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Mutation Polynomials and Oriented Matroids*

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Abstract. Several polynomials are of use in various enumeration problems concerning objects in oriented matroids. Chief among these is the Radon catalog. We continue to study these, as well as the total polynomials of uniform oriented matroids, by considering the effect on them of mutations of the uniform oriented matroid. The notion of a "mutation polynomial" is introduced to facilitate the study.

The affine spans of the Radon catalogs and the total polynomials in the appropriate rational vector spaces of polynomials are determined, and bases for the Z-modules generated by the mutation polynomials are found. The Radon polynomials associated with alternating oriented matroids are described; it is conjectured that a certain extremal property, like that held by cyclic polytopes among simplicial polytopes, is possessed by them.

1. Introduction

The combinatorial convex geometry of a finite set in R^d can be studied through an oriented matroid associated with the set. For example, the oriented matroid retains information about the facial structure of the polytope which is the convex hull of the set, and about intersections of simplexes determined by the set. By counting appropriate structures in the oriented matroid, one can enumerate the faces of various dimensions of the polytope, or count the intersections of simplexes.

The "total polynomial" and the "Radon catalog" of a uniform oriented matroid were introduced in [11], where the coefficients of these polynomials were seen to indicate or count several important structures in the uniform oriented matroid; and certain equations were seen to be satisfied by these polynomials. In this paper we continue to study these polynomials.

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We consider the effect of a "mutation" of the uniform oriented matroid upon the total polynomial and Radon catalog. The idea of a mutation in this context has a history which predates that of oriented matroids. Ringel considered them, for arrangements of lines and pseudolines, in [15], where it was proven that any two simple arrangements of pseudolines are connected by a sequence of such arrangements in which each consecutive pair differ by the "switching of a triangle." After [6], this is a result about uniform oriented matroids of rank 3; the switching of a triangle is a mutation. For a proof of Ringel's theorem, see Theorem 6.4.1 of [2]. For much more information on arrangements of lines and pseudolines see the monograph [8] of Grünbaum, which motivated much work in this area.

Various kinds of "local deformations" of oriented matroids have been defined and studied. See Las Vergnas's paper [9], where the idea is originally used to construct some interesting examples; and for more extensive studies see [4], [5], [7], and [18]. Using the fact that the appropriate Grassmann manifolds are connected, Roudneff and Sturmfels proved in [16] that any two realizable uniform oriented matroids of rank r on the same underlying set can be connected by a sequence of mutations. A mutation can be performed on a given uniform oriented matroid provided that it contains a tope which is a simplex. Las Vergnas [10] conjectured that each uniform oriented matroid has such a tope. This conjecture is still open; however, Richter-Gebert [14] has constructed many examples of uniform oriented matroids with many fewer simplex topes than is possible in the realizable case. A strengthened form of the conjecture of Las Vergnas is considered by Roudneff and Sturmfels in [16], where it is attributed to Cordovil and Las Vergnas. Cordovil and Las Vergnas conjecture that any two uniform oriented matroids on the same set of the same rank can be connected by a sequence of mutations. Roudneff and Sturmfels prove this, using connectivity of appropriate Grassmann manifolds, when the uniform oriented matroids are realizable. The rank 3 case of this conjecture is the theorem of Ringel, mentioned above.

In this paper we study the affine linear spaces spanned by the total polynomials and the Radon catalogs. It is shown, in Section 3, that the equations presented in [11] which are satisfied by the total polynomials determine the span of these polynomials. The difference between the total polynomial of a uniform oriented matroid and that of one of its mutations is termed a "mutation polynomial." We prove that the affine span of the total polynomials is determined already by the mutation polynomials. Indeed, in Section 4 it is found that the Z-module spanned by the differences of total polynomials (on a fixed set and of the same rank) is already spanned by the mutation polynomials. (This result would be an easy consequence of the Cordovil–Las Vergnas conjecture that any two uniform oriented matroids of rank r on a set E are connected by a sequence of mutations.) We describe a subset of the set of mutation polynomials which forms a basis for this Z-module

The equations presented in [11] which are satisfied by the Radon catalogs of uniform oriented matroids of rank r on a fixed set E of cardinality n determine the affine linear space spanned by these Radon catalogs, as is shown in Section 5. The dimension of this affine linear space is shown to be $[(n-r+1)/2][(r+1)/2]$. A particular basis for the linear space spanned by differences is chosen.

We pose a conjecture concerning the coefficients of the Radon catalogs with respect to our basis. Roughly, the conjecture states that the alternating uniform oriented matroids have extremal Radon catalogs, in much the same way that the cyclic polytopes have extremal f-vectors, among simplicial polytopes of the same dimension and number of vertices. A mutation of a uniform oriented matroid of rank r on a set E having n elements can be visualized as the flipping of a simplex σ of the corresponding arrangement. The "type" of this mutation is the pair (a, b) of integers, $0 \leq a \leq r$, $0 \leq b \leq n - r$, where a is the number of pseudospheres bounding the simplex σ and having the simplex on the positive side, and b is the number of pseudospheres of the arrangement not bounding the simplex which have the simplex on the positive side. Considering the simplex antipodal to σ, it is clear that a mutation of type (a, b) also has type $(r - a, n - r - b)$. Also, the reverse of a mutation of type (a, b) has type $(r - a, b)$, as well as $(a, n - r - b)$. Suppose $0 \leq a \leq [(r - 1)/2]$ and $0 \leq b \leq [(n - r - 1)/2]$. Suppose one starts with the alternating oriented matroid of rank r on E and performs a sequence of mutations, arriving at a second oriented matroid, and in the process computes the difference between the number of times mutations of type (a, b) are performed and the number of times the reverse of such mutations are performed. It is conjectured that this difference must be nonnegative. The conjecture is extended to all uniform oriented matroids by making use of knowledge of the affine span, and of the chosen basis.

In Sections 6 and 7 we study the Radon catalogs of the alternating oriented matroids. In Section 8 we consider further questions regarding these ideas.

In Section 2, before introducing the mutation polynomials, we present some background material from [11]. For terminology and notation regarding oriented matroids, see [2].

2. The Mutation Polynomials

A "mutation polynomial" is the difference between the total polynomial of a uniform oriented matroid and the total polynomial of a mutation of that uniform oriented matroid. The book [2] describes various kinds of mutations for oriented matroids in general and for uniform oriented matroids more particularly; see Section 7.3 of [2].

By E we denote a finite set, which is to be the underlying set of a uniform oriented matroid; n is its cardinality. In this paper we usually take $E = [n] = \{1, 2, \ldots, n\}$. A *signed subset* of E is an ordered partition $X = (X^+, X^-)$ of a subset of E. The union $X^+ \cup X^-$ is denoted by $\bar{X}$, and $|X|$ is synonymous with $|\bar{X}|$. A *sign vector* is an element of $\{-, +, 0\}^n$, which is to be identified with $\{-, +, 0\}^E$. (As in [2], we use $+$ and $-$ for $+1$ and -1 when to do so should cause no confusion.) Given a sign vector $U \in \{-, +, 0\}^E$ we denote $U^+ = \{e \in E\colon U_e = +\}$, $U^- = \{e \in E\colon U_e = -\}$, and $U^0 = \{e \in E\colon U_e = 0\}$.

There is a beautifully concise description of uniform oriented matroids, due to Folkman and included in [6] (using different notation, and under the name "positivity system"), which we state as the definition here. A *uniform oriented matroid of rank* r is a pair $\mathcal{O} = (E, \mathcal{C})$, where $\mathcal{C}$ is a collection of signed subsets of E, the set of *circuits* of $\mathcal{O}$, satisfying the following properties:

(a) $n \geq r$, and if $n > r$, then $\mathcal{C} \neq \emptyset$;
(b) if $C \in \mathcal{C}$, then $-C \in \mathcal{C}$;
(c) if $C, D \in \mathcal{C}$ and $\bar{D} \subseteq \bar{C}$, then $D = C$ or $D = -C$;
(d) if $C \in \mathcal{C}$, then $|C| = r + 1$; and

(e) if $C \in \mathcal{C}$, $p \in E$, and $p \notin C$, then there is $D \in \mathcal{C}$ such that $p \in D^+$, $D^+ \subseteq C^+ \cup \{p\}$, and $D^- \subseteq C^-$.

In this paper the set of "covectors" of the uniform oriented matroid $\mathcal{O}$ is important. A *covector* of $\mathcal{O}$ is a sign vector $U \in \{-, +, 0\}^E$ such that, for each circuit C of $\mathcal{O}$, there is $e \in C^+$ such that $U_e = +$ if and only if there is $f \in C^-$ such that $U_f = -$.

The set $\{+, -, 0\}^E$ is partially ordered by the relation

$$U \leq V \qquad \text{if whenever} \quad U_e \neq 0, \quad V_e = U_e.$$

The notation $[U, V]$, where U and V are sign vectors with $U \leq V$, denotes the interval in the partially ordered set $\{+, -, 0\}^E$:

$$[U, V] = \{W \in \{+, -, 0\}^E \colon U \leq W \leq V\}.$$

The set of covectors is given the induced partial ordering.

Clearly, $0 \in \{+, -, 0\}^E$ is a covector. The set of nonzero covectors of $\mathcal{O}$ will be denoted by $\mathcal{L}$.

The nonzero covectors of an oriented matroid correspond to the cells of a corresponding arrangement of pseudospheres, with $U \leq V$ if and only if the cell corresponding to U is a face of the cell corresponding to V.

For each element $e \in E$, let x_e and y_e be a pair of indeterminates. For each sign vector U let w_U denote the monomial

$$w_U = \prod_{e\colon U_e=+} x_e \prod_{e\colon U_e=-} y_e.$$

The *total polynomial* of the uniform oriented matroid $\mathcal{O}$ is

$$T_{\mathcal{O}}(x_e, y_e\colon e \in E) = \sum_{U\colon U \in \mathcal{L}} w_U.$$

We list some properties of total polynomials, from [11].

(1) $T_{\mathcal{O}}(x_e, y_e\colon e \in E)$ is a sum of monomial terms which are squarefree and not multiples of $x_e y_e$ for $e \in E$. These monomials have degree between $n - r + 1$ and n (inclusive).

(2) $T_{\mathcal{O}}(y_e, x_e\colon e \in E) = T_{\mathcal{O}}(x_e, y_e\colon e \in E)$.

(3) The identity

$$\prod_{e\in E}(1 + x_e + y_e) T_{\mathcal{O}}\left(\frac{-x_e}{1 + x_e + y_e}, \frac{-y_e}{1 + x_e + y_e}\colon e \in E\right) = (-1)^{n-r+1} T_{\mathcal{O}}(x_e, y_e\colon e \in E)$$

holds.

(4) The total polynomial of the dual $\widehat{\mathcal{O}}$ of $\mathcal{O}$ is given by

$$T_{\hat{\mathcal{O}}}(x_e, y_e\colon e \in E) = \prod_{e\in E}(1 + x_e + y_e) - (-1)^r - (-1)^n T_{\mathcal{O}}(-1 - x_e, -1 - y_e\colon e \in E).$$

In particular, the polynomial

$$T_{\mathcal{O}}(-1-x_e, -1-y_e\colon e \in E)$$

agrees with

$$(-1)^n \left(\prod_{e \in E} (1 + x_e + y_e) - (-1)^r \right)$$

on terms of degree at most r.

The maximal covectors under $\leq$ are called the *topes* of $\mathcal{O}$. It is easy to show that, for a tope U, $U_e \neq 0$ for each $e \in E$; so topes are maximal in $\{+, -, 0\}^E$, as well. A tope U is termed *simplicial* if the partially ordered set of covectors $V \leq U$ forms a boolean lattice. Since we are dealing with *uniform* oriented matroids, it is easy to describe this set of covectors explicitly. Given a set $F \subseteq E$ and a maximal element $U \in \{+, -, 0\}^E$, let $(U|F)$ and $(U * F)$ be the sign vectors defined by

$$(U|F)_e = \begin{cases} U_e & \text{if} \quad e \notin F, \\ 0 & \text{if} \quad e \in F, \end{cases}$$

and (for use in defining mutations)

$$(U * F)_e = \begin{cases} U_e & \text{if} \quad e \notin F, \\ -U_e & \text{if} \quad e \in F. \end{cases}$$

When U is a simplicial tope of $\mathcal{O}$, necessarily, by uniformity, there is a set $F \subseteq E$ having $|F| = r$ and such that the set of nonzero covectors V with $V \leq U$ is $[(U|F), U] \backslash \{(U|F)\}$.

If U is a simplicial tope in the uniform oriented matroid $\mathcal{O}$ and F is as above, and if we denote $I_1 = [(U|F), (U * F)]$, $I_2 = [-(U|F), -(U * F)]$, $I_3 = [(U|F), U]$, and $I_4 = [-(U|F), -U]$, then the collection

$$\mathcal{L}' = (\mathcal{L} \cup I_1 \cup I_2) \backslash (I_3 \cup I_4)$$

is the collection of covectors of another uniform oriented matroid $\mathcal{O}'$ of rank r on E, called a *mutation* of $\mathcal{O}$.

As noted in the Introduction, it has been conjectured by Las Vergnas [9] that each uniform oriented matroid has a simplicial tope; and a strengthened form of this conjecture, formulated by Cordovil and Las Vergnas, states that each pair of uniform oriented matroids of rank r on E is connected by a sequence of mutations.

We now consider the difference $T_{\mathcal{O}'} - T_{\mathcal{O}}$. From the expression for $\mathcal{L}'$ it is clear that this difference depends only on the element $U \in \{+, -, 0\}^E$ and the subset $F \subseteq E$.

Let X and Y be signed sets such that $\bar{X}, \bar{Y}$ form a partition of E. We denote by $N_{X,Y}$ the polynomial

$$N_{X,Y}(x_e, y_e\colon e \in E) = \prod_{e \in X^+} (1 + x_e) \prod_{e \in X^-} (1 + y_e) \prod_{e \in Y^+} x_e \prod_{e \in Y^-} y_e.$$

Lemma 1. *Given U and F as above, let the signed sets X and Y be defined by $X^+ = F \cap U^+$, $X^- = F \cap U^-$; and $Y^+ = U^+ \backslash F$, $Y^- = U^- \backslash F$. Then*

$$T_{\mathcal{O}'} = T_{\mathcal{O}} - (N_{X,Y} - N_{-X,Y} - N_{X,-Y} + N_{-X,-Y}).$$

Proof. This is clear when one notes that the polynomial $N_{X,Y}$ is

$$\sum_{V \in I_3} w_V,$$

and similarly

$$N_{-X,Y} = \sum_{V \in I_1} w_V,$$
$$N_{X,-Y} = \sum_{V \in I_2} w_V,$$

and

$$N_{-X,-Y} = \sum_{V \in I_4} w_V,$$

and that I_1 and I_3 have the common element $(U|F)$, I_2 and I_4 both contain $-(U|F)$, but the four intervals have, pairwise, no other covectors in common. □

The polynomial $N_{X,Y} - N_{-X,Y} - N_{X,-Y} + N_{-X,-Y}$ is the product of

$$\left(\prod_{e \in X^+} (1 + x_e) \prod_{e \in X^-} (1 + y_e) - \prod_{e \in X^+} (1 + y_e) \prod_{e \in X^-} (1 + x_e) \right)$$

and

$$\left(\prod_{e \in Y^+} x_e \prod_{e \in Y^-} y_e - \prod_{e \in Y^+} y_e \prod_{e \in Y^-} x_e \right).$$

This polynomial is termed a *mutation polynomial* and denoted by $M_{X,Y}$. We refer to the cardinality of $\bar{X}$ as the *order* of $M_{X,Y}$, as well as that of $N_{X,Y}$. In the present situation the order coincides with the rank r of $\mathcal{O}$.

The number of mutation polynomials of order r, up to sign, is $2^{n-2}\binom{n}{r}$.

If $\mathcal{O}$ and $\mathcal{O}'$ are realizable uniform oriented matroids of rank r on E, then there is a sequence of mutations connecting the two. It follows that the difference $T_{\mathcal{O}} - T_{\mathcal{O}'}$ is a sum of mutation polynomials. In the next two sections we generalize this by showing that, for any two uniform oriented matroids $\mathcal{O}$ and $\mathcal{O}'$ of rank r on E, the difference $T_{\mathcal{O}} - T_{\mathcal{O}'}$ of the total polynomials is of the form

$$\sum \gamma_{X,Y} M_{X,Y},$$

where the $\gamma_{X,Y}$'s are integers. The conjecture of Las Vergnas and Cordovil cited earlier would imply this. In the next section we study the affine span of the total polynomials of uniform oriented matroids of rank r on E.

3. Affine Span of Total Polynomials

In this section we examine the linear space spanned by the differences $T_{\mathcal{O}'} - T_{\mathcal{O}}$ of total polynomials of uniform oriented matroids of rank r on E. Obviously this space contains the mutation polynomials of order r. We will show that it is linearly spanned by these polynomials, and in the course of this we will see that the affine span of the total polynomials is determined by the equations derived in [11].

Let $\mathcal{R}$ denote the ring of polynomials with rational coefficients in the $2n$ indeterminates $x_1, y_1, \ldots, x_n, y_n$. We say that a polynomial $p \in \mathcal{R}$ is *linear in index i* if it has a representation

$$p = q_1 x_i + q_2 y_i + q_3,$$

where q_1, q_2, and q_3 are polynomials in the other $2(n-1)$ indeterminates.

We consider the vector space $\mathcal{W}$ of polynomials $p \in \mathcal{R}$ which are linear in each index and which satisfy the following two conditions, derived from conditions (1) and (4):

(1′) $p(x_e, y_e\colon e \in E)$ has no monomial terms of degree less than $n-r$.
(4′) $p(-1-x_e, -1-y_e\colon e \in E)$ has no monomial terms of degree less than r.

It is clear by (1) and (4) that the differences $T_{\mathcal{O}'} - T_{\mathcal{O}}$ satisfy these conditions. The mutation polynomials of order r lie in $\mathcal{W}$. (Indeed, these polynomials satisfy stronger conditions, there being no terms of degree r from (1), and none of degree $n-r$ from (4).)

We introduce new indeterminates $\bar{x}_i = -x_i - 1$ and $\bar{y}_i = -y_i - 1$ ($i = 1, \ldots, n$). More formally, let $\widetilde{\mathcal{R}}$ denote the ring of polynomials with rational coefficients in $4n$ indeterminates $x_1, \bar{x}_1, y_1, \bar{y}_1, \ldots, x_n, \bar{x}_n, y_n, \bar{y}_n$. Let $\eta\colon \widetilde{\mathcal{R}} \to \mathcal{R}$ be the algebra homomorphism taking the x_i's and y_i's to themselves and $\bar{x}_i$ and $\bar{y}_i$ to $-x_i - 1$ and $-y_i - 1$, respectively. The kernel of this epimorphism is the ideal $\mathcal{I}$ in $\widetilde{\mathcal{R}}$ generated by $\bar{x}_i + x_i + 1$, $\bar{y}_i + y_i + 1$ ($i = 1, \ldots, n$).

Let $\widetilde{\mathcal{W}}$ denote the polynomials p in $\widetilde{R}$ which are sums of monomial terms which are rational multiples of monomials of the form

$$m_{A,B,C,D} = \prod_{i \in A} \bar{x}_i \prod_{i \in B} \bar{y}_i \prod_{i \in C} x_i \prod_{i \in D} y_i,$$

where the sets A, B, C, D partition $[n]$, $|A \cup B| = r$, and $|C \cup D| = n - r$. It is clear that the dimension of $\widetilde{\mathcal{W}}$ as a rational vector space is $\binom{n}{r}2^n$, this being the number of monomials of the above form. Given sets A, B, C, D forming a partition of $[n]$, let

$$\tilde{g}_{A,B,C,D} = \prod_{i \in A}(x_i + y_i) \prod_{i \in B}(\bar{x}_i + \bar{y}_i) \prod_{i \in C}(x_i - y_i) \prod_{i \in D}(\bar{x}_i - \bar{y}_i).$$

Lemma 2. *The set of polynomials $\tilde{g}_{A,B,C,D}$ with $|A \cup C| = n - r$, $|B \cup D| = r$ forms a basis for $\widetilde{\mathcal{W}}$.*

Proof. We have noted that the $\binom{n}{r}2^n$ monomials in $\widetilde{\mathcal{W}}$ form a basis for it. Let $d_i = x_i - y_i$, $s_i = x_i + y_i$, $\bar{d}_i = \bar{x}_i - \bar{y}_i$, and $\bar{s}_i = \bar{x}_i + \bar{y}_i$, for $i = 1, \ldots, n$. Then $x_i = (d_i + s_i)/2$, $y_i = (d_i - s_i)/2$, $\bar{x}_i = (\bar{d}_i + \bar{s}_i)/2$, and $\bar{y}_i = (\bar{d}_i - \bar{s}_i)/2$. Taking products, one factor for each i, and expanding, it is possible to obtain any monomial in $\widetilde{\mathcal{W}}$, so it is spanned by the set of $\tilde{g}_{A,B,C,D}$'s, and indeed by those for which $|A \cup C| = n - r$ and $|B \cup D| = r$. Since there are $\binom{n}{r}2^n$ of these, it is clear that these expressions also form a basis for $\widetilde{\mathcal{W}}$. □

Also, given a partition A, B, C of $[n]$, we define

$$g_{A,B,C} = \prod_{i \in A}(x_i + y_i) \prod_{i \in B}(x_i + y_i + 2) \prod_{i \in C}(x_i - y_i) \in \mathcal{R}.$$

Lemma 3. *The polynomials $g_{A,B,C} \in \mathcal{R}$ such that $|A| \le n - r$ and $|B| \le r$ form a basis for $\mathcal{W}$. The dimension of $\mathcal{W}$ is*

$$\sum_{\substack{0 \le k \le r \\ 0 \le l \le n-r}} \binom{n}{k, l, n-k-l}.$$

Proof. Let $s_i = x_i + y_i$, $d_i = x_i - y_i$, and $c_i = x_i + y_i + 2$, for $i = 1, \ldots, n$. Then $x_i = (s_i + d_i)/2$, $y_i = (s_i - d_i)/2$, and $1 = (c_i - s_i)/2$, for $i = 1, \ldots, n$. Taking products, one for each index (as in the proof of Lemma 2), we see that the polynomials $g_{A,B,C}$ span the vector space of polynomials $p \in \mathcal{R}$ which are linear in each index; and, since there are 3^n $g_{A,B,C}$'s, which is the dimension of the vector space of such polynomials p, it is clear that they form a basis for this vector space.

Let $\mathcal{V} \subseteq \mathcal{W}$ be the vector subspace consisting of polynomials having no monomial term of degree less than $n - r$. The dimension of $\mathcal{V}$ is the number of such monomials. For each $k \ge n - r$, there are $\binom{n}{k}$ ways to choose a set of k indexes from $[n]$, and there are 2^n ways to choose an x_i or y_i for each index i of the k indexes chosen; therefore the dimension is given by $\sum_{k=n-r}^{n} \binom{n}{k}2^k$. Let $\varphi\colon \mathcal{W} \to \mathcal{W}/\mathcal{V}$ be the canonical map. If $|B| \le r$, then $\varphi(g_{A,B,C}) = 0$, so $\mathcal{W}/\mathcal{V}$ is spanned by the images of the $g_{A,B,C}$'s for which $|B| > r$. The number of such polynomials $g_{A,B,C}$ is $\sum_{l=r+1}^{n} \binom{n}{l}2^{n-l}$: For $l \ge r$, there are $\binom{n}{l}$ ways to choose a set B of l indexes, and there are 2^{n-l} ways to partition the remaining $n - l$ elements into two set A and C. This coincides with the dimension of $\mathcal{W}/\mathcal{V}$. Indeed, this latter is $3^n - \sum_{k=n-r}^{n} \binom{n}{k}2^k$. By the binomial theorem, $3^n = \sum_{k=0}^{n} \binom{n}{k}2^k$, so the difference is $\sum_{k=0}^{n-r-1} \binom{n}{k}2^k$, which upon changing the variable of summation to $l = n - k$ becomes $\sum_{l=r+1}^{n} \binom{n}{k}2^k$. These polynomials, whose images span $\mathcal{W}/\mathcal{V}$, must then form a basis for this space. From this it follows that a sum $p = \sum \gamma_{A,B,C}\, g_{A,B,C} \in \mathcal{R}$ has no monomial term of degree less than $n - r$, so that $\varphi(p) = 0$ if and only if, whenever $|B| \ge r + 1$, $\gamma_{A,B,C} = 0$.

Also, $g_{A,B,C}(-1 - x_e, -1 - y_e\colon e \in E) = (-1)^n g_{B,A,C}$, so a similar argument shows that $p = \sum \gamma_{A,B,C}\, g_{A,B,C} \in \mathcal{R}$ satisfies condition (4′) if and only if, whenever $|A| \ge n - r + 1$, $\gamma_{A,B,C} = 0$.

It follows that the $g_{A,B,C}$'s for which $|A| \le n - r$ and $|B| \le r$ form a basis for $\mathcal{W}$. The number of such $g_{A,B,C}$'s is given by the above sum, since the number of partitions of $[n]$ into sets A, B, C with $|A| = k$, $|B| = l$, and $|C| = n - k - l$ is given by $\binom{n}{k, l, n-k-l}$. □

Lemma 4. *The image of $\widetilde{\mathcal{W}}$ under the homomorphism η is $\eta(\widetilde{\mathcal{W}}) = \mathcal{W}$; equivalently, $\mathcal{W}$ is spanned (as a vector space) by the polynomials $N_{X,Y}$.*

Proof. Consider the monomial $m_{A,B,C,D}$. We have $\eta(m_{A,B,C,D}) = N_{X,Y}$ where X is the signed set (A, B) and $Y = (C, D)$, so $\eta(\widetilde{\mathcal{W}})$ is spanned by the $N_{X,Y}$'s. The rest follows from Lemmas 2 and 3, since $\eta(\tilde{g}_{A,B,C,D}) = (-1)^{|B\cup D|} g_{A,B,C\cup D}$. □

Next we consider involutions of the vector space $\mathcal{W}$, suggested by conditions (2) and (3).

The first of these, motivated by (2), can be simply defined on the whole ring $\mathcal{R}$ as the algebra homomorphism $\sigma\colon \mathcal{R} \to \mathcal{R}$ such that $\sigma(x) = y$ and $\sigma(y) = x$. It is an involutive algebra automorphism such that $\sigma(\mathcal{W}) = \mathcal{W}$. Note that σ permutes the polynomials $N_{X,Y}$: $\sigma(N_{X,Y}) = N_{-X,-Y}$. Also $\sigma(g_{A,B,C}) = (-1)^{|C|} g_{A,B,C}$.

Next consider the function which takes the polynomial p, assumed linear in each index, to

$$(-1)^{n-r} \prod_{e\in E} (1 + x_e + y_e)\, p \left(\frac{-x_e}{1 + x_e + y_e}, \frac{-y_e}{1 + x_e + y_e} \colon e \in E \right).$$

This mapping takes the polynomial $N_{X,Y}$ to $N_{-X,Y}$, so, since these span $\mathcal{W}$ by Lemma 4, it is an involutive automorphism of the vector space $\mathcal{W}$, $\rho\colon \mathcal{W} \to \mathcal{W}$. Also $\rho(g_{A,B,C}) = (-1)^{|B|} g_{A,B,C}$.

The mappings σ and ρ commute, and we set $\tau(p) = \sigma(\rho(p))$ for $p \in \mathcal{W}$. We have $\tau(N_{X,Y}) = N_{X,-Y}$ and $\tau(g_{A,B,C}) = (-1)^{n-r-|A|} g_{A,B,C}$.

In $\widetilde{\mathcal{R}}$ this is simpler. Define $\tilde{\rho}$, $\tilde{\sigma}$, and $\tilde{\tau}$, mapping $\widetilde{\mathcal{R}}$ to itself, to be the algebra homomorphisms such that $\tilde{\rho}$ switches $\bar{x}_i$ and $\bar{y}_i$ (so that $\tilde{\rho}(x_i) = x_i$, $\tilde{\rho}(x_i) = x_i$, $\tilde{\rho}(\bar{x}_i) = \bar{y}_i$, and $\tilde{\rho}(\bar{y}_i) = \bar{x}_i$) for each index i, $\tilde{\tau}$ switches x_i and y_i for each index i, and σ switches x_i with y_i and $\bar{x}_i$ with $\bar{y}_i$ for each index i. Then all of these involutive homomorphisms map $\widetilde{\mathcal{W}}$ to itself, and $\eta\tilde{\rho} = \rho\eta$, $\eta\tilde{\sigma} = \sigma\eta$, and $\eta\tilde{\tau} = \tau\eta$.

Based on these involutions we decompose $\mathcal{W}$ as a sum of four subspaces. For $\varepsilon_1, \varepsilon_2 \in \{1, -1\}$ let

$$\mathcal{W}^{\varepsilon_1,\varepsilon_2} = \{p \in \mathcal{W}\colon\ \rho(p) = \varepsilon_1 p,\ \tau(p) = \varepsilon_2 p\}.$$

The vector space $\mathcal{W}$ is the direct sum of the vector spaces $\mathcal{W}^{1,1}$, $W^{-1,1}$, $\mathcal{W}^{1,-1}$, and $\mathcal{W}^{-1,-1}$. Indeed, each element of the basis for $\mathcal{W}$ described in Lemma 3 lies in one of these. We are interested mainly in one of these four.

Theorem 1. *The mutation polynomials linearly span the vector space $\mathcal{W}^{-1,-1}$ of polynomials linear in each index which satisfy* (1′), (2), (3), *and* (4′). *Conditions* (1)–(4) *determine the affine linear span of the total polynomials, among polynomials in $\mathcal{R}$ which are linear in each index. The dimension of this affine space, as well as that of the vector space $\mathcal{W}^{-1,-1}$, is*

$$\sum_{k,l} \binom{n}{k, l, n-k-l},$$

where the k, l run over integers such that

(i) $0 \le k \le r$,
(ii) $0 \le l \le n - r$,
(iii) *$r - k$ is odd, and*
(iv) *$n - r - l$ is odd.*

Proof. Each polynomial $g_{A,B,C}$ having $|A| \le n - r$ and $|B| \le r$ from the basis for $\mathcal{W}$ of Lemma 3 lies in one of the $\mathcal{W}^{\varepsilon_1,\varepsilon_2}$'s, and it follows that each of these four subspaces has as a basis the set of $g_{A,B,C}$'s which lie in it. The polynomial $g_{A,B,C}$ is in $\mathcal{W}^{-1,-1}$ if and only if $n - r - |A|$ and $r - |B|$ are odd. The formula for the dimension follows.

For $p \in \mathcal{W}$ let

$$\varphi(p) = p - \rho(p) - \tau(p) + \sigma(p).$$

Then $\varphi(\mathcal{W}) \subseteq \mathcal{W}^{-1,-1}$ and, for $p \in \mathcal{W}^{-1,-1}$, $\varphi(p) = 4p$. It follows that $\varphi(\mathcal{W}) = \mathcal{W}^{-1,-1}$. Since the polynomials $N_{X,Y}$ of order r span $\mathcal{W}$, the mutation polynomials $M_{X,Y} = \varphi(N_{X,Y})$ of order r span $\mathcal{W}^{-1,-1}$. □

The vector spaces $\mathcal{W} = \mathcal{W}(n, r)$ and $\mathcal{W}(n, n - r)$ are isomorphic; an isomorphism is induced by the map taking $M_{X,Y}$ to $M_{Y,X}$. This is as it should be, considering oriented matroid duality.

Our next task is to show that the Z-module generated by the differences of pairs of total polynomials is already generated by the mutation polynomials.

4. The *Z*-module Spanned by the Mutation Polynomials

In this section we refine the result of Section 3 by considering the additive group spanned by the mutation polynomials rather than the vector space spanned by them.

We denote by $\mathcal{R}_Z$ the subring of $\mathcal{R}$ consisting of polynomials having integer coefficients, and by $\widetilde{\mathcal{R}}_Z$ the subring of $\widetilde{\mathcal{R}}$ of polynomials having integer coefficients. Also we write $\mathcal{W}_Z = \mathcal{W} \cap \mathcal{R}_Z$ and $\widetilde{\mathcal{W}}_Z = \widetilde{\mathcal{W}} \cap \widetilde{\mathcal{R}}_Z$.

Let $\mathcal{B}$ be the subset of $\mathcal{W}_Z$ consisting of the polynomials $N_{X,Y}$ of order r such that either $X^- = \emptyset$, $Y^- = \emptyset$, or the largest index $i \in X^-$ is less than the smallest index $j \in Y^-$. If disjoint subsets X^+ and Y^+ of $[n]$ such that $|X^+| \le r$ and $|Y^+| \le n - r$ are given, then X^- and Y^- are uniquely determined by the above conditions so it is clear that the number of such polynomials $N_{X,Y}$ is

$$\sum_{\substack{0 \le k \le r \\ 0 \le l \le n-r}} \binom{n}{k, l, n-k-l},$$

which is the rank of $\mathcal{W}_Z$.

Theorem 2. *The set $\mathcal{B}$ is a basis for the Z-module $\mathcal{W}_Z$.*

Proof. Given disjoint subsets A and B of $[n]$ let $m_{A,B}$ denote the monomial $m_{A,B} = \prod_{i \in A} x_i \prod_{j \in B} y_j$. Given such a monomial let $\nu = \nu(m_{A,B})$ be the vector $\nu = (\nu_1, \ldots, \nu_n)$, where

$$\nu_i = \begin{cases} 2 & \text{if} \quad i \in A, \\ 1 & \text{if} \quad i \notin A \cup B, \\ 0 & \text{if} \quad i \in B. \end{cases}$$

Write $m_{A,B} \prec m_{A',B'}$ if $m_{A,B} \neq m_{A',B'}$ and, letting $\nu(m_{A,B}) = (\nu_1, \ldots, \nu_n)$ and $\nu(m_{A',B'}) = (\nu'_1, \ldots, \nu'_n)$, for the smallest index i such that $\nu_i \neq \nu'_i$ we have $\nu_i < \nu'_i$. This is a linear ordering of the monomials $m_{A,B}$.

The *leading term* $\lambda(p)$ of a polynomial $p \in \mathcal{R}$ linear in each index is the nonzero monomial term of p which is smallest with respect to this ordering. For $N_{X,Y}$, the leading term is $\lambda(N_{X,Y}) = m_{A,B}$, where $A = Y^+$ and $B = X^- \cup Y^-$. If $N_{X,Y} \in \mathcal{B}$, then it is determined by $\lambda(N_{X,Y}) = m_{A,B}$: $Y^+ = A$, $X^+ = [n] \backslash (A \cup B)$, X^- consists of the first $r - |X^+|$ elements of B, and $Y^- = B \backslash X^-$.

Since distinct elements of $\mathcal{B}$ have distinct leading terms it is clear that $\mathcal{B}$ is an independent set. Since $\mathcal{B} \subseteq \mathcal{W}_Z \subseteq \mathcal{W}$ is independent and its cardinality is the dimension of $\mathcal{W}$, it forms a vector space basis for $\mathcal{W}$.

Suppose $\mathcal{B}$ is not a basis for the Z-module $\mathcal{W}_Z$. Then there is $p \in \mathcal{W}_Z$ such $p = \sum_{b \in \mathcal{B}} \gamma_b b$, where, for some $b_0 \in \mathcal{B}$, $\gamma_{b_0} \notin Z$. Choose p with a leading monomial as large as possible. There must be $b' \in \mathcal{B}$, necessarily unique, such that $\lambda(b') = \lambda(p)$. Since all the nonzero coefficients of $b' \in \mathcal{B}$ are 1 or -1, $\gamma_{b'} \in Z$. Then $p - \gamma_{b'} b'$ is again an element of $\mathcal{W}_Z$, and the coefficient of b_0 is again not an integer. Since $\lambda(p) \prec \lambda(p - \gamma_{b'} b')$, we have a contradiction. $\square$

It follows from the above that $\eta(\widetilde{\mathcal{W}}_Z) = \mathcal{W}_Z$. Since the kernel of the mapping η: $\widetilde{\mathcal{R}} \to \mathcal{R}$ is $\mathcal{I}$, we have that the kernel of the restriction η: $\widetilde{\mathcal{W}}_Z \to \mathcal{W}_Z$ is $\mathcal{I} \cap \widetilde{\mathcal{W}}_Z$; however, it is possible to replace the ideal $\mathcal{I}$ by a smaller ideal in this equation. Let $\mathcal{J} \subseteq \widetilde{\mathcal{R}}_Z$ be the ideal generated by the polynomials $(y_i - x_i)(\bar{y}_j - \bar{x}_j) - (y_j - x_j)(\bar{y}_i - \bar{x}_i)$, where $1 \leq i < j \leq n$.

Theorem 3. *We have*

$$\widetilde{\mathcal{W}}_Z \big/ (\widetilde{\mathcal{W}}_Z \cap \mathcal{J}) \simeq \mathcal{W}_Z.$$

Proof. We need only show that $\widetilde{\mathcal{W}}_Z \cap \mathcal{J}$ is the kernel of the restriction of η to $\widetilde{\mathcal{W}}_Z$ (which we also denote by η). That is, $\widetilde{\mathcal{W}}_Z \cap \mathcal{J} = \widetilde{\mathcal{W}}_Z \cap \mathcal{I}$.

Clearly, $\mathcal{J} \subseteq \mathcal{I}$, since each generator of $\mathcal{J}$ is mapped to 0 by η. Therefore $\widetilde{\mathcal{W}}_Z \cap \mathcal{J} \subseteq \widetilde{\mathcal{W}}_Z \cap \mathcal{I}$.

Consider again the monomial $m_{A,B,C,D}$. Suppose there are $i \in B$ and $j \in D$ such that $i > j$. Let $B' = B \backslash \{i\}$ and $D' = D \backslash \{j\}$. Since

$$\begin{aligned} m_{A,B'\cup\{i\},C,D'\cup\{j\}} &- m_{A,B'\cup\{i\},C\cup\{ji\},D'} + m_{A\cup\{i\},B',C\cup\{j\},D'} \\ &- m_{A\cup\{i\},B',C,D'\cup\{j\}} + m_{A,B'\cup\{j\},C,D'\cup\{i\}} \\ &- m_{A,B'\cup\{j\},C\cup\{i\},D'} + m_{A\cup\{j\},B',C\cup\{i\},D'} \\ &- m_{A\cup\{j\},B',C,D'\cup\{i\}} \end{aligned}$$

is a multiple of $(y_i - x_i)(\bar{y}_j - \bar{x}_j) - (y_j - x_j)(\bar{y}_i - \bar{x}_i)$, $m_{A,B,C,D}$ reduces modulo $\mathcal{J}$ to a combination of monomials $m_{\tilde{A},\tilde{B},\tilde{C},\tilde{D}}$ for which $\{(i, j)\colon i \in B, j \in D, i > j\}$ has smaller cardinality. It is clear that every such monomial is an integer combination of such monomials for which this set is empty. The images under η of such monomials are the $N_{X,Y}$'s in $\mathcal{B}$, so the reverse inclusion follows from Theorem 2. $\square$

The ring $\widetilde{\mathcal{R}}_Z$ is bigraded, with the bidegree of a monomial being defined as the pair (d_1, d_2), where d_1 is its degree in the x_i's and y_i's, and d_2 is its degree in the $\bar{x}_i$'s and $\bar{y}_i$'s. The ideal $\mathcal{J}$ is generated by bihomogeneous elements, so $\widetilde{\mathcal{R}}_Z/\mathcal{J}$ inherits this grading.

Given subsets A and A' of $[n]$, each of cardinality r, write $A \succeq A'$ if the (unique) order-preserving function $\varphi\colon A \to A'$ satisfies $\varphi(i) \geq i$ for each $i \in A$. This is a partial ordering relation on such subsets. If in addition $A \neq A'$, write $A \succ A'$.

Note that in the expression used in the proof of Theorem 3, the monomial $m_{A,B,C,D}$ reduces to a combination of monomials $m_{A',B',C',D'}$, where $A \cup B \succeq A' \cup B'$. The $N_{X,Y}$'s with X a given r-element set reduce to elements $N_{X',Y'}$ with $\bar{X} \succeq \bar{X}'$. It follows that, for a fixed set A of cardinality r, if we define $\mathcal{N}_A$ to be the Z-module spanned by the polynomials $N_{X,Y} = \eta(m_{X^+,X^-,Y^+,Y^-})$, where $\bar{X} = A$, and $\mathcal{B}_A$ to be $\mathcal{B} \cap \mathcal{N}_A$, then $\bigcup_{A'\colon A \succeq A'} \mathcal{B}_{A'}$ is a basis for $\sum_{A'\colon A \succeq A'} \mathcal{N}_{A'}$.

Let

$$\mathcal{N}'_A = \sum_{A'\colon A \succeq A'} \mathcal{N}_{A'} \Big/ \sum_{A'\colon A \succ A'} \mathcal{N}_{A'}$$

and let α denote the canonical map. Then $\alpha(\mathcal{B}_A)$ is a basis for $\mathcal{N}'_A$. If we write an arbitrary element q of $\mathcal{W}_Z$ as an integer combination of the basis elements,

$$q = \sum_{p \in \mathcal{B}} \gamma_p p,$$

then α can be extended to a function $\pi_A\colon \mathcal{W}_Z \to \mathcal{N}'_A$ by defining

$$\pi_A(q) = \sum_{p \in \mathcal{B}_A} \gamma_p \alpha(p).$$

Clearly, $\mathcal{W}_Z$ is isomorphic to the direct sum of these Z-modules $\mathcal{N}'_A$.

Since the functions ρ, σ, and τ preserve the submodules $\mathcal{N}_A$ of $\mathcal{W}_Z$, they induce involutions, also denoted by ρ, σ, and τ, on each $\mathcal{N}'_A$. We denote by $\mathcal{M}'_A$ the submodule of $\mathcal{N}'_A$ given by

$$\mathcal{M}'_A = \{p \in \mathcal{N}'_A\colon \rho(p) = -p = \tau(p)\}.$$

The Z-module $\mathcal{W}_Z^{-1,-1}$ is isomorphic to the direct sum of the Z-modules $\mathcal{M}'_A$.

Let $\mathcal{B}'$ be the subset of $\mathcal{W}_Z^{-1,-1} = \mathcal{W}^{-1,-1} \cap \mathcal{R}_Z$ consisting of mutation polynomials $M_{X,Y}$ of order r where, if a denotes the smallest element of X and b denotes the largest element of Y, then

(1) $a < b$,
(2) $a \in X^-$, $b \in Y^-$,
(3) if $i > b$, then $i \in X^+$, if $j < a$, then $j \in Y^+$, and
(4) if $a < j < i < b$, then either $j \notin Y^-$ or $i \notin X^-$.

We will soon see that $\mathcal{B}'$ is a basis for $\mathcal{W}_Z^{-1,-1}$.

Lemma 5. *The cardinality of $\mathcal{B}'$ equals the rank of the Z-module $\mathcal{W}_Z^{-1,-1}$.*

Proof. The rank of this Z-module equals the dimension of the vector space $\mathcal{W}^{-1,-1}$, which is given in Theorem 1. A basis for this vector space consists of the polynomials

$g_{A,B,C}$, where A, B, C partition $[n]$, $|A| \le n-r$, $|B| \le r$, $n-r-|A|$ is odd, and $r-|B|$ is odd. We need only exhibit a bijection δ: $\mathcal{B}' \to \{(A, B, C)$: the above conditions are satisfied$\}$.

For $M_{X,Y} \in \mathcal{B}'$ let $\delta(M_{X,Y}) = (A, B, C)$, where A equals X^+ or $X^+ \cup \{a\}$ (choosing so that $n-r-|A|$ is odd), B is Y^+ or $Y^+ \cup \{b\}$ (where $r-|B|$ is odd), and $C = [n]\backslash(A \cup B)$.

If $\delta(M_{X,Y}) = (A, B, C)$, then $M_{X,Y}$ is determined from (A, B, C) as follows. First, a is the smallest positive integer not in B and b is the largest integer, at most n, not in A. Then $X^+ = A\backslash\{a\}$, $Y^+ = B\backslash\{b\}$, X^- consists of the first $n-r-|X^+|$ elements of $[n]\backslash(X^+ \cup Y^+)$, and $Y^- = [n]\backslash(X^+ \cup Y^+ \cup X^-)$. □

For $A \subseteq [n]$, $|A| = r$, let $\mathcal{B}'_A = \mathcal{B}' \cap \mathcal{N}_A$. In order to show that $\mathcal{B}'$ is a basis for $\mathcal{W}_Z^{-1,-1}$, it is necessary and sufficient to show that $\pi_A(\mathcal{B}'_A)$ is a basis for $\mathcal{M}'_A$, for each such A.

When A is fixed, if $\bar{X} = A$, then $N_{X,Y}$ is determined by the set $T = X^- \cup Y^-$. We write z_T to denote the image $\pi_A(N_{X,Y})$. Note that z_T is an element of the image $\pi_A(\mathcal{B}_A)$ if and only if there are no integers i and j such that $i < j$, $i \in T \cap A$, and $j \in T\backslash A$. Let G be the graph having vertex set $[n]$ with i, j adjacent if $i < j$, $i \in [n]\backslash(A \cup T)$, and $j \in A\backslash T$. Then z_T is an element of the basis $\pi_A(\mathcal{B}_A)$ if and only if T is an independent set of the graph G.

The graph G is bipartite: No vertex in A is adjacent to one in $[n]\backslash A$.

Using the fact that the polynomial

$$\begin{aligned} m_{A,B\cup\{i\},C,D\cup\{j\}} &- m_{A,B\cup\{i\},C\cup\{j\},D} + m_{A\cup\{i\},B,C\cup\{j\},D} \\ &- m_{A\cup\{i\},B,C,D\cup\{j\}} + m_{A,B\cup\{j\},C,D\cup\{i\}} \\ &- m_{A,B\cup\{j\},C\cup\{i\},D} + m_{A\cup\{j\},B,C\cup\{i\},D} \\ &- m_{A\cup\{j\},B,C,D\cup\{i\}} \end{aligned}$$

is in the ideal $\mathcal{J}$ when the sets A, B, C, D, and $\{i, j\}$, partition $[n]$, one gets the identity

$$z_{T_0\cup\{i,j\}} = z_{T_0\cup\{i\}} + z_{T_0\cup\{j\}} - z_{T_0},$$

where $T_0 \subseteq [n]$, $i, j \notin T_0$, and i and j are adjacent in the graph G. This can be used repeatedly to reduce any z_T to an integer combination of $z_{T'}$'s with T' independent in G. Notice (for use in the proof of the next theorem) that, in this process, if T contains at most one of a, b, then the same is true of the $z_{T'}$'s to which z_T is reduced.

It is easy to describe the actions of ρ, σ, and τ on the z_T's. We have $\rho(z_T) = z_{T'}$, $\sigma(z_T) = z_{T''}$, and $\rho(z_T) = z_{T'''}$, where $T' = (A\backslash T) \cup (T\backslash A)$, $T'' = [n]\backslash T$, and $T''' = (A \cap T) \cup ([n]\backslash(A \cup T))$.

Theorem 4. *The set $\mathcal{B}'$ is a basis for $\mathcal{W}_Z^{-1,-1}$.*

Proof. We need only show that $\pi_A(\mathcal{B}'_A)$ is a basis for $\mathcal{M}'_A$, for each set $A \subseteq [n]$ having $|A| = r$.

Note that

$$\pi_A(\mathcal{B}'_A) = \{z_T - \rho(z_T) + \sigma(z_T) - \tau(z_T)\colon\ T \text{ is independent in } G \text{ and } a, b, \in T\}.$$

Let $\mathcal{H}_A$ denote the Z-module spanned by $\pi_A(\mathcal{B}'_A)$. Clearly, $\mathcal{H}_A \subseteq \mathcal{M}'_A$. We must show that these Z-modules are equal.

Consider the mapping $\zeta\colon \mathcal{N}'_A \to \mathcal{N}'_A$ taking z_T to itself if T is independent in G and $a, b \in T$, and taking z_T to 0 if T is independent but a, b are not both in T. The image $\zeta(\mathcal{N}'_A)$ is the submodule $\mathcal{K}$ spanned by the z_T's for which T is independent in G and $a, b \in T$.

If $T \subseteq [n]$ and T does not contain both a and b, then $z_T = \sum_{T'} \gamma_{T'} z_{T'}$, where the T''s are independent in G and do not contain both a and b; that is, $\zeta(z_T) = 0$. It follows that if we define the mapping $\omega\colon \mathcal{K} \to \mathcal{H}_A$ by $\omega(z_T) = z_T - \rho(z_T) + \sigma(z_T) - \tau(z_T)$ for indepndent T such that $a, b \in T$, then ω is the inverse of the restriction of the mapping ζ to $\mathcal{H}_A$.

It follows that $\operatorname{rank}(\mathcal{H}_A)$ is $|\mathcal{B}'_A|$. Since $\mathcal{M}'_A \supseteq \mathcal{H}_A$, $\operatorname{rank}(\mathcal{M}'_A) \ge \operatorname{rank}(\mathcal{H}_A)$. Since also $\sum_A \operatorname{rank}(\mathcal{M}'_A) = \sum_A |\mathcal{B}'_A|$, $\operatorname{rank}(\mathcal{M}'_A) = |\mathcal{B}'_A| = \operatorname{rank}(\mathcal{H}_A)$ for each A.

Clearly, $\mathcal{N}'_A \big/ \mathcal{K}$ is torsion-free. Suppose $p \in \mathcal{N}'_A$ and $mp \in \mathcal{H}_A$, where m is a positive integer. Then $\zeta(mp) = m\zeta(p) \in \mathcal{K}$, so $\zeta(p) \in \mathcal{K}$. Therefore $p = \omega(\zeta(p)) \in \mathcal{H}_A$. It follows that $\mathcal{N}'_A \big/ \mathcal{H}_A$ is torsion-free.

Finally, since $\mathcal{H}_A \subseteq \mathcal{M}'_A$, $\operatorname{rank}(\mathcal{M}'_A) = \operatorname{rank}(\mathcal{H}_A)$, and $\mathcal{M}'_A \big/ \mathcal{H}_A$ is torsion-free, $\mathcal{M}'_A = \mathcal{H}_A$. □

5. Affine Span of Radon Catalogs

Let $\mathcal{O}$ be a uniform oriented matroid of rank r, as above, and let $T_{\hat{\mathcal{O}}}$ be the total polynomial of its dual $\widehat{\mathcal{O}}$. The *Radon catalog* of $\mathcal{O}$ is

$$R_{\mathcal{O}}(x, y) = T_{\hat{\mathcal{O}}}(x_e = x, y_e = y\colon\ e \in E).$$

Radon catalogs were introduced in [11]. Properties (1)–(4) of total polynomials yield the following properties of Radon catalogs:

(a) $R_{\mathcal{O}}(x, y) = \sum \gamma_{a,b} x^a y^b$ for some nonnegative integers $\gamma_{a,b}$, where $\gamma_{a,b} = 0$ unless $a, b \ge 0$ and $r + 1 \le a + b \le n$.
(b) $R_{\mathcal{O}}(y, x) = R_{\mathcal{O}}(x, y)$.
(c) $(1 + x + y)^n R_{\mathcal{O}}(-x/(1 + x + y), -y/(1 + x + y)) = (-1)^{r+1} R_{\mathcal{O}}(x, y)$.
(d) $R_{\hat{\mathcal{O}}}(x, y) = (1+x+y)^n - (-1)^{n-r} - (-1)^n R_{\mathcal{O}}(-1-x, -1-y)$; consequently, this polynomial has no terms of degree less than or equal to $n - r$.

Suppose a and b are integers such that $0 \le a \le r$ and $0 \le b \le n - r$. We define polynomials

$$n_{a,b}(x, y) = x^a y^{r-a} (1 + x)^b (1 + y)^{n-r-b}$$

and

$$\begin{aligned} m_{a,b}(x, y) &= n_{a,b}(x, y) - n_{r-a,b}(x, y) - n_{a,n-r-b}(x, y) + n_{r-a,n-r-b}(x, y) \\ &= (x^a y^{r-a} - x^{r-a} y^a)((1 + x)^b (1 + y)^{n-r-b} - (1 + x)^{n-r-b} (1 + y)^b). \end{aligned}$$

The $n_{a,b}$'s are obtained from the $N_{X,Y}$'s of order $n - r$ by substituting x's for x_i's and y's for y_i's, and the $m_{a,b}$'s are similarly derived from the $M_{X,Y}$'s.

We are primarily interested in the $m_{a,b}$'s, which we dub the *little mutation polynomials*. Here are some facts about them.

For integers a, b such that $0 \le a \le r$ and $0 \le b \le n-r$, $m_{a,b} = -m_{r-a,b} = -m_{a,n-r-b} = m_{r-a,n-r-b}$. Consequently, if $r = 2a$ or if $n-r = 2b$, then $m_{a,b} = 0$.

The polynomials $m_{a,b}$, where $0 \le a \le [r/2]-1 = [(r-1)/2]$ and $0 \le b \le [(n-r)/2]-1 = [(n-r-1)/2]$, are independent. Indeed, we have the following.

Lemma 6. *The little mutation polynomials $m_{a,b}(x,y)$ with indexes in the range $0 \le a \le [(r-1)/2]$ and $0 \le b \le [(n-r-1)/2]$ form a basis for the Z-module of polynomials $p(x,y)$ which*

(a) *have integer coefficients, and*
(b) *are in the vector space spanned by the $m_{a,b}$'s.*

Proof. Suppose

$$p(x,y) = \sum_{a=0}^{[(r-1)/2]} \sum_{b=0}^{[(n-r-1)/2]} \gamma_{a,b} m_{a,b}(x,y),$$

and that the coefficients of $p(x,y)$, with respect to the ordinary basis of monomials $x^k y^l$, are integers; that is,

$$p(x,y) = \sum_{k,l} \beta_{k,l} x^k y^l,$$

where the $\beta_{k,l}$'s are in Z. We need only show that the $\gamma_{a,b}$'s are integers. Suppose not. Choose a and b such that $\gamma_{a,b}$ is not an integer, with a as small as possible, and given this, with b as small as possible. Consider the coefficient $\beta_{n-a-b,a}$ of $x^{n-a-b}y^a$. When the little mutation polynomial $m_{a,b}$ is written in terms of the ordinary basis, the monomial $x^{n-a-b}y^a$ has coefficient 1. If $x^{n-a-b}y^a$ has nonzero coefficient in the expansion of $m_{c,d}$, then (since $a \le [(r-1)/2]$ and $b \le [(n-r-1)/2]$) $c \le a \le c+d$ and $r-c \le n-a-b \le n-c-d$, so $c \le a$ and if $c = a$, then $d \le b$. This means that $\gamma_{c,d}$ fails to be an integer precisely when $c = a$ and $d = b$; but then $\beta_{n-a-b,a} = \sum \gamma_{c,d}$ is not an integer, contrary to our assumption. □

Theorem 5. *If $\mathcal{O}_1$ and $\mathcal{O}_2$ are uniform oriented matroids of rank r on E, then there are unique integers $\gamma_{a,b}$ ($0 \le a \le [r/2]-1 = [(r-1)/2]$, $0 \le b \le [(n-r)/2]-1 = [(n-r-1)/2]$) such that*

$$R_{\mathcal{O}_2}(x,y) - R_{\mathcal{O}_1}(x,y) = \sum \gamma_{a,b} m_{a,b}.$$

Proof. The existence of the integers $\gamma_{a,b}$ follows immediately from Theorems 1 and 4. Uniqueness follows from Lemma 6. □

The matrix $\mathcal{M}(\mathcal{O}_1, \mathcal{O}_2)$ of $\gamma_{a,b}$'s is called the *mutation count matrix*.

Using condition (d) above, one gets for the duals $\widehat{\mathcal{O}}_1$ and $\widehat{\mathcal{O}}_2$ of $\mathcal{O}_1$ and $\mathcal{O}_2$ that, under the circumstances of Theorem 5,

$$R_{\hat{\mathcal{O}}_1}(x,y) - R_{\hat{\mathcal{O}}_2}(x,y) = \sum \gamma_{b,a} m_{b,a}.$$

It would certainly be nice to have a complete characterization of the Radon catalogs. Along these lines we state a conjecture concerning some inequalities involving them.

Recall that the *alternating oriented matroid* $\mathcal{A} = \mathcal{A}_{n,r}$ is the oriented matroid of affine dependencies of the points $p_k = (k, k^2, k^3, \ldots, k^{r-1}) \in R^{r-1}$, for $1 \le k \le n$. It is a uniform oriented matroid of rank r on $[n]$.

Conjecture 1. *If $\mathcal{O}$ is a uniform oriented matroid on $[n]$ of rank r and*

$$R_{\mathcal{O}}(x, y) - R_{\mathcal{A}}(x, y) = \sum_{a,b} \gamma_{a,b} m_{a,b}(x, y),$$

as in Theorem 5, then the $\gamma_{a,b}$'s are nonnegative.

Roughly, this conjecture asserts that the alternating oriented matroids have an extremal property with respect to Radon catalogs much like the well-known extremal property of cyclic polytopes with respect to f-vectors, for simplicial convex polytopes. In terms of the mutation count matrix, the conjecture asserts that the entries are nonnegative:

$$\mathcal{M}(\mathcal{A}, \mathcal{O}) \ge 0.$$

If $\mathcal{A}' = \widehat{\mathcal{A}}_{n,n-r}$ is the dual of $\mathcal{A}_{n,n-r}$, then the rank of $\mathcal{A}'$ is r. Using oriented matroid duality, the above conjecture can be reformulated as follows, letting $\delta_{a,b}$ be the integers such that

$$R'_{\mathcal{A}} - R_{\mathcal{A}} = \sum_{a,b} \delta_{a,b} m_{a,b}.$$

Note that $\mathcal{M}(\mathcal{A}, \widehat{\mathcal{A}})$ is the matrix of $\delta_{a,b}$'s and that

$$\mathcal{M}(\mathcal{A}, \mathcal{O}) + \mathcal{M}(\mathcal{O}, \widehat{\mathcal{A}}) = \mathcal{M}(\mathcal{A}, \widehat{\mathcal{A}}).$$

Conjecture 1 asserts the nonnegativity of $\mathcal{M}(\mathcal{A}, \mathcal{O})$, and by duality this implies the nonnegativity of $\mathcal{M}(\mathcal{O}, \widehat{\mathcal{A}})$.

Conjecture 1$'$. *The $\gamma_{a,b}$'s of Conjecture 1 satisfy $\gamma_{a,b} \le \delta_{a,b}$.*

Yet another version is:

Conjecture 1$''$. *If $\mathcal{O}_1$, $\mathcal{O}_2$, and the $\gamma_{a,b}$'s are as in Theorem 5, then $\gamma_{a,b} \le \delta_{a,b}$ for $0 \le a \le [(r-1)/2]$ and $0 \le b \le [(n-r-1)/2]$.*

We compute the $\delta_{a,b}$'s in Section 7. First, in Section 6, some polynomials which will help with this are introduced.

6. Some Auxiliary Polynomials

In this section we introduce some other polynomials which will be of use in the next section in computing the mutation count matrix $\mathcal{M}(\mathcal{A}(n, r), \widehat{\mathcal{A}}(n, n-r))$. First, the polynomial $K_{\mathcal{O}}(u, v)$ which is closely related to $R_{\mathcal{O}}(x, y)$ makes some symmetries transparent. Define

$$K_{\mathcal{O}}(u, v) = (v-u)^n R_{\mathcal{O}}\left(\frac{u+1}{v-u}, \frac{u}{v-u}\right).$$

Then $K_{\mathcal{O}}(u, v)$ is a polynomial of degree n in u, v, having no terms in v of degree r or more. Also,

$$R_{\mathcal{O}}(x, y) = (x-y)^n K_{\mathcal{O}}\left(\frac{y}{x-y}, \frac{1+y}{x-y}\right).$$

This follows by inverting the equations $x = (u+1)/(v-u)$, $y = u/(v-u)$ to get $u = y/(x-y)$, $v = (1+y)/(x-y)$.

From the identities (b) and (c) for $R_{\mathcal{O}}(x, y)$ we get simple identities for $K_{\mathcal{O}}(u, v)$.

Theorem 6. *We have*

$$\begin{aligned} K_{\mathcal{O}}(u, v) &= (-1)^{n-r-1} K_{\mathcal{O}}(u, -1-v) = (-1)^{r-1} K_{\mathcal{O}}(-1-u, v) \\ &= (-1)^n K_{\mathcal{O}}(-1-u, -1-v). \end{aligned}$$

Proof. We have

$$\begin{aligned} (-1)^{n-r-1} K_{\mathcal{O}}(u, -1-v) &= (-1)^{n-r-1}(-1-u-v)^n R_{\mathcal{O}}\left(\frac{u+1}{-1-u-v}, \frac{u}{-1-u-v}\right) \\ &= (-1)^n (1+u+v)^n R_{\mathcal{O}}\left(\frac{-1-u}{1+u+v}, \frac{-u}{1+u+v}\right). \end{aligned}$$

By (c) we may continue,

$$\begin{aligned} &= (1+u+v)^n \left(\frac{v-u}{1+u+v}\right)^n R_{\mathcal{O}}\left(\frac{u+1}{v-u}, \frac{u}{v-u}\right) \\ &= K_{\mathcal{O}}(u, v). \end{aligned}$$

Also we have

$$(-1)^n K_{\mathcal{O}}(-1-u, -1-v) = (v-u)^n R_{\mathcal{O}}\left(\frac{u}{v-u}, \frac{1+u}{v-u}\right),$$

which by symmetry of $R_{\mathcal{O}}$ in x and y is

$$(v-u)^n R_{\mathcal{O}}\left(\frac{1+u}{v-u}, \frac{u}{v-u}\right) = K_{\mathcal{O}}(u, v).$$

Finally using the just proven identities we have also

$$(-1)^{r-1} K_{\mathcal{O}}(-1-u, v) = (-1)^{r-1} K_{\mathcal{O}}(u, -1-v) = K_{\mathcal{O}}(u, v). \qquad \square$$

Also, letting $\widehat{\mathcal{O}}$ be the oriented matroid dual to $\mathcal{O}$, we get from (d) a similar equation yielding $K_{\widehat{\mathcal{O}}}$ from $K_{\mathcal{O}}$.

Theorem 7. *We have*

$$K_{\hat{\mathcal{O}}}(u,v) = (1+u+v)^n - (-1)^r(u-v)^n - K_{\mathcal{O}}(v,u).$$

Proof. By definition,

$$K_{\hat{\mathcal{O}}}(u,v) = (v-u)^n R_{\hat{\mathcal{O}}}\left(\frac{u+1}{v-u}, \frac{u}{v-u}\right).$$

By (d), this is

$$\begin{aligned}
&(v-u)^n\left(\left(1+\frac{u+1}{v-u}+\frac{u}{v-u}\right)^n - (-1)^{n-r} - (-1)^n R_{\mathcal{O}}\left(-1-\frac{u+1}{v-u}, -1-\frac{u}{v-u}\right)\right)\\
&\quad = (1+u+v)^n-(-1)^{n-r}(v-u)^n-(-1)^n(v-u)^n R_{\mathcal{O}}\left(-\frac{v+1}{v-u}, -\frac{v}{v-u}\right)\\
&\quad = (1+u+v)^n-(-1)^{n-r}(v-u)^n\\
&\qquad -(u-v)^n R_{\mathcal{O}}\left(\frac{v+1}{u-v}, \frac{v}{u-v}\right)(1+u+v)^n-(-1)^r(u-v)^n-K_{\mathcal{O}}.
\end{aligned}$$

□

The relationship between the Radon catalog $R_{\mathcal{O}}$ and the total polynomial $T_{\mathcal{O}}$ is paralleled by that between $K_{\mathcal{O}}$ and the polynomial

$$H_{\mathcal{O}}(u_1,v_1,\ldots,u_n,v_n) = \prod_{i=1}^{n}(v_i-u_i)T_{\mathcal{O}}\left(\frac{u_1+1}{v_1-u_1}, \frac{u_1}{v_1-u_1}, \ldots, \frac{u_n+1}{v_n-u_n}, \frac{u_n}{v_n-u_n}\right);$$

that is, $K_{\mathcal{O}}(u,v) = H_{\hat{\mathcal{O}}}(u,v,\ldots,u,v)$. The polynomial H is of degree n jointly in the u_i's and v_i's, and it has no terms of degree r or more in the v_i's. Also $T_{\mathcal{O}}$ (and hence $\mathcal{O}$ itself) is determined by H:

$$T_{\mathcal{O}}(x_1,y_1,\ldots,x_n,y_n) = \prod_{i=1}^{n}(x_i-y_i)H\left(\frac{y_1}{x_1-y_1}, \frac{1+y_1}{x_1-y_1}, \ldots, \frac{y_n}{x_n-y_n}, \frac{1+y_n}{x_n-y_n}\right).$$

There are identities which can be derived from (2)–(4) which specialize to those of the theorems above for K which can be proven in the same way. We omit the proofs.

Theorem 8. *The polynomial H satisfies the equations*

$$\begin{aligned}
H(u_1,v_1,\ldots,u_n,v_n) &= (-1)^{r-1}H(u_1,-1-v_1,\ldots,u_n,-1-v_n)\\
&= (-1)^{n-r-1}H(-1-u_1,v_1,\ldots,-1-u_n,v_n)\\
&= (-1)^n H(-1-u_1,-1-v_1,\ldots,-1-u_n,-1-v_n).
\end{aligned}$$

Theorem 9. *One has*

$$\begin{aligned}
H_{\hat{\mathcal{O}}}(u_1,v_1,\ldots,u_n,v_n) &= \prod_{i=1}^{n}(1+u_i+v_i)^n - (-1)^{n-r}\prod_{i=1}^{n}(u_i-v_i)\\
&\quad - H_{\mathcal{O}}(v_1,u_1,\ldots,v_n,u_n).
\end{aligned}$$

It is not difficult to determine the combinatorial significance of the coefficients of $H_{\mathcal{O}}$. Here we assume that the underlying set of the oriented matroid $\mathcal{O}$ is $[n]$. Consider an arrangement $\mathcal{A}$ of pseudospheres corresponding to $\mathcal{O}$. Then $\mathcal{A} = \{S_1^0, S_2^0, \ldots, S_n^0\}$, where each set S_i^0 is a topological $(r-2)$-sphere contained in the $(r-1)$-sphere $S^{r-1} \subseteq R^r$ bounding two closed pseudohemispheres S_i^+ and S_i^-. A *cell* of the arrangement is an atom of the boolean lattice generated by the sets S_i^+, S_i^- $(i = 1, \ldots, n)$ under intersection, union, and complementation. The nonzero covectors of $\mathcal{O}$ correspond to the cells: if U is such a covector and C is the corresponding cell, then

(i) $i \in U^+$ if and only if $C \subseteq S_i^+ \backslash S_i^0$,
(ii) $i \in U^-$ if and only if $C \subseteq S_i^- \backslash S_i^0$, and
(iii) $i \in U^0$ if and only if $C \subseteq S_i^0$.

Since $\mathcal{O}$ is uniform and has rank r, the dimension $\dim(C)$ of the cell corresponding to U is $r - 1 - |U^0|$.

If P is a union of cells of the arrangement (so that it is an element of the boolean lattice generated by the sides), then its Euler characteristic is

$$\chi(P) = \sum_{\text{cells } C \subseteq P} (-1)^{\dim(C)} = (-1)^{r-1} \sum_{\substack{U \in \mathcal{L} \\ \text{corresponding to} \\ C \subseteq P}} (-1)^{|U^0|}.$$

For a given pair A, B of disjoint subsets of $[n]$ let

$$P_{A,B} = \bigcap_{i \in B} S_i^0 \cap \bigcap_{i \notin A \cup B} (S_i^+ \backslash S_i^0).$$

Theorem 10. *We have*

$$H(u_1, v_1, \ldots, u_n, v_n) = \sum \alpha_{A,B} \prod_{i \in A} u_i \prod_{i \in B} v_i,$$

where the sum extends over pairs A, B of disjoint subsets of $[n]$ and

$$\begin{aligned} \alpha_{A,B} &= |\chi(P_{A,B})| \\ &= \begin{cases} 1 & \textit{if} \quad P_{A,B} \neq \emptyset \quad \textit{and} \quad A \cup B \neq [n], \\ 1 + (-1)^{r-1-|B|} & \textit{if} \quad A \cup B = [n], \\ 0 & \textit{otherwise.} \end{cases} \end{aligned}$$

Proof. By definition

$$\begin{aligned} H_{\mathcal{O}}(u_1, v_1, \ldots, u_n, v_n) &= \prod_{i=1}^{n} (v_i - u_i) T_{\mathcal{O}}\left(\frac{u_1+1}{v_1-u_1}, \frac{u_1}{v_1-u_1}, \ldots, \frac{u_n+1}{v_n-u_n}, \frac{u_n}{v_n-u_n}\right) \\ &= \sum_{U \in \mathcal{L}} \prod_{i \in U^+} \frac{u_i+1}{v_i-u_i} \prod_{i \in U^-} \frac{u_i}{v_i-u_i} \\ &= \sum_{U \in \mathcal{L}} \prod_{i \in U^+} (u_i+1) \prod_{i \in U^-} u_i \prod_{i \in U^0} (v_i - u_i). \end{aligned}$$

Clearly, when expanded to a sum of monomials in the u_i's and v_i's, each monomial term will be squarefree and no terms will have both u_i and v_i as factors. We wish to determine the coefficient of the monomial $\prod_{i\in A} u_i \prod_{i\in B} v_i$, where $A, B \subseteq [n]$ and $A \cup B = \emptyset$. The coefficient is

$$\sum_{\substack{U\in\mathcal{L}\\ B\subseteq U^0\\ [n]\setminus(A\cup B)\subseteq U^+}} (-1)^{|U^0\setminus B|}.$$

It is the Euler characteristic of the set $P_{A,B}$ multiplied by the factor $(-1)^{r-1-|B|}$. Since $\mathcal{O}$ is uniform, if this set is nonempty and $A \cup B \neq [n]$, then it is an open ball of dimension $r-1-|B|$, so its Euler characteristic is $(-1)^{r-1-|B|}$, and $\alpha_{A,B} = 1$. If $A \cup B = [n]$ and $|B| \geq r$, then $P_{A,B} = \emptyset$, so $\alpha_{A,B} = \chi(P_{A,B}) = 0$. Finally, if $A \cup B = [n]$ and $|B| \leq r-1$, then $P_{A,B}$ is a sphere of dimension $r-1-|B|$, its Euler characteristic is $1 + (-1)^{r-1-|B|}$, and this is also the value of $\alpha_{A,B}$. □

Corollary. *One has*

$$K_{\hat{\mathcal{O}}}(u, v) = \sum_{\substack{0\leq a\leq n\\ 0\leq b\leq r-1}} \alpha_{a,b} u^a v^b,$$

where the $\alpha_{a,b}$'s are nonnegative integers. Indeed, if $a+b<n$, then $\alpha_{a,b}$ is the number of pairs of disjoint subsets $A, B \subseteq [n]$ having $|A| = a$, $|B| = b$, such that $P_{A,B} \neq \emptyset$. Furthermore, if $a+b=n$ and $0 \leq b \leq r-1$, then

$$\alpha_{a,b} = \begin{cases} 2\binom{n}{b} & \text{if } \ r-b \text{ is odd,} \\ 0 & \text{if } \ r-b \text{ is even,} \end{cases}$$

and if $a+b<n$ and $b \leq r-1$, then

$$\alpha_{a,b} = \binom{n}{a, b, n-a-b}.$$

Proof. This follows from the theorem by counting: when $a+b=n$ then $P_{A,B}$ is empty if $|B| = b \geq r$; it is a sphere having Euler characteristic $1+(-1)^{r-1-b}$ when $b \leq r-1$, so the number $\binom{n}{b}$ of pairs of disjoint subsets of $[n]$ having cardinalities a and $b = n-a$ is multiplied by the Euler characteristic; when $a+b<n$, $\alpha_{a,b}$ is the number of pairs of disjoint subsets A, B of $[n]$ with $|A| = a$, $|B| = b$, and $P_{A,B} \neq \emptyset$. This is certainly a nonnegative integer; and, if $|B| \leq r-1$, $P_{A,B}$ must be nonempty, so if $b \leq r-1$, $\alpha_{a,b}$ counts the number of pairs of disjoint subsets of $[n]$ having cardinalities a and b, which is $\binom{n}{a,b,n-a-b}$. □

7. Mutation Count Matrices

The last theorem of this paper gives the $\delta_{a,b}$'s of Section 5.

Consider the definition of $K_{\mathcal{O}}$ in terms of $R_{\mathcal{O}}$, in Section 6, and define

$$\begin{aligned}\tilde{m}_{a,b}(u,v) &= (v-u)^n m\left(\frac{u+1}{v-u}, \frac{u}{v-u}\right)\\ &= ((u+1)^a u^{r-a} - (u+1)^{r-a}u^a)((v+1)^b v^{n-r-b} - (v+1)^{n-r-b}v^b).\end{aligned}$$

Then, if $\mathcal{O}_1$ and $\mathcal{O}_2$ are two oriented matroids of rank r on $[n]$,

$$R_{\mathcal{O}_2}(x,y) - R_{\mathcal{O}_1}(x,y) = \sum_{\substack{0\le a\le[(r-1)/2]\\0\le b\le[(n-r-1)/2]}} \gamma_{a,b} m_{a,b}(x,y)$$

if and only if

$$K_{\mathcal{O}_2}(u,v) - K_{\mathcal{O}_1}(u,v) = \sum_{\substack{0\le a\le[(r-1)/2]\\0\le b\le[(n-r-1)/2]}} \gamma_{a,b} \tilde{m}_{a,b}(u,v).$$

Also, whenever $\mathcal{O}_1$ and $\mathcal{O}_2$ are oriented matroids of rank r on $[n]$, it follows that there are $\gamma_{a,b}$'s such that

$$K_{\mathcal{O}_2}(u,v) - K_{\mathcal{O}_1}(u,v) = \sum_{\substack{0\le a\le[(r-1)/2]\\0\le b\le[(n-r-1)/2]}} \gamma_{a,b} \tilde{m}_{a,b}(u,v),$$

since the same holds for the Radon catalogs, by Theorem 5.

The next lemma shows how to get the coefficients $\gamma_{a,b}$ if $K_{\mathcal{O}_2} - K_{\mathcal{O}_1}$ is known.

Lemma 7. *Let*

$$p(u,v) = \sum_{\substack{0\le k\le r\\0\le l\le n-r}} \beta_{k,l} u^k v^l.$$

If also

$$p(u,v) = \sum_{\substack{0\le i\le[(r-1)/2]\\0\le j\le[(n-r-1)/2]}} \gamma_{i,j} \tilde{m}_{i,j}(u,v),$$

then

$$\gamma_{i,j} = \sum_{\substack{0\le k\le i\\0\le l\le j}} (-1)^{i-k+j-l} \binom{r-k}{i-k}\binom{n-r-l}{j-l} \beta_{k,l}.$$

Proof. Let $q(x,y) = (1-x)^r(1-y)^{n-r} p(x/(1-x), y/(1-y))$. Then

$$\begin{aligned}q(x,y) &= \sum_{\substack{0\le i\le[(r-1)/2]\\0\le j\le[(n-r-1)/2]}} \gamma_{i,j}(1-x)^r(1-y)^{n-r}\tilde{m}_{i,j}\left(\frac{x}{1-x}, \frac{y}{1-y}\right)\\ &= \sum_{\substack{0\le i\le[(r-1)/2]\\0\le j\le[(n-r-1)/2]}} \gamma_{i,j}(x^i - x^{r-i})(y^j - y^{n-r-l}).\end{aligned}$$

Also

$$\begin{aligned} q(x,y) &= (1-x)^r(1-y)^{n-r} \sum_{\substack{0\le k\le r\\ 0\le l\le n-r}} \beta_{k,l}\left(\frac{x}{1-x}\right)^k\left(\frac{y}{1-y}\right)^l \\ &= \sum_{\substack{0\le k\le r\\ 0\le l\le n-r}} \beta_{k,l}x^k(1-x)^{r-k}y^l(1-y)^{n-r-l} \\ &= \sum_{\substack{0\le i\le r\\ 0\le j\le n-r}} \sum_{\substack{0\le k\le i\\ 0\le l\le j}} \beta_{k,l}(-1)^{i-k+j-l}\binom{r-k}{i-k}\binom{n-r-l}{j-l}x^iy^j. \end{aligned}$$

Equating coefficients we get the desired result. □

A uniform oriented matroid $\mathcal{O}$ of rank r on $[n]$ is termed *neighborly* (see [19]) if there is no circuit C of $\mathcal{O}$ such that $|C^+| > (r+1)/2$, or (as follows since $-C$ is also a circuit) $|C^-| > (r+1)/2$. Equivalently, if $K_{\mathcal{O}}$ is written as a sum of monomials

$$K_{\mathcal{O}}(u,v) = \sum_{k,l} \beta_{k,l}u^kv^l,$$

the coefficients $\beta_{k,l}$ are 0 when $k \le [(r-1)/2]$.

A uniform oriented matroid $\mathcal{O}$ of rank r on $[n]$ is termed *dual-neighborly* if its dual oriented matroid is neighborly. Equivalently, using Theorem 7, $\mathcal{O}$ is dual-neighborly if, with $K_{\mathcal{O}}$ as above, $\beta_{k,l} = \binom{n}{k,l,n-k-l}$ when $l \le [(n-r-1)/2]$.

Examples of neighborly oriented matroids are provided by the alternating oriented matroids $\mathcal{A}_{n,r}$. The dual $\widehat{\mathcal{A}}_{n,n-r}$ has rank r and is dual-neighborly.

Theorem 11. *If $\mathcal{O}_1$ is neighborly and $\mathcal{O}_2$ is dual-neighborly, then the mutation count matrix $\mathcal{M}(\mathcal{O}_1, \mathcal{O}_2)$ is $(\delta_{i,j})$, where*

$$\delta_{i,j} = \sum_{\substack{0\le k\le i\\ 0\le l\le j}} (-1)^{i-k+j-l}\binom{r-k}{i-k}\binom{b-l}{j-l}\binom{n}{k,l,n-k-l}$$

for $0 \le i \le [(r-1)/2]$ and $0 \le j \le [n-r-1]$.

Proof. Let $p(u,v) = K_{\mathcal{O}_2}(u,v) - K_{\mathcal{O}_1}(u,v)$. By neighborliness and dual-neighborliness of the oriented matroids,

$$p(u,v) = \sum_{\substack{0\le k\le r\\ 0\le l\le n-r}} \beta_{k,l}u^kv^l,$$

where $\beta_{k,l} = \binom{n}{k,l,n-k-l}$, when $0 \le k \le [(r-1)/2]$ and $0 \le l \le [(n-r-1)/2]$. The result now follows from Lemma 7. □

This expression for $\delta_{i,j}$ can be written

$$\delta_{i,j} = \sum_{0 \le k \le i} (-1)^{i-k} \binom{n}{k} \binom{r-k}{i-k} \binom{r+j-k-1}{j},$$

which is simpler when r is small.

Here are the mutation count matrices for the alternating oriented matroids and their duals, of ranks 1, 2, 3, and 4:

$$\mathcal{M}(\mathcal{A}_{n,1}, \widehat{\mathcal{A}}_{n,n-1}) = \begin{pmatrix} 1 & 1 & 1 & \ldots & 1 \end{pmatrix} \left(\text{with } \left\lceil \frac{n}{2} \right\rceil \text{ 1's}\right),$$

$$\mathcal{M}(\mathcal{A}_{n,2}, \widehat{\mathcal{A}}_{n,n-2}) = \left(1 \quad 2 \quad 3 \quad \ldots \quad \left\lceil \frac{n-1}{2} \right\rceil\right),$$

$$\mathcal{M}(\mathcal{A}_{n,3}, \widehat{\mathcal{A}}_{n,n-3}) = \begin{pmatrix} 1 & 3 & 6 & \ldots & \left\lceil \frac{n}{2} \right\rceil \\ n-3 & 2n-9 & 3n-18 & \ldots & n\left\lceil \frac{n-2}{2} \right\rceil - 3\binom{\left\lceil \frac{n}{2} \right\rceil}{2} \end{pmatrix},$$

and

$$\mathcal{M}(\mathcal{A}_{n,4}, \widehat{\mathcal{A}}_{n,n-4}) = \begin{pmatrix} 1 & 4 & 10 & \ldots & \binom{\left\lceil \frac{n+1}{2} \right\rceil}{3} \\ n-4 & 3n-16 & 6n-40 & \ldots & n\binom{\left\lceil \frac{n-1}{2} \right\rceil}{2} - 4\binom{\left\lceil \frac{n+1}{2} \right\rceil}{3} \end{pmatrix}.$$

For ranks $n-1, n-2, n-3$, and $n-4$, the mutation count matrices for the appropriate alternating oriented matroids and their duals are the transposes of the above matrices.

8. Further Questions

Conjecture 1 would be a step toward characterization of the mutation matrices and Radon catalogs. Is there a full characterization, say, along the lines of the characterization of face vectors of simplicial polytopes accomplished by Billera, Lee, and Stanley (see [1] and [17]) by establishing the conjectured characterization of McMullen ([13])?

Conjecture 1 is open even for realizable uniform oriented matroids of rank 3. However, in the realizable case more can be said about the first column (and first row, by duality) of $\mathcal{M}(\mathcal{O}, \widehat{\mathcal{A}}_{n,n-r})$. Indeed, the first column consists of the g-vector of the oriented matroid polytope, and these vectors were characterized, in the realizable case, and used in the determination of the f-vectors. It is not known if simplicial oriented matroid polytopes (or sphere triangulations more generally) must satisfy these conditions.

For not necessarily uniform oriented matroids the Radon catalog does not have the nice properties described in Section 5. Perhaps there is a slightly more complicated polynomial, having useful properties, and refining the Radon catalog. Is there an analogue of the mutation count matrix, for oriented matroids in general?

In [12] we describe a collection of linear inequalities on the coefficients of the polynomials, such that T is the total polynomial of some uniform oriented matroid of rank r on $[n]$ if and only if the inequalities are satisfied and the coefficients are integers. This is a new characterization of uniform oriented matroids. However, the inequalties in general do not delimit the convex hull of the total polynomials; the polyhedron they determine is larger. It would certainly be nice to know the inequalities which determine the convex hull of the total polynomials.

We have no example at present of a uniform oriented matroid whose Radon catalog is not the Radon catalog of some realizable uniform oriented matroid. On the basis of this sparse evidence we state the following conjecture.

Conjecture 2. *If R is the Radon catalog of some uniform oriented matroid of rank r on $[n]$, then it is the Radon catalog of a realizable uniform oriented matroid of rank r on $[n]$.*

If E is a set of n points in the plane, no three on a line, and $\mathcal{O}$ is the oriented matroid of Radon partitions, then $\mathcal{O}$ is a uniform oriented matroid of rank 3, and the number of pairs of crossing edges in the drawing of the complete graph with n vertices obtained by connecting each pair of points of E by a line segment is half the coefficient of x^2y^2 in the Radon catalog $R_{\mathcal{O}}(x, y)$. The same holds if the points are on the unit sphere in R^3, no three are on a common great circle, $\mathcal{O}$ is the oriented matroid of linear dependencies, and the drawing is obtained by joining each pair of points by the shorter segment of the great circle through them. For $r = 3$, the coefficient of x^2y^2 in $m_{a,b}(x, y)$ is $-2(n - 3 - 2b)$ when $a = 1$, and 0 when $a = 0$. Therefore, if the mutation count matrix $\mathcal{M}(\mathcal{O}, \widehat{\mathcal{A}}_{n,n-3})$ is $(\gamma_{a,b})$ then the number of crossings is given by

$$c + \sum_{0 \le b \le [(n-4)/2]} (n - 3 - 2b)\gamma_{1,b},$$

where c is the number of crossings in the drawing corresponding to the oriented matroid $\widehat{\mathcal{A}}_{n,n-3}$. According to Conjecture 1, the $\gamma_{a,b}$'s should be nonnegative. Therefore Conjecture 1 implies the spherical crossing number conjecture for the complete graph, which states that the number of crossings in any such drawing is at least as large as in the drawing corresponding to $\widehat{\mathcal{A}}_{n,n-3}$. See [3] for a discussion of the crossing number, the spherical crossing number, and the rectilinear crossing number.

It would be interesting to know if there are analogues to the numbers $\gamma_{i,j}$ which refine the unrestricted crossing number conjecture in the way that those of the mutation matrix refine the spherical crossing number. Also, what is the similar refinement, for the crossing numbers of the complete bipartite graphs?

Conjecture 1 would in the same way imply statements analogous to the spherical crossing number conjecture for the complete graph, in higher dimensions.

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