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A PRACTICAL ALGORITHM FOR COUNTING LATTICE POINTS IN A CONVEX POLYTOPE

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Abstract. We provide a practical algorithm for counting lattice points in the convex polytope $\{x \in \mathbb{R}^n | Ax = b, x \geq 0\}$. It is based on an exact (tractable) formula for the case $A \in \mathbb{Z}^{m\times(m+1)}$ that we repeatedly use for the general case $A \in \mathbb{Z}^{m\times n}$.

1. Introduction

Consider the (not necessarily compact) polyhedron

$$\Omega(y) \;=\; \{x \in \mathbb{R}^n \,|\, Ax = y; \quad x \geq 0\}, \tag{1.1}$$

with $y \in \mathbb{Z}^m$ and $A \in \mathbb{Z}^{m\times n}$ for $n > m$, and the function $f : \mathbb{Z}^m \to \mathbb{R}$

$$y \;\mapsto\; f(y) \;:=\; \sum_{x\in\Omega(y)\cap\mathbb{N}^n} \mathrm{e}^{c'x}, \tag{1.2}$$

where the vector $c \in \mathbb{R}^n$ is chosen small enough (even negative) to ensure that $f(y)$ is well defined even when $\Omega(y)$ is not compact. If $\Omega(y)$ is compact, then $f(y)$ provides us with the exact number of points of the set $\Omega(y) \cap \mathbb{N}^n$ by either choosing $c := 0$, or taking $\lim_{c\to 0} f(y)$ (or even rounding up $f(y)$ to the nearest integer for c sufficiently close to zero).

In recent works, Barvinok [2], Barvinok and Pommersheim [3], Brion and Vergne [6], Pukhlikov and Khovanskii [7] have provided nice exact (theoretical) formulas for $f(y)$. For instance, Brion and Vergne [6] (using generating functions along with a generalized residue formula), or Barvinok [2] (also working with generating functions, but with different arguments) express $f(y)$ in terms of a weighted sum of $\mathrm{e}^{c'x}$ over the vertexes of $\Omega(y)$. However, despite of its theoretical interest, Brion and Vergne 's formula is not directly *tractable* because it contains many products with complex coefficients (roots of unity) which makes the formula difficult to evaluate numerically. However, in some cases, this formula can be exploited as e.g. in Baldoni-Silva and Vergne [1] for flow polytopes. Similarly, Beck [4], and Beck, Diaz and Robins [5] provided a complete analysis based on residue techniques for the case of a tetrahedron ($m = 1$) and mentioned the possibility of evaluating $f(b)$ for general polytopes by means of residues as well. In Lasserre and Zeron [8], we provided two algorithms based on Cauchy residue techniques to invert the generating function; and an alternative algebraic technique

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based on partial fraction expansion of the generating function (using the Hilbert NullStellenSatz). A nice feature of the latter technique is to avoid computing residues.

Contribution. The goal of this paper, as a sequel to [8], is to provide a practical algorithm to compute $f(y)$ in the spirit of the algebraic technique briefly outlined in [8, §7]; but now in a more constructive and explicit way. We use the same generating function as in Brion and Vergne, and we provide a decomposition into simpler rational fractions whose "inversion" is easy to obtain. To avoid handling complex roots of unity, we do not use residues "explicitly" but build up the required decomposition in a recursive manner. Properly speaking, we inductively calculate real constants $Q_{\sigma,\beta}$ and a fix positive integer M, all of them completely independent of y, such that the *counting* function f is given by the finite sum :

$$f(y) = \sum_{A_\sigma} \sum_{\beta \in \mathbb{Z}^m,\, \|\beta\| \le M} Q_{\sigma,\beta} \times \begin{cases} e^{c'_\sigma x} & \text{if } x := A_\sigma^{-1}[y-\beta] \in \mathbb{N}^m, \\ 0 & \text{otherwise;} \end{cases}$$

where the first finite sum is computed over all A_σ invertible $[m \times m]$-square sub-matrices of A. This formula is presented in Theorem 2.6, and all the necessary notation is introduced in next section \$2.

Crucial in our algorithm is an explicit decomposition in *closed form* (and thus, an explicit formula for $f(y)$) for the case $n = m + 1$, that we next repeatedly use for the general case $n > m + 1$.

Our closed form expression for the case $n = m + 1$ is immediately computable and *tractable* as it does *not* contain complex coefficients as the roots of unity in Brion and Vergne's formula.

The paper is organized as follows: In §2 we provide our main result which states an exact expression of $f(y)$ provided its generating function has a decomposition into certain rational fractions. In §3 we provide this explicit decomposition for the case $n = m+1$, as well as the corresponding expression for $f(y)$. In §4 we present a recursive algorithm that provides the required decomposition for the general case $n \ge m+1$. The computational complexity is $\mathrm{O}[(m+1)^{n-m}\Lambda]$, where the coefficient Λ depends only on the matrix A and *not* on the magnitude of y (cf. (4.11)).

Thus, the formulas presented in section §3 give us a very efficient procedure of calculating $f(y)$ in the case $n = m + 1$. Moreover, the recursive algorithm presented in section §4 is also very efficient for calculating $f(y)$ for relatively small values of $n - m$, *no matter* the magnitude of y. However, this algorithm becomes less efficient when we consider the case $n = m + k$ for large values of k.

Analyzing the algorithm presented in §4 against the algorithm (via integration) that we introduce in [8], we can conclude that they are both complementary in the sense that the algorithm presented in [8] is very efficient when m is small, and the algorithm presented in this paper is very efficient when $n - m$ is small, no matter how large are m and n.

2. Main result

2.1. **Notation and definitions.** The notation $\mathbb{R}$ and $\mathbb{Z}$ stand for the usual sets of real and integer numbers respectively; moreover, the set of natural number $\{0,1,2,\ldots\}$ is denoted by $\mathbb{Z}_+$ or $\mathbb{N}$. The notation c' and A' stand for the respective transpose of the vector $c\in\mathbb{R}^n$ and the matrix $A\in\mathbb{Z}^{m\times n}$. Moreover, the k-th column of the matrix $A\in\mathbb{Z}^{m\times n}$ is denoted by

$$A_k := (A_{1,k},\ldots,A_{m,k})'.$$

When $y=0$, $\Omega(0)$ in (1.1) is a convex cone with *dual* cone

$$\Omega(0)^* := \{b\in\mathbb{R}^n \,|\, b'x\geq 0 \quad \text{for every} \quad x\in\Omega(0)\}. \tag{2.1}$$

We may now define the following open set

$$\Gamma := \{c\in\mathbb{R}^n \,|\, -c>b \quad \text{for some} \quad b\in\Omega(0)^*\}. \tag{2.2}$$

Notice that Γ and $\Omega(0)^*$ are both equal to $\mathbb{R}^n$ whenever $\Omega(0)$ is the singleton $\{0\}$, which is the case if $\Omega(y)$ is compact.

On the other hand, and with no loss of generality, we may and will suppose from now on that the matrix $A\in\mathbb{Z}^{m\times n}$ has maximal rank (see the beginning of §2.2).

Definition 2.1. Let $p\in\mathbb{N}$ satisfy $m\leq p\leq n$, and let $\eta=\{\eta_1,\eta_2,\ldots\eta_p\}\subset\mathbb{N}$ be an ordered set with cardinality $|\eta|=p$ and $1\leq\eta_1<\eta_2<\ldots<\eta_p\leq n$. Then

(i) η is said to be a *basis* of order p if the $[m\times p]$ sub-matrix

$$A_\eta := \left[A_{\eta_1}|A_{\eta_2}|\cdots|A_{\eta_p}\right]$$

has maximal rank, that is, $\mathrm{rank}(A_\eta)=m$.

(ii) For $m\leq p\leq n$, let

$$\mathbb{J}_p := \{\eta\subset\{1,\ldots,n\} \,|\quad \eta \text{ is a basis of order } p\} \tag{2.3}$$

be the set of bases of order p.

Notice that $\mathbb{J}_n=\{\{1,2,\ldots n\}\}$ because A has maximal rank. Moreover,

Lemma 2.2. *Let η be any subset of $\{1,2,\ldots n\}$ with cardinality $|\eta|$.*

(i) *If $|\eta|=m$ then $\eta\in\mathbb{J}_m$ if and only if A_η is invertible.*

(ii) *If $|\eta|=q$ with $m<q\leq n$, then $\eta\in\mathbb{J}_q$ if and only if there exists a basis $\sigma\in\mathbb{J}_m$ such that $\sigma\subset\eta$.*

Proof. (i) is immediate because A_η is a square matrix, and A_η is invertible if and only if A_η has maximal rank.

On the other hand, (ii) also follows from the fact that A_η has maximal rank if and only if A_η contains a square invertible sub-matrix. □

Lemma 2.2 automatically implies $\mathbb{J}_m\neq\emptyset$ because the matrix A must contain at least one square invertible sub-matrix (we are supposing that A has maximal rank). Besides, $\mathbb{J}_p\neq\emptyset$ for $m<p\leq n$, because $\mathbb{J}_m\neq\emptyset$.

Finally, given a basis $\eta \in \mathbb{J}_p$ for $m \le p \le n$, and three vectors $z \in \mathbb{C}^m$, $c \in \mathbb{R}^n$ and $w \in \mathbb{Z}^m$, we introduce the following notation

$$\begin{aligned} z^w &:= z_1^{w_1} z_2^{w_2} \cdots z_m^{w_m}, \\ c_\eta &:= (c_{\eta_1}, c_{\eta_2}, \ldots c_{\eta_p})', \\ \|w\| &:= \max\{|w_1|, |w_2|, \ldots |w_m|\}. \end{aligned} \tag{2.4}$$

Definition 2.3. The vector $c \in \mathbb{R}^n$ is said to be *regular* if for every basis $\sigma \in \mathbb{J}_{m+1}$, there exist a non-zero vector $v(\sigma) \in \mathbb{Z}^{m+1}$ such that :

$$A_\sigma v(\sigma) = 0 \quad \text{and} \quad c'_\sigma v(\sigma) \neq 0. \tag{2.5}$$

Notice that $c \neq 0$ whenever c is regular. Moreover, there are infinitely many vectors $v \in \mathbb{Z}^{m+1}$ such that $A_\sigma v = 0$, because $\text{rank}(A_\sigma) = m < n$. Thus, the vector $c \in \mathbb{R}^n$ is regular if and only if

$$c_j - c'_\pi A_\pi^{-1} A_j \neq 0, \quad \forall \pi \in \mathbb{J}_m, \quad \forall j \notin \pi;$$

which is the regularity condition used in Brion and Vergne [6], except we do not require $c_j \neq 0$ for all $j = 1, \ldots, n$.

2.2. **Generating function.** As already mentioned, and with no loss of generality, we may and will suppose that the matrix $A \in \mathbb{Z}^{m\times n}$ in (1.1)–(1.2) has maximal rank. That is, the m rows of A, $v(j) = (A_{j,1}, \ldots, A_{j,n})$, $j = 1, \ldots, m$, are linearly independent. For suppose that A has not maximal rank. Then we can find a no null vector $\beta \in \mathbb{Z}^m$ such that $0 = \beta_1 v(1) + \cdots + \beta_m v(m)$ and $\beta \neq 0$. Assume that $\beta_1 \neq 0$. The equation $y = Ax$ has a solution $x \in \mathbb{N}^n$ if and only if x is a solution of the system of equations

$$\begin{aligned} y_j &= v(j)x \quad \text{for} \quad 2 \le j \le m, \quad \text{and} \\ y_1 &= v(1)x \;=\; -\sum_{j=2}^m \beta_j v(j)x/\beta_1 \;=\; \sum_{j=2}^m y_j\beta_j/\beta_1. \end{aligned}$$

So, if $y_1 \neq \sum_{j=2}^m y_j\beta_j/\beta_1$ then $f(y) = 0$; otherwise we can eliminate the equation $y_1 = v(1)x$ from $y = Ax$ (because it does not depend on the free variable x) and use instead the trivial relationship $\beta_1 y(1) + \cdots + \beta_m y(m) = 0$.

On the other hand. An appropriate tool for computing the exact value of $f(y)$ is the *generating function* $F : \mathbb{C}^m \to \mathbb{C}$,

$$z \mapsto F(z) := \sum_{y\in\mathbb{Z}^m} f(y) z^y, \tag{2.6}$$

with z^y defined in (2.4). This generating function was already considered in Brion and Vergne [6], with $\lambda := (\ln z_1, \ldots \ln z_m)$.

Proposition 2.4. *Let f and $\mathcal{F}$ be like in (1.2) and (2.6) respectively, and let $c \in \Gamma$. Then :*

$$F(z) \,=\, \prod_{k=1}^{n} \frac{1}{(1-e^{c_k}\, z_1^{A_{1,k}} z_2^{A_{2,k}} \cdots z_m^{A_{m,k}})}, \tag{2.7}$$

on the domain

$$\begin{array}{l} (\,|z_1|, \dots |z_m|\,) \,\in\, D, \quad \textit{with} \\ D \,:=\, \{\rho \in \mathbb{R}^m \mid \rho > 0; \quad e^{c_k} \rho^{A_k} < 1,\ k = 1, \dots n\}. \end{array} \tag{2.8}$$

Proof. Apply the definition (2.6) of F to obtain :

$$F(z) \,=\, \sum_{y \in \mathbb{Z}^m} z^y \left[\sum_{x \in \mathbb{N}^n,\, Ax = y} \mathrm{e}^{c'x} \right] \,=\, \sum_{x \in \mathbb{N}^n} \mathrm{e}^{c'x}\, z^{Ax}.$$

On the other hand,

$$\mathrm{e}^{c'x}\, z^{Ax} \,=\, \prod_{k=1}^{n} \left(\mathrm{e}^{c_k}\, z_1^{A_{1,k}} \cdots z_m^{A_{m,k}}\right)^{x_k}.$$

The domain D in (2.8) is not empty because $c \in \Gamma$. Indeed, a variant of Farkas' Lemma (see Corollary 7.1e in Schrijver [9, p. 89]) states that the system $A'u \le b$ has a solution if and only if $b'x \ge 0$ for every vector $x \ge 0$ with $Ax = 0$. Whence, the system $A'u \le b$ will have a solution whenever b is in the dual cone $\Omega(0)^*$. Moreover, recalling the definition (2.2) of Γ, we can deduce that $A'u < -c$ has indeed a solution $\breve{u} \in \mathbb{R}^m$ because $c \in \Gamma$. Thus, we also have that $(\mathrm{e}^{\breve{u}_1}, \mathrm{e}^{\breve{u}_2}, \dots \mathrm{e}^{\breve{u}_m})^{A_k} < \mathrm{e}^{-c_k}$ for every $1 \le k \le n$, and so $\rho := (\mathrm{e}^{\breve{u}_1}, \dots \mathrm{e}^{\breve{u}_m}) \in D$.

Thus, the condition $\left|\mathrm{e}^{c_k}\, z_1^{A_{1,k}} \dots z_m^{A_{m,k}}\right| < 1$ holds whenever $1 \le k \le n$ and $(|z_1|, \dots |z_m|) \in D$, so

$$\begin{aligned} F(z) \;&=\; \prod_{k=1}^{n} \sum_{x_k=0}^{\infty} \left(\mathrm{e}^{c_k}\, z_1^{A_{1,k}} \cdots z_m^{A_{m,k}}\right)^{x_k} \\ &=\; \prod_{k=1}^{n} \frac{1}{(1-\mathrm{e}^{c_k}\, z_1^{A_{1,k}} \cdots z_m^{A_{m,k}})}. \end{aligned}$$

□

2.3. **Inverting the generating function.** We will compute the exact value of $f(y)$ by first determining an appropriate expansion of the generating function in the form

$$F(z) \,=\, \sum_{\sigma \in \mathbb{J}_m} \frac{Q_\sigma(z)}{\prod_{k \in \sigma} (1-\mathrm{e}^{c_k}\, z^{A_k})}, \tag{2.9}$$

where the coefficients $Q_\sigma : \mathbb{C}^m \to \mathbb{C}$ are *rational* functions with a finite Laurent series

$$z \mapsto Q_\sigma(z) = \sum_{\beta\in\mathbb{Z}^m,\, \|\beta\|\leq M} Q_{\sigma,\beta} z^\beta. \tag{2.10}$$

In (2.10), the strictly positive integer M is fixed and each $Q_{\sigma,\beta}$ is a real constant.

Remark 2.5. The decomposition (2.9) is *not* unique (at all) and there are several ways to obtain such a decomposition. For instance, Brion and Vergne [6, §2.3, p. 815] provide an explicit decomposition of $F(z)$ into elementary rational fractions of the form

$$F(z) = \sum_{\sigma\in\mathbb{J}_m} \sum_{g\in G(\sigma)} \frac{1}{\prod_{j\in\sigma}\left(1-\gamma_j(g)(\mathrm{e}^{c_j}\, z^{A_j})^{1/q}\right)} \frac{1}{\prod_{k\notin\sigma}\delta_k(g)}, \tag{2.11}$$

where $G(\sigma)$ is a certain set of cardinality q, and the coefficients $\{\gamma_j(g), \delta_k(g)\}$ involve certain roots of unity. The fact that c is *regular* ensures that (2.11) is well-defined. Thus, in principle, we could obtain (2.9) from (2.11), but this would require a highly nontrivial analysis and manipulation of the coefficients $\{\gamma_j(g), \delta_k(g)\}$. In the sequel, we provide an alternative algebraic approach that avoids manipulating these complex coefficients.

If F satisfies (2.9) then we get the following result.

Theorem 2.6. *Let $A \in \mathbb{Z}^{m\times n}$ be of maximal rank, f be as in (1.2) with $c \in \Gamma$, and assume that the generating function F in (2.6) satisfies (2.9)–(2.10). Then :*

$$f(y) = \sum_{\sigma\in\mathbb{J}_m} \sum_{\beta\in\mathbb{Z}^m,\, \|\beta\|\leq M} Q_{\sigma,\beta}\, E_\sigma(y-\beta) \tag{2.12}$$

with

$$E_\sigma(y-\beta) = \begin{cases} \mathrm{e}^{c'_\sigma x} & \text{if} \quad x := A_\sigma^{-1}[y-\beta] \in \mathbb{N}^m, \\ 0 & \text{otherwise;} \end{cases} \tag{2.13}$$

where $c_\sigma \in \mathbb{R}^m$ was defined in (2.4).

Proof. Recall that $z^{A_k} = z_1^{A_{1,k}} \cdots z_m^{A_{m,k}}$, according to (2.4). On the other hand, in view of (2.8), the inequality $|\mathrm{e}^{c_k}\, z^{A_k}| < 1$ holds for every $1 \leq k \leq n$; and so the following expansion holds as well for each $\sigma \in \mathbb{J}_m$:

$$\prod_{k\in\sigma} \frac{1}{1-\mathrm{e}^{c_k}\, z^{A_k}} = \prod_{k\in\sigma}\left[\sum_{x_k\in\mathbb{N}} \mathrm{e}^{c_k x_k}\, z^{A_k x_k}\right] = \sum_{x\in\mathbb{N}^m} \mathrm{e}^{c'_\sigma x}\, z^{A_\sigma x}.$$

Next, suppose that a decomposition (2.9)–(2.10) exists. Then the following relationship is easy to establish.

$$
\begin{aligned}
(2.14)\qquad F(z) &= \sum_{\sigma\in\mathbb{J}_m} \sum_{x\in\mathbb{N}^m} Q_\sigma(z)\, \mathrm{e}^{c'_\sigma x}\, z^{A_\sigma x} \\
&= \sum_{\sigma\in\mathbb{J}_m} \sum_{\beta\in\mathbb{Z}^m,\, \|\beta\|\le M} \sum_{x\in\mathbb{N}^m} Q_{\sigma,\beta}\, \mathrm{e}^{c'_\sigma x}\, z^{\beta+A_\sigma x}.
\end{aligned}
$$

Notice that both equations in (2.6) and (2.14) are equal. Hence, if we want to obtain the exact value of $f(y)$ from (2.14), we only have to sum up all the terms whose exponent $\beta + A_\sigma x$ is equal to y. That is, recalling that A_σ is invertible for every $\sigma \in \mathbb{J}_m$ (see Lemma 2.2),

$$
f(y) = \sum_{\sigma\in\mathbb{J}_m} \sum_{\beta\in\mathbb{Z}^m,\, \|\beta\|\le M} Q_{\sigma,\beta} \times \begin{cases} \mathrm{e}^{c'_\sigma x} & \text{if } x := A_\sigma^{-1}[y-\beta] \in \mathbb{N}^m; \\ 0 & \text{otherwise;} \end{cases}
$$

which is exactly(2.12). $\square$

Remark 2.7. Observe that function $f(y)$ in Theorem 2.6 can be rewritten as a weighted sum of $\mathrm{e}^{c'x}$ at some integral points $x \in \mathbb{N}^n$, namely

$$
(2.15)\qquad f(y) = \sum_{\sigma\in\mathbb{J}_m} \left(\sum_{\beta} Q_{\sigma,\beta}\, \mathrm{e}^{c'\breve{x}(\sigma,\beta)} \right),
$$

where the second finite sum is calculated over all $\beta \in \mathbb{Z}^m$ such that $\|\beta\| \le M$ and $A_\sigma^{-1}[y-\beta] \in \mathbb{N}^m$. Moreover, each vector $\breve{x}(\sigma,\beta) \in \mathbb{N}^n$ is an *integral* point. Indeed, given $x := A_\sigma^{-1}(y-\beta)$ inside $\mathbb{N}^m$ like in (2.13), we define the integral vector $\breve{x}(\sigma,\beta) \in \mathbb{N}^n$ by setting the entries:

$$
\breve{x}(\sigma,\beta)_j = \begin{cases} x_k & \text{if } j = \sigma_k \text{ for some } 1 \le k \le m, \\ 0 & \text{if } j \notin \sigma; \end{cases}
$$

for $j = 1,\dots n$. Clearly, we have that $\mathrm{e}^{c'_\sigma x} = \mathrm{e}^{c'\breve{x}(\sigma,\beta)}$ from which equation (2.15) follows. In addition, these integral points $\breve{x}(\sigma,\beta) \in \mathbb{N}^n$ have at most m nontrivial coordinates and their convex hull defines an integral polyhedron (that is, a polyhedron with integral vertices).

In view of Theorem 2.6, $f(y)$ is easily obtained once the rational functions $Q_\sigma(z)$ in the decomposition (2.9) are available. As already pointed out, the decomposition (2.9)–(2.10) is not unique and the purpose of the next section (§3) is to provide :

- a simple decomposition (2.9) for which the expression of the coefficients Q_σ are easily calculated in the case $n = m + 1$;

whereas in §4 we present :

- a recursive algorithm to provide the Q_σ in the general case $n > m + 1$.

3. The case $n = m + 1$

In this section we completely solve the case $n = m+1$, that is, we provide an explicit expression of $f(y)$. We first need some essential intermediate algebraic calculations, in order to deduce the decomposition (2.9)–(2.10) of $F(z)$ when $n = m + 1$.

3.1. **Some auxiliary rational functions.** Let sgn : $\mathbb{R} \to \mathbb{Z}$ be the *sign* function defined by

$$t \mapsto \operatorname{sgn}(t) := \begin{cases} 1 & \text{if} \quad t > 0, \\ -1 & \text{if} \quad t < 0, \\ 0 & \text{otherwise.} \end{cases}$$

Now, given a fix integer $q > 0$ and for every $k = 1, \dots q$, we are going to construct auxiliary functions $P_k : \mathbb{Z}^q \times \mathbb{C}^q \to \mathbb{C}$, such that each $w \mapsto P_k(\theta, w)$ is a rational function of the variable $w \in \mathbb{C}^q$. Given a vector $\theta \in \mathbb{Z}^q$ whose entries $\theta_k \neq 0$, for $k = 1, \dots, q$, we define :

$$\begin{aligned} P_1(\theta, w) &:= \sum_{r=0}^{|\theta_1|-1} w_1^{\operatorname{sgn}(\theta_1)\, r}, \\ P_2(\theta, w) &:= \left[w_1^{\theta_1} \right] \sum_{r=0}^{|\theta_2|-1} w_2^{\operatorname{sgn}(\theta_2)\, r}, \\ P_3(\theta, w) &:= \left[w_1^{\theta_1} w_2^{\theta_2} \right] \sum_{r=0}^{|\theta_3|-1} w_3^{\operatorname{sgn}(\theta_3)\, r}, \\ \vdots \quad &:= \quad \vdots \\ P_q(\theta, w) &:= \left[\prod_{j=1}^{q-1} w_j^{\theta_j} \right] \sum_{r=0}^{|\theta_q|-1} w_q^{\operatorname{sgn}(\theta_q)\, r}. \end{aligned} \tag{3.1}$$

We claim that

Lemma 3.1. *Let $\theta \in \mathbb{Z}^q$ and $w \in \mathbb{C}^q$. The functions P_k defined in (3.1) satisfy*

$$\sum_{k=1}^{q} \left(1 - w_k^{\operatorname{sgn}(\theta_k)} \right) P_k(\theta, w) \;=\; 1 \;-\; w^{\theta}. \tag{3.2}$$

Proof. Firstly, notice that

$$\left(1 - w_1^{\operatorname{sgn}(\theta_1)} \right) P_1(\theta, w) \;=\; \left(1 - w_1^{\operatorname{sgn}(\theta_1)} \right) \sum_{r=0}^{|\theta_1|-1} w_1^{\operatorname{sgn}(\theta_1)\, r} \;=\; 1 - w_1^{\theta_1}.$$

We have similar formulas for $2 \le k \le q$,

$$\left(1 - w_k^{\operatorname{sgn}(\theta_k)}\right) P_k(\theta, w) \;=\; \left(1 - w_k^{\theta_k}\right) \prod_{j=1}^{k-1} w_j^{\theta_j} \;=\; \prod_{j=1}^{k-1} w_j^{\theta_j} \;-\; \prod_{j=1}^{k} w_j^{\theta_j}.$$

Therefore, adding together all the terms in equation (3.2) yields

$$\sum_{k=1}^{q} \left(1 - w_k^{\operatorname{sgn}(\theta_k)}\right) P_k(\theta, w) \;=\; 1 \;-\; \prod_{j=1}^{q} w_j^{\theta_j}.$$

□

3.2. **Solving the case $n = m+1$.** We now use the algebraic expansions of §3.1 to calculate the function $f(y)$ in (1.2) where $\Omega(y)$ is given in (1.1) and $A \in \mathbb{Z}^{m\times(m+1)}$ is a maximal rank matrix.

Theorem 3.2. *Let $n = m+1$ be fixed, $A \in \mathbb{Z}^{m\times n}$ a maximal rank matrix and let $c \in \Gamma$ be regular. Let $v \in \mathbb{Z}^n$ be a non-zero vector such that $Av = 0$ and $c'v \neq 0$ (cf. Definition 2.3).*

Denote by $\{v_{j_k}\}$ the q non-zero entries of v, with $1 \le j_1 < j_2 < \cdots < j_q \le m+1$, and define the pair of vectors

$$\theta := (v_{j_1}, \dots v_{j_q}), \quad w := (e^{c_{j_1}} z^{A_{j_1}}, e^{c_{j_2}} z^{A_{j_2}}, \dots e^{c_{j_q}} z^{A_{j_q}}). \tag{3.3}$$

Then :

(i) *The generating function $F(z)$ in (2.6) has the expansion*

$$F(z) \;=\; \sum_{k=1}^{q} \frac{Q_k(z)}{\prod_{j\neq j_k}(1 - e^{c_j} z^{A_j})} \;=\; \sum_{\sigma\in\mathbb{J}_m} \frac{Q_\sigma(z)}{\prod_{j\in\sigma}(1 - e^{c_j} z^{A_j})}, \tag{3.4}$$

where the rational functions $z \mapsto Q_k(z)$ are defined by :

$$Q_k(z) := \begin{cases} P_k(\theta, w)/(1 - e^{c'v}) & \text{if } v_{j_k} > 0, \\ -w_k^{-1} P_k(\theta, w)/(1 - e^{c'v}) & \text{if } v_{j_k} < 0; \end{cases} \tag{3.5}$$

for $1 \le k \le q$. Each function P_k in (3.5) is defined as in (3.1).

(ii) *Given $y \in \mathbb{Z}^m$, the function $f(y)$ in (1.2) is directly obtained by applying Theorem 2.6.*

Proof. (i) Since c is regular, let $v \in \mathbb{Z}^n$ be a vector such that $Av = 0$ and $c'v \neq 0$ (see (2.5) in Definition 2.3). Let $\theta \in \mathbb{Z}^q$ and $w \in \mathbb{C}^q$ be the vectors defined in (3.3). We can easily deduce that

$$w^\theta \;=\; \prod_{j=1}^{m+1} \left(e^{c_j} z^{A_j}\right)^{v_j} \;=\; e^{c'v} z^{Av} \;=\; e^{c'v} \;\neq\; 1. \tag{3.6}$$

Next, let $z \mapsto Q_k(z)$ be the rational function defined in (3.5). Then, from Lemma 3.1,

$$\begin{aligned} \sum_{k=1}^{q} \left(1 - \mathrm{e}^{c_{j_k}} z^{A_{j_k}}\right) Q_k(z) \;&=\; \sum_{k=1}^{q} \left(1 - w_k^{\operatorname{sgn}(\theta_k)}\right) \frac{P_k(\theta, w)}{1 - \mathrm{e}^{c'v}} \\ &=\; \frac{1 - w^{\theta}}{1 - \mathrm{e}^{c'v}} \;=\; 1. \end{aligned} \tag{3.7}$$

Multiplying the generating function (2.7) and the end sides of (3.7) together yields the expansion

$$F(z) \;=\; \sum_{k=1}^{q} \frac{Q_k(z)}{\prod_{j \neq j_k} (1 - \mathrm{e}^{c_j} z^{A_j})}; \tag{3.8}$$

which gives us the first equality in (3.4).

(ii) As $c \in \Gamma$, $F(z)$ is the generating function of $f(y)$. Next, consider the ordered sets

$$\sigma(k) \;=\; \{1 \le j \le m+1 \,|\quad j \neq j_k\} \quad \text{for} \quad k = 1, \dots, q. \tag{3.9}$$

In order to apply Theorem 2.6, we only need to prove that each square sub-matrix $A_{\sigma(k)}$ is indeed invertible for every $k = 1, \dots, q$. Recall that $\sigma(k)$ is an element of $\mathbb{J}_m$ precisely when $A_{\sigma(k)}$ is invertible.

We know that $A \in \mathbb{R}^{m \times (m+1)}$ has maximal rank, so A has m linearly independent columns. With no loss of generality, we may assume that the first m columns A_k are linearly independent, for $k = 1, \dots, m$. Hence, since $v \in \mathbb{Z}^n$ satisfies $Av = 0$ with $v \neq 0$, we must have $v_{m+1} \neq 0$. Recall that $\{v_{j_k}\}$ are the q non-zero entries of v, such that $1 \le j_1 < \cdots < j_q \le m+1$. We already know that $j_q = m+1$, so that the matrix

$$A_{\sigma(q)} \;=\; [A_1 | A_2 | \cdots | A_m]$$

is nonsingular. That is, the set $\sigma(q) = \{1, \dots, m\}$ defined in (3.9) is an element of $\mathbb{J}_m$. On the other hand, since $Av = 0$ and $v_{m+1} \neq 0$, we that the $m+1$ column of A is equal to $A_{m+1} = \sum_{j=1}^{m} \frac{-v_j}{v_{m+1}} A_j$. Whence, for every $1 \le k < q$, the square matrix

$$A_{\sigma(k)} \;=\; [A_1 | \dots | A_{j_k - 1} | A_{j_k + 1} | \dots | A_m | A_{m+1}]$$

is clearly nonsingular because the column A_{j_k} of $A_{\sigma(q)}$ has been substituted with the linear combination $A_{m+1} = \sum_{j=1}^{m} \frac{-v_j}{v_{m+1}} A_j$ whose coefficient $-v_{j_k}/v_{m+1}$ is different from zero. Thus, the set $\sigma(k)$ in (3.9) is an element of $\mathbb{J}_m$ for every $1 \le k < q$.

Therefore, the expansion (3.8) can be re-written

$$F(z) \;=\; \sum_{k=1}^{q} \frac{Q_k(z)}{\prod_{j \in \sigma(k)} (1 - \mathrm{e}^{c_j}\, z^{A_j})} \;=\; \sum_{\sigma \in \mathbb{J}_m} \frac{Q_\sigma(z)}{\prod_{j \in \sigma} (1 - \mathrm{e}^{c_j}\, z^{A_j})},$$

with $Q_\sigma \equiv Q_k$ if $\sigma = \sigma(k)$, and $Q_\sigma \equiv 0$ whenever $\sigma \neq \sigma(k)$, for $k = 1, \dots q$. And so, a closed form of $f(y)$ is obtained by applying Theorem 2.6. $\square$

Remark 3.3. In the case where $n = m+1$ and $\Omega(y)$ is compact, a naive way to evaluate $f(y)$ is as follows. Suppose that $B := [A_1|\ldots|A_m]$ is invertible. One may then calculate $\rho := \max\{x_{m+1} \,|\, Ax = y,\, x \geq 0\}$. Thus, the evaluation of $f(y)$ reduces to summing up $\sum_x \mathrm{e}^{c'x}$ over all vectors $x = (\hat{x}, x_{m+1}) \in \mathbb{N}^{m+1}$ such that $x_{m+1} \in [0, \rho] \cap \mathbb{N}$ and $\hat{x} := B^{-1}[y - A_{m+1}x_{m+1}]$. This procedure may work very well for reasonable values of ρ, which clearly depends on the magnitude of y. On the other hand, the computational complexity of the algorithm presented in §3 does *not* depend on y. Indeed, the bound M in (2.12) of Theorem 2.6, does not depend at all on y. Moreover, the algorithm also applies to the case where $\Omega(y)$ is not compact.

To illustrate the difference, consider the following trivial example where $n = 2$, $m = 1$, $A = [1, 1]$ and $c = [0, a]$ with $a \neq 0$. The generating function $F(z)$ in (2.6) and (2.7) is the rational function

$$F(z) \;=\; \frac{1}{(1-z)(1-\mathrm{e}^a z)}.$$

Setting $\theta = v = (-1, 1)$ and $w = (z, \mathrm{e}^a z)$, we obtain the following Hilbert's decomposition of the unit

$$\begin{aligned} 1 \;&=\; (1-z)\, Q_1(z) \;+\; (1-\mathrm{e}^a z)\, Q_2(z) \\ &=\; (1-z)\, \frac{-z^{-1}}{1-\mathrm{e}^a} \;+\; (1-\mathrm{e}^a z)\, \frac{z^{-1}}{1-\mathrm{e}^a}. \end{aligned}$$

And so, the generating function $F(z)$ gets expanded to

$$F(z) \;=\; \frac{-z^{-1}}{(1-\mathrm{e}^a)(1-\mathrm{e}^a z)} \;+\; \frac{z^{-1}}{(1-\mathrm{e}^a)(1-z)}. \tag{3.10}$$

Finally, using Theorem 2.6, we obtain $f(y)$ in closed form by

$$f(y) \;=\; \sum_{x_k \in \mathbb{N},\, x_1 + x_2 = y} \mathrm{e}^{x_2 a} \;=\; \frac{1 - \mathrm{e}^{(y+1)a}}{1 - \mathrm{e}^a}. \tag{3.11}$$

Looking back at (2.10) we may see that $M = 1$ (which obviously does not depend on y) and so the evaluation of $f(y)$ via (2.12) in Theorem 2.6 is done in 4 elementary steps, no matter the magnitude of y. On the other hand, the naive procedure would require y elementary steps.

Remark 3.4. We have already mentioned that the expansion of the generating function $F(z)$ is not unique. In the trivial example of Remark 3.3 we may also expand $F(z)$ as the following sum of linear fractions

$$F(z) \;=\; \frac{\mathrm{e}^a}{(\mathrm{e}^a - 1)(1-\mathrm{e}^a z)} \;-\; \frac{1}{(\mathrm{e}^a - 1)(1-z)},$$

which is not the same as the expansion in (3.10). However, applying Theorem 2.6 again yields the same formula (3.11) for $f(y)$.

4. The general case $n > m + 1$

We now consider the case $n > m+1$ and obtain the decomposition (2.9) that permits to compute $f(y)$ by invoking Theorem 2.6. The idea is to use the results of §3 recursively and we exhibit a decomposition (2.9) in the general case $n > m+1$, by induction.

The following result is proved with same arguments like in the proof of Theorem 3.2.

Proposition 4.1. *Let $A \in \mathbb{Z}^{m\times n}$ be a maximal rank matrix and $c \in \Gamma$ be regular. Suppose that the generating function F in (2.6)–(2.7) has the expansion*

$$F(z) = \sum_{\pi\in\mathbb{J}_{p+1}} \frac{Q_\pi(z)}{\prod_{k\in\pi}(1 - e^{c_k} z^{A_k})}, \tag{4.1}$$

for some integer p with $m \leq p < n$, and for some rational functions $z \mapsto Q_\pi(z)$, explicitly known, and with a finite Laurent's series expansion (2.10).

Then, F also has the expansion

$$F(z) = \sum_{\breve{\pi}\in\mathbb{J}_p} \frac{Q^*_{\breve{\pi}}(z)}{\prod_{k\in\breve{\pi}}(1 - e^{c_k} z^{A_k})}, \tag{4.2}$$

where the rational functions $z \mapsto Q^*_{\breve{\pi}}(z)$ are constructed explicitly, and have a finite Laurent's series expansion (2.10).

Proof. Let $\pi \in \mathbb{J}_{p+1}$ be a given basis with $m \leq p < n$ and such that $Q_\pi(z) \not\equiv 0$ in (4.1). We are going to build up simple rational functions $z \mapsto R^\pi_\eta(z)$, where $\eta \in \mathbb{J}_p$, such that the expansion

$$\frac{1}{\prod_{k\in\pi}(1 - \mathrm{e}^{c_k} z^{A_k})} = \sum_{\eta\in\mathbb{J}_p} \frac{R^\pi_\eta(z)}{\prod_{k\in\eta}(1 - \mathrm{e}^{c_k} z^{A_k})} \tag{4.3}$$

holds.

Invoking Lemma 2.2, there exists a basis $\breve{\sigma} \in \mathbb{J}_m$ such that $\breve{\sigma} \subset \pi$. Pick any subset $\sigma \subset \pi$ such that $|\sigma| = m+1$ and $\breve{\sigma} \subset \sigma$. From Lemma 2.2 again, $\sigma \in \mathbb{J}_{m+1}$. Next, as c is regular, pick $v \in \mathbb{Z}^{m+1}$ such that $A_\sigma v = 0$ and $c'_\sigma v \neq 0$, as in (2.5), and let $\{v_{j_k}\}$ be the q non-zero entries of v, with $1 \leq j_1 < \cdots < j_q \leq m+1$.

The statements below follow from the same arguments as in the proof Theorem 3.2(i), so we only make a sketch of the proof. Define the vectors

$$\begin{array}{rcl} \theta & := & (v_{j_1}, v_{j_2}, \dots v_{j_q}), \\ g & := & (\sigma_{j_1}, \sigma_{j_2}, \dots \sigma_{j_q}), \\ \text{and} \quad w & := & (\mathrm{e}^{c_{g_1}} z^{A_{g_1}}, \mathrm{e}^{c_{g_2}} z^{A_{g_2}}, \dots \mathrm{e}^{c_{g_q}} z^{A_{g_q}}). \end{array} \tag{4.4}$$

Like in (3.6), we may deduce that $w^\theta = \mathrm{e}^{c'_\sigma v} \neq 1$. Moreover, define the rational functions

$$R_k(z) := \left\{ \begin{array}{ll} P_k(\theta, w)/(1 - \mathrm{e}^{c'_\sigma v}) & \text{if } v_{j_k} > 0, \\ -w_k^{-1} P_k(\theta, w)/(1 - \mathrm{e}^{c'_\sigma v}) & \text{if } v_{j_k} < 0; \end{array} \right. \tag{4.5}$$

where the functions P_k are defined as in (3.1), and with $1 \leq k \leq q$. Therefore,

$$1 \;=\; \sum_{k=1}^{q} \left(1 - \mathrm{e}^{c_{g_k}} z^{A_{g_k}}\right) R_k(z),$$

like in (3.7). The latter automatically implies that

$$\frac{1}{\prod_{j\in\pi}(1 - \mathrm{e}^{c_j} z^{A_j})} \;=\; \sum_{k=1}^{q} R_k(z) \left[\prod_{j\in\pi,\, j\neq g_k} \frac{1}{1 - \mathrm{e}^{c_j} z^{A_j}}\right], \tag{4.6}$$

where g is defined in (4.4). Next, we use the same arguments as in the proof of Theorem 3.2(ii). Consider the ordered sets

$$\eta(k) \;=\; \{j \in \pi \,|\quad j \neq \sigma_{j_k}\} \quad \text{for} \quad k = 1, \ldots, q. \tag{4.7}$$

We are going to show that each sub-matrix $A_{\eta(k)}$ has maximal rank for $k = 1, \ldots, q$. Notice that $|\eta(k)| = p$ because $|\pi| = p + 1$; hence, the set $\sigma(k)$ is indeed an element of $\mathbb{J}_p$ precisely when $A_{\eta(k)}$ has maximal rank.

Recall that the pair $(\breve{\sigma}, \sigma) \in \mathbb{J}_m \times \mathbb{J}_{m+1}$ is such that $\breve{\sigma} \subset \sigma \subset \pi$. Thus, with no loss of generality, we may and will assume that the ordered sets

$$\begin{array}{rcl} \pi & = & \{\pi(1), \pi(2), \ldots \pi(p+1)\}, \\ \sigma & = & \{\sigma_1, \sigma_2, \ldots \sigma_{m+1}\} \subset \pi, \\ \breve{\sigma} & = & \{\sigma_2, \sigma_3, \ldots \sigma_{m+1}\}. \end{array} \tag{4.8}$$

¿From Lemma 2.2, the square sub-matrix $A_{\breve{\sigma}}$ is invertible because $\breve{\sigma} \in \mathbb{J}_m$. Moreover, the vector $v \in \mathbb{Z}^{m+1}$ satisfies $A_\sigma v = 0$ with $v \neq 0$. Whence, we may conclude that the first entry $v_1 \neq 0$, after noticing (4.8). Recall that $\{v_{j_k}\}$ are the q non-zero entries of v, with $1 \leq j_1 < \cdots < j_q \leq m + 1$. We already know that $j_1 = 1$, so $\eta(1) = \pi \setminus \{\sigma_1\}$ is in $\mathbb{J}_p$ because $\breve{\sigma} \subset \eta(1)$ and Lemma 2.2. The matrix $A_{\eta(1)}$ has maximal rank as well.

On the other hand, set k to be an integer such that $2 \leq k \leq q$. Recall that $v_{j_k} \neq 0$, and suppose that $\pi(s) = \sigma_{j_k}$ following the notation introduced in (4.8). Since $A_\sigma v = 0$ and $v_1 \neq 0$, we have that the first column of A_σ is equal to $A_{\sigma_1} = \sum_{j=2}^{m+1} -\frac{v_j}{v_1} A_{\sigma_j}$. Whence, recalling (4.7), we have that the matrix

$$A_{\eta(k)} \;=\; \left[A_{\pi(1)}|\cdots|A_{\sigma_1}|\cdots|A_{\pi(s-1)}|A_{\pi(s+1)}|\cdots|A_{\pi(p+1)}\right]$$

has maximal rank, for the ($\pi(s) = \sigma_{j_k}$) column $A_{\pi(s)}$ of $A_{\eta(1)}$ has been substituted with the linear combination $A_{\sigma_1} = \sum_{j=2}^{m+2} -\frac{v_j}{v_1} A_{\sigma_j}$ whose coefficient $-v_{\pi(s)}/v_{m+1}$ is different from zero.

Therefore, each matrix $A_{\eta(k)}$ has maximal rank and each $\eta(k) \in \mathbb{J}_p$. Expansion (4.6) can then be re-written

$$\frac{1}{\prod_{j\in\pi}(1 - \mathrm{e}^{c_j} z^{A_j})} \;=\; \sum_{k=1}^{q} \frac{R_k(z)}{\prod_{j\in\eta(k)}(1 - \mathrm{e}^{c_j}\, z^{A_j})} \;=\; \sum_{\eta\in\mathbb{J}_p} \frac{R^\pi_\eta(z)}{\prod_{j\in\eta}(1 - \mathrm{e}^{c_j}\, z^{A_j})},$$

with $R^\pi_\eta \equiv R_k$ if $\eta = \eta(k)$, and $R^\pi_\eta \equiv 0$ whenever $\eta \neq \eta(k)$, for $k = 1, \dots, q$. The latter identity automatically yields (4.3), as desired.

On the other hand, it is easy to see that all rational functions R_k and R^π_η have finite Laurent's series (2.10), because each R_k is defined in terms of P_k in (4.5), and each rational function P_k, as defined in (3.1), also has a finite Laurent series. Finally, (4.2) follows easily. Compounding (4.1) and (4.3) together, yields

$$F(z) \;=\; \sum_{\eta\in\mathbb{J}_p} \quad \sum_{\pi\in\mathbb{J}_{p+1}} \frac{R^\pi_\eta(z)\, Q_\pi(z)}{\prod_{k\in\eta}(1-\mathrm{e}^{c_k} z^{A_k})}, \tag{4.9}$$

so that the decomposition (4.2) holds by setting Q^*_η identically equal to the finite sum $\sum_{\pi\in\mathbb{J}_{p+1}} R^\pi_\eta Q_\pi$ for every $\eta \in \mathbb{J}_p$. $\square$

Notice that the sum in (4.1) runs over the bases of order $p+1$, whereas the sum in (4.2) runs over the bases of order p. Hence, repeated applications of Proposition 4.1 yields a decomposition of the generating function F into a sum over the bases of order m, which is the decomposition described in (2.9)–(2.10). Namely,

Corollary 4.2. *Let $A \in \mathbb{Z}^{m\times n}$ be a maximal rank matrix, and let $c \in \Gamma$ be regular. Let f be as in (1.2) and F be its generating function (2.6)–(2.7). Then :*

(i) *$F(z)$ has the expansion*

$$F(z) \;=\; \sum_{\sigma\in\mathbb{J}_m} \frac{Q_\sigma(z)}{\prod_{k\in\sigma}(1-e^{c_k} z^{A_k})}, \tag{4.10}$$

for some rational functions $z \mapsto Q_\sigma(z)$ which can be built up explicitly, and with finite Laurent series (2.10).

(ii) *For every $y \in \mathbb{Z}^m$, the function $f(y)$ is obtained from Theorem 2.6.*

Proof. The point (i) is proved by induction. Notice that (2.7) can be rewritten

$$F(z) \;=\; \sum_{\pi\in\mathbb{J}_n} \frac{1}{\prod_{k\in\pi}(1-\mathrm{e}^{c_k} z^{A_k})},$$

because $\mathbb{J}_n = \{\{1, 2, \dots n\}\}$ and A has maximal rank (see (2.3)). Thus, from Proposition 4.1, (4.2) holds for $p = n-1$ as well. And more generally, repeated applications of Proposition 4.1 show that (4.2) holds for all $m \leq p < n$. However, (4.10) is precisely (4.2) with $p = m$.

On the other hand, (ii) follows because as $c \in \Gamma$, $F(z)$ is the generating function of $f(y)$, and has the decomposition (4.10) required to apply Theorem 2.6. $\square$

4.1. **Computational complexity.** The computational complexity is essentially determined by the number of coefficients $\{Q_{\sigma,\beta}\}$ in formula (2.12); or equivalently, by the number of nonzero coefficients of the polynomials $\{Q_\sigma\}$ in the decomposition (2.9)–(2.10). Define

$$\Lambda := \max_{\sigma \in \mathbb{J}_{m+1}} \left\{ \min\{ \|v\| \quad | \quad A_\sigma v = 0,\ c'v \neq 0,\ v \in \mathbb{Z}^{m+1}\} \right\}. \tag{4.11}$$

In the case $n = m + 1$ (see §3.1), each polynomial Q_σ has at most Λ terms. It is only a question of analyzing equations (3.1) and (3.5).

For $n = m + 2$, we have at most $(m + 1)^2$ polynomials Q_σ in (2.9); and again, each one of them has at most Λ non-zero coefficients. Therefore, in the general case $n > m$, we end up with at most $(m + 1)^{n-m}\Lambda$ terms in (2.12).

5. Illustrative Example

Consider the following example with $n = 6, m = 3$ and data

$$A := \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 2 & 1 & 0 & 0 & 1 & 0 \\ 0 & 2 & 1 & 0 & 0 & 1 \end{bmatrix}, \quad c := (c_1, \ldots, c_6),$$

so that $F(z)$ is equal to the rational fraction

$$\frac{1}{(1 - \mathrm{e}^{c_1} z_1 z_2^2)(1 - \mathrm{e}^{c_2} z_1 z_2 z_3^2)(1 - \mathrm{e}^{c_3} z_1 z_3)(1 - \mathrm{e}^{c_4} z_1)(1 - \mathrm{e}^{c_5} z_2)(1 - \mathrm{e}^{c_6} z_3)}.$$

First Step: Setting $\pi = \{1, 2, \ldots, 6\} \in \mathbb{J}_6$, choose $\breve{\sigma} := \{4, 5, 6\}$ and $\sigma := \{3, 4, 5, 6\}$. Let $v := (-1, 1, 0, 1) \in \mathbb{Z}^4$ solve $A_\sigma v = 0$. We obviously have that $q = 3$, $\theta = (-1, 1, 1)$ and $w = (\mathrm{e}^{c_3} z_1 z_3, \mathrm{e}^{c_4} z_1, \mathrm{e}^{c_6} z_3)$, so we get

$$\begin{aligned} R_1^\pi(z) &= \frac{-(\mathrm{e}^{c_3} z_1 z_3)^{-1}}{1 - \mathrm{e}^{c_4 + c_6 - c_3}}, \\ R_2^\pi(z) &= \frac{(\mathrm{e}^{c_3} z_1 z_3)^{-1}}{1 - \mathrm{e}^{c_4 + c_6 - c_3}}, \\ R_3^\pi(z) &= \frac{\mathrm{e}^{(c_4 - c_3)} z_3^{-1}}{1 - \mathrm{e}^{c_4 + c_6 - c_3}}. \end{aligned}$$

Hence

$$1 = (1 - \mathrm{e}^{c_3} z_1 z_3) R_1^\pi(z) + (1 - \mathrm{e}^{c_4} z_1) R_2^\pi(z) + (1 - \mathrm{e}^{c_6} z_3) R_3^\pi(z).$$

Notice that the term $(1 - \mathrm{e}^{c_3} z_1 z_3) R_1^\pi(z)$ will *kill* the element 3 in the base π. Moreover, the terms $(\cdots) R_2^\pi(z)$ and $(\cdots) R_3^\pi(z)$ will also *kill* the respective entries 4 and 6 in the base π, so

$$\begin{aligned}
F(z) &= \frac{R_1^{\pi}(z)}{(1-\mathrm{e}^{c_1}z_1z_2^2)(1-\mathrm{e}^{c_2}z_1z_2z_3^2)(1-\mathrm{e}^{c_4}z_1)(1-\mathrm{e}^{c_5}z_2)(1-\mathrm{e}^{c_6}z_3)}\\
&+ \frac{R_2^{\pi}(z)}{(1-\mathrm{e}^{c_1}z_1z_2^2)(1-\mathrm{e}^{c_2}z_1z_2z_3^2)(1-\mathrm{e}^{c_3}z_1z_3)(1-\mathrm{e}^{c_5}z_2)(1-\mathrm{e}^{c_6}z_3)}\\
&+ \frac{R_3^{\pi}(z)}{(1-\mathrm{e}^{c_1}z_1z_2^2)(1-\mathrm{e}^{c_2}z_1z_2z_3^2)(1-\mathrm{e}^{c_3}z_1z_3)(1-\mathrm{e}^{c_4}z_1)(1-\mathrm{e}^{c_5}z_2)}\\
&= \sum_{j=1}^{3}\frac{Q_{\eta_5(j)}(z)}{\prod_{k\in\eta_5(j)}(1-\mathrm{e}^{c_k}z^{A_k})},
\end{aligned}$$

where $\eta_5(1)=\{1,2,4,5,6\}$, $\eta_5(2)=\{1,2,3,5,6\}$, $\eta_5(3)=\{1,2,3,4,5\}$ and $Q_{\eta_5(j)}(z)=R_j^{\pi}(z)$ for $j=1,2,3$.

Second Step: Analyzing $\eta_5(1)=\{1,2,4,5,6\}\in\mathbb{J}_5$, choose $\breve{\sigma}=\{4,5,6\}$ and $\sigma:=\{1,4,5,6\}$. Let $v:=(-1,1,2,0)\in\mathbb{Z}^4$ solve $A_\sigma v=0$. We have that $q=3$, $\theta=(-1,1,2)$ and $w=(\mathrm{e}^{c_1}z_1z_2^2,\mathrm{e}^{c_4}z_1,\mathrm{e}^{c_5}z_2)$, so we get

$$\begin{aligned}
R_1^{\eta_5(1)}(z) &= \frac{-(\mathrm{e}^{c_1}z_1z_2^2)^{-1}}{1-\mathrm{e}^{-c_1+c_4+2c_5}},\\
R_2^{\eta_5(1)}(z) &= \frac{(\mathrm{e}^{c_1}z_1z_2^2)^{-1}}{1-\mathrm{e}^{-c_1+c_4+2c_5}},\\
R_3^{\eta_5(1)}(z) &= \frac{(\mathrm{e}^{c_4-c_1}z_2^{-2})(1+\mathrm{e}^{c_5}z_2)}{1-\mathrm{e}^{-c_1+c_4+2c_5}}.
\end{aligned}$$

Notice that the terms associated to $R_1^{\eta_5(1)}(z)$, $R_2^{\eta_5(1)}(z)$ and $R_3^{\eta_5(1)}(z)$ *kill* the respective entries 1, 4 and 5 in the base $\eta_5(1)$.

Analyzing $\eta_5(2)=\{1,2,3,5,6\}\in\mathbb{J}_5$, choose $\breve{\sigma}=\{3,5,6\}$ and $\sigma:=\{2,3,5,6\}$. Let $v:=(-1,1,1,1)\in\mathbb{Z}^4$ solve $A_\sigma v=0$. We have that $q=4$, $\theta=v$ and $W=(\mathrm{e}^{c_2}z_1z_2z_3^2,\mathrm{e}^{c_3}z_1z_3,\mathrm{e}^{c_5}z_2,\mathrm{e}^{c_6}z_3)$, so we get

$$\begin{aligned}
R_1^{\eta_5(2)}(z) &= \frac{-(\mathrm{e}^{c_2}z_1z_2z_3^2)^{-1}}{1-\mathrm{e}^{-c_2+c_3+c_5+c_6}},\\
R_2^{\eta_5(2)}(z) &= \frac{(\mathrm{e}^{c_2}z_1z_2z_3^2)^{-1}}{1-\mathrm{e}^{-c_2+c_3+c_5+c_6}},\\
R_3^{\eta_5(2)}(z) &= \frac{\mathrm{e}^{c_3-c_2}z_2^{-1}z_3^{-1}}{1-\mathrm{e}^{-c_2+c_3+c_5+c_6}},\\
R_4^{\eta_5(2)}(z) &= \frac{\mathrm{e}^{c_5+c_3-c_2}z_3^{-1}}{1-\mathrm{e}^{-c_2+c_3+c_5+c_6}}.
\end{aligned}$$

Notice that the terms associated to $R_1^{\eta_5(2)}(z)$, $R_2^{\eta_5(2)}(z)$, $R_3^{\eta_5(2)}(z)$ and $R_4^{\eta_5(2)}(z)$ *kill* the respective entries 2, 3, 5 and 6 in the base $\eta_5(2)$.

Analyzing $\eta_5(3)=\{1,2,3,4,5\}\in\mathbb{J}_5$, choose $\breve{\sigma}=\{3,4,5\}$ and $\sigma:=\{2,3,4,5\}$. Let $v:=(-1,2,-1,1)\in\mathbb{Z}^4$ solve $A_\sigma v=0$. We have that $q=4$,

$\theta = v$ and $w = (\mathrm{e}^{c_2} z_1 z_2 z_3^2, \mathrm{e}^{c_3} z_1 z_3, \mathrm{e}^{c_4} z_1, \mathrm{e}^{c_5} z_2)$, so we get

$$
\begin{aligned}
R_1^{\eta_5(3)}(z) &= \frac{-(\mathrm{e}^{c_2} z_1 z_2 z_3^2)^{-1}}{1 - \mathrm{e}^{-c_2+2c_3-c_4+c_5}},\\
R_2^{\eta_5(3)}(z) &= \frac{(\mathrm{e}^{c_2} z_1 z_2 z_3^2)^{-1}(1 + \mathrm{e}^{c_3} z_1 z_3)}{1 - \mathrm{e}^{-c_2+2c_3-c_4+c_5}},\\
R_3^{\eta_5(3)}(z) &= \frac{-(\mathrm{e}^{c_4} z_1)^{-1}(\mathrm{e}^{2c_3-c_2} z_1 z_2^{-1})}{1 - \mathrm{e}^{-c_2+2c_3-c_4+c_5}},\\
R_4^{\eta_5(3)}(z) &= \frac{\mathrm{e}^{2c_3-c_2-c_4} z_2^{-1}}{1 - \mathrm{e}^{-c_2+2c_3-c_4+c_5}}.
\end{aligned}
$$

Notice that the terms associated to $R_1^{\eta_5(3)}(z)$, $R_2^{\eta_5(3)}(z)$, $R_3^{\eta_5(3)}(z)$ and $R_4^{\eta_5(3)}(z)$ *kill* the respective entries 2, 3, 4 and 5 in the base $\eta_5(3)$.

Therefore, we have the following expansion of $F(z)$.

$$
\begin{aligned}
F(z) &= \frac{Q_{\eta_5(1)}(z) R_1^{\eta_5(1)}(z)}{(1-\mathrm{e}^{c_2} z_1 z_2 z_3^2)(1-\mathrm{e}^{c_4} z_1)(1-\mathrm{e}^{c_5} z_2)(1-\mathrm{e}^{c_6} z_3)}\\
&+ \frac{Q_{\eta_5(1)}(z) R_2^{\eta_5(1)}(z) + Q_{\eta_5(2)}(z) R_2^{\eta_5(2)}(z)}{(1-\mathrm{e}^{c_1} z_1 z_2^2)(1-\mathrm{e}^{c_2} z_1 z_2 z_3^2)(1-\mathrm{e}^{c_5} z_2)(1-\mathrm{e}^{c_6} z_3)}\\
&+ \frac{Q_{\eta_5(1)}(z) R_3^{\eta_5(1)}(z)}{(1-\mathrm{e}^{c_1} z_1 z_2^2)(1-\mathrm{e}^{c_2} z_1 z_2 z_3^2)(1-\mathrm{e}^{c_4} z_1)(1-\mathrm{e}^{c_6} z_3)}\\
&+ \frac{Q_{\eta_5(2)}(z) R_1^{\eta_5(2)}(z)}{(1-\mathrm{e}^{c_1} z_1 z_2^2)(1-\mathrm{e}^{c_3} z_1 z_3)(1-\mathrm{e}^{c_5} z_2)(1-\mathrm{e}^{c_6} z_3)}\\
&+ \frac{Q_{\eta_5(2)}(z) R_3^{\eta_5(2)}(z)}{(1-\mathrm{e}^{c_1} z_1 z_2^2)(1-\mathrm{e}^{c_2} z_1 z_2 z_3^2)(1-\mathrm{e}^{c_3} z_1 z_3)(1-\mathrm{e}^{c_6} z_3)}\\
&+ \frac{Q_{\eta_5(2)}(z) R_4^{\eta_5(2)}(z) + Q_{\eta_5(3)}(z) R_3^{\eta_5(3)}(z)}{(1-\mathrm{e}^{c_1} z_1 z_2^2)(1-\mathrm{e}^{c_2} z_1 z_2 z_3^2)(1-\mathrm{e}^{c_3} z_1 z_3)(1-\mathrm{e}^{c_5} z_2)}\\
&+ \frac{Q_{\eta_5(3)}(z) R_1^{\eta_5(3)}(z)}{(1-\mathrm{e}^{c_1} z_1 z_2^2)(1-\mathrm{e}^{c_3} z_1 z_3)(1-\mathrm{e}^{c_4} z_1)(1-\mathrm{e}^{c_5} z_2)}\\
&+ \frac{Q_{\eta_5(3)}(z) R_2^{\eta_5(3)}(z)}{(1-\mathrm{e}^{c_1} z_1 z_2^2)(1-\mathrm{e}^{c_2} z_1 z_2 z_3^2)(1-\mathrm{e}^{c_4} z_1)(1-\mathrm{e}^{c_5} z_2)}\\
&+ \frac{Q_{\eta_5(3)}(z) R_4^{\eta_5(3)}(z)}{(1-\mathrm{e}^{c_1} z_1 z_2^2)(1-\mathrm{e}^{c_2} z_1 z_2 z_3^2)(1-\mathrm{e}^{c_3} z_1 z_3)(1-\mathrm{e}^{c_4} z_1)}\\
&= \sum_{j=1}^{9} \frac{Q_{\eta_4(j)}(z)}{\prod_{k\in\eta_4(j)}(1-\mathrm{e}^{c_k} z^{A_k})}.
\end{aligned}
$$

Final Step: At this last step step we obtain the required decomposition (4.10), that is, we will be able to express $F(z)$ as the sum

$$F(z) = \sum_j \frac{Q_{\eta_3(j)}(z)}{\prod_{k\in\eta_3(j)}(1-\mathrm{e}^{c_k}z^{A_k})}. \tag{5.1}$$

The exact values of $f(y)$ can be then calculating by using Theorem 2.6. Moreover, we must make the observation that, out of the potentially $\binom{6}{3} = 20$ terms, the above sum contains 16 terms. We are going to conclude this paper providing the term $Q_{\eta_3(j)}(z)$ relative to the basis $\eta_3(j) = \{2,5,6\} \in \mathbb{J}_3$.

Setting $\eta_4(1) = \{2,4,5,6\} \in \mathbb{J}_4$, choose $\breve{\sigma} := \{4,5,6\}$ and $\sigma := \{2,4,5,6\}$. Let $v := (-1,1,1,2) \in \mathbb{Z}^4$ solve $A_\sigma v = 0$. We have that $q = 4$, $\theta = v$ and $w = (\mathrm{e}^{c_2}z_1z_2z_3^2, \mathrm{e}^{c_4}z_1, \mathrm{e}^{c_5}z_2, \mathrm{e}^{c_6}z_3)$, so we get

$$\begin{aligned}
R_1^{\eta_4(1)}(z) &= \frac{-(\mathrm{e}^{c_2}z_1z_2z_3^2)^{-1}}{1-\mathrm{e}^{2c_6+c_5+c_4-c_2}},\\
R_2^{\eta_4(1)}(z) &= \frac{(\mathrm{e}^{c_2}z_1z_2z_3^2)^{-1}}{1-\mathrm{e}^{2c_6+c_5+c_4-c_2}},\\
R_3^{\eta_4(1)}(z) &= \frac{\mathrm{e}^{c_4-c_2}(z_2z_3^2)^{-1}}{1-\mathrm{e}^{2c_6+c_5+c_4-c_2}},\\
R_4^{\eta_4(1)}(z) &= \frac{(\mathrm{e}^{c_4+c_5-c_2}z_3^{-2})(1+\mathrm{e}^{c_6}z_3)}{1-\mathrm{e}^{2c_6+c_5+c_4-c_2}}.
\end{aligned}$$

Notice that the term associated to $R_2^{\eta_4(1)}$ *kills* the entry 4 in the base $\eta_4(1) = \{2,4,5,6\}$, so we are getting the desired base $\eta_3(1) = \{2,5,6\}$.

Setting $\eta_4(2) = \{1,2,5,6\} \in \mathbb{J}_4$, choose $\breve{\sigma} := \{2,5,6\}$ and $\sigma := \{1,2,5,6\}$. Let $v := (-1,1,1,-2) \in \mathbb{Z}^4$ solve $A_\sigma v = 0$. We have that $q = 4$, $\theta = v$ and $w = (\mathrm{e}^{c_1}z_1z_2^2, \mathrm{e}^{c_2}z_1z_2z_3^2, \mathrm{e}^{c_5}z_2, \mathrm{e}^{c_6}z_3)$, so we get

$$\begin{aligned}
R_1^{\eta_4(2)}(z) &= \frac{-(\mathrm{e}^{c_1}z_1z_2^2)^{-1}}{1-\mathrm{e}^{c_2+c_5-c_1-2c_6}},\\
R_2^{\eta_4(2)}(z) &= \frac{(\mathrm{e}^{c_1}z_1z_2^2)^{-1}}{1-\mathrm{e}^{c_2+c_5-c_1-2c_6}},\\
R_3^{\eta_4(2)}(z) &= \frac{\mathrm{e}^{c_2-c_1}z_2^{-1}z_3^2}{1-\mathrm{e}^{c_2+c_5-c_1-2c_6}},\\
R_4^{\eta_4(2)}(z) &= \frac{-(\mathrm{e}^{c_6}z_3)^{-1}(\mathrm{e}^{c_2-c_1+c_5}z_3^2)(1+(\mathrm{e}^{c_6}z_3)^{-1})}{1-\mathrm{e}^{c_2+c_5-c_1-2c_6}}.
\end{aligned}$$

Notice that the term associated to $R_1^{\eta_4(2)}$ *kills* the entry 1 in the base $\eta_4(2) = \{1,2,5,6\}$, so we are getting the desired base $\eta_3(1) = \{2,5,6\}$.

Therefore, working on the base $\eta_3(1)$, we obtain the numerator

$$\begin{aligned}
Q_{\eta_3(1)}(z) &= Q_{\eta_4(1)}R_2^{\eta_4(1)} + Q_{\eta_4(2)}R_1^{\eta_4(2)}\\
&= \left[Q_{\eta_5(1)}(z)R_1^{\eta_5(1)}(z)\right]R_2^{\eta_4(1)} +\\
&+ \left[Q_{\eta_5(1)}(z)R_2^{\eta_5(1)}(z) + Q_{\eta_5(2)}(z)R_2^{\eta_5(2)}(z)\right]R_1^{\eta_4(2)}.
\end{aligned}$$

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