Oriented Matroids
and
Associated Valuations

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It is possible to associate a valuation on the "orthant lattice" with each oriented matroid. In the case of uniform oriented matroids, it is not difficult to provide a characterization of the corresponding valuations. This is done here, thereby establishing a new characterization of the uniform oriented matroids themselves. Additionally, a connection between the valuations and the total polynomials associated with uniform oriented matroids is described.

1. Introduction.

In this paper we show how to associate with each oriented matroid a valuation on the "orthant lattice" which characterizes it. We are able to characterize the valuations which arise in this way from *uniform* oriented matroids in terms of certain linear inequalities and integrality constraints. This gives a new characterization of uniform oriented matroids.

The *orthant lattice* $\mathcal{Q}^n$ is the collection of sets $Q(A,B) \subseteq R^n$, where for sets $A, B \subseteq [n]$,

$$Q(A,B) = \{(x_1,\ldots,x_n) \in R^n : \begin{cases} x_i \geq 0 & \text{if } i \in A \\ x_i \leq 0 & \text{if } i \in B \end{cases} \}.$$

There are 4^n such sets. The intersection of two such sets is another.

In this paper the word "orthant" as defined above has a more liberal meaning than is normally assigned to it. We use the phrase *pointed orthant* to designate the 2^n subsets $Q(A,B)$, where $A \cup B = [n]$ and $A \cap B = \emptyset$, of R^n which are usually called (closed) orthants. Also, we call the 2^n coordinate subspaces $Q(A,A)$, for $A \subseteq [n]$, *linear orthants.*

The coordinate hyperplanes are the linear orthants $Q(\{i\},\{i\})$, which will also be denoted by H_i^0. The two closed halfspaces bounded by H_i^0 are $H_i^+ = Q(\{i\},\emptyset)$ and $H_i^- = Q(\emptyset,\{i\})$.

A *valuation* on $\mathcal{Q}^n$ is a function $v : \mathcal{Q}^n \to A$, where A is an abelian group (written additively), which satisfies

$$v(Q \cap H_i^0) + v(Q) = v(Q \cap H_i^+) + v(Q \cap H_i^+),$$

for each orthant Q and $i \in [n]$.

2. Some Valuations on the Orthant Lattice.

In this section we describe two classes of valuations on $\mathcal{Q}^n$.

First, suppose that P is a convex set in R^n. Define

$$\nu_P(Q) = \begin{cases} 1 & \text{if } P \cap Q \neq \emptyset \\ 0 & \text{if } P \cap Q = \emptyset. \end{cases}$$

If Q is an orthant and $i \in [n]$, then clearly

$$\nu_P(Q) = \max(\nu_P(Q \cap H_i^+), \nu_P(Q \cap H_i^-))$$

and also

$$\nu_P(Q \cap H_i^0) = \min(\nu_P(Q \cap H_i^+), \nu_P(Q \cap H_i^-)).$$

Clearly these properties imply that ν_P is indeed a valuation on the orthant lattice. A valuation satisfying these stronger properties will be called a *max-min valuation.*

If χ is the topological Euler characterisitic and the convex set P is compact, we can write $\nu_P(Q) = \chi(P \cap Q)$. Then the valuation property follows from the well-known property of the Euler characteristic.

Here are some properties of ν_P:

ν_P is a valuation;

ν_P is monotone; that is, if $Q_1, Q_2 \in \mathcal{Q}^n$ and $Q_1 \subseteq Q_2$ then $\nu_P(Q_1) \leq \nu_P(Q_2)$;

and

ν_P is 0, 1-valued.

These properties are equivalent to the assertion that ν_P is a $\{0,1\}$-valued max-min valuation on $\mathcal{Q}^n$.

For small values of n these conditions characterize the valuations that can be obtained in this way. This is not the case, however, for $n \geq 7$. We will see in Section 3 that the $\{0,1\}$-valued max-min valuations correspond to "lopsided sets." For an example of a lopsided subset of the 7-cube which is not realizable by a convex set as above, see [9].

Next, let W denote a linear subspace of R^n. Let ∂B^n denote the boundary of the n-dimensional cross-polytope,

$$B^n = \{x = (x_1, \ldots, x_n) : |x_i| \leq 1, i = 1, \ldots, n\}.$$

We define a valuation $\mu_W : \mathcal{Q}^n \to Z$ by

$$\mu_W(Q) = \chi(Q \cap W \cap \partial B^n).$$

That μ_W is a valuation on $\mathcal{Q}^n$ is immediate from the similar property of χ. Also it is clear that

$$\mu_W(Q) = \begin{cases} 1 & \text{if } Q \cap W \text{ is not a linear subspace} \\ 1 + (-1)^{d+1} & \text{if } Q \cap W \text{ is a linear subspace of dimension } d. \end{cases}$$

(In particular, $\mu_W(\{0\}) = 0$.) It is easy to verify that μ_W is a valuation directly from this.

We will see in Section 4 that any oriented matroid on $[n]$ yields a valuation on $\mathcal{Q}^n$ in a similar way.

If $0 \leq r \leq n$ and we impose the further resctrictions on W that

(a) $\dim(W) = r$, and

(b) W is in *general position* with respect to the coordinate axes, so that, for each linear orthant Q of dimension at most $n - r$, $W \cap Q = \{0\}$,

then we get somewhat stronger restrictions on μ_W. By (b), if $Q \in \mathcal{Q}^n$ is not a linear orthant and $Q \cap W$ is a linear subspace then $Q \cap W = \{0\}$; and if Q is a linear orthant of dimension $d \geq n - r$ then the dimension of the linear subspace $Q \cap W$ is $r + d - n$. Several conditions are therefore satisfied by the function μ_W, and we list these.

The function μ_W is a valuation.

The valuation μ_W is *symmetric*; that is, if Q is an orthant and $-Q$ is the opposite orthant then $\mu_W(-Q) = \mu_W(Q)$.

The valuation μ_W is *partly monotone*; that is, if $Q_1, Q_2 \in \mathcal{Q}^n$ are not linear and $Q_1 \subseteq Q_2$ then $\mu_W(Q_1) \leq \mu_W(Q_2)$.

If Q is an orthant which is not linear, then $\mu_W(Q)$ is 0 or 1; for a linear orthant Q, the value is given by:

$$\mu_W(Q) = \begin{cases} 1 + (-1)^{r+d-n-1} & \text{if } Q \text{ is a linear orthant of dimension } d \geq n-r \\ 1 & \text{if } Q = Q(A,B), \text{ where } A \neq B \text{ and } |A \cup B| \leq r \\ 0 & \text{if } Q \text{ is of dimension less than } n-r. \end{cases}$$

In particular, $\mu_W(\{0\}) = 0$.

We will show in Section 5 that any uniform oriented matroid on $[n]$ yields a valuation on $\mathcal{Q}^n$ satisfying these properties, and that all such valuations are obtained in this way: The uniform oriented matroids correspond to those valuations which are symmetric, partly monotone, $\{0,1\}$-valued on linear orthants which are not linear, and have value 0 on the smallest orthant.

3. Valuations and Lopsided Sets.

If sets A, B form a partition of $[n]$ then $\epsilon(A, B)$ denotes the vector $(x_1, \ldots, x_n)$, where $\epsilon_i = 1$ if $i \in A$ and $\epsilon_i = -1$ if $i \in B$.

From [5], a *lopsided set* S is a subset of the set $\{-1, 1\}^n$ of vertices of the cube $[-1, 1]^n \subseteq R^n$ satisfying the following condition: Whenever $A, B \subseteq [n]$, either

(a) there are sets $A' \subseteq [n] \setminus B$, $B' \subseteq [n] \setminus A$ partitioning $[n] \setminus (A \cap B)$ such that for each pair of sets $A'' \supseteq A'$, $B'' \supseteq B'$ partitioning $[n]$, $\epsilon(A'', B'') \notin S$, or

(b) there are sets $A' \subseteq A$, $B' \subseteq B$ partitioning $A \cup B$ such that for each pair of sets $A'' \supseteq A'$, $B'' \supseteq B'$ partitioning $[n]$, $\epsilon(A'', B'') \in S$.

Both cannot hold: If there are sets A', B' for which (a) holds, and sets $\tilde{A}', \tilde{B}'$ for which (b) holds, then, setting $A'' = A' \cup \tilde{A}'$ and $B'' = B' \cup \tilde{B}'$ then A'' and B'' partition $[n]$, (a) implies $\epsilon(A'', B'') \notin S$, and (b) implies $\epsilon(A'', B'') \in S$, a contradiction.

Given a lopsided set $S \subseteq \{-1, 1\}^n$, we may define a function v on Q^n having values in $\{0, 1\}$ by

$$v(Q(A, B)) = \begin{cases} 1 & \text{if (a) holds for } A \text{ and } B \\ 0 & \text{if (b) holds for } A \text{ and } B. \end{cases}$$

Theorem 1. *Suppose S is a lopsided set and v is derived as above. Then v is a max-min valuation on the orthant lattice.*

Proof. Suppose $Q = Q(A, B)$, where $A, B \subseteq [n]$, and suppose $i \notin A \cup B$.

If $v(Q(A, B)) = 0$ then there exist $\tilde{A}'$, $\tilde{B}'$ as described in (b). Since $i \notin \tilde{A}'$ and $i \notin \tilde{B}'$, both pairs $A \cup \{i\}$, B and A, $B \cup \{i\}$ satisfy this condition as well, so $v(Q(A \cup \{i\}, B)) = v(Q(A, B \cup \{i\})) = 0$.

If $v(Q(A, B)) = 1$, then there exist A', B' as in (a). One of these must contain i. Therefore either $v(Q(A \cup \{i\}, B)) = 1$ or $v(Q(A, B \cup \{i\})) = 1$.

In each case, $v(Q(A, B)) = \max(v(Q(A \cup \{i\}, B)), v(Q(A, B \cup \{i\})))$.

If $v(Q(A \cup \{i\}, B \cup \{i\})) = 0$ then there exist $\tilde{A}'$, $\tilde{B}'$ as in (b). One of these contains i. Therefore $v(Q(A \cup \{i\}, B) = 0$ or $v(Q(A, B \cup \{i\})) = 0$.

If $v(Q(A \cup \{i\}, B \cup \{i\})) = 1$ then there are sets A', B' as in (a). Neither contains i. It follows that $A' \cup \{i\}$, B' and A', $B \cup \{i\}$ both satisfy condition (a), so $v(Q(A \cup \{i\}, B)) = v(Q(A, B \cup \{i\})) = 1$.

In each case, $v(Q(A \cup \{i\}, B \cup \{i\})) = \min\{v(Q(A \cup \{i\}, B)), v(Q(A, B \cup \{i\}))\}$. □

It is clear that the set S can be retrieved from v:

$$S = \{\epsilon(A, B) : A, B \text{ partition } [n] \text{ and } v(Q(A, B) = 1\}.$$

For a max-min valuation v it is clear that, when $A, B \subseteq [n]$,

$$v(Q(A, B)) = \max_{\substack{A' \supseteq A, B' \supseteq B \\ A \cup B = [n]}} v(Q(A', B')),$$

and also that

$$v(Q(A, B)) = \min_{\substack{A' \subseteq A, B' \subseteq B \\ A \cap B = \emptyset}} v(Q(A', B')).$$

It follows that a max-min valuation is determined by its values on the pointed orthants using either of the formulas

$$v(Q(A, B)) = \max_{\substack{A' \subseteq A, B' \subseteq B \\ \text{partioning } A \cup B}} \min_{\substack{A'' \supseteq A', B'' \supseteq B' \\ \text{partioning } [n]}} v(Q(A'', B'')),$$

$$v(Q(A, B)) = \min_{\substack{A' \subseteq [n] \setminus A, B' \subseteq [n] \setminus B \\ \text{partioning } [n] \setminus (A \cap B)}} \max_{\substack{A'' \supseteq A', B'' \supseteq B' \\ \text{partioning } [n]}} v(Q(A'', B'')).$$

Theorem 2. *Suppose v is a $\{0, 1\}$-valued max-min valuation on $\mathcal{Q}^n$. Then $S = \{\epsilon(A, B) : A, B$ partition $[n]$, and $v(Q(A, B)) = 1\}$ forms a lopsided set.*

Proof. Indeed, it is immediate from the last two equations that, for S so defined and for any subsets $A, B \subseteq [n]$, one of the alternatives above must hold. □

If S is a lopsided set then so is the complementary set of vertices of the cube. If v is the $\{0, 1\}$-valued max-min valuation on $\mathcal{Q}^n$ corresponding to S, then the valuation v' corresponding to its complement satisfies $v'(Q) = 1 - v(Q^+)$, for each orthant Q, where, if $Q = Q(A, B)$, then $Q^+ = Q([n] \setminus B, [n] \setminus A)$. To see this, simply note that v' is a $\{0, 1\}$-valued max-min valuation which has the correct values on the pointed orthants.

4. Valuations from Arbitrary Oriented Matroids.

By making use of a certain topological representation for oriented matroids, it is not difficult to extend the derivation of μ_W of Section 2 to one relevant for arbitrary oriented matroids.

First, recall the following terminology from [1]. Given a sphere X of dimension d, a subset Y such that there exists a homeomorphism taking X to the unit sphere $S^d = \{(x_1, \ldots, x_{d+1}) : \sum x_i^2 = 1\}$ in R^{d+1} and Y to its equator $\{(x_1, \ldots, x_{d+1}) \in S^d : x_{d+1} = 0\}$ is called a *pseudo-sphere of* X. The *sides* of Y (in X) are the sets which correspond to $\{(x_1, \ldots, x_{d+1}) \in S^d : x_{d+1} \geq 0\}$ and $\{(x_1, \ldots, x_{d+1}) \in S^d : x_{d+1} \leq 0\}$ under this homeomorphism.

The sphere $W \cap \partial B^n$ of Section 2 can be replaced by a sphere $X \subseteq R^n \setminus \{0\}$ for which the following conditions are satisfied. The sphere X is symmetric about the origin: $X = -X$. The sphere X is a piecewise linear subset of R^n. The remaining conditions refer to the relationship between the sphere X and the orthants of $\mathcal{Q}^n$. For each linear orthant Q, the set $Q \cap X$ is a (possibly empty) sphere. The sides of $X \cap H_i^0$ in X are $X \cap H_i^+$ and $X \cap H_i^-$; and more generally, if Q is a linear orthant such that $X \cap Q \not\subseteq H_i^0$, then the sides of $X \cap Q$ are the sets $X \cap Q \cap H_i^+$ and $X \cap Q \cap H_i^-$. A consequence of these conditions is that, for each orthant Q, the set $Q \cap X$ is either a ball or a sphere. (See Mandel's thesis, [8].) We call a sphere X satisfying these conditions a piecewise linear *OM-sphere*.

The topological representation theorem of [3] describes a bijective correspondence between the loopless oriented matroids of rank r on $[n]$ and the homeomorphism classes of arrangements of n distinct pseudospheres on a sphere of dimension $r-1$. This theorem has been improved upon in various ways. See in particular [8] and [1]. After [8], the result can be formulated in terms of piecewise linear topology. See [1] for more details.

For a loopless oriented matroid $\mathcal{O}$ and corresponding arrangement of pseudospheres on a sphere Z, by using a suitable mapping of Z into R^n, one obtains the OM-sphere X as the image of Z satisfying the condition that the vector $U \in \{-1, 0, 1\}^n$ is a covector if and only if it is the zero vector or if $Q(A, B) \cap X \neq \emptyset$, where $A = \{i \in [n] : U_i = 1\}$ and $B = \{i \in [n] : u_i = -1\}$.

Making use of the Euler characteristic χ, we may define a valuation $v_{\mathcal{O}}$ on $\mathcal{Q}^n$ by the equation $v_{\mathcal{O}}(Q) = \chi(Q \cap X)$, where $Q \in \mathcal{Q}^n$.

The valuation $v_{\mathcal{O}}$ has the following additional properties.

The value $v_{\mathcal{O}}(\{0\})$ is 0; also, $v_{\mathcal{O}}(Q)$ is 0, 1, or 2, for each $Q \in \mathcal{Q}^n$.

The value on an orthant is the same as the value on its reflection through the origin: $v_{\mathcal{O}}(-Q) = v_{\mathcal{O}}(Q)$ for each $Q \in \mathcal{Q}^n$.

The remaining properties pertain to sets $A, B \subseteq [n]$.

The value $v_{\mathcal{O}}(Q(A,B))$ is 0 if $X \cap Q(A,B) = \emptyset$, or, equivalently, if there is no nonzero covector U of $\mathcal{O}$ having $U^+ \subseteq A$ and $U^- \subseteq B$.

The value $v_{\mathcal{O}}(Q(A,B))$ is $1 + (-1)^d$ if $X \cap Q(A,B)$ is a sphere of dimension d, or, equivalently, if there exists a covector U such that $U^+ \subseteq A$, $U^- \subseteq B$, the set of such covectors U is closed under negation, and the rank of the set $A \cup B$ in the underlying matroid is $r - d - 1$.

Finally, $v_{\mathcal{O}}(Q(A,B)) = 1$ otherwise, in which case $X \cap Q(A,B)$ is a ball.

We see that the values of $v_{\mathcal{O}}$ are indeed determined by the oriented matroid $\mathcal{O}$. It is not difficult to prove that the function so determined is a valuation, without resorting to the use of the topological representation theorem.

Given a loopless oriented matroid $\mathcal{O}$, the function $v_{\mathcal{O}}$ determines it. Indeed, from the properties above, we see that U is a covector of $\mathcal{O}$ if and only if $v_{\mathcal{O}}(Q([n] \setminus U^+, [n] \setminus U^-)) = 1$ and for any sets $A, B \subseteq [n]$ such that $A \supseteq [n] \setminus U^+$ and $B \supseteq [n] \setminus U^-$, $v_{\mathcal{O}}(Q(A,B)) = 1$ implies $A = [n] \setminus U^+$ and $B = [n] \setminus U^-$.

In the next section we characterize the functions $v_{\mathcal{O}}$ which arise from uniform oriented matroids $\mathcal{O}$. It would certainly be nice to have a similar characterization of the functions $v_{\mathcal{O}}$ arising from oriented matroids in general.

5. Valuations from Uniform Oriented Matroids.

Let $\mathcal{O}$ be a uniform oriented matroid and let $v_{\mathcal{O}}$ be the associated valuation, as in Section 4. In addition to the properties listed there, $v_{\mathcal{O}}$ satisfies the following.

The valuation $v_{\mathcal{O}}$ is partly monotone.

The valuation $v_{\mathcal{O}}$ has values in $\{0, 1\}$ on orthants which are not linear.

If r is the rank of $\mathcal{O}$ then

$$v_{\mathcal{O}}(Q(A, B)) = \begin{cases} 0 & \text{if } |A \cap B| \geq r \\ 1 & \text{if } |A \cap B| \leq r - 1 \text{ and } A \neq B \\ 1 + (-1)^d & \text{if } B = A \text{ and } d = r - 1 - |A| \geq 0. \end{cases}$$

Using a result of [5], we show that, given a valuation v which has value 0 or 1 on orthants which are not linear, has the property of symmetry, is partly monotone, and for which $v(\{0\}) = 0$, then there is a uniform oriented matroid $\mathcal{O}$ such that $v = v_{\mathcal{O}}$. The other properties listed above are therefore consequences.

The following lemma will be of use.

Lemma 1. *Suppose v_0 is defined on orthants Q which are not linear and satisfies*

$$v_0(Q) + v_0(Q \cap H_i^0) = v_0(Q \cap H_i^+) + v_0(Q \cap H_i^-)$$

whenever Q is not linear, $Q \not\subseteq H_i^+$, and $Q \not\subseteq H_i^-$. Then v_0 is the restriction of a unique valuation v on $\mathcal{Q}^n$ with $v(\{0\}) = 0$.

Proof. Suppose such a valuation v exists. Let $Q = Q(A, A)$ be a linear orthant. For $A = [n]$ we have $Q = \{0\}$, so $v(Q) = 0$; if $A \neq [n]$, so that there is $i \in [n] \setminus A$, we have that $v(Q) = v_0(Q \cap H_i^+) + v_0(Q \cap H_i^-) - v(Q \cap H_i^0)$ is determined by the value of v on a smaller linear orthant. It is clear that v is unique.

Given v_0, we define a function $t(A, B)$, for subsets A and B of $[n]$ with $A \subseteq B$, as follows.

We set $t(A, A) = 0$ for each subset $A \subseteq [n]$; if $|B| = |A| + 1$, then we set $t(A, B) = (-1)^{|A|}(v_0(Q(A, B)) + v_0(Q(B, A)))$; and if $B \setminus A = \{b_1, b_2, \ldots, b_k\}$, where $k > 1$, then we set $t(A, B) = t(A, A \cup \{b_1\}) + t(A \cup \{b_1\}, A \cup \{b_1, b_2\}) + \cdots + t(A \cup \{b_1, b_2, \ldots, b_{k-1}\}, A \cup \{b_1, b_2, \ldots, b_k\})$.

We must show that this is well-defined, that is, that the ordering of the elements of $B \setminus A$ is irrelevant. Clearly, for this, we need only show that if $A \subseteq [n]$ and $i, j \in [n] \setminus A$, then $t(A, A \cup \{i\}) + t(A \cup \{i\}, A \cup \{i,j\}) = t(A, A \cup \{j\}) + t(A \cup \{j\}, A \cup \{i,j\})$.

Using the property of v_0, we have:

$$\begin{aligned}
&t(A, A \cup \{i\}) + t(A \cup \{i\}, A \cup \{i,j\}) \\
&\quad = (-1)^{|A|}(v_0(Q(A, A \cup \{i\})) + v_0(Q(A \cup \{i\}, A)) \\
&\qquad - v_0(Q(A \cup \{i\}, A \cup \{i,j\})) - v_0(Q(A \cup \{i,j\}, A \cup \{i\}))) \\
&\quad = (-1)^{|A|}(v_0(Q(A \cup \{j\}, A \cup \{i\})) + v_0(Q(A, A \cup \{i,j\})) \\
&\qquad\qquad - v_0(Q(A \cup \{j\}, A \cup \{i,j\})) \\
&\qquad + v_0(Q(A \cup \{i,j\}, A)) + v_0(Q(A \cup \{i\}, A \cup \{j\})) \\
&\qquad\quad - v_0(Q(A \cup \{i,j\}, A \cup \{j\})) \\
&\qquad - v_0(Q(A, A \cup \{i,j\})) - v_0(Q(A \cup \{i\}, A \cup \{j\})) \\
&\qquad\quad + v_0(Q(A, A \cup \{j\})) \\
&\qquad - v_0(Q(A \cup \{j\}, A \cup \{i\})) - v_0(Q(A \cup \{i,j\}, A)) \\
&\qquad\quad + v_0(Q(A \cup \{j\}, A)) \\
&\quad = (-1)^{|A|}\big(- v_0(Q(A \cup \{j\}, A \cup \{i,j\})) - v_0(Q(A \cup \{i,j\}, A \cup \{j\})) \\
&\qquad + v_0(Q(A, A \cup \{j\})) + v_0(Q(A \cup \{j\}, A))\big) \\
&\quad = t(A, A \cup \{j\}) + t(A \cup \{j\}, A \cup \{i,j\}).
\end{aligned}$$

Let

$$v(Q(A,B)) = \begin{cases} v_0(Q(A,B)) & \text{if } A \neq B \\ (-1)^{|A|+1} t(\emptyset, A) & \text{if } A = B. \end{cases}$$

Suppose $Q = Q(A,B) \in \mathcal{Q}^n$. If $i \in A$ then $Q \cap H_i^+ = Q$ and $Q \cap H_i^- = Q \cap H_i^0$, so the required equality

$$v(Q) + v(Q \cap H_i^0) = v(Q \cap H_i^+) + v(Q \cap H_i^-)$$

holds. Similarly, if $i \in B$, equality holds. If $i \notin A$ and $i \notin B$ then neither $Q \cap H_i^+$ nor $Q \cap H_i^-$ is linear, and Q is linear if and only if $Q \cap H_i^0$ is linear. If these are not linear then equality holds, since in this case the values are also the values of v_0. If Q and $Q \cap H_i^0$ are linear with $Q = Q(A, A)$, then

$$\begin{aligned}
v(Q) + v(Q \cap H_i^0) &= (-1)^{|A|+1} t(\emptyset, A) + (-1)^{|A|+2} t(\emptyset, A \cup \{i\}) \\
&= (-1)^{|A|} t(A, A \cup \{i\}) \\
&= v(Q \cap H_i^+) + v(Q \cap H_i^-). \qquad \square
\end{aligned}$$

Theorem 3. *The map $\mathcal{O} \mapsto v_{\mathcal{O}}$ of Section 4, restricted to uniform oriented matroids, is a bijective correspondence between the uniform oriented matroids on $[n]$ and the valuations on the orthant lattice which are partly monotone, symmetric, $\{0,1\}$-valued on the orthants which are not linear, and having value 0 on $\{0\}$.*

Proof. We describe how to retrieve the oriented matroid $\mathcal{O}$, given a valuation v which satisfies these properties. Let

$$S = \{\epsilon(A, B) \in \{-1, 1\}^n : v(Q(A, B)) = 1\}.$$

For fixed i, consider the sets

$$S_i = \{\epsilon \in S : \epsilon_i = -1\}$$

and

$$S^i = \{\epsilon \in S : \epsilon_i = 1\}.$$

These sets are lopsided, by Theorem 2. Indeed, taking $i = n$, for example, it is clear that the set S_n corresponds to the max-min, $\{0,1\}$-valued valuation v' on $\mathcal{Q}^{n-1}$ (which is the collection of orthants $Q(A, B)$, where $n \in A \cap B$) given by $v'(Q(A, B)) = v(Q(A \setminus \{n\}, B)$; and the set S^n corresponds to the max-min, $\{0,1\}$-valued valuation v'' on $\mathcal{Q}^{n-1}$ given by $v''(Q(A, B)) = v(Q(A, B \setminus \{n\})$. Therefore the intersection of S with any facet of $[-1, 1]^n$ is lopsided.

Theorem 9 of [5] states that a set of vertices of the cube which is symmetric, but such that its intersection with each facet of the cube is lopsided, corresponds to the set of topes of some uniform oriented matroid. It follows that S corresponds to the set of topes of a uniform oriented matroid, $\mathcal{O}$.

Theorem 1, applied to each facet of the cube, yields a function v_0, defined on nonlinear orthants and satisfying the hypotheses of Lemma 1. It is clear that its unique extension must be the valuation v with which we began, and that $v = v_{\mathcal{O}}$. ∎

6. Valuations and Total Polynomials.

The notion of the "total polynomial" of a uniform oriented matroid was introduced in [7], as follows. Suppose that $\mathcal{L}$ denotes the collection of nonzero covectors of the uniform oriented matroid $\mathcal{O}$ on $[n]$. For each element $i \in [n]$, let x_i and y_i be a pair of indeterminates. Then the *total polynomial* of $\mathcal{O}$ is the following sum of monomials:

$$T_{\mathcal{O}}(x_i, y_i : i \in [n]) = \sum_{U:U \in \mathcal{L}} \Big(\prod_{i \in U^+} x_i \prod_{i \in U^-} y_i \Big).$$

It is clear that this may be reformulated in terms of the valuation $v_{\mathcal{O}}$ as:

$$T_{\mathcal{O}}(x_i, y_i : i \in [n]) = \sum_{\substack{A,B \subseteq [n] \\ A \cap B = \emptyset}} v_{\mathcal{O}}(Q(A,B)) \prod_{i \in A} x_i \prod_{i \in B} y_i.$$

Clearly the function taking $v_{\mathcal{O}}$ to $T_{\mathcal{O}}$ acts in a linear fashion.

Given $T_{\mathcal{O}}$, the values $v_{\mathcal{O}}(Q(A,B))$ can be retrieved. If $A \cap B$ is empty, then $v_{\mathcal{O}}(Q(A,B))$ is a coefficient of the polynomial $T_{\mathcal{O}}$. If $i \in A \cap B$ then, since $v_{\mathcal{O}}$ is a valuation, we have

$$v_{\mathcal{O}}(Q(A,B)) = v_{\mathcal{O}}(Q(A_0,B)) + v_{\mathcal{O}}(Q(A,B_0)) - v_{\mathcal{O}}(Q(A_0,B_0)),$$

where $A_0 = A \setminus \{i\}$ and $B_0 = B \setminus \{i\}$. Proceeding in this way, we get

$$v_{\mathcal{O}}(Q(A,B)) = \sum (-1)^{|A \cup B| - |C \cup D|} v_{\mathcal{O}}(Q(C,D)),$$

where the summation extends over pairs C, D such that $A \setminus B \subseteq C \subseteq A$, $B \setminus A \subseteq D \subseteq B$, and $C \cap D = \emptyset$. Therefore not only the coefficients but also the other values $v_{\mathcal{O}}(Q(A,B))$ can be retrieved linearly from $T_{\mathcal{O}}$.

We now describe a relationship between certain valuations on polyhedral cones in R^n and three identities, proven in [7], involving the polynomials $T_{\mathcal{O}}$. Considering the linear equivalence between the valuations and the total polynomials associated with uniform oriented matroids, it is certainly possible to formulate the identities in the valuation-theoretic setting. The object of this section is to accomplish this. Even though it seems irrelevant for the results of this paper that these valuations are defined on the larger class of polyhedral cones rather than simply the orthants, we will describe them in this generality.

The three identities are:

$$T_{\mathcal{O}}(y_e, x_e : e \in E) = T_{\mathcal{O}}(x_e, y_e : e \in E); \tag{1}$$

$$\prod_{e \in E}(1 + x_e + y_e)T_{\mathcal{O}}(\frac{-x_e}{1 + x_e + y_e}, \frac{-y_e}{1 + x_e + y_e} : e \in E) \tag{2}$$

$$= (-1)^{n-r+1}T_{\mathcal{O}}(x_e, y_e : e \in E)$$

and

$$T_{\widehat{\mathcal{O}}}(x_e, y_e : e \in E) = \prod_{e \in E}(1 + x_e + y_e) - (-1)^r \tag{3}$$

$$-(-1)^n T_{\mathcal{O}}(-1 - x_e, -1 - y_e : e \in E),$$

where $\widehat{\mathcal{O}}$ is the total polynomial of the dual of $\mathcal{O}$.

The corresponding relations for valuations are (1'), (2'), and (3'), below.

Let $\mathcal{P}^n$ denote the collection of closed convex polyhedral cones emanating from the origin in R^n. A *valuation* on $\mathcal{P}^n$ is a function $v : \mathcal{P}^n \to A$, where A is an additive abelian group, satisfying

$$v(P) + v(P \cap H^0) = v(P \cap H^+) + v(P \cap H^-),$$

whenever $P \in \mathcal{P}^n$ and H^0 is a hyperplane which bounds the two closed halfspaces H^+ and H^-.

Given a set $S \subseteq R^n$, we denote by $[S]$ its *indicator function*:

$$[S](x) = \begin{cases} 1 & \text{if } x \in S, \\ 0 & \text{if } x \notin S. \end{cases}$$

We denote by $\mathcal{S}(\mathcal{P}^n)$ the additive group of Z-valued functions on R^n which is generated by the indicator functions $[P]$ of $P \in \mathcal{P}^n$. We denote by $\mathcal{S}(\mathcal{Q}^n)$ the subgroup of $\mathcal{S}(\mathcal{P}^n)$ generated by the indicator functions of the orthants.

Each valuation $v : \mathcal{P}^n \to A$ determines uniquely a homomorphism $\bar{v} : \mathcal{S}(\mathcal{P}^n) \to A$ such that, for each $P \in \mathcal{P}^n$, $v(P) = \bar{v}([P])$.

We describe three valuations r, s, and t on $\mathcal{P}^n$, each related to one of the identities involving total polynomials.

The first of these, $r : \mathcal{P}^n \to \mathcal{S}(\mathcal{P}^n)$, takes a cone $P \in \mathcal{P}^n$ to the indicator function of its reflection through the origin: $r(P) = [-P]$.

The next, $s : \mathcal{P}^n \to \mathcal{S}(\mathcal{P}^n)$, is essentially the Sallee-Shephard mapping of [6]. If $P \in \mathcal{P}^n$, we denote by P^{ri} its relative interior. If $P \in \mathcal{P}^n$ and P is of dimension d, then $s(P) = (-1)^d[P^{\mathrm{ri}}]$ for $P \in \mathcal{P}^n$.

Finally, $t : \mathcal{P}^n \to \mathcal{S}(\mathcal{P}^n)$ is the normal cone mapping of [6]: $t(P) = [P^{\perp}]$.

Each of these valuations maps orthants to elements of the subgroup $\mathcal{S}(\mathcal{Q}^n)$.

We will also need two special valuations, ϵ_1, ϵ_2, on $\mathcal{Q}^n$. On orthants Q which are not linear, each of these has value 1. For linear orthants Q, $\epsilon_1(Q) = 1+(-1)^d$, where d is the dimension of Q; and $\epsilon_2(Q) = 1+(-1)^{d+1}$, where d is the dimension of Q. Clearly $\epsilon_1(Q) + \epsilon_2(Q) = 2$, for each orthant Q.

Now, if v is any valuation on $\mathcal{Q}^n$, we get three new valuations $\tilde{r}(v)$, $\tilde{s}(v)$, and $\tilde{t}(v)$ on $\mathcal{Q}^n$ by setting $\tilde{r}(v)(P) = \bar{v}(r(P))$, $\tilde{s}(v)(P) = \bar{v}(s(P))$, and $\tilde{t}(v)(P) = \bar{v}(t(P))$.

Given a uniform oriented matroid $\mathcal{O}$ on $[n]$ having rank r, we have the three identities

$$\tilde{r}(v_{\mathcal{O}}) = v_{\mathcal{O}}, \tag{1'}$$

$$\tilde{s}(v_{\mathcal{O}}) = (-1)^{n-r-1} v_{\mathcal{O}}, \tag{2'}$$

and

$$v_{\widehat{\mathcal{O}}} = \epsilon_i - \tilde{t}(v_{\mathcal{O}}), \tag{3'}$$

where $i = 1$ if r is odd, and $i = 2$ otherwise.

7. Notes.

A pleasing characterization of the valuations associated with oriented matroids in general isn't apparent. In this paper, we have relied on results of [5] in our characterization of the valuations arising from uniform oriented matroids. Da Silva [2] has improved upon the results of [5] by giving a related characterization of the subsets of the vertex sets of the cubes which correspond to topes of oriented matroids in general. Perhaps this work would be useful in the search for a pleasing characterization of the valuations.

Also, concerning the characterization in [5] of uniform oriented matroids, it is worth noting that another nice characterization of these is described by Gärtner and Welzl in [4]. Also in that paper, a connection between lopsided sets and the notion of "Vapnik-Chervonenkis dimension" is noted.

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