THE INTEGER HULL OF A CONVEX RATIONAL POLYTOPE

JEAN B. LASSERRE

Abstract. Given $A \in \mathbb{Z}^{m\times n}$ and $b \in \mathbb{Z}^m$, we consider the integer program $\max\{c'x | Ax = b; x \in \mathbb{N}^n\}$ and provide an *equivalent* and *explicit* linear program $\max\{\widehat{c}'q | \mathrm{M}q = r; q \geq 0\}$, where $\mathrm{M}, r, \widehat{c}$ are easily obtained from A, b, c with no calculation. We also provide an explicit algebraic characterization of the integer hull of the convex polytope $\mathrm{P} = \{x \in \mathbb{R}^n | Ax = b; x \geq 0\}$. All strong valid inequalities can be obtained from the generators of a convex cone whose definition is explicit in terms of M.

1. Introduction

Let $A \in \mathbb{Z}^{m\times n}, b \in \mathbb{Z}^m, c \in \mathbb{R}^n$ and consider the integer program

$$\mathbb{P} \rightarrow \; := \; \max\{\; c'x \quad | \quad Ax \; = \; b; \quad x \in \mathbb{N}^n\}, \tag{1.1}$$

where the convex polyhedron $\mathrm{P} := \{x \in \mathbb{R}^n \,|\, Ax = b;\; x \geq 0\}$ is compact. If P_1 denotes the *integer hull* of P, then solving $\mathbb{P}$ is equivalent to solving the linear program $\max\{c'x \,|\, x \in \mathrm{P}_1\}$.

However, finding the integer hull P_1 of P is a difficult problem. As mentioned in Wolsey [8, p. 15]), and to the best of our knowledge, no *explicit* (or "simple") characterization (or description) of P_1 has been provided so far. In the general *cutting plane* methods originated by Gomory and Chvátal in the early sixties, and the *lift-and-project* methods described in e.g. Laurent [4], one obtains P_1 as the final iterate of a *finite* nested sequence $\mathrm{P} \supseteq \mathrm{P}' \supseteq \mathrm{P}'' \cdots \supseteq \mathrm{P}_1$ of polyhedra. However, in all those procedures, P_1 has no explicit description in terms of the initial data A, b. On the other hand, for specific polytopes P, one is often able to provide some *strong valid inequalities* in explicit form, but very rarely all of them (as for the matching polytope of a graph). For more details the interested reader is referred to Cornuejols and Li [1], Jeroslow [2], Laurent [4], Nemhauser and Wolsey [6], Schrijver [7, §23], Wolsey [8, §8,9], and the many references therein.

Contribution. The main goal of this paper is to provide a *structural* result on the integer hull P_1 of a convex rational polytope P, in the sense that we obtain an explicit algebraic characterization of the defining hyperplanes of P_1, in terms of generators of a convex cone C which is itself described directly from the initial data A, with *no* calculation.

We first show that the integer program $\mathbb{P}$ is equivalent to a linear program in the *explicit* form

$$\max_{q\in\mathbb{R}^s}\{\,\widehat{c}'q \quad | \quad \mathrm{M}\,q = r; \quad q \geq 0\}. \tag{1.2}$$

By *explicit* we mean that the data $\mathrm{M}, r, \widehat{c}$ of the linear program (1.2) are constructed *explicitly* and *easily* from the initial data A, b, c. In particular, *no* calculation is needed and M, r have all their entries in $\{0, \pm 1\}$. In addition M is very sparse. Of course, and as expected, the dimension of the matrix M is in general exponential in the problem size. However, for the class of problems where A has nonnegative integral entries, and b and the column sums of A are bounded, then (1.2) is solvable in time polynomial in the problem size.

There is a simple linear relation $x = \mathrm{E}q$ linking x and q, but q is not a *lifting* of x like in the the lift-and-project procedures described in Laurent [4]. It is more appropriate to say that q is a *disaggregation* of x, as will become clear in the sequel. Moreover, with *each* extreme point q of the convex polyhedron $\Omega := \{q \in \mathbb{R}^s \,|\, \mathrm{M}\,q = r, q \geq 0\}$ is associated an integral point $x = \mathrm{E}q$ of P (i.e. $x \in \mathrm{P} \cap \mathbb{Z}^n$).

Using the latter result, and when P is compact, we provide the integer hull P_1 in the *explicit* form $\{x \in \mathbb{R}^n |\, Ux \geq u\}$ for some matrix U and vector u. By this we mean that U, u are obtained from the generators of a convex cone C which has a very simple and explicit description in terms of A (via M). Hence, all strong valid inequalities for P_1 can be obtained from the generators of the cone C. Of course, in view of the potentially large size of M, one cannot expect to get all generators of C in general. However, we hope that this structural result on the characterization of P_1 will be helpful in either deriving strong valid inequalities, or validating some candidates inequalities, at least for some specific polytopes P.

2. Notation and preliminary results

Let $\mathbb{N}$ denote the natural numbers or, equivalently, $\mathbb{Z}_+$. For a vector $b \in \mathbb{R}^m$ and a matrix $A \in \mathbb{R}^{m\times n}$, denote by b' and $A' \in \mathbb{R}^{n\times m}$ their respective transpose. Denote by $e_m \in \mathbb{R}^m$ the vector with all entries equal to 1. Let $\mathbb{R}[x_1, \ldots, x_n]$ be the ring of real-valued polynomials in the variables $x_1, \ldots, x_n$. A polynomial $f \in \mathbb{R}[x_1, \ldots, x_n]$ is written

$$x \mapsto f(x) = \sum_{\alpha\in\mathbb{N}^n} f_\alpha x^\alpha = \sum_{\alpha\in\mathbb{N}^n} f_\alpha x_1^{\alpha_1} \cdots x_n^{\alpha_n},$$

for *finitely many* real coefficients $\{f_\alpha\}$, in the (usual) basis of monomials.

Given a matrix $A \in \mathbb{Z}^{m\times n}$, let $A_j \in \mathbb{Z}^m$ denote its j-th column (equivalently, the j-th row of A'); then z^{A_j} stands for

$$z^{A_j} := z_1^{A_{1j}} \cdots z_m^{A_{mj}} = \mathrm{e}^{\langle A_j, \ln z\rangle} = \mathrm{e}^{(A' \ln z)_j},$$

and if $A_j \in \mathbb{N}^m$ then z^{A_j} is a monomial of $\mathbb{R}[z_1, \ldots, z_m]$.

2.1. **Preliminary result.** We first recall the following result :

Theorem 2.1 (A discrete Farkas lemma)**.** *Let $A \in \mathbb{N}^{m\times n}, b \in \mathbb{N}^m$. Then the following two statements (i) and (ii) are equivalent :*

(i) *The linear system $Ax = b$ has a solution $x \in \mathbb{N}^n$.*

(ii) *The real-valued polynomial $z \mapsto z^b - 1 := z_1^{b_1} \cdots z_m^{b_m} - 1$ can be written*

$$z^b - 1 \;=\; \sum_{j=1}^{n} Q_j(z)(z^{A_j} - 1) \tag{2.1}$$

for some real-valued polynomials $Q_j \in \mathbb{R}[z_1, \ldots, z_m]$, $j = 1, \ldots, n$, all of them with nonnegative coefficients.

In addition, the degree of the Q_j's in (2.1) is bounded by

$$b^* \;:=\; \sum_{j=1}^{m} b_j - \min_k \sum_{j=1}^{m} A_{jk}. \tag{2.2}$$

A proof based on counting techniques via generating functions and inverse $\mathbb{Z}$-transform can be found in [3]. However, thanks to an anonymous referee's remark, a self-contained and simpler proof is provided in §2.3 below. Before, we make some useful remarks and introduce some additional material.

2.2. **Discussion.** (a) With b^* as in (2.2) denote by $s := s(b^*) := \binom{m+b^*}{b^*}$ the dimension of the vector space of polynomials of degree b^* in m variables. In view of Theorem 2.1, and given $b \in \mathbb{N}^m$, checking the existence of a solution $x \in \mathbb{N}^n$ to $Ax = b$ reduces to checking whether or not there exists a nonnegative solution q to a system of linear equations

$$\mathrm{M}\, q \;=\; r; \quad q \geq 0 \tag{2.3}$$

for some matrix $\mathrm{M} \in \mathbb{Z}^{p\times ns}$, and vector $r \in \mathbb{Z}^p$, with :

- $n\,s$ variables $\{q_{j\alpha}\}$, the nonnegative coefficients of the Q_j's.
- p equations to identify the terms of same power in both sides of (2.1); obviously one has $p \leq s(b^*+a) := \binom{m+b^*+a}{b^*+a}$ (with $a := \max_k \sum_{j=1}^{m} A_{jk}$).

  In fact we may and will take $p = s(b^* + a)$.

This in turn reduces to solving a linear programming (LP) problem. Observe that in view of (2.1), the matrix of constraints $\mathrm{M} \in \mathbb{Z}^{p\times ns}$ which has only 0 and ± 1 coefficients, is *easily* deduced from A with *no* calculation (and is very sparse). The same is true for $r \in \mathbb{Z}^p$ which has only two non zero entries (equal to -1 and 1).

(b) In fact, from the proof of Theorem 2.1, it follows that one may even enforce the weights Q_j in (2.1) to be polynomials in $\mathbb{Z}[z_1, \ldots, z_m]$ (instead of $\mathbb{R}[z_1, \ldots, z_m]$) with nonnegative coefficients (and even with coefficients in $\{0, 1\}$) However, (a) above shows that the strength of Theorem 2.1 is precisely to allow $Q_j \in \mathbb{R}[z_1, \ldots, z_m]$ as it permits to check feasibility by solving a (continuous) linear program. Enforcing $Q_j \in \mathbb{Z}[z_1, \ldots, z_m]$ would result in an *integer* program of size larger than that of the original problem.

(c) Theorem 2.1 reduces the issue of existence of a solution $x \in \mathbb{N}^n$ to a particular *ideal membership problem*, that is, $Ax = b$ has a solution $x \in \mathbb{N}^n$ if and only if the polynomial $z^b - 1$ belongs to the *binomial ideal* $I = \langle z^{A_j} - 1\rangle_{j=1,\ldots,n} \subset \mathbb{R}[z_1, \ldots, z_m]$ *and* for some weights Q_j all with *non-negative coefficients.* In fact, one could prove Theorem 2.1 by an appropriate reduction of the initial problem of existence of a solution $x \in \mathbb{N}^n$ to $Ax = b$, to a polynomial ideal membership problem (with special features) in the framework developed in Mayr and Meyer [5, §3], another alternative to the proof in [3].

Next, with $A \in \mathbb{N}^{m\times n}, b \in \mathbb{N}^m$ let $\mathrm{P} \subset \mathbb{R}^n$ be the convex polyhedron

$$\mathrm{P} := \{x \in \mathbb{R}^n \quad | \quad Ax = b; \quad x \geq 0\}. \tag{2.4}$$

Similarly, with $\mathrm{M} \in \mathbb{Z}^{p\times ns}, r \in \mathbb{Z}^p$ as in (2.3), let

$$\Omega := \{q \in \mathbb{R}^{ns} \quad | \quad \mathrm{M}\, q = r; \quad q \geq 0\}, \tag{2.5}$$

be the convex polyhedron of feasible solutions $q \in \mathbb{R}^{ns}$ of (2.3). So, obviously, (2.1) holds if and only if $\Omega \neq \emptyset$.

Define the row vector $e_s := (1, \ldots, 1) \in \mathbb{R}^s$ and let $\mathrm{E} \in \mathbb{N}^{n\times ns}$ be the block diagonal matrix, whose each diagonal block is the row vector e_s, that is,

$$\mathrm{E} := \left[\begin{array}{cccc} e_s & 0 & \ldots & 0 \\ 0 & e_s & 0 & \ldots \\ \ldots & \ldots & \ldots & \ldots \\ 0 & \ldots & 0 & e_s \end{array}\right]. \tag{2.6}$$

Proposition 2.2. *Let $A \in \mathbb{N}^{m\times n}, b \in \mathbb{N}^m$ be given and* M *be as in (1.2). Let* P, Ω *be the convex polyhedra defined in (2.4)-(2.5).*

(a) *Let $q \in \Omega$. Then $x := \mathrm{E}q \in \mathrm{P}$. In particular, if $q \in \Omega \cap \mathbb{Z}^{ns}$ then $x \in \mathrm{P} \cap \mathbb{Z}^n$.*

(b) *Let $x \in \mathrm{P} \cap \mathbb{Z}^n$. Then $x = \mathrm{E}q$ for some $q \in \Omega \cap \mathbb{Z}^{ns}$.*

(c) *The matrix* M *is totally unimodular*

(d) *Whenever $\Omega \neq \emptyset$, each vertex of Ω is integral.*

Proof. (a) With $q \in \Omega$, let $\{Q_j\}_{j=1}^n \subset \mathbb{R}[z_1, \ldots, z_m]$ be the set of polynomials (with vector of nonnegative coefficients q) which satisfy (2.1). Taking the derivative of both sides of (2.1) with respect to z_k, at the point $z = (1, \ldots, 1)$, yields

$$b_k = \sum_{j=1}^{n} Q_j(1, \ldots, 1)A_{kj} = \sum_{j=1}^{n} A_{kj}x_j \quad k = 1, \ldots, n,$$

with $x_j := Q_j(1, \ldots, 1)$ for all $j = 1, \ldots, n$. Next, use the facts that (a) all the Q_j's have nonnegative coefficients $\{q_{j\alpha}\}$, and (b), $Q_j(1, \ldots, 1) = \sum_{\alpha\in\mathbb{N}^m} q_{j\alpha} = (\mathrm{E}q)_j$ for all $j = 1, \ldots, n$, to obtain $x := \mathrm{E}q \in \mathrm{P}$. Moreover, if $q \in \Omega \cap \mathbb{Z}^{ns}$ then obviously $x \in \mathrm{P} \cap \mathbb{Z}^n$.

(b) Let $x \in \mathrm{P} \cap \mathbb{Z}^n$ so that $x \in \mathbb{N}^n$ and $Ax = b$; write

$$z^b - 1 \,=\, z^{A_1 x_1} - 1 + z^{A_1 x_1}(z^{A_2 x_2} - 1) + \cdots + z^{\sum_{j=1}^{n-1} A_j x_j}(z^{A_n x_n} - 1),$$

and, whenever $x_j \neq 0$,

$$z^{A_j x_j} - 1 \,=\, (z^{A_j} - 1)\left[1 + z^{A_j} + \cdots + z^{A_j(x_j - 1)}\right] \quad j = 1, \ldots, n,$$

to obtain (2.1) with

$$z \,\mapsto\, Q_j(z) \,:=\, z^{\sum_{k=1}^{j-1} A_k x_k}\left[1 + z^{A_j} + \cdots + z^{A_j(x_j - 1)}\right],$$

and $Q_j \equiv 0$ if $x_j = 0$, $j = 1, \ldots, n$. We immediately see that each Q_j has all its coefficients $\{q_{j\alpha}\}$ nonnegative (and even in $\{0, 1\}$). Moreover, $Q_j(1, \ldots, 1) = x_j$ for all $j = 1, \ldots, n$, or equivalently, $x = \mathrm{E}q$ with $q \in \Omega \cap \mathbb{Z}^{ns}$.

(c) That M is totally unimodular follows from the fact that M is a *network matrix*, that is, a matrix with $\{0, \pm 1\}$ entries and with *exactly two* nonzero entries 1 and -1 in each column (see Schrijver [7, p. 274]). Indeed, from the identity (2.1), and the definition of M, each row of M is associated with a monomial z^α, with $\sum_j \alpha_j \leq b^* + a$. Thus, consider a particular column of M associated with the variable $q_{k\alpha}$ (the coefficient of the monomial z^α of the polynomial Q_k in (2.1), with $\sum_j \alpha_j \leq b^*$). From (2.1), the variable $q_{k\alpha}$ is only involved :

- in the row (or, equation) associated with the monomial z^α (with coefficient -1), and
- in the row (or, equation) associated with the monomial $z^{\alpha + A_k}$ (with coefficient $+1$).

(d) The right-hand-side r in the definition of Ω is integral. Therefore, as M is totally unimodular, whenever $\Omega \neq \emptyset$ each vertex of Ω is integral. $\square$

2.3. **Proof of Theorem 2.1.** (i) $\Rightarrow$ (ii) follows directly from Proposition 2.2(b) and the fact that (2.1) holds if and only if $\Omega \neq \emptyset$.

(ii) $\Rightarrow$ (i). Suppose (2.1) holds for some polynomials $\{Q_j\} \subset \mathbb{R}[z_1, \ldots, z_m]$. Then, $\Omega \neq \emptyset$ and so pick any vertex $\widehat{q}$ of Ω. By Proposition 2.2(d), $\widehat{q} \in \Omega \cap \mathbb{Z}^{ns}$ and by Proposition 2.2(a), $x := \mathrm{E}\widehat{q} \in \mathrm{P} \cap \mathbb{Z}^n$, that is, $Ax = b$ and $x \in \mathbb{N}^n$. $\square$

From Proposition 2.2(b) and its proof, one sees that q is a disaggregation of $x \in \mathbb{N}^n$. Indeed, if we write $q = (q_1, \ldots, q_n)$ then each q_j has exactly x_j nontrivial entries, all equal to 1. So q is *not* a *lifting* of x as in the lift-and-project procedures described in Laurent [4]. In the latter, x is part of the vector q in the augmented space, and is obtained by *projection* of q.

3. Main result

We first prove our results in the case $A \in \mathbb{N}^{m \times n}$ and then in §3.3, we show that the general case $A \in \mathbb{Z}^{m \times n}$ reduces to the former by adding one variable and one contraint to the original problem.

So let $A \in \mathbb{N}^{m\times n}, b \in \mathbb{N}^m$, and with no loss of generality, we may and will assume that every column of A has at least one non zero entry in which case P in (2.4) is a polytope.

Recall that with every solution $0 \leq q \in \mathbb{R}^{ns}$ of the linear system $\mathrm{M}q = r$ in (2.3) we may associate a set of polynomials $\{Q_j\} \subset \mathbb{R}[z_1, \ldots, z_m]$, with nonnegative coefficients, such that (2.1) is satisfied, and conversely to such a set of polynomials $\{Q_j\}$ with nonnegative coefficients, is associated a vector $0 \leq q \in \mathbb{R}^{ns}$ that satisfies (2.3). In fact, $q = \{q_{j\alpha}\}$ is the vector of coeffcients of the polynomials Q_j's in the (usual) basis of monomials.

3.1. **An equivalent linear program.** We now consider the integer program $\mathbb{P}$. For every $c \in \mathbb{R}^n$ let $\widehat{c} \in \mathbb{R}^{ns}$ be defined as

$$\widehat{c}' = (\widehat{c_1}', \ldots, \widehat{c_n}') \quad \text{with} \quad \widehat{c}_j' = c_j(1, \ldots, 1) \in \mathbb{R}^s \quad \forall j = 1, \ldots, n \tag{3.1}$$

Equivalently, $\widehat{c}' = c'\mathrm{E}$ with E as in (2.6). It also follows that $\widehat{c}'q = c'x$ whenever $x = \mathrm{E}q$. As a consequence of Theorem 2.1 we obtain immediately

Corollary 3.1. *Let $A \in \mathbb{N}^{m\times n}, b \in \mathbb{N}^m, c \in \mathbb{R}^n$ be given. Let $\mathrm{M} \in \mathbb{Z}^{p\times ns}, r \in \mathbb{Z}^p$ and $\mathrm{E} \in \mathbb{N}^{n\times ns}$, be as in (2.3) and (2.6), respectively.*

(a) *The integer program*

$$\mathbb{P} \rightarrow \quad \max_x \{\, c'x \quad | \quad Ax = b; \quad x \in \mathbb{N}^n\} \tag{3.2}$$

has same optimal value as the linear program

$$\mathbb{Q} \rightarrow \quad \max_{q\in\mathbb{R}^{ns}} \{\, \widehat{c}'\, q \quad | \quad \mathrm{M}\, q \,=\, r; \quad q \geq 0\} \tag{3.3}$$

(including the case $-\infty$).

(b) *In addition, let $q^* \in \mathbb{R}^{ns}$ be a vertex of Ω in (2.5), optimal solution of the linear program $\mathbb{Q}$. Then $x^* := \mathrm{E}q^* \in \mathbb{N}^n$ and x^* is an optimal solution of the integer program $\mathbb{P}$.*

Proof. Let $\max \mathbb{P}$ and $\max \mathbb{Q}$ denote the respective optimal values of $\mathbb{P}$ and $\mathbb{Q}$. We first treat the case $-\infty$. $\max \mathbb{P} = -\infty$ only if $\mathrm{P} \cap \mathbb{Z}^n = \emptyset$. But then $\Omega = \emptyset$ as well, which in turn implies $\max \mathbb{Q} = -\infty$. Indeed, by Theorem 2.1, if $\mathrm{P} \cap \mathbb{Z}^n = \emptyset$, i.e., if $Ax = b$ has no solution $x \in \mathbb{N}^n$, then one cannot find polynomials $\{\mathbb{Q}_j\} \subset \mathbb{R}[z_1, \ldots, z_m]$ with nonnegative coefficients, that satisfy (2.1). Therefore, from the definition of Ω, if $\Omega \neq \emptyset$ one would have a contradiction.

Conversely, if $\Omega = \emptyset$ (so that $\max \mathbb{Q} = -\infty$) then by definition of Ω, one cannot find polynomials $\{\mathbb{Q}_j\} \subset \mathbb{R}[z_1, \ldots, z_m]$ with nonnegative coefficients, that satisfy (2.1). Therefore, by Theorem 2.1, $Ax = b$ has no solution $x \in \mathbb{N}^n$ which in turn implies $\max \mathbb{P} = -\infty$, i.e., $\mathrm{P} \cap \mathbb{Z}^n = \emptyset$.

In the case when $\max \mathbb{P} \neq -\infty$, we necessarily have $\max \mathbb{P} < \infty$ because the convex polyhedron P is compact. Next, consider a feasible solution $q \in \Omega$ of $\mathbb{Q}$. From Proposition 2.2(a) $x := \mathrm{E}q \in \mathrm{P}$. Therefore, as x is bounded then so is $\mathrm{E}q$, which, in view of the definition (2.6) of E, also implies that q is bounded. Hence Ω is compact which in turn implies that the optimal value of $\mathbb{Q}$ is finite and attained at some vertex q^* of Ω.

Now, let $x^* \in \mathbb{N}^n$ be an optimal solution of $\mathbb{P}$. By Proposition 2.2(b) there exists some $q \in \Omega$ with $\mathrm{E}q = x^*$. From the definition (3.1) of the vector $\widehat{c}$ we have

$$\widehat{c}'q \,=\, c'\mathrm{E}q \,=\, c'x^*,$$

which implies $\max \mathbb{Q} \geq \max \mathbb{P}$.

On the other hand, let $q^* \in \Omega$ be a vertex of Ω, optimal solution of $\mathbb{Q}$. By Proposition 2.2(d), $q^* \in \Omega \cap \mathbb{Z}^{ns}$ and by Proposition 2.2(a), $x := \mathrm{E}q^* \in \mathrm{P} \cap \mathbb{Z}^n$, that is, $x \in \mathbb{N}^n$ is a feasible solution of $\mathbb{P}$. Again, from the definition (3.1) of the vector $\widehat{c}$ we have

$$c'x \,=\, c'\mathrm{E}q^* \,=\, \widehat{c}'q^*,$$

which, in view of $\max \mathbb{P} \leq \max \mathbb{Q}$, implies $\max \mathbb{P} = \max \mathbb{Q}$, and $x \in \mathbb{N}^n$ is an optimal solution of $\mathbb{P}$. This completes the proof of (a) and (b). □

Remark 3.2. Let $b^* := \sum_j b_j - \min_k \sum_{j=1}^m A_{jk}$ and $a := \max_k \sum_{j=1}^m A_{jk}$. From the discussion right after Theorem 2.1, $\mathrm{M} \in \mathbb{Z}^{p \times ns}$ where

$$p \,=\, \binom{m + b^* + a}{b^* + a} \,=\, p_1(m) \qquad \text{and} \quad s \,=\, \binom{m + b^*}{b^*} \,=\, p_2(m).$$

The polynomial $m \mapsto p_1(m)$ has degree $b^* + a$ whereas the polynomial $m \mapsto p_2(m)$ has degree b^*. Moreover, all the entries of M, r are $0, \pm 1$. Let $\mathcal{M}$ be the class of integer programs $\mathbb{P}$ with $A \in \mathbb{N}^{m \times n}, b \in \mathbb{N}^m$, and where, uniformly in $\mathbb{P} \in \mathcal{M}$,

- the column sums of A are bounded (i.e., $\sup_k \sum_j A_{jk}$ is bounded), and
- $\sum_j b_j$ is bounded,

so that a and b^* above are bounded, uniformly in $\mathbb{P} \in \mathcal{M}$. Then one may solve the integer programs $\mathbb{P}$ of the class $\mathcal{M}$ in time polynomial in the problem size, because it suffices to solve the linear program $\mathbb{Q}$ which has $p_1(m)$ constraints and $np_2(m)$ variables. One may consider this result as a dual counterpart of the known result which states that integer programs are solvable in time polynomial in the problem size when the dimension n is fixed. (A dual counterpart would not be that integer programs are solvable in time polynomial in the problem size when the number of constraints m is fixed. Just think of the knapsack problem where $m = 1$.)

3.2. **The integer hull.** We are now interested in describing the integer hull P_1 of P, i.e., the convex hull of $\mathrm{P} \cap \mathbb{Z}^n$.

Theorem 3.3. *Let $A \in \mathbb{N}^{m \times n}, b \in \mathbb{N}^m$, and let $\mathrm{E} \in \mathbb{N}^{n \times ns}, \mathrm{M} \in \mathbb{Z}^{p \times ns}, r \in \mathbb{Z}^p$ be as in (2.6) and (2.3), respectively.*

Let $\{(u^k, v^k)\}_{k=1}^t \subset \mathbb{R}^{n \times p}$ be a (finite) set of generators of the convex cone $C \subset \mathbb{R}^{n \times p}$ defined by

$$C \,:=\, \{(u, v) \in \mathbb{R}^{n \times p} \,|\, \mathrm{E}'u + \mathrm{M}'v \,\geq\, 0\}. \tag{3.4}$$

(a) *The integer hull P_1 of P is the convex polyhedron defined by the linear constraints*

$$\langle u^k, x \rangle + \langle v^k, r \rangle \,\geq\, 0 \qquad \forall k = 1, \ldots, t, \tag{3.5}$$

or, equivalently,

$$\mathrm{P}_1 := \{x \in \mathbb{R}^n \quad | \quad U\,x \,\geq\, u\}, \tag{3.6}$$

where the matrix $U \in \mathbb{R}^{t\times n}$ *has row vectors* $\{u^k\}$*, and the vector* $u \in \mathbb{R}^t$ *has coordinates* $u_k = \langle -v^k, r\rangle$*,* $k = 1,\ldots,t$*.*

(b) *Equivalently* $\mathrm{P}_1 = \mathrm{E}\,(\Omega)$*.*

Proof. (a) Given $x \in \mathbb{R}^n$, consider the following linear system :

$$\left\{\begin{array}{lcl} \mathrm{E}\,q & = & x \\ \mathrm{M}\,q & = & r \\ q & \geq & 0 \end{array}\right. \tag{3.7}$$

where M, E are defined in (2.3) and (2.6) respectively. Invoking the celebrated Farkas lemma (see e.g. Schrijver [7]), the system (3.7) has a solution $q \in \mathbb{R}^{ns}$ if and only if (3.5) holds.

Therefore, let $x \in \mathbb{R}^n$ satisfy $Ux \geq u$ with U, u as in (3.6). By Farkas lemma, the system (3.7) has a solution $q \in \mathbb{R}^{ns}$, that is, $\mathrm{M}q = r, q \geq 0$ and $x = \mathrm{E}q$. As $q \in \Omega$ and Ω is compact, q is a convex combination $\sum_k \gamma_k \widehat{q}^k$ of the vertices $\{\widehat{q}^k\}$ of Ω. By Proposition 2.2(d) and (a), for each vertex $\widehat{q}^k$ of Ω we have $\widehat{x}^k := \mathrm{E}\widehat{q}^k \in \mathrm{P} \cap \mathbb{Z}^n$. Therefore,

$$x \,=\, \mathrm{E}q \,=\, \sum_k \gamma_k \mathrm{E}\widehat{q}^k \,=\, \sum_k \gamma_k \widehat{x}^k, \tag{3.8}$$

that is, x is a convex combination of points $\widehat{x}^k \in \mathrm{P} \cap \mathbb{Z}^n$, i.e., $x \in \mathrm{P}_1$; hence $\{x \in \mathbb{R}^n \,|\, Ux \geq u\} \subseteq \mathrm{P}_1$.

Conversely, let $x \in \mathrm{P}_1$, i.e., $x \in \mathbb{R}^n$ is a convex combination $\sum_k \gamma_k \widehat{x}^k$ of points $\widehat{x}^k \in \mathrm{P} \cap \mathbb{Z}^n$. By Proposition 2.2(b), for each k, $\widehat{x}^k = \mathrm{E}q^k$ for some vector $q^k \in \Omega \cap \mathbb{Z}^{ns}$. Therefore, as each $(\widehat{x}^k, q^k)$ satisfies (3.7), then so does their convex combination $(x, q) := \sum_k \gamma_k (\widehat{x}^k, q^k)$. By Farkas lemma again, we must have $Ux \geq u$, and so, $\mathrm{P}_1 \subseteq \{x \in \mathbb{R}^n \,|\, Ux \geq u\}$, which completes the proof.

(b) This follows directly from (a) and

$$\mathrm{E}(\Omega) \,=\, \{x \in \mathbb{R}^n \,|\, x \,=\, \mathrm{E}\,q\,; \quad \mathrm{M}\,q \,=\, r; \quad q \geq 0\}.$$

$\square$

Observe that the convex cone C in (3.4) of Theorem 3.3 is defined *explicitly* in terms of the initial data A, and with *no* calculation. Indeed, the matrix M in (2.3) is easily obtained from A and E is explictly given in (2.6). Thus, the interest of Theorem 3.3 is that we obtain an algebraic characterization (3.6) of P_1 via generators of a cone C simply related to A.

From the proof of Theorem 3.3, every element (u, v) of the cone C produces a valid inequality for P_1, and clearly, all strong valid inequalities can be obtained from *generators* of C.

Next suppose that for some $a \in \mathbb{R}^n, w \in \mathbb{R}$, we want to test whether $a'x \geq w$ is a valid inequality. If there is some $v \in \mathbb{R}^p$ such that $\mathrm{M}'v \geq -\mathrm{E}'a$

and $-v'r \geq w$, then indeed, $a'x \geq w$ is a valid inequality. In fact w can be improved to $\tilde{w}$ with

$$\tilde{w} := \max_v \{-v'r \quad | \quad \mathrm{M}'v \geq -\mathrm{E}'a\}.$$

3.3. **The general case $A \in \mathbb{Z}^{m\times n}$.** In this section we consider the case where $A \in \mathbb{Z}^{m\times n}$, that is, A may have negative entries. We will assume that the convex polyhedron $\mathrm{P} \subset \mathbb{R}^n$, defined in (2.4) is compact.

Let $\alpha \in \mathbb{N}^n, \beta \in \mathbb{N}$ be such that for all $j = 1, \dots, m$,

$$\widehat{b}_j := b_j + \beta \geq 0; \quad \widehat{A}_{jk} := A_{jk} + \alpha_k \geq 0; \quad k = 1, \dots, n. \tag{3.9}$$

As P is compact we have

$$\max_{x\in\mathbb{N}^n} \{\sum_{j=1}^n \alpha_j x_j \,|\, Ax = b\} \leq \max_{x\in\mathbb{R}^n; x\geq 0} \{\sum_{j=1}^n \alpha_j x_j \,|\, Ax = b\} =: \rho^*(\alpha) < \infty.$$

Given $\alpha \in \mathbb{N}^n$, the scalar $\rho^*(\alpha)$ is easily calculated by solving a LP problem. Note that we can choose $\beta \in \mathbb{N}$ as large as desired. Therefore, choose $\rho^*(\alpha) \leq \beta \in \mathbb{N}$. Let $\widehat{A} \in \mathbb{N}^{m\times n}, \widehat{b} \in \mathbb{N}^m$ be as in (3.9) with $\beta \geq \rho^*(\alpha)$.

The feasible solutions $x \in \mathbb{N}^n$ of $Ax = b$, i.e., the points of $\mathrm{P} \cap \mathbb{Z}^n$, are in one-to-one correspondance with the solutions $(x, u) \in \widehat{\mathrm{P}} \cap \mathbb{Z}^{n+1}$ where $\widehat{\mathrm{P}} \subset \mathbb{R}^{n+1}$ is the convex polytope

$$\widehat{\mathrm{P}} := \left\{(x,u) \in \mathbb{R}^n \times \mathbb{R} \,|\, \left\{ \begin{array}{rcl} \widehat{A}x + e_m u & = & \widehat{b} \\ \alpha' x + u & = & \beta \end{array} \right. ; \quad x, u \geq 0 \right\}; \tag{3.10}$$

Indeed, if $x \in \mathrm{P} \cap \mathbb{Z}^n$, i.e., $Ax = b$ with $x \in \mathbb{N}^n$, then

$$Ax + e_m \sum_{j=1}^n \alpha_j x_j - e_m \sum_{j=1}^n \alpha_j x = b + (\beta - \beta)e_m,$$

or equivalently,

$$\widehat{A}x + \left(\beta - \sum_{j=1}^n \alpha_j x_j\right) e_m = \widehat{b},$$

and thus, as $\beta \geq \rho^*(\alpha) \geq \alpha' x$, letting $u := \beta - \alpha' x \in \mathbb{N}$, yields $(x, u) \in \widehat{\mathrm{P}} \cap \mathbb{Z}^{n+1}$. Conversely, let $(x, u) \in \widehat{\mathrm{P}} \cap \mathbb{Z}^{n+1}$. Using the definitions of $\widehat{A}$ and $\widehat{b}$, it then follows immediately that

$$Ax + e_m \sum_{j=1}^n \alpha_j x_j + u e_m = b + \beta e_m; \quad \sum_{j=1}^n \alpha_j x_j + u = \beta,$$

so that $Ax = b$ with $x \in \mathbb{N}^n$, i.e., $x \in \mathrm{P} \cap \mathbb{Z}^n$. In other words,

$$x \in \mathrm{P} \cap \mathbb{Z}^n \quad \Leftrightarrow \quad (x, \beta - \alpha' x) \in \widehat{\mathrm{P}} \cap \mathbb{Z}^{n+1}. \tag{3.11}$$

The convex polytope $\widehat{\mathrm{P}}$ can be written

$$\widehat{\mathrm{P}} := \{(x,u) \in \mathbb{R}^{n+1} \,|\, B \begin{bmatrix} x \\ u \end{bmatrix} = (\widehat{b}, \beta); \quad x, u \geq 0\}, \tag{3.12}$$

with

$$B := \left[\begin{array}{ccc} \widehat{A} & \mid & e_m \\ - & & - \\ \alpha' & \mid & 1 \end{array}\right].$$

As $B \in \mathbb{N}^{(m+1)\times(n+1)}$, we are back to the case analyzed in §3.1 and §3.2.

In particular, the integer program $\mathbb{P} \to \max\{c'x \mid Ax = b; x \in \mathbb{N}^n\}$ is equivalent to the integer program

$$\text{(3.13)} \qquad \widehat{\mathbb{P}} \quad \to \max\left\{ c'x \quad \mid \quad B\left[\begin{array}{c} x \\ u \end{array}\right] = \left[\begin{array}{c} \widehat{b} \\ \beta \end{array}\right]; \quad (x,u) \in \mathbb{N}^n \times \mathbb{N}\right\}.$$

Hence, Theorem 2.1, Proposition 2.2, Corollary 3.1 and Theorem 3.3 are still valid with $B \in \mathbb{N}^{(m+1)\times(n+1)}$ in lieu of $A \in \mathbb{N}^{m\times n}$, $(\widehat{b}, \beta) \in \mathbb{N}^m \times \mathbb{N}$ in lieu of $b \in \mathbb{N}^m$, and $\widehat{\mathrm{P}} \subset \mathbb{R}^{n+1}$ in lieu of $\mathrm{P} \subset \mathbb{R}^n$.

So again, as in previous sections, the polytope $\widehat{\Omega}$ associated with $\widehat{\mathrm{P}}$ is explicitly defined from the initial data A, because $\widehat{A}$ is simply defined from A and α. In turn, as the convex cone C in Theorem 3.3 is also defined explicitly from A via M, again one obtains a simple characterization of the integer hull $\widehat{\mathrm{P}}_1$ of $\widehat{\mathrm{P}}$ via the generators of C.

If we are now back to the initial data A, b then P_1 is easily obtained from $\widehat{\mathrm{P}}_1$. Indeed, by Theorem 3.3, let

$$\widehat{\mathrm{P}}_1 = \{(x,u) \in \mathbb{R}^{n+1} \mid \langle w^k, x\rangle + \delta^k u \geq \rho^k; \quad k = 1, \ldots, t\},$$

for some $\{(w^k, \delta^k) \in \mathbb{R}^n \times \mathbb{R}\}_{k=1}^t$, and some $t \in \mathbb{N}$. Then from (3.11) it immediately follows that

$$\mathrm{P}_1 = \{x \in \mathbb{R}^n \mid \langle w^k - \delta^k \alpha, x\rangle \geq \rho^k - \beta\delta^k; \quad k = 1, \ldots, t\}.$$

3.4. **0-1 integer programs.** The extension to 0-1 integer programs

$$\max_x \{ c'x \quad \mid \quad Ax = b; \quad x \in \{0,1\}^n\},$$

is straightforward by considering the equivalent integer program

$$\max_{x,u} \{ c'x \quad \mid \quad Ax = b; \quad x_j + u_j = 1\, \forall j = 1, \ldots, n; \quad (x,u) \in \mathbb{N}^n \times \mathbb{N}^n\},$$

which is an integer program in the form (1.1). However, the resulting linear equivalent program $\mathbb{Q}$ of Corollary 3.1 is now more complicated. For instance, if $A \in \mathbb{N}^{m\times n}$, then $q \in \mathbb{R}^{2ns}$ and s is now the dimension of the vector space of polynomials in $n+m$ variables and of degree at most $n + \sum_j b_j$.

4. Conclusion

We have presented an explicit algebraic characterization of the integer hull P_1 of a convex polytope $\mathrm{P} \subset \mathbb{R}^n$. Indeed, the defining hyperplanes of P_1 are obtained from the generators of a convex cone whose description is obtained from the data A, b with no calculation. Of course, and as expected, this convex cone is in a space of large dimension (exponential in the problem size). However, this structural result shows that all *strong valid inequalities* can be obtained in this manner. Therefore, we hope this result to be helpful in deriving strong valid inequalities, or in validating some candidate inequalities, at least for some specific polytopes P.

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LAAS-CNRS, 7 avenue du Colonel Roche, 31 077 Toulouse cédex 4, France
E-mail address: lasserre@laas.fr