Referee report

Wolsey!!

Integer programming, duality and superadditive functions

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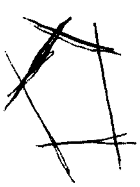
Abstract. Given $A \in \mathbb{Z}^{m\times n}$, $b \in \mathbb{Z}^m, c \in \mathbb{R}^n$, we consider the integer program $\mathbb{P}_d$: $\max\{c'x|Ax = b; x \in \mathbb{Z}^n_+\}$ which has a well-known abstract dual optimization problem stated in terms of superadditive functions. Using a linear program $\mathbb{Q}$ equivalent to $\mathbb{P}_d$ that we have introduced recently, we show that its dual $\mathbb{Q}^*$ can be interpreted as a simplified and tractable form of the abstract dual, and identifies a subclass of superadditive functions, sufficient to consider in the abstract dual.

1. Introduction

Let $A \in \mathbb{Z}^{m\times n}, b \in \mathbb{Z}^m, c \in \mathbb{R}^n$ and consider the integer program



$$\text{(1.1)} \qquad \mathbb{P}_d \rightarrow := \max\{\, c'x \mid Ax = b; \quad x \geq 0; \quad x \in \mathbb{Z}^n\},$$

where the convex polyhedron $\Omega(b) := \{x \in \mathbb{R}^n \mid Ax = b;\ x \geq 0\}$ is compact.

Related to $\mathbb{P}_d$ is the *dual* optimization problem

$$\text{(1.2)} \qquad \min_{f\in\Gamma}\{\, f(b) \mid \quad f(A_j) \geq c_j,\ j = 1,\dots,n\}$$

fact that

where Γ is a certain set of functions $f : \mathbb{R}^m \rightarrow \mathbb{R}$ that are *superadditive* and such that $f(0) = 0$; see e.g. Wolsey [**11**] (who considers the case $Ax \leq b$). Despite the dual problem (1.2) is rather conceptual in nature, one still retrieves several concepts already available in standard linear programming (LP) duality (see [**11**, p. 175] still for the case $Ax \leq b$). More importantly, and this our main motivation to better understand (1.2), the fundamental and basic Gomory (fractional) *cuts* for integer programs, which are crucial for the efficiency of today 's most powerful codes for solving $\mathbb{P}_d$, have an interpretation in terms of superadditive functions f in (1.2). For more details see Wolsey [**10**, §7]. Therefore, besides its theoretical interest, any insight on the dual problem (1.2) is of potential interest as it could provide useful information for deriving efficient cuts in standard solving procedure for $\mathbb{P}_d$. Moreover, problem (1.2) can be transformed into an equivalent finite LP. For instance, for the case $Ax \leq b$ and when $A \in \mathbb{N}^{m\times n}, b \in \mathbb{N}^m$, introducing the

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Next, nice and complete duality results available for $\mathbb{P}, \mathbb{I}$ and $\mathbb{I}_d$, extend in a natural way to $\mathbb{P}_d$.

The dual problem $\mathbb{P}^*$ of $\mathbb{P}$ is obtained from the *Fenchel-transform* duality applied to the concave function $b \mapsto f(b,c)$, whereas the *dual* problem $\mathbb{I}^*$ of $\mathbb{I}$ is obtained from the *Laplace-transform* duality applied to the function $b \mapsto \widehat{f}(b,c)$. Namely, we obtain the dual problems

$$\mathbb{P}^*: \quad \begin{array}{rll} f(b,c) = & \min & b'\lambda \\ & \text{s.t.} & A'\lambda \geq c \; ; \\ & & \lambda \in \mathbb{R}^m \end{array} \qquad \mathbb{I}^*: \quad \begin{array}{l} \widehat{f}(b,c) = \displaystyle\int_{\gamma - i\infty}^{\gamma + i\infty} \frac{e^{b'\lambda}\, d\lambda}{\prod_{j=1}^n (A'\lambda - c)_j} \\ \text{with } \gamma \in \mathbb{R}^m \text{ fixed; } A'\gamma > c. \end{array}$$

A

A_j

Similarly, the dual problem $\mathbb{I}_d^*$ of $\mathbb{I}_d$ is obtained from the $\mathbb{Z}$-*transform* duality applied to the function $b \mapsto \widehat{f}_d(b,c)$, and reads

$$\mathbb{I}_d^*: \quad \widehat{f}_d(b,c) = \int_{|z|=e^\gamma} \frac{z^{b-e_m}\, dz}{\prod_{j=1}^n (1 - z^{-A_j} e^{c_j})}, \quad \text{with } \gamma \in \mathbb{R}^m \text{ fixed; } A'\gamma > c.$$

(and where $e_m \in \mathbb{R}^m$ is a vector of ones).

For a detailed account on several approaches on how to compute $\widehat{f}(b,c)$ and $\widehat{f}_d(b,c)$, the interested reader is referred to e.g. Barvinok [**1, 2**], Brion and Vergne [**3**] and the many refererences therein.

Despite both dual problems $\mathbb{I}^*$ and $\mathbb{I}_d^*$ are of same nature (a complex integral), a key feature distinguishes $\mathbb{I}^*$ from $\mathbb{I}_d^*$. In $\mathbb{I}^*$, the data A, b appear as *coefficients* of the dual variables λ, whereas in $\mathbb{I}_d^*$ they appear as *exponents* of the dual variables z (or, e^λ). As a consequence, the integrand in $\mathbb{I}^*$ has only *real* poles, whereas the integrand in $\mathbb{I}_d^*$ has many more (complex) poles, which makes problem $\mathbb{I}_d^*$ harder to solve (e.g. by Cauchy' s residue technique). This fact is also reflected in the continuous and periodic formulae of Brion and Vergne [**3**, Theor. p. 820-821] which provide $\widehat{f}(b,c)$ and $\widehat{f}_d(b,c)$ in closed form, both in terms of a weighted summation of $e^{c'x}$ over the vertices of $\Omega(b)$.

The obvious analogies between $\mathbb{I}^*$ and $\mathbb{I}_d^*$ and the relationship (1.4) linking respectively $\mathbb{P}$ and $\mathbb{P}_d$ with $\mathbb{I}$ and $\mathbb{I}_d$, were our motivation to develop a duality framework for $\mathbb{P}_d$ in [**5**] (where Brion and Vergne' s periodic formula plays a central role). By a detailed analysis of this formula and using (1.4) which relates problems $\mathbb{I}_d$ and $\mathbb{P}_d$, we showed that each basis $A_\sigma \in \mathbb{Z}^{m\times m}$ of the linear program $\mathbb{P}$ provides exactly $\det(A_\sigma)$ complex *dual* vectors $z \in \mathbb{C}^m$, the complex (periodic) analogues for $\mathbb{P}_d$ of the unique dual vector $\lambda \in \mathbb{R}^m$ for $\mathbb{P}$, associated with the basis A_σ. This allowed us to define a dual problem in $\mathbb{C}^m$, an analogue of $\mathbb{P}^*$ defined in $\mathbb{R}^m$. Using $\mathbb{I}_d$, we have also provided a *discrete* Farkas Lemma for the existence of nonnegative integral solutions $x \in \mathbb{N}^n$ to $Ax = b$. Its form (Theorem 3.1 below) also confirms the central role of the $\mathbb{Z}$-transform (or generating function) of the function $b \mapsto \widehat{f}_d(b,c)$. For more details, the interested reader is referred to Lasserre [**5**].

Contribution: The goal of this paper is to relate duality results obtained in Lasserre [**5, 6, 7**] with the abstract dual problem (1.2), and to provide some insights for the latter problem. Namely, we present a dual problem $\mathbb{P}_d^*$ of $\mathbb{P}_d$ in the form (1.2), where a subclass of superadditive functions is identified and also yields a finite LP, simpler than (1.3). In fact, we obtain our result the other way around. From the discrete Farkas lemma proposed in [**6**], we have also obtained a linear programming

Well,
all maximal
valid inequalities
for S can be
obtained from
these subadditive
functions
— linear combination
— rounding

Thus, let $f \in \Gamma$ be any feasible solution of (1.2). Then, if $x \in \mathbb{N}^n$ is any feasible solution of $Ax = b$,

$$\begin{aligned} f(b) = f(\sum_{j=1}^n A_j x_j) &\geq \sum_{j=1}^n f(A_j x_j) \quad \text{[by superadditivity]} \\ &\geq \sum_{j=1}^n f(A_j) x_j \quad \text{[by superadditivity]} \\ &\geq \sum_{j=1}^n c_j x_j = c'x, \end{aligned}$$

and so, $f(b) \geq c'x$, that is, the weak duality property holds. Strong duality holds with $f^*(b) = \max \mathbb{P}_d = c'x^*$ for any optimal solution $x^* \in \mathbb{N}^n$ of $\mathbb{P}_d$, and also whenever $Ax = b$ has no solution $x \in \mathbb{N}^n$, in which case $f^*(b) = \max \mathbb{P}_d = -\infty$.

2.2. A class of superadditive functions. Let $\mathcal{D} \subset \mathbb{N}^m$ be a finite set such that

(2.1) $$0 \in \mathcal{D}; \quad \text{and} \quad \alpha \in \mathcal{D} \Rightarrow \beta \in \mathcal{D} \quad \forall \beta \leq \alpha,$$ a simplicial complex

and let Δ be the set of functions $\pi : \mathbb{N}^m \to \mathbb{R} \cup \{+\infty\}$, such that

(2.2) $$\pi \in \Delta \quad \text{if } \pi(0) = 0 \quad \text{and} \quad \pi(\alpha) = +\infty \quad \text{only if } \alpha \notin \mathcal{D}.$$

Next, given $\pi \in \Delta$ let $f_\pi : \mathbb{N}^m \to \mathbb{R} \cup \{+\infty\}$ be defined as

(2.3) $$x \mapsto f_\pi(x) := \inf_{\alpha \in \mathcal{D}} \{\pi(\alpha + x) - \pi(\alpha)\}, \qquad x \in \mathbb{N}^m.$$

Observe that $f_\pi \in \Delta$ whenever $\pi \in \Delta$, that is,

$$f_\pi(0) = 0 \quad \text{and} \quad f_\pi(x) = +\infty \quad \text{only if } x \notin \mathcal{D}.$$

Indeed, $f_\pi(0) = 0$ follows from the definition (2.3) of f_π. Next, if $x \notin \mathcal{D}$ then so does $\alpha + x$ for every $\alpha \in \mathbb{N}^m$; thus $\pi(\alpha + x) = +\infty$ and from (2.3) $f_\pi(x) = +\infty$ because $\pi(\alpha) < \infty$ whenever $\alpha \in \mathcal{D}$. Next, let $x \in \mathcal{D}$ so that $\pi(x) < +\infty$. From (2.3) $f_\pi(x) \leq \pi(x) - \pi(0) = \pi(x) < +\infty$.

LEMMA 2.1. *For every $\pi \in \Delta$:*

(i) *$f_\pi \leq \pi$, and f_π is superadditive.*

(ii) *If $\pi \in \Delta$ is superadditive then $\pi = f_\pi$.*

PROOF. (i) We have just seen that $f_\pi \leq \pi$. Next, let $\pi \in \Delta$, and let $x, y \in \mathbb{N}^m$ be fixed, arbitrary. First consider the case where $x + y \in \mathcal{D}$ so that $x, y \in \mathcal{D}$. Observe that if $\alpha \in \mathcal{D}$ and $\alpha + x + y \notin \mathcal{D}$ then $\pi(\alpha + x + y) - \pi(\alpha) = +\infty$; hence

$$\inf_{\alpha \in \mathcal{D}} \{\pi(\alpha + x + y) - \pi(\alpha)\} = \inf_{\alpha + x + y \in \mathcal{D}} \{\pi(\alpha + x + y) - \pi(\alpha)\}.$$

Therefore, we have:

$$\begin{aligned} f_\pi(x + y) &= \inf_{\alpha \in \mathcal{D}} \{\pi(\alpha + x + y) - \pi(\alpha)\} \\ &= \inf_{\alpha + x + y \in \mathcal{D}} \{[\pi(\alpha + x + y) - \pi(\alpha + x)] + [\pi(\alpha + x) - \pi(\alpha)]\} \\ &\geq \inf_{\alpha + x \in \mathcal{D}} \{\pi(\alpha + x + y) - \pi(\alpha + x)\} + \inf_{\alpha \in \mathcal{D}} \{\pi(\alpha + x) - \pi(\alpha)\} \\ &= f_\pi(y) + f_\pi(x), \end{aligned}$$

where we have used that $\alpha + x + y \in \mathcal{D}$ implies $\alpha + x \in \mathcal{D}$ (see (2.1)).

and so,

$$\widehat{f}_d(b,0) \;=\; \int_{|z_m|=e^{\gamma_m}} \cdots \int_{|z_1|=e^{\gamma_1}} \frac{z^{b-e_m}}{\prod_{j=1}^{n}(1-z^{-A_j})}\,dz_1\cdots dz_m,$$

where $\gamma \in \mathbb{R}^m$ satisfies $A'\gamma > 0$. Then, using (3.1) in the above integral, one may show that $\widehat{f}_d(b,0) \geq 1$.

Therefore, as the degree of each Q_j is bounded, let $q \geq 0$ be the vector of (nonnegative) coefficients of all the polynomials Q_j's. Then, checking the existence of such polynomials $\{Q_j\}$ in (3.1) reduces to solve a linear system

$$\mathrm{M}q \;=\; r, \quad q \geq 0, \tag{3.3}$$

for some matrix M and vector r with only 0 and ± 1 entries. The constraints $\mathrm{M}q = r$ state that the polynomials $z \mapsto z^b - 1$ and $z \mapsto \sum_j Q_j(z)(z^{A_j} - 1)$ are identical by equating their respective coefficients; see Lasserre [**6**] for more details.

In fact, from the proof of Theorem 3.1 in [**7**], each polynomial Q_j in (3.1) may be restricted to contain only monomials z^α with $\alpha \in \mathbb{N}^m$ such that $\alpha \leq b - A_j$. Indeed, if $x \in \mathbb{N}^n$ solves $Ax = b$ then (3.1) holds with the Q_j's as in (3.2). And so, the monomials z^α of Q_j satisfy $\alpha \leq \sum_{k=1}^{j} A_k x_k - A_j \leq b - A_j$.

Therefore, in the constraints $\mathrm{M}q = r$ which states that the polynomials $z \mapsto z^b - 1$ and $z \mapsto \sum_j Q_j(z)(z^{A_j} - 1)$ are identical, we only need to equate their respective coefficients of same monomials z^α for those $\alpha \in \mathbb{N}^m$ that satisfy $\alpha \leq b$. This is because, as each Q_j contains only monomials z^α with $\alpha \leq b - A_j$, each polynomial $Q_j(z)(z^{A_j} - 1)$ contains only monomials z^β with $\beta \leq b$.

Hence, in the LP (3.3), the vector q and the matrix M can be taken in $\mathbb{R}^s$ and $\mathbb{R}^{p\times s}$, respectively, where :

- $p = \prod_{i=1}^{m}(b_i + 1)$, i.e., the number of monomials z^α with $\alpha \leq b$.
- $s = \sum_{j=1}^{n} s_j$ with $s_j = \prod_{i=1}^{m}(b_i - A_{ij} + 1)$ for all $j = 1, \ldots, n$ (the number of monomials z^α with $\alpha - A_j \leq b$.

Note that the matrix M is *totally unimodular* because it is a network matrix (each column has only two nonzero entries $+1$ and -1).

Define the row vectors $\mathrm{e}_{s_j} := (1, \ldots, 1) \in \mathbb{R}^{s_j}$, for every $j = 1, \ldots, n$, and let $\mathrm{E} \in \mathbb{N}^{n\times ns}$ be the n-block diagonal matrix, whose each diagonal block is a row vector e_{s_j}, that is,

$$\mathrm{E} := \begin{bmatrix} \mathrm{e}_{s_1} & 0 & \ldots & 0 \\ 0 & \mathrm{e}_{s_2} & 0 & \ldots \\ \ldots & \ldots & \ldots & \ldots \\ 0 & \ldots & 0 & \mathrm{e}_{s_n} \end{bmatrix}. \tag{3.4}$$

Given $c \in \mathbb{R}^n$, let $\widehat{c} \in \mathbb{R}^s$ be the vector $\mathrm{E}'c$. Then, in the same manner as we did in Lasserre [**7**], we have :

THEOREM 3.2. *Let $A \in \mathbb{N}^{m\times n}, b \in \mathbb{N}^m, c \in \mathbb{R}^n$. Let $\mathrm{M} \in \mathbb{R}^{p\times s}, r \in \mathbb{R}^p, \mathrm{E} \in \mathbb{R}^{n\times s}$ be as in (3.3) and (3.4), and let $\widehat{c} := \mathrm{E}'c$. Then :*

(a) *The optimal value* $\max \mathbb{P}_d$ *of the integer program* $\mathbb{P}_d$ *is the same as the optimal value* $\max \mathbb{Q}$ *of the linear program*

$$\mathbb{Q} \quad \to \max\,\{\,\widehat{c}'q \mid \; \mathrm{M}\,q \;=\; r; \quad q \geq 0\}. \tag{3.5}$$

(b) *With every optimal vertex $q^* \in \mathbb{R}^s$ of the linear program $\mathbb{Q}$, is associated an optimal solution $x^* := \mathrm{E}\,q^* \in \mathbb{N}^n$ of the integer program $\mathbb{P}_d$.*

ones. The integer program $\mathbb{P}_d$ is equivalent to the integer program

$$(3.7) \qquad \widehat{\mathbb{P}}_d \quad \left\{ \begin{array}{lll} \max\limits_{x,u} & c'x & \\ \text{s.t.} & \widehat{A}x + \mathrm{e}_m u & = \widehat{b} \\ & \alpha' x + u & = \beta \\ & (x,u) & \in \mathbb{N}^n \times \mathbb{N}. \end{array} \right.$$

It is straightforward to check that $\max \mathbb{P}_d = \max \widehat{\mathbb{P}}_d$ and if $x^* \in \mathbb{N}^n$ is an optimal solution of $\mathbb{P}_d$ then so is (x^*, u^*) for $\widehat{\mathbb{P}}_d$ (with $u^* = \beta - \alpha' x^* \in \mathbb{N}$). As the matrix of the constraints of $\widehat{\mathbb{P}}_d$ has nonnegative entries, we are back to to the case analyzed in §3.1, with matrix and vector

$$\left[\begin{array}{cc} \widehat{A} & \mathrm{e}_m \\ \alpha' & 1 \end{array} \right] \in \mathbb{N}^{(m+1)\times(n+1)}, \qquad \left[\begin{array}{c} \widehat{b} \\ \beta \end{array} \right] \in \mathbb{N}^{m+1},$$

in lieu of A and b, respectively.

3.3. The link with superadditivity. To relate the above LP with superadditive functions and with the abstract dual (1.2), we proceed as follows. As we have just seen, we may restrict to the case $A \in \mathbb{N}^{m\times n}, b \in \mathbb{N}^m$. Let $\mathcal{D} \subset \mathbb{N}^m$ be the set

$$(3.8) \qquad \mathcal{D} := \prod_{j=1}^{m} \{0, 1, \ldots, b_j\}.$$

In view of the simple form of the matrix M of the linear program $\mathbb{Q}$ in (3.5), its LP dual $\mathbb{Q}^*$ is easy to state. Namely,

$$(3.9) \qquad \mathbb{Q}^* \left\{ \begin{array}{lll} \min\limits_{\gamma} & \gamma(b) - \gamma(0) & \\ \text{s.t.} & \gamma(\alpha + A_j) - \gamma(\alpha) & \geq c_j,\ \alpha + A_j \in \mathcal{D};\ j = 1, \ldots, n, \end{array} \right.$$

with optimal value denoted $\min \mathbb{Q}^*$.

Clearly, by the change of variable $\pi(\alpha) := \gamma(\alpha) - \gamma(0)$, $\alpha \in \mathcal{D}$, $\mathbb{Q}^*$ also reads

$$(3.10) \qquad \left\{ \begin{array}{lll} \min\limits_{\pi} & \pi(b) & \\ \text{s.t.} & \pi(\alpha + A_j) - \pi(\alpha) & \geq c_j, \quad \alpha + A_j \in \mathcal{D};\ j = 1, \ldots, n \\ & \pi(0) & = 0. \end{array} \right.$$

Now, extend π to $\mathbb{N}^m$ by $\pi(\alpha) = +\infty$ whenever $\alpha \notin \mathcal{D}$. Then with Δ as in (2.2), the linear program $\mathbb{Q}^*$ is equivalent to the optimization problem

$$(3.11) \qquad \rho_1 := \left\{ \begin{array}{lll} \min\limits_{\pi \in \Delta} & \pi(b) & \\ \text{s.t.} & \pi(\alpha + A_j) - \pi(\alpha) & \geq c_j, \quad \alpha \in \mathcal{D};\ j = 1, \ldots, n, \end{array} \right.$$

that is, $\min \mathbb{Q}^* = \rho_1$.

THEOREM 3.3. *Let $A \in \mathbb{N}^{m\times n}, b \in \mathbb{N}^m, c \in \mathbb{R}^n$ and let $\mathbb{Q}^*$ be the linear program defined in (3.9) or (3.10), so that $\min \mathbb{Q}^* = \max \mathbb{P}_d$. Consider the optimization problem*

$$(3.12) \qquad \mathbb{P}_d^* : \quad \rho_2 := \left\{ \begin{array}{lll} \inf\limits_{\pi \in \Delta} & f_\pi(b) & \\ \text{s.t.} & f_\pi(A_j) & \geq c_j \quad j = 1, \ldots, n. \end{array} \right.$$

where $f_\pi : \mathbb{N}^m \to \mathbb{R}$ is the superadditive function defined in (2.3), for every $\pi \in \Delta$. If $\mathbb{P}_d$ is solvable, i.e., if $\max \mathbb{P}_d > -\infty$, then

$$(3.13) \qquad \max \mathbb{P}_d = \min \mathbb{Q}^* = \rho_2 = f_{\pi^*}(b) \quad \textit{for some } \pi^* \in \Delta.$$

EXAMPLE 3.4. Consider the following simple illustrative example where $A := [2,\ 3] \in \mathbb{N}^{1\times 2}$ and $b := 5$ so that $Ax = b$ has only one solution $x^* = (1,1)$. Let the cost vector be $c = [c_1, c_2]$, and $\mathcal{D} := \{0, 1, \ldots, b\}$. The dual problem (3.10) reads

$$\mathbb{Q}^*\ (\text{or } \mathbb{P}_d^*)\quad \left\{ \begin{array}{lllll} & \min\limits_{\pi} & \pi(5) & & \\ \text{s.t.} & \pi(2) & \geq c_1; & \pi(3) & \geq c_2 \\ & \pi(3)-\pi(1) & \geq c_1; & \pi(4)-\pi(1) & \geq c_2 \\ & \pi(5)-\pi(3) & \geq c_1; & \pi(5)-\pi(2) & \geq c_2 \\ & \pi(4)-\pi(2) & \geq c_1, & \pi(0) & = 0, \end{array} \right.$$

with optimal value $\min \mathbb{Q}^* = \pi^*(5) = c_1 + c_2$, and optimal solution

$$\pi^*(1) = c_2 - c_1,\ \pi^*(2) = c_1,\ \pi^*(3) = c_2,\ \pi^*(4) = \max[2c_1, 2c_2 - c_1].$$

The superadditive function $f_{\pi^*} : \mathbb{N}^2 \to \mathbb{R}$ defined in (2.3) (with $\pi^*(x) = +\infty$ if $x > 5$) satisfies $f_{\pi^*}(5) = c_1 + c_2$.

If $b = 1$ instead of $b = 5$, the system $Ax = b$ has no solution $x \in \mathbb{N}^n$. As now $\mathcal{D} = \{0, 1\}$, the LP dual (3.10) reads

$$\min \mathbb{Q}^* \ =\ \min_{\pi} \{\, \pi(1) \mid \quad \pi(0) = 0 \} \ =\ -\infty,$$

because $\alpha + A_j \notin \mathcal{D}$ for every $\alpha \in \mathcal{D}$, which is consistent with $\max \mathbb{P}_d = -\infty$.

4. Conclusion

In view of the above results and back to the four problems $\mathbb{P}, \mathbb{I}, \mathbb{I}_d$ and $\mathbb{P}_d$, displayed in Table 1, their respective duals $\mathbb{P}^*, \mathbb{I}^*, \mathbb{I}_d^*$ and $\mathbb{P}_d^*$ are now displayed in Table 2 (with $\mathcal{D}$ as in (3.8) and $A \in \mathbb{N}^{m\times n}$).

TABLE 2. The four dual problems

Continuous Optimization	**Discrete Optimization**
$\mathbb{P}^*: \quad \begin{array}{ll} \min\limits_{\lambda\in\mathbb{R}^m} & b'\lambda \\ \text{s.t.} & A'\lambda \ \geq c. \end{array}$	$\mathbb{P}_d^*: \quad \begin{array}{ll} \min\limits_{\pi\in\mathbb{R}^p} & \pi(b) \\ \text{s.t.} & \pi(\alpha + A_j) - \pi(\alpha) \geq c_j, \\ & \alpha + A_j \in \mathcal{D};\ j = 1, \ldots, n, \\ & \pi(0) = 0. \end{array}$
Integration	**Summation**
$\mathbb{I}^*: \quad \displaystyle\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{e^{b'\lambda}\, d\lambda}{\prod_{j=1}^n (A_j'\lambda - c_j)}$ with $\gamma \in \mathbb{R}^m$ fixed, s.t. $A'\gamma > c$.	$\mathbb{I}_d^*: \quad \displaystyle\int_{\vert z\vert=\gamma} \frac{z^{b-e_m}\, dz}{\prod_{j=1}^n (1 - z^{-A_j} e^{c_j})}$ with $\gamma \in \mathbb{R}^m$ fixed, s.t. $A' \ln \gamma > c$.