Integer programming, Barvinok's counting algorithm and Gomory relaxations

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Abstract

We consider the integer program $\mathbf{P}\rightarrow \max\{c'x|Ax \leq b; x \in \mathbf{Z}^n\}$, and propose an algorithm based on Barvinok's counting algorithm, which runs in time polynomial in the input size of the polyhedron $\{x \in \mathbf{R}^n \,|\, Ax \leq b\}$ when the dimension n is fixed. Under a condition on the vector c, it provides the optimal value of $\mathbf{P}$ and an upper bound in general. We also relate Barvinok's counting formula and Gomory relaxations of integer programs.

Keywords: Integer programming; generating functions.

1 Introduction

With $A \in \mathbf{Z}^{m\times n}, c \in \mathbf{R}^n, b \in \mathbf{Q}^m$, we consider the integer program

$$\mathbf{P}\rightarrow \quad p^* := \max\{c'x \,|\, Ax \leq b; \quad x \in \mathbf{Z}^n\}. \tag{1.1}$$

This discrete analogue of linear programming (LP) is a fundamental NP-hard problem with numerous important applications. Solving $\mathbf{P}$ remains in general a formidable computational challenge. For a standard reference on integer programming, the reader is referred to e.g. Schrijver [8], Nemhauser and Wolsey [7], Wolsey [10].

The first integer programming algorithm with polynomial time complexity when the dimension n is fixed, is due to H.W. Lenstra [6], and uses lattice reduction technique along with a *rounding* of a convex body. As underlined in Barvinok and Pommersheim [3, p. 21], this rounding can be quite time-consuming. On the other hand, Barvinok [2] was the first to propose an algorithm to *count* the integral points of a convex rational polytope $\Omega(b) := \{x \in \mathbf{R}^n \,|\, Ax \leq b\}$,

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with polynomial time complexity when the dimension n is fixed. The main step is to produce a compact form of the generating function $f : \mathbf{C}^n \to \mathbf{C}$,

$$z \mapsto f(z) := \sum_{x \in \Omega(b) \cap \mathbf{Z}^n} z^x; \tag{1.2}$$

see also Barvinok and Pommersheim [3]. That is, Barvinok's algorithm reduces $f(z)$ to the rational function

$$f(z) = \sum_{i \in I} \epsilon_i \frac{z^{a_i}}{\prod_{k=1}^n (1 - z^{b_{ik}})}, \tag{1.3}$$

for some vectors $\{a_i, b_{ik}\}$ in $\mathbf{Z}^n$, and I is some index set. In (1.3), $f(z)$ encodes all the information about the set $\Omega(b) \cap \mathbf{Z}^n$ of integer points in $\Omega(b)$.

He then suggested that his algorithm, coupled with a standard dichotomy procedure, would yield an alternative to Lenstra 's algorithm for integer programming, with no rounding procedure (see the discussion in Barvinok and Pommersheim [3, p. 21]). However, in this scheme, one must redo Barvinok's calculation to obtain the compact form (1.3) of the new generating function associated with the (new) polyhedron $\Omega(b')$ considered at each step in the dichotomy, which can be also quite time consuming.

In this paper :

- We provide an upper bound ρ^* on the optimal value of $\mathbf{P}$ by a simple inspection of Barvinok's formula; under some (easy to check) condition on the reward vector c, ρ^* is also the optimal value of $\mathbf{P}$.

- We relate Barvinok's counting formula and Gomory relaxations of integer programs, and provide a simplified procedure for large values of b.

2 Notation and preliminaries

2.1 Notation and definitions

We consider the integer program

$$\mathbf{P} \to p^* = \max \{c'x \mid Ax \leq b; \quad x \in \mathbf{Z}^n\}, \tag{2.1}$$

where $A \in \mathbf{Z}^{m \times n}, c \in \mathbf{R}^n, b \in \mathbf{Q}^m$. With any two vectors $z \in \mathbf{C}^n, u \in \mathbf{Z}^n$ we use the standard notation z^u for the monomial

$$z^u := z_1^{u_1} \cdots z_n^{u_n}.$$

Also, the usual scalar product of two vectors $u, v \in \mathbf{C}^m$ is denoted $u'v$, where u' stands for the transpose of u.

With the integer program $\mathbf{P}$ is associated the convex rational polyhedron

$$\Omega(b) \;:=\; \{x \in \mathbf{R}^n \,|\, Ax \;\leq\; b\} \tag{2.2}$$

and the generating function $f(z) : \mathbf{C}^n \to \mathbf{C}$, given by

$$z \;\mapsto\; f(z) \;:=\; \sum_{x \in \Omega(b) \cap \mathbf{Z}^n} z^x, \tag{2.3}$$

for any $z \in \mathbf{C}^n$ such that the series converges absolutely. For each vertex $v \in \mathbf{R}^n$ of the rational polyhedron $\Omega(b)$, denote by C_v the rational pointed cone with vertex v, that supports $\Omega(b)$, also called the *supporting* or *tangent* cone of $\Omega(b)$ at v. Let $f_v : \mathbf{C}^n \to \mathbf{C}$ be given by

$$z \;\mapsto\; f_v(z) \;:=\; \sum_{x \in C_v \cap \mathbf{Z}^n} z^x, \tag{2.4}$$

for any $z \in \mathbf{C}^n$ such that the series converges absolutely.

2.2 Barvinok's formula

With f, f_v as in (2.3)-(2.4), Brion [4] proved that

$$f(z) \;=\; \sum_{v:\ \text{vertex of } \Omega(b)} f_v(z); \tag{2.5}$$

see also Barvinok and Pommersheim [3, Theor. 3.5, p. 12]. Next, using Brion's formula (2.5), Barvinok showed that

$$f(z) \;=\; \sum_{i \in I} \epsilon_i \frac{z^{a_i}}{\prod_{k=1}^{n}(1 - z^{b_{ik}})}, \tag{2.6}$$

where I is a certain index set, and for all $i \in I$, $\epsilon_i \in \{-1, +1\}$, $a_i \in \mathbf{Z}^n$ and $\{b_{ik}\}_{k=1}^n$ form basis of the lattice $\mathbf{Z}^n$. Each $i \in I$ is associated with a *unimodular cone* in the decomposition of the tangent cones of $\Omega(b)$ (at its vertices), into unimodular cones.

From Barvinok and Pommersheim [3, Theor. 4.4., p. 18], the number $|I|$ of unimodular cones in such a decomposition, is $\mathcal{L}^{O(n)}$ where $\mathcal{L}$ is the input size of $\Omega(b)$, and the overall computational complexity to obtain the coefficients $\{a_i, b_{ik}\}$ in (2.6) is also $\mathcal{L}^{O(n)}$. Crucial for the latter property is the *signed* decomposition (triangulation alone into unimodular cones does not guarantee this polynomial time complexity). For more details, the interested reader is referred to Barvinok [2], and Barvinok and Pommersheim [3].

3 Solving P via Barvinok's algorithm

With $c \in \mathbf{R}^n$ as in (2.1) and $r \in \mathbf{N}$, let $z := \mathrm{e}^{rc} = (\mathrm{e}^{rc_1}, \ldots, \mathrm{e}^{rc_n}) \in \mathbf{R}^n$ so that with $f(z)$ as in (2.6), and assuming $c'b_{ik} \neq 0$ for all $i \in I$, $k = 1, \ldots, n$,

$$f(\mathrm{e}^{rc}) \;=\; \sum_{i \in I} \epsilon_i \frac{(\mathrm{e}^r)^{c'a_i}}{\prod_{k=1}^n (1 - (\mathrm{e}^r)^{c'b_{ik}})}. \tag{3.1}$$

Next, doing the change of variable $u := \mathrm{e}^r \in \mathbf{R}$, (3.1) reads

$$f(\mathrm{e}^{rc}) \;=\; \sum_{i \in I} \epsilon_i \frac{u^{c'a_i}}{\prod_{k=1}^n (1 - u^{c'b_{ik}})} \;=\; \sum_{i \in I} \epsilon_i \frac{u^{c'a_i}}{Q_i(u)} \;=:\; g(u) \tag{3.2}$$

for some functions $\{Q_i\}$ of u. For every $i \in I$, let Γ_i be the set

$$\Gamma_i \;:=\; \{k \in \{1, \ldots, n\} \,|\, c'b_{ik} > 0\}, \quad i \in I, \tag{3.3}$$

with cardinal $|\Gamma_i|$, and define the vector $v_i \in \mathbf{Z}^n$ by

$$v_i \;:=\; a_i - \sum_{k \in \Gamma_i} b_{ik}, \quad i \in I. \tag{3.4}$$

If $\Gamma_i = \emptyset$ then we let $|\Gamma_i| = 0$ and $v_i := a_i$.

Theorem 3.1. *Let $f(z)$ be as in (2.6), and let $c \in \mathbf{R}^n$ be such that $c'b_{ik} \neq 0$ for all $i \in I$, $k = 1, \ldots, n$. Assume that $\mathbf{P}$ in (2.1) has a feasible point $x \in \mathbf{Z}^n$ and a finite optimal value p^*.*

(a) *The optimal value p^* of the integer program $\mathbf{P}$ is given by*

$$p^* \;=\; \lim_{r \to \infty} \frac{1}{r} \ln f(e^{rc}) \;=\; \lim_{r \to \infty} \frac{1}{r} \ln g(e^r). \tag{3.5}$$

(b) *With v_i as in (3.4), let S^* be the set*

$$S^* \;:=\; \{i \in I \,|\, c'v_i \;=\; \rho^* \;:=\; \max_{j \in I} c'v_j\}. \tag{3.6}$$

Then $\rho^* \geq p^*$, and

$$\rho^* \;=\; p^* \quad \textit{if} \quad \sum_{i \in S^*} \epsilon_i (-1)^{|\Gamma_i|} \neq 0. \tag{3.7}$$

Proof. (a) With $z := \mathrm{e}^{rc}$ in the definition of $f(z)$, we have

$$f(\mathrm{e}^{rc}) \;=\; \sum_{x \in \Omega(b) \cap \mathbf{Z}^n} \mathrm{e}^{rc'x},$$

and thus,

$$\begin{aligned}
\mathrm{e}^{p^*} \;=\; \mathrm{e}^{\max\{c'x \,|\, x\in\Omega(b)\cap\mathbf{Z}^n\}} \;&=\; \max\{\mathrm{e}^{c'x} \,|\, x\in\Omega(b)\cap\mathbf{Z}^n\} \\
&=\; \lim_{r\to\infty}\left(\sum_{x\in\Omega(b)\cap\mathbf{Z}^n} \mathrm{e}^{rc'x}\right)^{1/r} \\
&=\; \lim_{r\to\infty} f(\mathrm{e}^{rc})^{1/r},
\end{aligned}$$

and by continuity of the logarithm,

$$p^* \;=\; \max\{c'x \,|\, x\in\Omega(b)\cap\mathbf{Z}^n\} \;=\; \lim_{r\to\infty}\frac{1}{r}\ln f(\mathrm{e}^{rc}) \;=\; \lim_{r\to\infty}\frac{1}{r}\ln g(\mathrm{e}^{r}).$$

(b) From (a), one may hope to obtain p^* by just considering the leading terms (as $u\to\infty$) of the functions $u^{c'a_i}/Q_i(u)$ in (3.2). If the sum in (3.2) of the leading terms (with same power of u) does not vanish, then one obtains p^* by a simple limit argument as $u\to\infty$. From (3.1)-(3.2) it follows that

$$\frac{u^{c'a_i}}{Q_i(u)} \;\approx\; \frac{u^{c'a_i}}{\alpha_i u^{\rho_i}} \;=\; \frac{u^{c'a_i-\rho_i}}{\alpha_i}, \quad \text{as } u\to\infty,$$

where $\alpha_i u^{\rho_i}$ is the leading term of the function $Q_i(u)$ as $u\to\infty$. Again from the definition of Q_i, its leading term $\alpha_i u^{\rho_i}$ as $u\to\infty$ is obtained with

$$\rho_i \;=\; \begin{cases} \sum_{k\in\Gamma_i} c'b_{ik} & \text{if } \Gamma_i\neq\emptyset \\ 0 & \text{otherwise,} \end{cases}$$

and its coefficient α_i is 1 if $\rho_i=0$ and $(-1)^{|\Gamma_i|}$ otherwise.

Remembering the convention that $\sum_{k\in\Gamma_i} c'b_{ik}=0$ and $(-1)^{|\Gamma_i|}=1$ if $\Gamma_i=\emptyset$, we obtain

$$\epsilon_i\frac{u^{c'a_i}}{Q_i(u)} \;\approx\; \epsilon_i(-1)^{|\Gamma_i|}u^{c'(a_i-\sum_{k\in\Gamma_i} b_{ik})} \quad \text{as } u\to\infty.$$

Therefore, with S^* and ρ^* as in (3.6), if $\sum_{i\in S^*}\epsilon_i(-1)^{|\Gamma_i|}\neq 0$ then

$$g(u) \;\approx\; u^{\rho^*}\sum_{i\in S^*}\epsilon_i(-1)^{|\Gamma_i|} \quad \text{as } u\to\infty,$$

so that $\lim_{u\to\infty}\frac{1}{r}\ln g(\mathrm{e}^r)=\rho^*$. This and (3.5) yields $p^*=\rho^*$, the desired result. From the above analysis it easily follows that if $\sum_{i\in S^*}\epsilon_i(-1)^{|\Gamma_i|}=0$ then ρ^* is only an upper bound on p^*. $\square$

The interest of Theorem 3.1 is that the value ρ^* is obtained by simple inspection of (3.2), which in turn is obtained in time polynomial in the input size of the polyhedron $\Omega(b)$ when the dimension n is fixed.

When $\sum_{i\in S^*}\epsilon_i(-1)^{|\Gamma_i|}\neq 0$ then it also yields the optimal value p^* of $\mathbf{P}$ in (2.1). On the other hand, if $\sum_{i\in S^*}\epsilon_i(-1)^{|\Gamma_i|}=0$, i.e., the sum of the leading terms of the functions $u^{c'a_i}/Q_i(u)$ (with same power of u) cancel, then one needs to examine the "next" leading terms which requires a further and nontrivial analysis of each function $u^{c'a_i}/Q_i(u)$.

An alternative would be to adopt the standard dichotomy trick suggested in Barvinok and Pommersheim [3]. But now, at each step of the dichotomy, one recomputes ρ^* as in Theorem 3.1 for the new polyhedron considered at this step, until the condition in Theorem 3.1 is met, in which case one stops because the optimal value of $\mathbf{P}$ is obtained.

Observe that the vectors $a_i,\{b_{ik}\}$ in Barvinok's formula depend only on the polyhedron $\Omega(b)$. Therefore, (3.7) in Theorem 3.1(b) provides a simple (and easy to check) necessary and sufficient condition on the vector $c\in\mathbf{R}^n$, to ensure that the optimal value p^* of $\mathbf{P}$ is equal to ρ^* in (3.6), obtained directly from Barvinok's formula.

4 The link with Gomory relaxations

Let us consider an integer program $\mathbf{P}$ in equality form, that is,

$$\mathbf{P}\rightarrow p^*\;:=\;\max\{c'x\,|\,Ax\;=\;b,\quad x\in\mathbf{N}^n\},\tag{4.1}$$

where $A\in\mathbf{Z}^{m\times n}, b\in\mathbf{Z}^m, c\in\mathbf{R}^n$, with associated polyhedron

$$\Omega(b)\;:=\;\max\{x\in\mathbf{R}^n\,|\,Ax\;=\;b,\quad x\geq 0\}.\tag{4.2}$$

Let $\mathbf{L}$ be the linear programming (LP) problem associated with $\mathbf{P}$, that is,

$$\mathbf{L}\rightarrow\max\{c'x\,|\,Ax\;=\;b,\quad x\geq 0;\quad x\in\mathbf{R}^n\}.\tag{4.3}$$

The *Gomory relaxation* of $\mathbf{P}$ is defined with respect to the optimal basis σ^* of the LP (4.3). That is, if A_j denote the j-th column of A, and $\sigma^*=(\sigma_1^*,\ldots,\sigma_m^*)\in\{1,\ldots,n\}^m$, let $A_{\sigma^*}=[A_{\sigma_1^*}|\ldots|A_{\sigma_m^*}]\in\mathbf{Z}^{m\times m}$ be the submatrix of A associated with the optimal basis of the LP (4.3), and let $\lambda^*\in\mathbf{R}^m$ be an optimal solution of the LP dual of $\mathbf{L}$. Then the Gomory relaxation is the integer program

$$\mathbf{G}_{\sigma^*}\left\{\begin{array}{ll} & b'\lambda^*+\max\sum_{j\notin\sigma^*}(c_j-A_j'\lambda^*)x_j\\ \text{s.t.} & A_{\sigma^*}x_{\sigma^*}+\sum_{j\notin\sigma^*}A_jx_j\;=\;b\\ & x_{\sigma^*}\in\mathbf{Z}^m;\;x_j\in\mathbf{N},\;j\notin\sigma^*.\end{array}\right.\tag{4.4}$$

That is, $\mathbf{G}_{\sigma^*}$ is obtained from $\mathbf{P}$ by relaxing the nonnegativity constraint on the vector $x_{\sigma^*}\in\mathbf{Z}^m$. For more details and various extensions of this approach, the interested reader is referred to Gomory [5], Wolsey [9], and Aardal et al. [1].

If $\mathbf{G}_{\sigma^*}$ has an optimal solution $x = (x_{\sigma^*}, \{x_j\}) \in \mathbf{Z}^m \times \mathbf{N}^{n-m}$ with $x_{\sigma^*} \geq 0$, then x is an optimal solution of $\mathbf{P}$ and the Gomory relaxation is *exact*. In fact, when b is sufficiently "large", the Gomory relaxation is exact (see Gomory [5, Theor. 4, Theor. 5, p. 462]). Observe that the criterion in $\mathbf{G}_{\sigma^*}$ is easily seen to be $c'x$, with $x = (x^{\sigma^*}, \{x_j\})$.

Consider the associated counting problem

$$\delta_{\sigma^*} := \{\sum \mathrm{e}^{c'x} \,|\, Ax = b; \quad x_j \in \mathbf{Z},\, j \in \sigma^*; \quad x_j \in \mathbf{N},\, j \notin \sigma^*\}, \tag{4.5}$$

which sums up $\mathrm{e}^{c'x}$ over all integral points $x \in \mathbf{Z}^n$ of the set

$$C_{\sigma^*} \;:=\; \{x \in \mathbf{R}^n \,|\, Ax \;=\; b; \quad x_j \geq 0 \;\forall j \notin \sigma^*\}. \tag{4.6}$$

Let $x(\sigma^*) \in \mathbf{R}^n_+$ be the optimal vertex of $\Omega(b)$ associated with the optimal basis σ^* of the LP (4.3). The set C_{σ^*} is nothing less than the tangent cone of $\Omega(b)$, at the vertex $x(\sigma^*)$.

Let Δ be the set of feasible bases σ of the LP (4.3), and let $x(\sigma) \in \mathbf{R}^n_+$ be the corresponding vertex of $\Omega(b)$ in (4.2). For every $\sigma \in \Delta$, let C_σ be the tangent cone of $\Omega(b)$ at the vertex $x(\sigma)$ (that is, in (4.6) replace σ^* with σ).

Brion's formula (2.5) applied to the polyhedron $\Omega(b)$ in (4.2), reads

$$f(z) \;=\; \sum_{\sigma \in \Delta} f_\sigma(z) \;=\; \sum_{\sigma \in \Delta} \;\sum_{x \in C_\sigma \cap \mathbf{Z}^n} z^x. \tag{4.7}$$

The above summation or (4.2) is *formal* in the sense that some terms $f_\sigma(z)$ may not be defined for the same values of $z \in \mathbf{C}^m$ (see e.g. Example 3.2 in Barvinok and Prommersheim [3, p. 10]).

Note that $f_\sigma(\mathrm{e}^c) = \delta_\sigma$, where δ_σ is as in (4.5), with σ in lieu of σ^*.

So, $C_\sigma \cap \mathbf{Z}^n$ is the *feasible set* of the Gomory relaxation associated with the basis σ (usually defined for σ^* only). Then, as the Gomory relaxation $\mathbf{G}_{\sigma^*}$ povides an upper bound on p^* (and exactly p^* when b is sufficiently large), we can apply Theorem 3.1 to the integer program $\mathbf{G}_{\sigma^*}$ in (4.4), instead of $\mathbf{P}$ in (4.1).

So, when the dimension n is fixed, Barvinok's algorithm produces in time polynomial in the input size of C_{σ^*}, the equivalent compact form of $f_{\sigma^*}(z)$,

$$f_{\sigma^*}(z) \;=\; \sum_{i \in I_{\sigma^*}} \epsilon_i \frac{z^{a_i}}{\prod_{k=1}^n (1 - z^{b_{ik}})}, \tag{4.8}$$

where the above summation is over the unimodular cones in Barvinok's decomposition of C_{σ^*} into unimodular cones. There is much less work to do because now, in Brion's formula (4.7), we have only considered the term f_{σ^*} relative to the optimal basis σ^* of the LP (4.3).

When the condition on c in Theorem 3.1(b) is satisfied, one obtains the optimal value of the Gomory relaxation $\mathbf{G}_{\sigma^*}$ (and the optimal value of $\mathbf{P}$ for sufficiently large b), in time polynomial in the input size of $\Omega(b)$ when the dimension n is fixed. Hence, this technique could provide a viable alternative to the dynamic programming based algorithms for solving group relaxations, as discussed in Wolsey [9].

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