Applied Mathematics and Computation xxx (2009) xxx–xxx

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

Extension of the Lasserre–Avrachenkov theorem on the integral of multilinear forms over simplices

Mohammadali Khosravifard [a], Morteza Esmaeili [b,*], Hossein Saidi [a]

[a] Department of Electrical and Computer Engineering, Isfahan University of Technology, 84156-83111 Isfahan, Iran
[b] Department of Mathematical Sciences, Isfahan University of Technology, 84156-83111 Isfahan, Iran

ARTICLE INFO

Keywords:
Integration over simplices
Homogeneous polynomial
Multilinear symmetric Form
Quasilinear form

ABSTRACT

The Lasserre–Avrachenkov theorem on integration of symmetric multilinear forms over simplices establishes a method (called LA) for integrating homogeneous polynomials over simplices. Although the computational complexity of LA is generally much higher than that of the other known methods (e.g. Grundmann–Moller formula), it is still useful in deriving closed-form expressions for the value of such integrals. However, LA cannot be directly applied for nonhomogeneous polynomials. It is shown in this paper that Lasserre–Avrachenkov theorem holds for a wider class of symmetric forms, to be called *quasilinear forms*. This extension can substantially facilitate derivation of a closed-form expression (not computation) for integral of some nonhomogeneous polynomials (such as $\prod_{j=1}^{q}(b_j + \sum_{i=1}^{n} c_{i,j}x_i)$) over simplices.

1. Introduction

The problem of integrating a real function over a subset of $\mathbb{R}^n$ arises in a wide range of theoretical and applied problems. Among the important subsets of $\mathbb{R}^n$, n-dimensional simplices convey special significance from a pragmatic point of view, as complex volumes can be decomposed to and/or approximated by simplices. Triangles and tetrahedrons play such a role in $\mathbb{R}^2$ and $\mathbb{R}^3$ respectively.

On the other hand, besides the natural appearance of polynomials in many applications, other real functions can also be approximated by polynomials of proper degrees. Therefore, the integral of polynomials over simplices are of great importance. Besides the well-known numerical methods [1,2,7], one may use the *exact* formulas [3,8] for computing such integrals. Moreover, the recent results due to Lasserre and Avrachenkov [5] can be employed for formulating these integrals.

In order to develop an n-dimensional counterpart for the well-known formula

$$\int_a^b x^q dx = \frac{b^{q+1} - a^{q+1}}{1+q} = \frac{b-a}{1+q}\left(a^q + a^{q-1}b + \cdots + ab^{q-1} + b^q\right),$$

Lasserre and Avrachenkov proved Theorem 1. Let us briefly remind the terminology before stating the theorem.

A polynomial $f : \mathbb{R}^n \to \mathbb{R}$ is called q-homogeneous if $f(\mu X) = \mu^q f(X)$ for all $\mu \in \mathbb{R}$ and $X \in \mathbb{R}^n$. A form $\mathbf{M} : (\mathbb{R}^n)^q \to \mathbb{R}$ is said to be symmetric if the value $\mathbf{M}(X_1, X_2, \ldots, X_q)$ is invariant under any permutation of the variables $X_1, X_2, \ldots, X_q \in \mathbb{R}^n$, and it is called a multilinear form (q-linear) [6] if for all $a, b \in \mathbb{R}$ and $1 \leqslant i \leqslant q$ we have

$$\mathbf{M}(X_1, \ldots, aX_i' + bX_i'', \ldots, X_q) = a\mathbf{M}(X_1, \ldots, X_i', \ldots, X_q) + b\mathbf{M}(X_1, \ldots, X_i'', \ldots, X_q).$$

* Corresponding author.
E-mail addresses: khosravi@cc.iut.ac.ir (M. Khosravifard), emorteza@cc.iut.ac.ir (M. Esmaeili), hsaidi@cc.iut.ac.ir (H. Saidi).

0096-3003/$ - see front matter © 2009 Elsevier Inc. All rights reserved.
doi:10.1016/j.amc.2009.02.005

Theorem 1 (Lasserre–Avrachenkov [5]). *Let $V_0, V_1, V_2, \ldots, V_n$ be the vertices of an n-dimensional simplex Ψ_n. Then, for a symmetric q-linear form $\mathbf{M} : (\mathbb{R}^n)^q \to \mathbb{R}$, we have*

$$\int_{\Psi_n} \mathbf{M}(X, X, X, \ldots, X) d_X = \frac{vol(\Psi_n)}{\binom{n+q}{q}} \sum_{0 \leqslant i_1 \leqslant i_2 \leqslant \cdots \leqslant i_q \leqslant n} \mathbf{M}(V_{i_1}, V_{i_2}, \ldots, V_{i_q}), \tag{1}$$

where $vol(\Psi_n) = \int_{\Psi_n} d_X$ stands for the volume of the simplex Ψ_n. □

To an arbitrary q-homogeneous polynomial $f(X)$, one can associate a symmetric q-linear form $\mathbf{M}_f(X_1, X_2, \ldots, X_q)$ satisfying $f(X) = \mathbf{M}_f(X, X, \ldots, X)$ [5,9]. Hence, if we can find the associated symmetric multilinear form $\mathbf{M}_f$, then $\int_{\Psi_n} f(X) d_X$ is evaluated using Theorem 1. Hereafter, this approach is referred to by LA.

Example 1. The integral of the 2-homogeneous polynomial $g(x, y, z, u) = x^2 + yz + u^2$ over a 4-dimensional simplex Θ with vertices $T_0, T_1, T_2, T_3, T_4 \in \mathbb{R}^4$, can be evaluated by applying Theorem 1 to the associated symmetric multilinear form

$$\mathbf{M}_g\left(\begin{bmatrix} x_1 \\ y_1 \\ z_1 \\ u_1 \end{bmatrix}, \begin{bmatrix} x_2 \\ y_2 \\ z_2 \\ u_2 \end{bmatrix}\right) = x_1x_2 + \frac{1}{2}(y_1z_2 + y_2z_1) + u_1u_2,$$

that is

$$\int_{\Theta} (x^2 + yz + u^2) dxdydzdu = \frac{vol(\Theta)}{\binom{6}{2}} \sum_{i_1=0}^{4} \sum_{i_2=i_1}^{4} \mathbf{M}_g(T_{i_1}, T_{i_2}).$$

It is noticeable that even if finding the associated symmetric multilinear form $\mathbf{M}_f$ is an easy task, the computational complexity of LA might be much higher than other exact formulas. In particular, with Grundmann–Moller formula [3] the exact value of $\int_{\Psi_n} f(X) d_X$ is evaluated in terms of the values of $f(X)$ at $\binom{n + \lceil \frac{q-1}{2} \rceil + 1}{n+1}$ definite points, while with LA approach the exact value of $\int_{\Psi_n} f(X) d_X$ for a q-homogeneous polynomial $f(X)$ is given in terms of the values of $\mathbf{M}_f(X_1, X_2, \ldots, X_q)$ at $\binom{n+q}{q}$ groups of vertices. But sometimes we need a closed-form expression for the value of such integrals in terms of some parameters. In such cases LA approach would be helpful provided that $\mathbf{M}_f$ can be obtained easily.

Example 2. Consider the q-homogeneous polynomials of the form $w(X) = w(x_1, x_2, \ldots, x_n) = \prod_{j=1}^{q} L_j(X)$ where $L_j(X)$'s are linear combinations of the variables, i.e. $L_j(X) = \sum_{i=1}^{n} c_{i,j} x_i$. Linearity of $L_j(X)$ (i.e. $L_j(aX + bX') = aL_j(X) + bL_j(X')$) implies multilinearity of the symmetric form

$$\mathbf{M}_w(X_1, X_2, \ldots, X_q) = \frac{1}{q!} \sum_{\sigma \in S_q} \prod_{j=1}^{q} L_j(X_{\sigma(j)}),$$

where S_q is the set of all permutations $(\sigma(1), \sigma(2), \ldots, \sigma(q))$ on q objects ($|S_q| = q!$). Hence, it follows from $\mathbf{M}_w(X, X, \ldots, X) = w(X)$ and Theorem 1 that

$$\int_{\Psi_n} w(X) d_X = \frac{vol(\Psi_n)}{\binom{n+q}{q}} \times \frac{1}{q!} \sum_{0 \leqslant i_1 \leqslant i_2 \leqslant \cdots \leqslant i_q \leqslant n} \sum_{\sigma \in S_q} \prod_{j=1}^{q} L_j\left(V_{i_{\sigma(j)}}\right), \tag{2}$$

where V_i, $0 \leqslant i \leqslant n$, are vertices of the n-dimensional simplex Ψ_n. Therefore, Theorem 1 offers a closed-form expression for the integral of q-homogeneous polynomial $w(X)$.

The beauty and importance of Theorem 1 is that it gives the *exact* value of such integrals directly in terms of the vertices of the underlying simplex. Furthermore, one can decompose an arbitrary nonhomogeneous polynomial into homogeneous ones and then apply this technique to each of them. However, an arbitrary nonhomogeneous polynomial of high order may be decomposed into lots of homogeneous polynomials and consequently formulating its integral would not be an easy task. In such cases a direct approach (rather than decomposing the polynomial), which may simplify this formulation, is desired. In this paper we attempt to achieve such a direct approach.

In order to design some codes in an information theory problem [4] we were to derive closed-form expression for the value of the integrals

$$I(n, k, m) := \int_{\Lambda^n} \left(1 - \sum_{i=k}^{n} x_i\right)^m d_X, \tag{3}$$

for $m, n \in IN$ and $1 \leqslant k \leqslant n$, where Λ^n is the n-dimensional simplex defined by

$$\Lambda^n = \left\{ (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n \middle| 1 - \sum_{i=1}^n x_i \geqslant x_1 \geqslant x_2 \geqslant \cdots \geqslant x_n \geqslant 0 \right\}.$$

In information theory Λ^n represents the set of monotone sources with $n+1$ symbols. In spite of the intensive similarity between the integrand $(1 - \sum_{i=k}^n x_i)^m$ and $w(X)$ (given in Example 2), $(1 - \sum_{i=k}^n x_i)^m$ is not homogeneous. Therefore, at the first glance, Theorem 1 cannot be directly applied to this case (i.e. (3)). This problem was the motivation behind extending Theorem 1 which makes it applicable to arbitrary polynomials.

It should be stressed that our goal in this paper is not computation of the exact value of the integrals (such as (3)) and hence the computational complexity is of no importance. By contrast, we need a closed-form expression for the value of such integrals which will be used for the next analytical purposes.

In the next section, quasilinear forms are defined and the Lasserre–Avrachenkov theorem is extended for them. The usefulness of this extension is described in Section 3 followed by a conclusion in Section 4.

2. Extension of Lasserre–Avrachenkov theorem

In the proof of Theorem 1 Lasserre and Avrachenkov used q-linearity of the form $\mathbf{M}$ to write

$$\begin{aligned} &\mathbf{M}\left(Y_1, Y_2, \ldots, Y_{i-1}, \left(1 - \sum_{j=1}^n \lambda_j\right) V_0 + \sum_{j=1}^n \lambda_j V_j, Y_{i+1}, \ldots, Y_q\right) \\ &= \left(1 - \sum_{j=1}^n \lambda_j\right) \mathbf{M}(Y_1, Y_2, \ldots, Y_{i-1}, V_0, Y_{i+1}, \ldots, Y_q) + \sum_{j=1}^n \lambda_j \mathbf{M}(Y_1, Y_2, \ldots, Y_{i-1}, V_j, Y_{i+1}, \ldots, Y_q), \end{aligned} \tag{4}$$

for $1 \leqslant i \leqslant q$, where

$$\lambda_j \geqslant 0 \text{ for } 1 \leqslant j \leqslant n, \quad \text{and} \quad \sum_{j=1}^n \lambda_j \leqslant 1, \tag{5}$$

and Y_k $(1 \leqslant k \leqslant q)$ are linear combinations of the vertices $V_0, V_1, V_2, \ldots, V_n$.

Setting $c_0 = 1 - \sum_{j=1}^n \lambda_j$ and $c_j = \lambda_j$ for $1 \leqslant j \leqslant n$, we may rewrite (4) and (5) as

$$\mathbf{M}\left(Y_1, Y_2, \ldots, Y_{i-1}, \sum_{j=0}^n c_j V_j, Y_{i+1}, \ldots, Y_q\right) = \sum_{j=0}^n c_j \mathbf{M}(Y_1, Y_2, \ldots, Y_{i-1}, V_j, Y_{i+1}, \ldots, Y_q), \tag{6}$$

where

$$c_j \geqslant 0 \text{ for } 0 \leqslant j \leqslant n \quad \text{and} \quad \sum_{j=0}^n c_j = 1. \tag{7}$$

Therefore, in the proof process of Theorem 1 we do not really need the multilinearity of $\mathbf{M}$. What we need is to assume that $\mathbf{M}$ satisfies (6) under the constraints given by (7). Compared to multilinearity, this is trivially a *weaker* condition on $\mathbf{M}$ (since the sum of coefficients c_j's is 1). Hence Theorem 1 can be applied to a wider class of symmetric forms.

Example 3. Given the coefficients α, β and γ, the following symmetric form $\mathbf{R}$ is multilinear iff $\alpha = \beta = 0$,

$$\mathbf{R}\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \right) = \alpha + \beta(x_2 + y_2) + \gamma(x_3 y_1 + x_1 y_3).$$

However, it is not hard to show that if $\sum_{j=0}^n c_j = 1$ then

$$\mathbf{R}\left(\sum_{j=0}^n c_j X_j, Y\right) = \sum_{j=0}^n c_j \mathbf{R}(X_j, Y) \quad \text{and} \quad \mathbf{R}\left(X, \sum_{j=0}^n c_j Y_j\right) = \sum_{j=0}^n c_j \mathbf{R}(X, Y_j).$$

Therefore, Theorem 1 can be applied to $\mathbf{R}$ for arbitrary values of α, β and γ.

Definition 1 (*Quasilinear form*). A form $\mathbf{Q} : (\mathbb{R}^n)^q \to \mathbb{R}$ is called q-quasilinear if for all $0 \leqslant c \leqslant 1$ and $1 \leqslant i \leqslant q$ we have

$$\begin{aligned} \mathbf{Q}(X_1, \ldots, X_{i-1}, cX_i' + (1-c)X_i'', X_{i+1}, \ldots, X_q) &= c\mathbf{Q}(X_1, \ldots, X_{i-1}, X_i', X_{i+1}, \ldots, X_q) \\ &\quad + (1-c)\mathbf{Q}(X_1, \ldots, X_{i-1}, X_i'', X_{i+1}, \ldots, X_q). \end{aligned} \tag{8}$$

It is clear that any q-linear form is q-quasilinear. Noting the following lemma we can conclude that a q-quasilinear form $\mathbf{Q}$ satisfies (8) for all $c \in \mathbb{R}$.

Please cite this article in press as: M. Khosravifard et al., Extension of the Lasserre–Avrachenkov theorem on the integral ..., Appl. Math. Comput. (2009), doi:10.1016/j.amc.2009.02.005

Lemma 1. *If a function $f:\mathbb{R}^n \to \mathbb{R}$ satisfies*

$$f(cX_1 + (1-c)X_2) = cf(X_1) + (1-c)f(X_2), \tag{9}$$

for all $X_1, X_2 \in IR^n$ and $0 \leqslant c \leqslant 1$, then it satisfies (9) for all $c \in \mathbb{R}$.

Proof. Let $t < 0$ and define $X_3 = tX_1 + (1-t)X_2$. Then we may write $X_2 = c'X_3 + (1-c')X_1$ where $c' = \frac{1}{1-t}$. Since $0 < c' < 1$, we can use (9) for X_1 and X_3 and write $f(X_2) = c'f(X_3) + (1-c')f(X_1)$ which implies $f(X_2) = \frac{1}{1-t}f(tX_1 + (1-t)X_2) + (1 - \frac{1}{1-t})f(X_1)$ whence

$$f(tX_1 + (1-t)X_2) = tf(X_1) + (1-t)f(X_2), \tag{10}$$

for $t < 0$. Similarly, for $t > 1$ we can define $X_3 = tX_1 + (1-t)X_2$ and write $X_1 = c''X_3 + (1-c'')X_2$ (where $c'' = \frac{1}{t}$) and conclude that (10) is satisfied. □

Moreover, it is easy to prove by induction on m that

$$\mathbf{Q}\left(X_1, \ldots, X_{i-1}, \sum_{j=1}^{m} c_j X'_j, X_{i+1}, \ldots, X_q\right) = \sum_{j=1}^{m} c_j \mathbf{Q}\left(X_1, \ldots, X_{i-1}, X'_j, X_{i+1}, \ldots, X_q\right),$$

is satisfied for a q-quasilinear form $\mathbf{Q}$, $1 \leqslant i \leqslant q$, $m \in IN$ and $(c_1, c_2, \ldots, c_m) \in IR^m$ with the constraint $\sum_{i=1}^{m} c_i = 1$. Accordingly, under the constraint (7), a q-quasilinear form satisfies (6) and hence the following theorem is proven.

Theorem 2 (Extension of Lasserre–Avrachenkov Theorem). *The q-Linearity condition in Theorem 1 can be replaced by q-quasilinearity condition which is a weaker condition.*

The relation between Theorem 2 and the integral of polynomials over the simplices is clarified by the next theorem, which guarantees the existence of an associated symmetric q-quasilinear form for any polynomial of order q.

Theorem 3. *For an arbitrary polynomial $f(X):\mathbb{R}^n \to \mathbb{R}$ of order q, there exists a symmetric q-quasilinear form $\mathbf{Q}_f(X_1, X_2, \ldots, X_q)$ for which*

$$\mathbf{Q}_f(X, X, \ldots, X) = f(X).$$

Proof. Clearly, a given polynomial $f(X)$ of order q can be decomposed into a constant term f_0 and q homogeneous polynomials $f_1(X), f_2(X), \ldots, f_q(X)$ of orders 1, 2, …, q, respectively, that is $f(X) = f_0 + \sum_{i=1}^{q} f_i(X)$. Using a polarization formula [5,9], each $f_i(X)$ can be associated with an i-linear form $\mathbf{M}_i$ such that $\mathbf{M}_i(X, X, \ldots, X) = f_i(X)$. It is easy to show that the form $\mathbf{H}_f(X_1, X_2, \ldots, X_q) = f_0 + \sum_{i=1}^{q} \mathbf{M}_i(X_1, X_2, \ldots, X_i)$ is a q-quasilinear one for which

$$\mathbf{H}_f(X, X, \ldots, X) = f(X).$$

Therefore,

$$\mathbf{Q}_f(X_1, X_2, \ldots, X_q) = \frac{1}{q!} \sum_{\sigma \in S_q} \mathbf{H}_f\left(X_{\sigma(1)}, X_{\sigma(2)}, \ldots, X_{\sigma(q)}\right),$$

is a symmetric q-quasilinear form and

$$\mathbf{Q}_f(X, X, \ldots, X) = \mathbf{H}_f(X, X, \ldots, X) = f(X). \qquad \square$$

Theorem 3 is proved just to assure us that such a symmetric q-quasilinear form $\mathbf{Q}_f\left(X_1, X_2, \ldots, X_q\right)$ does exist for any arbitrary polynomial $f(X)$. Although the proof of Theorem 3 follows a constructive approach and presents a method to construct $\mathbf{Q}_f$, it does not necessarily provide the simplest way for finding $\mathbf{Q}_f$. For some polynomials it can be easily written by inspection (See Example 4).

3. Integral of nonhomogeneous polynomials over simplices

We usually encounter the integral of a nonhomogeneous polynomial $f(X)$ over a simplex Ψ_n. When the exact numerical value of $\int_{\Psi_n} f(X) d_X$ is desired, conventional formulas [3,8] can be employed with an acceptable computational complexity. But, sometimes (e.g. our information theory problem) a closed-form expression for $\int_{\Psi_n} f(X) d_X$ is required in terms of some parameters of $f(X)$. In such cases, the mentioned formulas may not result in simple expressions. Therefore, one may think of the following two alternatives:

- **Decomposition approach**: As mentioned earlier, a given nonhomogeneous polynomial $f(X)$ of order q can be presented as a sum of homogeneous polynomials $f_i(X)$ of order i, $1 \leqslant i \leqslant q$, and a constant term (i.e. $f(X) = f_0 + \sum_{i=1}^{q} f_i(X)$). The method given in [5] for treating the integral of $f(X)$ over an n-dimensional simplex Ψ_n is to consider

$$\int_{\Psi_n} f(X)d_X = f_0 \cdot vol(\Psi_n) + \sum_{i=0}^{q} \int_{\Psi_n} f_i(X)d_X$$

and use Theorem 1 to formulate integrals $\int_{\Psi_n} f_i(X)d_X$, $1 \leqslant i \leqslant n$.

- **Direct approach**: We showed that Theorem 1 does hold for symmetric quasilinear forms (Theorem 2). On the other hand, Theorem 3 guarantees the existence of a symmetric q-quasilinear form $\mathbf{Q}_f(X_1, X_2, \ldots, X_q)$ satisfying $\mathbf{Q}_f(X, X, \ldots, X) = f(X)$. If $\mathbf{Q}_f$ can be found easily in a simple form, then one may apply Theorem 2 to formulate $\int_{\Psi_n} f(X)d_X = \int_{\Psi_n} \mathbf{Q}_f(X, X, \ldots, X)d_X$ directly.

In some cases the Direct Approach, results in a simpler formulation than that of the Decomposition Approach. The following illustrates a remarkable instance.

Example 4. Remind Example 2 and suppose $L_j(X)$ is replaced by $b_j + L_j(X)$ where b_j is a constant. As

$$w'(x_1, x_2, \ldots, x_n) = \prod_{j=1}^{q} (b_j + L_j(X)),$$

is not homogeneous anymore, the similar formula to (2) for $w'(X)$,

$$\int_{\Psi_n} w'(X)d_X = \frac{vol(\Psi_n)}{\binom{n+q}{q}} \times \frac{1}{q!} \sum_{0 \leqslant i_1 \leqslant i_2 \leqslant \cdots \leqslant i_q \leqslant n} \sum_{\sigma \in S_q} \prod_{j=1}^{q} \left(b_j + L_j\left(V_{i_{\sigma(j)}}\right)\right), \tag{11}$$

cannot be concluded directly via the first approach (though it is valid).

For instance, in a simple case with $q = 3$ we have

$$w'(X) = \prod_{j=1}^{3} (b_j + L_j(X)) = b_1 b_2 b_3 + w'_1(X) + w'_2(X) + w'_3(X),$$

where $w'_1(X), w'_2(X), w'_3(X)$ are homogeneous components of order $1, 2, 3$:

$$\begin{aligned} w'_1(X) &= b_1 b_2 L_3(X) + b_1 b_3 L_2(X) + b_2 b_3 L_1(X), \\ w'_2(X) &= b_1 L_2(X) L_3(X) + b_2 L_1(X) L_3(X) + b_3 L_1(X) L_2(X), \\ w'_3(X) &= L_1(X) L_2(X) L_3(X). \end{aligned}$$

Hence, the complication of integrating $w'(X)$ is the sum of that of $w'_i(X), 1 \leqslant i \leqslant q$.

However, the situation considerably changes if the notion of quasilinear forms is applied. This is due to the fact that the associated symmetric quasilinear form for $w'(X)$ can be written by inspection as

$$\mathbf{Q}_{w'}(X_1, X_2, \ldots, X_q) = \frac{1}{q!} \sum_{\sigma \in S_q} \prod_{j=1}^{q} (b_j + L_j(X_{\sigma(j)})).$$

It is because we have

$$\mathbf{Q}_{w'}(X, X, \ldots, X) = \frac{1}{q!} \sum_{\sigma \in S_q} \prod_{j=1}^{q} (b_j + L_j(X)) = \prod_{j=1}^{q} (b_j + L_j(X)) \frac{1}{q!} \sum_{\sigma \in S_q} 1 = w'(X),$$

and the equality

$$b_j + L_j(cX + (1-c)X') = cb_j + (1-c)b_j + cL_j(X) + (1-c)L_j(X') = c(b_j + L_j(X)) + (1-c)(b_j + L_j(X')),$$

implies quasilinearity of the symmetric form $\mathbf{Q}_{w'}$. Now, applying Theorem 2 to $\mathbf{Q}_{w'}$ verifies (11). Using (11), complication of integrating $w'(X)$ is the same as that of $w'_q(X)$ or $w(X)$. Hence, in this case, direct approach gives a simple closed-form expression for the integral (i.e. Eq. (11)).

Now we use the proposed approach to derive a closed-form expression for the desired integrals.

Example 5. It is not hard to see that the associated symmetric m-quasilinear form for the integrand of our information theory problem, i.e. $p(X) = (1 - \sum_{i=k}^{n} x_i)^m$, is given by

$$\mathbf{Q}_p\left(\begin{bmatrix} x_{1,1} \\ x_{2,1} \\ \vdots \\ x_{n,1} \end{bmatrix}, \begin{bmatrix} x_{1,2} \\ x_{2,2} \\ \vdots \\ x_{n,2} \end{bmatrix}, \ldots, \begin{bmatrix} x_{1,m} \\ x_{2,m} \\ \vdots \\ x_{n,m} \end{bmatrix}\right) = \prod_{h=1}^{m} \left(1 - \sum_{j=k}^{n} x_{j,h}\right). \tag{12}$$

The vertices of the simplex Λ^n are in the form of

Please cite this article in press as: M. Khosravifard et al., Extension of the Lasserre–Avrachenkov theorem on the integral ..., Appl. Math. Comput. (2009), doi:10.1016/j.amc.2009.02.005

$$\Phi_\ell = \begin{bmatrix} \varphi_{1,\ell} \\ \varphi_{2,\ell} \\ \vdots \\ \varphi_{n,\ell} \end{bmatrix} = \Bigg(\overbrace{\frac{1}{\ell},\frac{1}{\ell},\ldots,\frac{1}{\ell}}^{\ell},\overbrace{0,0,\ldots,0}^{n-\ell}\Bigg)^T,$$

for $0 \leqslant \ell \leqslant n$. In other words we have

$$\varphi_{j,h} = \begin{cases} \frac{1}{h} & j \leqslant h \\ 0 & j > h \end{cases}. \tag{13}$$

Thus we can write

$$\begin{aligned}
I(n,k,m) &= \int_{\Lambda^n}\left(1-\sum_{i=k}^{n}x_i\right)^m d_X \overset{(a)}{=} \frac{vol(\Lambda^n)}{\binom{n+m}{m}} \sum_{0\leqslant i_1\leqslant i_2\leqslant\cdots\leqslant i_m\leqslant n} \mathbf{Q}_p(\Phi_{i_1},\Phi_{i_2},\ldots,\Phi_{i_m}) \\
&\overset{(b)}{=} \frac{vol(\Lambda^n)}{\binom{n+m}{m}} \sum_{0\leqslant i_1\leqslant i_2\leqslant\cdots\leqslant i_m\leqslant n} \prod_{h=1}^{m}\left(1-\sum_{j=k}^{n}\varphi_{j,i_h}\right) = \frac{vol(\Lambda^n)}{\binom{n+m}{m}} \sum_{\ell=0}^{m} \sum_{\substack{0\leqslant i_1\leqslant i_2\leqslant\cdots\leqslant i_\ell\leqslant k-1 \\ k\leqslant i_{\ell+1}\leqslant i_{\ell+2}\leqslant\cdots\leqslant i_m\leqslant n}} \prod_{h=1}^{m}\left(1-\sum_{j=k}^{n}\varphi_{j,i_h}\right) \\
&= \frac{vol(\Lambda^n)}{\binom{n+m}{m}} \sum_{\ell=0}^{m} \sum_{0\leqslant i_1\leqslant i_2\leqslant\cdots\leqslant i_\ell\leqslant k-1} \prod_{h=1}^{\ell}\left(1-\sum_{j=k}^{n}\varphi_{j,i_h}\right) \sum_{k\leqslant i_{\ell+1}\leqslant i_{\ell+2}\leqslant\cdots\leqslant i_m\leqslant n} \prod_{h=\ell+1}^{m}\left(1-\sum_{j=k}^{n}\varphi_{j,i_h}\right) \\
&\overset{(c)}{=} \frac{vol(\Lambda^n)}{\binom{n+m}{m}} \sum_{\ell=0}^{m} \sum_{0\leqslant i_1\leqslant i_2\leqslant\cdots\leqslant i_\ell\leqslant k-1} 1 \sum_{k\leqslant i_{\ell+1}\leqslant i_{\ell+2}\leqslant\cdots\leqslant i_m\leqslant n} \prod_{h=\ell+1}^{m}\left(1-\sum_{j=k}^{i_h}\frac{1}{i_h}\right) \\
&\overset{(d)}{=} \frac{vol(\Lambda^n)}{\binom{n+m}{m}} \sum_{\ell=0}^{m} \binom{k+\ell-1}{k-1} \sum_{\leqslant i_{\ell+1}\leqslant i_{\ell+2}\leqslant\cdots\leqslant i_m\leqslant n} \prod_{h=\ell+1}^{m} \frac{k-1}{i_h} \\
&= \frac{vol(\Lambda^n)}{\binom{n+m}{m}} \sum_{\ell=0}^{m} \binom{k+\ell-1}{k-1} (k-1)^{m-\ell} \sum_{k\leqslant i_1\leqslant i_2\leqslant\cdots\leqslant i_{m-\ell}\leqslant n} \frac{1}{\prod_{h=1}^{m-\ell} i_h},
\end{aligned}$$

where we have written (a) from Theorem 2, (b) from (12), (c) from (13) and (d) from the identity $\sum_{0\leqslant i_1\leqslant i_2\leqslant\cdots\leqslant i_j\leqslant k} 1 = \binom{k+j}{k}$. Note that such an expression cannot be easily derived by Decomposition Approach or other conventional formulas.

4. Conclusion

Deriving a closed-form expression for the integral of some nonhomogeneous polynomials over some simplices motivated us to extend the Lasserre–Avrachenkov theorem. It was done by defining the concept of quasilinearity of a form. Quasilinearity is a weaker condition than the multilinearity. The given extension provides another option for integrating an arbitrary polynomial (not necessarily homogeneous) over simplices in condition that a simple symmetric quasilinear form associate with the polynomial is easily obtained. For example the integral of the polynomial $w'(X) = \prod_{j=1}^{q}\left(b_j + \sum_{i=1}^{n} c_{i,j}x_i\right)$ over a simplex is simply formulated.

References

[1] R. Cools, An encyclopedia of cubature formulas, J. Complex. 19 (3) (2003) 445–453.
[2] A. Genz, R. Cools, An adaptive cubature algorithm for simplices, ACM Trans. Math. Soft. 29 (2003) 297–308.
[3] A. Grundmann, H.M. Moller, Invariant integration formulas for the N–Simplex by combinatorial methods, SIAM J. Numer. Anal. 15 (1978) 282–290.
[4] M. Khosravifard, Some codes for monotone sources, Ph.D. Thesis, Department of Electrical and Computer Engineering, Isfahan University of Technology, June 2004.
[5] J.B. Lasserre, K.E. Avrachenkov, The multi-dimensional version of $\int_a^b x^p dx$, Amer. Math. Mon. 108 (2001) 151–154.
[6] R. Merris, Multilinear Algebra, Gordon and Breach Science Publishers, 1997.
[7] A.H. Stroud, Approximate Calculation of Multiple Integrals, Prentice Hall, Englewood Cliffs, NJ, 1971.
[8] P. Sylvester, Symmetric quadrature formulae for simplexes, Math. Comp. 24 (1970) 95–100.
[9] Erik G.F. Thomas, A Polarization Identity For Multilinear Maps, <http://www.citeseer.ist.psu.edu/56937.html>.