 z_2^2, \mathrm{e}^{c_2} z_1 z_2 z_3^2, \mathrm{e}^{c_5} z_2, \mathrm{e}^{c_6} z_3)$, so we get

$$\begin{aligned}
R_1^{\eta_4[2]}(z) &= \frac{-(\mathrm{e}^{c_1} z_1 z_2^2)^{-1}}{1-\mathrm{e}^{c_2+c_5-c_1-2c_6}},\\
R_2^{\eta_4[2]}(z) &= \frac{(\mathrm{e}^{c_1} z_1 z_2^2)^{-1}}{1-\mathrm{e}^{c_2+c_5-c_1-2c_6}},\\
R_3^{\eta_4[2]}(z) &= \frac{\mathrm{e}^{c_2-c_1} z_2^{-1} z_3^2}{1-\mathrm{e}^{c_2+c_5-c_1-2c_6}},\\
R_4^{\eta_4[2]}(z) &= \frac{-(\mathrm{e}^{c_6} z_3)^{-1}(\mathrm{e}^{c_2-c_1+c_5} z_3^2)(1+(\mathrm{e}^{c_6} z_3)^{-1})}{1-\mathrm{e}^{c_2+c_5-c_1-2c_6}}.
\end{aligned}$$

$$\begin{aligned}
1 = (1-\mathrm{e}^{c_1} z_1 z_2^2) R_1^{\eta_4[2]}(z) + (1-\mathrm{e}^{c_2} z_1 z_2 z_3^2) R_2^{\eta_4[1]}(z) + \\
+(1-\mathrm{e}^{c_5} z_2) R_3^{\eta_4[1]}(z) + (1-\mathrm{e}^{c_6} z_3) R_4^{\eta_4[1]}(z).
\end{aligned}$$

Notice that the term associated to $R_1^{\eta_4[2]}$ kills the entry 1 in the base $\eta_4[2] = \{1,2,5,6\}$, so we are getting the desired base $\eta_3[1] = \{2,5,6\}$.

Therefore, working on the base $\eta_3[1]$, we obtain the numerator

$$
\begin{aligned}
Q_{\eta_3[1]}(z) &= Q_{\eta_4[1]} R_2^{\eta_4[1]} + Q_{\eta_4[2]} R_1^{\eta_4[2]} \\
&= \left[Q_{\eta_5[1]}(z) R_1^{\eta_5[1]}(z) \right] R_2^{\eta_4[1]} + \\
&+ \left[Q_{\eta_5[1]}(z) R_2^{\eta_5[1]}(z) + Q_{\eta_5[2]}(z) R_2^{\eta_5[2]}(z) \right] R_1^{\eta_4[2]}.
\end{aligned}
$$

Step 2: The value of $f(y)$ is obtained from (2.12) in Theorem 2.6, using the expression (5.1) of $F(z)$.

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