Oriented Matroids and Multiply Ordered Sets

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ABSTRACT

The many different axiomatizations for matroids all have their uses. In this paper we show that Gutierrez Novoa's n-ordered sets are cryptomorphically the same as the oriented matroids, thereby establishing the existence of an axiomatization for oriented matroids in which the "oriented" bases of the matroid are the objects of paramount importance.

I. INTRODUCTION

Many axiom systems for oriented matroids exist, as is also the case for (ordinary) matroids. For example, several descriptions of oriented matroids, analogous to well-known circuit descriptions of matroids, are given by Bland and Las Vergnas [1]. Still others are given in Folkman and Lawrence [2], including one in terms of hull operators.

In [3], Gutierrez Novoa introduced the notion of "d-ordered sets." Here we show that finite d-ordered sets are oriented matroids in disguise (and *vice versa*), and that these objects yield a treatment of oriented matroids analogous to a treatment of matroids relying on their collections of bases. We feel that such a treatment will be of much use for oriented matroids, as it has been for matroids; in particular, such a description can be used to extend the "union" operation for matroids to oriented matroids. (See Lawrence and Weinberg [7].)

A *d-ordered set* (for $d \geqslant 0$) is a pair (X, Φ), where X is a set and Φ is a function on $(d+1)$-tuples $(x_0, \dots, x_d)$ of elements of X with values in

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$\{-1,0,1\}$, not identically zero, satisfying:

(A1) Φ is *alternating*; i.e., if the $(d+1)$-tuple σ is obtained from the $(d+1)$-tuple τ by interchanging two entries, then $\Phi(\sigma) = -\Phi(\tau)$;

(A2) If $s_0,\ldots,s_d$ and $t_0,\ldots,t_d$ are elements of X and if
(i) $\Phi(t_i,s_1,\ldots,s_d)\Phi(t_0,\ldots,t_{i-1},s_0,t_{i+1},\ldots,t_d) \geq 0$ for each i with $0 \leq i \leq d$, then
(ii) $\Phi(s_0,s_1,\ldots,s_d)\Phi(t_0,t_1,\ldots,t_d) \geq 0$.

Note that a 0-ordered set is simply a pair (X,Φ), where Φ is a function $\Phi: X \to \{-1,0,1\}$, not identically zero. Note also that if (X,Φ) is a d-ordered set, so is $(X,-\Phi)$.

Note, in (A2), that if none of the products in (i) are positive, then the product in (ii) cannot be positive either. (This is easily verified by considering the effect of interchanging two of the t_i's.) It follows that if all of the products in (i) are zero, so is that in (ii).

It will prove convenient to have the following notation for the manipulation of tuples. If $\sigma = (x_0,\ldots,x_k)$ is a $(k+1)$-tuple, let $|\sigma| = \{x_0,\ldots,x_k\}$. For $0 \leq i \leq k$, let $L^i\sigma = (x_0,\ldots,x_{i-1})$, $R^i\sigma = (x_{i+1},\ldots,x_k)$, and $E^i\sigma = x_i$. Note that if $i=0$, then $L^i\sigma$ is the unique 0-tuple, as is $R^i\sigma$ if $i=k$. Finally, if $\sigma_1,\ldots,$ and σ_m are tuples—say, σ_i is a k_i-tuple—then we denote by $(\sigma_1,\ldots,\sigma_m)$ the $(k_1+\cdots+k_m)$-tuple obtained by their concatenation. In particular, $\sigma = (L^i\sigma, E^i\sigma, R^i\sigma)$.

Now (A2) may be stated as follows. If (X,Φ) is a d-ordered set, s is in X, σ is a d-tuple from X, τ is a $(d+1)$-tuple from X, and $\Phi(s,\sigma)\Phi(\tau) = -1$, then for some i between 0 and d, $\Phi(E^i\tau,\sigma)\Phi(L^i\tau,s,R^i\tau) = -1$.

We describe how some examples of d-ordered sets arise. Suppose A is a $(d+1)\times(n+1)$ matrix of real numbers, of rank $d+1$, and let X be the set consisting of its columns. Tuples from X may be viewed as matrices. For a $(d+1)$-tuple σ from X, let $D(\sigma)$ denote the determinant of the $(d+1)\times(d+1)$ matrix σ. Let $\phi(\sigma)$ be the sign, 1, -1, or 0, of $D(\sigma)$. Then (X,ϕ) is a d-ordered set. Indeed, the function D is alternating (so that ϕ is, as well), and for any $(d-1)$-tuples (s,σ) and τ from X, we have

$$\text{(B)}\quad D(s,\sigma)D(\tau) = \sum_{i=0}^{d} D(E^i\tau,\sigma)D(L^i\tau,s,R^i\tau).$$

(For a proof see Hodge and Pedoe [4, Vol. 1, Chapter VII].) It follows that if the signs of each of the terms on the right in this equality are nonnegative, then so is the term on the left, so ϕ satisfies (A2).

In Section 2, we describe the manner in which each (finite) d-ordered set determines an oriented matroid of rank $d+1$. In Section 3, we describe the

reverse process, showing that each oriented matroid of rank $d+1$ gives rise to a pair, (X,ϕ) and $(X,-\phi)$, of d-ordered sets.

In fact, Las Vergnas [5,6] has already shown how to construct an alternating function ϕ from a given oriented matroid, and (in [5]) has asked for a characterization of the functions ϕ arising by his construction. Our results give such a characterization: (X,ϕ) arises in this way if and only if (X,ϕ) is a d-ordered set.

In Section 4 we describe the d-ordered sets corresponding to digraphoids (Minty [8]), and mention some classes of problems.

In this paper we rely heavily on simple results about matroids. A good reference is Welsh [11].

II. GETTING ORIENTED MATROIDS FROM d-ORDERED SETS

Recall that a non-empty collection $\mathfrak{B}$ of subsets of a finite set X is the set of bases of a matroid provided

(C) If B_1 and B_2 are in $\mathfrak{B}$ and s is an element of B_1, then there is an element t of B_2 such that $(B_2 \sim \{t\})\cup\{s\}$ is in $\mathfrak{B}$.

If (X,ϕ) is a d-ordered set and $S = \{s_0,\ldots,s_d\}$ is a subset of X of cardinality $d+1$, let $\bar\phi(S) = |\phi(s_0,\ldots,s_d)|$. Since ϕ is alternating, this function $\bar\phi$ is well defined. Let $\mathfrak{B} = \{S \subseteq X : \bar\phi(S) = 1\}$.

Theorem 1. *$\mathfrak{B}$ is the collection of bases of a matroid M.*

Proof. Clearly $\mathfrak{B}$ is nonempty.

Suppose B_1 and B_2 are in $\mathfrak{B}$ and that s is in B_1. Let σ be a d-tuple and τ a $(d+1)$-tuple with $|(s,\sigma)| = B_1$ and $|\tau| = B_2$. Then

$$1 = \bar\phi(B_1)\bar\phi(B_2) = |\phi(s,\sigma)\phi(\tau)|.$$

We have seen in the discussion following (A1) that for some i with $0 \leq i \leq d$ we must have

$$\phi(E^i\tau,\sigma)\phi(L^i\tau,s,R^i\tau) \neq 0.$$

With $t = E^i\tau$, we have that both $(B_1 \sim \{s\})\cup\{t\}$ and $(B_2 \sim \{t\})\cup\{s\}$ lie in $\mathfrak{B}$.

This is evidently stronger than the conclusion required by (C), so $\mathfrak{B}$ is the collection of bases of a matroid. (Actually, this apparently stronger condition holds in any matroid. See Welsh [11, p. 15, Problem 2(b)].) ■

We will call M the *underlying matroid* of (X, ϕ). Note that its rank, the common cardinality of the elements of $\mathfrak{B}$, is $d+1$.

Before we describe the oriented matroid determined by (X, ϕ), we will describe a duality for multiply ordered sets. Suppose $|X| = n+1$, and let (X, ε) be one of the two n-orderings of X. Given the alternating function ϕ mapping $(d+1)$-tuples from X to $\{-1, 0, 1\}$, where $0 \leqslant d \leqslant n-1$, define the function $\acute{\phi}$ as follows. (An equivalent definition is given in [6, Proposition 3].) Suppose σ is an $(n-d)$-tuple from X. Let $\acute{\phi}(\sigma) = 0$ if σ has repeated entries. Otherwise, let τ be a $(d+1)$-tuple from X with $|\tau| = X \sim |\sigma|$, and let $\acute{\phi}(\sigma) = \phi(\tau)\varepsilon(\sigma, \tau)$. Clearly, this number will not depend on the ordering in τ of the elements of $X \sim |\sigma|$, so that $\acute{\phi}(\sigma)$ is well defined by this expression. Note, however, that the construction does depend on which of the two n-orderings is chosen. If (X, ϕ) is a d-ordered set, we call $(X, \acute{\phi})$ the *ε-dual* of (X, ϕ). [This is not strictly a "duality." If d and $n-d-1$ are both even, then, if one takes the ε-dual of the ε-dual of (X, ϕ), one obtains $(X, -\phi)$ instead of (X, ϕ). What is important is that one gets either (X, ϕ) or $(X, -\phi)$.]

THEOREM 2. *$(X, \acute{\phi})$ is an $(n-d-1)$-ordered set.*

Proof. Clearly $\acute{\phi}$ is not identically zero, and it is alternating. It remains to show that it also satisfies property (A2).

Suppose s is an element of X, σ is an $(n-d-1)$-tuple from X, and τ is an $(n-d)$-tuple from X. Suppose $\acute{\phi}(s, \sigma)\acute{\phi}(\tau) = -1$. We must show that, for some i with $0 \leqslant i \leqslant n-d-1$,

$$\acute{\phi}(E^i\tau, \sigma)\acute{\phi}(L^i\tau, s, R^i\tau) = -1.$$

Note that if $s \in |\tau|$, then the i with $s = E^i\tau$ works. We may assume $s \notin |\tau|$. Since $\acute{\phi}(s, \sigma) \neq 0$, the entries of (s, σ) are distinct. Let $\bar{\sigma}$ be a $(d+1)$-tuple with $|\bar{\sigma}| = X \sim |(s, \sigma)|$. Similarly, let $\bar{\tau}$ be a d-tuple with $|\bar{\tau}| = X \sim (|\tau| \cup \{s\})$. Then

$$\phi(\bar{\sigma})\phi(s, \bar{\tau}) = \acute{\phi}(s, \sigma)\varepsilon(s, \sigma, \bar{\sigma})\acute{\phi}(\tau)\varepsilon(\tau, s, \bar{\tau})$$
$$= -\varepsilon(s, \sigma, \bar{\sigma})\varepsilon(\tau, s, \bar{\tau}). \quad (1)$$

Since (X, ϕ) is a d-ordered set, for some i with $0 \leqslant i \leqslant d$ we must have

$$\phi(L^i\bar{\sigma}, s, R^i\bar{\sigma})\phi(E^i\bar{\sigma}, \bar{\tau}) = -\varepsilon(s, \sigma, \bar{\sigma})\varepsilon(\tau, s, \bar{\tau}). \quad (2)$$

For this i, $\phi(E^i\bar{\sigma}, \bar{\tau}) \neq 0$, so $E^i\bar{\sigma}$ is not among the entries of $\bar{\tau}$. Then $E^i\bar{\sigma}$ is in $|\tau| \cup \{s\}$. Also, $E^i\bar{\sigma} \neq s$, so $E^i\bar{\sigma} = E^j\tau$, for some j with $0 \leqslant j \leqslant n-d-1$. Then

$$\acute{\phi}(E^j\tau, \sigma) = \phi(L^i\bar{\sigma}, s, R^i\bar{\sigma})\varepsilon(E^j\tau, \sigma, L^i\bar{\sigma}, s, R^i\bar{\sigma})$$
$$= -\phi(L^i\bar{\sigma}, s, R^i\bar{\sigma})\varepsilon(s, \sigma, \bar{\sigma}), \quad (3)$$

and

$$\acute{\phi}(L^j\tau, s, R^j\tau) = \phi(E^j\tau, \bar{\tau})\varepsilon(L^j\tau, s, R^j\tau, E^j\tau, \bar{\tau}) = -\phi(E^j\tau, \bar{\tau})\varepsilon(\tau, s, \bar{\tau}). \quad (4)$$

By (2), the product of (3) and (4) is -1. ■

We note, in passing, that if (X, ϕ) arises from a $(d+1)\times(n+1)$ matrix A, as in Section 1, then $(X, \acute{\phi})$ arises in the same way from any $(n-d)\times(n+1)$ matrix $\acute{A}$ whose rows span the orthogonal complement of the row-space of A. (This follows, for example, from Theorem I on p. 294 of [4].)

Consider the underlying matroid $\acute{M}$ of the ε-dual $(X, \acute{\phi})$ of (X, ϕ). Suppose B is a subset of X of cardinality $d+1$. Suppose $B = \{s_0, \ldots, s_d\}$ and $X - B = \{t_0, \ldots, t_{n-d-1}\}$. Clearly $\phi(s_0, \ldots, s_d) \neq 0$ if and only if $\acute{\phi}(t_0, \ldots, t_{n-d-1}) \neq 0$. It follows that B is a basis of M if and only if $X \sim B$ is a basis of $\acute{M}$; i.e., M and $\acute{M}$ are dual matroids.

We turn now to *oriented* matroids. Here it is convenient to view them as being certain triples $(X, \mathcal{C}, P)$, where X is the finite set, $\mathcal{C}$ is the collection of circuits of a matroid $M = (X, \mathcal{C})$, and P is a function which assigns to each circuit C an (unordered) partition $P(C) = \{A, B\}$, so that $A \cup B = C$ and $A \cap B = \varnothing$. (It is not required that A and B be nonempty. The sets A and B are called the *classes* of C.) To describe which triples are oriented matroids, we use a characterization due to Bland and Las Vergnas [1]. (The function P here determines ordered partitions (A, B) and (B, A) of circuits C, termed "circuit signatures" in [1]. We use unordered partitions, thereby avoiding the use of "signed sets.") The characterization of triples $(X, \mathcal{C}, P)$ which are oriented matroids runs as follows. $(X, \mathcal{C}, P)$ is an *oriented matroid* if there is another such triple, $(X, \hat{\mathcal{C}}, \hat{P})$, such that the following two conditions are satisfied. First, $(X, \mathcal{C})$ and $(X, \hat{\mathcal{C}})$ are dual matroids. Second, if C is a circuit of $(X, \mathcal{C})$, U is a circuit of $(X, \hat{\mathcal{C}})$, $P(C) = \{A, B\}$, and $\hat{P}(U) = \{A', B'\}$, then either $C \cap U = \varnothing$ or both of the sets $(A \cap A') \cup (B \cap B')$ and $(A \cap B') \cup (B \cap$

A') are nonempty. If such a triple $(X, \hat{\mathcal{C}}, \hat{P})$ exists, it can be shown that it is unique. It is called the oriented matroid *dual* to $(X, \mathcal{C}, P)$.

We rephrase the second condition. Call a pair $\{s, t\}$ *mixed* with respect to C and U if s and t occur in the same class of one of C and U and in different classes of the other. The second condition is equivalent to requiring that if s is an element of $C \cap U$, then there is an element t of $C \cap U$ such that $\{s, t\}$ is mixed with respect to C and U.

For an example, let X be a finite subset of R^d. Let $\mathcal{C}$ be the collection of minimal affinely dependent subsets of X. If C is an element of $\mathcal{C}$, then C has a unique *Radon partition* $P(C) = \{A, B\}$, a partition into two sets A and B whose convex hulls intersect. (See Peterson [10].) Then $(X, \mathcal{C}, P)$ is an oriented matroid.

If X affinely spans R^d, so that $(X, \mathcal{C})$ has rank $d+1$, then the corresponding triple $(X, \hat{C}, \hat{P})$ can be described as follows. A set $U \subseteq X$ is a circuit of the matroid $(X, \hat{\mathcal{C}})$ dual to $(X, \mathcal{C})$ if $X \sim U$ affinely spans a hyperplane H in R^d and $U \cap H = \varnothing$. Then U has the natural partition $P(U)$ into the two sets of elements of U lying on the one side or the other side of H.

We verify the second property. Suppose C is in $\mathcal{C}$, U is in $\hat{\mathcal{C}}$, and s is in $C \cap U$. Let $P(C) = \{A', B'\}$, $\hat{P}(U) = \{A, B\}$, and suppose s is in $A \cap A'$. Let H be the hyperplane affinely spanned by $X \sim U$. Let H_A be the open half space of R^d determined by H in which A lies, so that $A = H_A \cap X$. Similarly, let H_B be the other open half space determined by H, so that $B = H_B \cap X$. Let $\bar{H}_A$ and $\bar{H}_B$ denote the closures of H_A and H_B. Since $P(C) = \{A', B'\}$, the convex hulls, $\operatorname{conv} A'$ and $\operatorname{conv} B'$, intersect in a single point. (See [10, Corollary 4.2, p. 953].) Let u be the point of intersection. Then $\operatorname{conv} A'$ and $\operatorname{conv} B'$ are simplexes and u is in their relative interiors. If u is in H_A, then $B' \cap H_A \neq \varnothing$, since otherwise $u \in \operatorname{conv} B' \subseteq \bar{H}_B$. Then if t is an element of $B' \cap H_A$, $\{s, t\}$ is mixed with respect to C and U. If u is in $\bar{H}_B$, then since u is in the relative interior of $\operatorname{conv} A'$ and A' is not contained in H, we have $A' \cap H_B \neq \varnothing$. If t is in $A' \cap H_B$, then $\{s, t\}$ is mixed with respect to C and U.

Given a d-ordered set (X, ϕ), we have seen that (X, ϕ) determines a matroid M, and that the ϵ-dual $(X, \hat{\phi})$ similarly determines the matroid $\hat{M}$ which is dual to M. Let $\mathcal{C}$ be the set of circuits of M, and let $\hat{\mathcal{C}}$ be the set of circuits of $\hat{M}$. We must describe the partition functions P and $\hat{P}$, in order to complete the description of the corresponding oriented matroid.

Suppose C is a circuit of M. Let τ be a $(d+2)$-tuple with $C \subseteq |\tau|$ and such that $|\tau|$ is a spanning set of M. Let

$$A = \{E^i\tau : \phi(L^i\tau, R^i\tau) = (-1)^i\},$$

$$B = \{E^i\tau : \phi(L^i\tau, R^i\tau) = (-1)^{i-1}\}.$$

Note that C is the unique circuit of M contained in $|\tau|$. Therefore, $|\tau| \sim \{E^i\tau\} = |(L^i\tau, R^i\tau)|$ is a basis for M if and only if $E^i\tau$ is in C; i.e., $\phi(L^i\tau, R^i\tau)$ is nonzero if and only if $E^i\tau$ is in C. It follows that $\{A, B\}$ is a partition of C. We must show that this partition is not dependent on which $(d+2)$-tuple τ is chosen. This is the content of the following lemma.

LEMMA. *Any $(d+2)$-tuple τ such that $|\tau|$ is a spanning set of M with $C \subseteq |\tau|$ yields the same partition of C.*

Proof. Suppose σ is another such $(d+2)$-tuple. Let

$$A = \{E^i\tau : \phi(L^i\tau, R^i\tau) = (-1)^i\},$$

$$B = \{E^i\tau : \phi(L^i\tau, R^i\tau) = (-1)^{i-1}\},$$

$$A' = \{E^i\sigma : \phi(L^i\sigma, R^i\sigma) = (-1)^i\},$$

$$B' = \{E^i\sigma : \phi(L^i\sigma, R^i\sigma) = (-1)^{i-1}\}.$$

We must show that $\{A, B\}$ and $\{A', B'\}$ are identical partitions of C.

If σ may be obtained from τ by permuting two entries of τ, it is easily verified that $A' = B$ and $A' = B'$. It follows that the result holds whenever σ is obtained by permuting entries of τ.

Suppose s and t are two elements of C. We must show that they are in the same class in each of the partitions $\{A, B\}$ and $\{A', B'\}$, or they are in different classes in each of these partitions. We may assume that s and t are the first two elements in each of σ and τ, so that $\sigma = (s, t, \sigma_0)$ and $\tau = (s, t, \tau_0)$, for appropriate choices of σ_0 and τ_0. We need only show that $\phi(s, \sigma_0)\phi(t, \tau_0) = \phi(t, \sigma_0)\phi(s, \tau_0)$.

Since (X, ϕ) is a d-ordered set, there is an i with $0 \leqslant i \leqslant d$ such that

$$\phi(E^i(t, \tau_0), \sigma_0)\phi(L^i(t, \tau_0), s, R^i(t, \tau_0)) \tag{1}$$

is equal to $\phi(s, \sigma_0)\phi(t, \tau_0)$. If $i > 0$, (1) is

$$\phi(t, E^{i-1}\tau_0, \sigma_0)\phi(t, L^{i-1}\tau_0, s, R^{i-1}\tau_0) \tag{2}$$

If $E^{i-1}\tau_0$ is in C, then it occurs in σ_0, so the left factor in (2) is zero. It follows that $i = 0$. For $i = 0$, (1) is $\phi(t, \sigma_0)\phi(s, \tau_0)$, and we have the required conclusion. ■

Using $\hat{\phi}$, we similarly obtain a partition function $\hat{P}$ for the dual matroid.

It will be convenient to be able to determine $\hat{P}$ directly from ϕ. (See, also, the Remark on p. 287 of [6].) Suppose U is an element of $\hat{C}$. Let τ be an $(n-d+1)$-tuple with $U \subseteq |\tau|$ and such that $|\tau|$ spans in M; i.e., such that $X \sim |\tau|$ is independent in $\hat{M}$. Choose a d-tuple σ with $|\sigma| = X \sim |\tau|$. Then

$$\hat{\phi}(L^i\tau, R^i\tau) = \phi(E^i\tau, \sigma)\varepsilon(L^i\tau, R^i\tau, E^i\tau, \sigma)$$

$$= \phi(E^i\tau, \sigma)\varepsilon(\tau, \sigma)(-1)^{n-i-d-1}$$

$$= \phi(E^i\tau, \sigma)(-1)^i\eta,$$

where $\eta = \pm 1$ is a constant. It follows that $\hat{P}(U) = \{A, B\}$, where

$$A = \{u \in X : \phi(u, \sigma) = 1\},$$

$$B = \{u \in X : \phi(u, \sigma) = -1\}.$$

Furthermore, any d-tuple σ with $|\sigma|$ independent (in M) and disjoint from U will yield the same partition.

THEOREM 3. *$(X, \mathcal{C}, P)$ is an oriented matroid.*

Proof. We have described $(X, \hat{\mathcal{C}}, \hat{P})$ and noted that $(X, \hat{\mathcal{C}})$ is the matroid dual to $(X, \mathcal{C})$. It remains to show that if C is in $\mathcal{C}$, U is in $\hat{\mathcal{C}}$, and s is in $C \cap U$, then there is an element t of $C \cap U$ such that $\{s, t\}$ is mixed with respect to C and U.

Choose a d-tuple σ and a $(d+1)$-tuple τ such that $|\sigma|$ is an independent set of $(X, \mathcal{C})$ not intersecting U, and $|\tau|$ is a basis with $C \subseteq \{s\} \cup |\tau|$. For some i with $0 \leqslant i \leqslant d$, $\phi(E^i\tau, \sigma)\phi(L^i\tau, s, R^i\tau)$ must equal $\phi(s, \sigma)\phi(\tau)$. Let $t = E^i\tau$. ■

III. GETTING d-ORDERED SETS FROM ORIENTED MATROIDS

Suppose $\mathcal{C} = (X, \mathcal{C}, P)$ is an oriented matroid, and suppose the matroid $M = (X, \mathcal{C})$ has rank $d+1$. An *admissible sign function* for $\mathcal{C}$ is a function ϕ mapping $(d+1)$-tuples of X to $\{-1, 0, 1\}$ such that.

(i) ϕ is alternating;
(ii) $\phi(\sigma)$ is nonzero if and only if $|\sigma|$ is a basis for M; and

(iii) if C is in $\mathcal{C}$ and σ is a $(d+2)$-tuple with $C \subseteq |\sigma|$ and such that $|\sigma|$ is a spanning set of M, then $P(C) = \{A, B\}$, where

$$A = \{E^i\sigma : \phi(L^i\sigma, R^i\sigma) = (-1)^i\},$$

$$B = \{E^i\sigma : \phi(L^i\sigma, R^i\sigma) = (-1)^{i+1}\}.$$

In the remainder of this section, $\mathcal{C} = (X, \mathcal{C}, P)$ will be an oriented matroid and $(X, \hat{\mathcal{C}}, \hat{P})$ will be its dual. The following theorem is a reformulation of Proposition 3 in [6].

THEOREM 4. *If ϕ is admissible for $(X, \mathcal{C}, P)$, then the function $\hat{\phi}$ (determined by ϕ as in Section 2) is admissible for $(X, \hat{\mathcal{C}}, \hat{P})$.*

Proof. Suppose U is in $\hat{\mathcal{C}}$, and let τ be a d-tuple such that $|\tau|$ is independent in $(X, \mathcal{C})$ and disjoint from U. Considering the discussion preceding Theorem 3, we need only show that elements u and v of U are in the same class of U if and only if $\phi(u, \tau) = \phi(v, \tau)$. Let C be the circuit of M contained in $\{u, v\} \cup |\tau|$. Then u and v are both in C, and $\{u, v\} = C \cap U$. It follows that $\{u, v\}$ must be mixed with respect to C and U. Then u and v are in the same class of U if and only if they are in different classes of C. Since ϕ is admissible for $\mathcal{C}$, (iii) applies with $\sigma = (u, v, \tau)$, and we see that u and v are in different classes of C if any only if $\phi(u, \tau) = \phi(v, \tau)$. ■

THEOREM 5. *If ϕ is admissible for $\mathcal{C}$, then (X, ϕ) is a d-ordered set.*

Proof. Let σ be a d-tuple and τ be a $(d+1)$-tuple of X. Suppose $\phi(s, \sigma)\phi(\tau) = -1$. We must show that there is an i with $0 \leqslant i \leqslant d$ such that $\phi(E^i\tau, \sigma)\phi(L^i\tau, s, R^i\tau) = -1$. Let $C = \{s\} \cup \{E^i\tau : \phi(s, L^i\tau, R^i\tau) \neq 0\}$ and let $U = \{u : \phi(u, \sigma) \neq 0\}$. Then C is the unique circuit of $(X, \mathcal{C})$ contained in the spanning set $\{s\} \cup |\tau|$, and U is the unique circuit of $(X, \hat{\mathcal{C}})$ which misses $|\sigma|$. Furthermore, s is in $C \cap U$, so there must be an element t of $C \cap U$ such that $\{s, t\}$ is mixed with respect to C and U. Then t is in $C \sim \{s\} \subseteq |\tau|$. Let i be such that $E^i\tau = t$. This is the i required. ■

Finally, we show that for any oriented matroid there is an admissible sign function. In fact this was proved already by Las Vergnas in [6]. We include a proof here, for completeness.

THEOREM 6. *If β is a $(d+1)$-tuple such that $|\beta|$ is a basis for M, then $\mathcal{C}$ has exactly one admissible sign function ϕ with $\phi(\beta) = 1$.*

Proof. We prove this by induction on $n = |X|$. The statement is trivially true if $n = d + 1$. Suppose $n > d + 1$ and that the result holds for sets of smaller cardinality.

Suppose b is an element of $X \sim |\beta|$. Let $X_0 = X \sim \{b\}$, let $\mathcal{C}_0 \subseteq \mathcal{C}$ be the set of circuits $C \subseteq X_0$, and let P_0 be the restriction of P to $\mathcal{C}_0$. Then $(X_0, \mathcal{C}_0, P_0)$ is an oriented matroid, a *minor* of $(X, \mathcal{C}, P)$ [2]. Furthermore, $|\beta|$ is a basis for $(X_0, \mathcal{C}_0)$, and $|X_0| < n$. Therefore, there is exactly one admissible sign function ϕ_0 for $(X_0, \mathcal{C}_0, P_0)$ with $\phi_0(\beta) = 1$. It is clear that if ϕ is an admissible sign function for $\mathcal{C}$ with $\phi(\beta) = 1$, then its restriction must be admissible for $(X_0, \mathcal{C}_0, P_0)$, so this restriction must be ϕ_0.

We now define a function ϕ on $(d+1)$-tuples τ from x. If two entries of τ are equal, or if $|\tau|$ is dependent, let $\phi(\tau) = 0$. If $|\tau| \subseteq X_0$, let $\phi(\tau) = \phi_0(\tau)$. Finally, if $|\tau|$ is a basis with $E^i\tau = b$, choose an element x in X_0 with $(|\tau| \sim \{b\}) \cup \{x\}$ independent, and let C be the element of $\mathcal{C}$ contained in $|\tau| \cup \{x\}$. Note that $\{x, b\} \subseteq \mathcal{C}$. Let

$$\phi(\tau) = \begin{cases} \phi_0(L^i\tau, x, R^i\tau) & \text{if } x \text{ and } b \text{ are in different classes of } C, \\ -\phi_0(L^i\tau, x, R^i\tau) & \text{otherwise.} \end{cases} \tag{1}$$

Note that this assignment is dictated by (iii), with $\sigma = (L^i\tau, x, b, R^i\tau)$. It follows that if an admissible sign function for $\mathcal{C}$ exists, it must be ϕ. Note, also, that (1) does not depend on the choice of x, since, if U is the element of $\hat{\mathcal{C}}$ with

$$U = \{y : (|\tau| \sim \{b\}) \cup \{y\} \text{ is independent}\}$$

then $U \cap C = \{x, b\}$, so that $\{x, b\}$ is mixed with respect to U and C, and we may write

$$\phi(\tau) = \begin{cases} \phi_0(L^i\tau, x, R^i\tau) & \text{if } x \text{ and } b \text{ are in the same class of } U, \\ -\phi_0(L^i\tau, x, R^i\tau) & \text{otherwise.} \end{cases} \tag{2}$$

This clearly does not depend on which x in $U \sim \{b\}$ is chosen.

We must show that ϕ is admissible. It is easily verified that ϕ satisfies (i) and (ii). For (iii), suppose σ is a $(d+2)$-tuple such that $|\sigma|$ spans, and suppose that the circuit C is contained in $|\sigma|$. If $|\sigma| \subseteq X_0$, then (iii) holds for σ, since ϕ_0 is admissible.

Suppose that $E^i\sigma = b$ and that $|\sigma| \sim \{b\}$ is independent, so that b is in C. Suppose $x = E^j\sigma$ is another element of C. We must show that x and b are in the same class of C if and only if $(-1)^i\phi(L^i\sigma, R^i\sigma) = (-1)^j\phi(L^j\sigma, R^j\sigma)$. If $i < j$, then, with $\tau = (L^i\sigma, b, R^iL^j\sigma, R^j\sigma)$, the conclusion follows at once from (1); a similar construction yields the result for $i > j$.

Finally, suppose $E^i\sigma = b$ and that $|\sigma| \sim \{b\}$ is dependent, so that $C \subseteq |\sigma| \sim \{b\}$. Let $U = \{x : (|\sigma| \sim \{b\}) \cup \{x\}$ is spanning$\}$. Then U is in $\hat{\mathcal{C}}$. Let x be an element of $U \sim \{b\}$, and let $\sigma' = (L^i\sigma, x, R^i\sigma)$. Then (iii) holds for C and σ', since $|\sigma'| \subseteq X_0$. Using (2), we see that (iii) holds for σ as well. Indeed $\phi(L^j\sigma, R^j\sigma) = \phi(L^j\sigma', R^j\sigma') = 0$ if $j = i$ or if $E^j\sigma = E^j\sigma'$ is not in C; and if $E^j\sigma = E^j\sigma'$ is in C, then the set U for (2) is the set U we have here, and either $\phi(L^j\sigma, R^j\sigma) = \phi(L^j\sigma', R^j\sigma')$ for all j, or $\phi(L^j, R^j\sigma) = -\phi(L^j\sigma', R^j\sigma')$ for all j, depending on whether or not x and b are in the same class of U. ■

IV. RELATED QUESTIONS AND RESULTS

These results can be used to connect questions concerning signs of determinants of submatrices of a matrix to questions about oriented matroids. For instance, the problem of determining which patterns of signs of minors of a matrix can occur is equivalent to that of determining which oriented matroids are "realizable," in the sense of [2]. See Motzkin [9] for some results concerning signs of minors.

If A is a $(d+1)\times(n+1)$ matrix of real numbers, A determines an oriented matroid $(X, \mathcal{C}, P)$, where X is the set of columns of A, $\mathcal{C}$ is the set of minimal linearly dependent subsets of X, and, for $C \in \mathcal{C}$, $P(C) = \{U, V\}$ is the unique partition of C for which the origin is in the convex hull of $U \cup (-V)$. Assuming that A has rank $d+1$, the two admissible sign functions for $(X, \mathcal{C}, P)$ are ϕ and $-\phi$, where, for a $(d+1)$-tuple σ of columns of A, $D(\sigma)$ is the determinant, and $\phi(\sigma) = \operatorname{sgn} D(\sigma)$, as in Section 1.

If A is unimodular, so that $D(\sigma)$ is always 1, 0, or -1, the oriented matroid $(X, \mathcal{C}, P)$ is an *oriented regular matroid*. (We will refer to these as *digraphoids*, abusing terminology slightly. See Minty [8].) In this case the corresponding function ϕ is identical to D, so ϕ satisfies condition (B) of Section 1. Indeed, this condition characterizes the d-ordered sets corresponding to digraphoids, since, given a nonzero alternating function ϕ (or D) on a set X with $|X| = n+1$, satisfying (B), there is a $(d+1)\times(n+1)$ matrix A (unique up to multiplication on the left by a matrix of determinant 1) whose columns may be identified with X in such a way that D is the determinant function. (See [4, Chapter VII].)

This suggests a problem. Fix a set S of real numbers, with $S = -S$. Let $D(S)$ denote the set of d-ordered sets (or oriented matroids, or matroids) arising from matrices A having the property that all of the maximal square submatrices have determinants in S. It is possible to show that the class $D(S)$ is closed under the operations of taking minors and duals.

If $S = \{0, \pm 1\}$, $D(S)$ is the set of digraphoids. The digraphoids can also be characterized in terms of "excluded minors." Indeed, they are the oriented matroids which do not have the "four point line" as the underlying matroid of any minor. (See [1, Section VI], and [11, Theorem 10.2.1]. The *four point line* is the matroid $(X, \mathcal{C})$ with $|X| = 4$ and with $\mathcal{C} = \{C \leqslant X : |C| = 3\}$.)

If $S = \{0, \pm 1, \pm a\}$, where $a > 0$, then if $D(S)$ contains the four point line, the number a must be $\sqrt{2}$, 2, $(1+\sqrt{5})/2$, or the reciprocal of one of these. [This can easily be seen, using (B) of Section 1.] It follows that if a is not one of these, $D(S)$ is again the set of digraphoids.

PROBLEM. *Characterize* $D(\{0, \pm 1, \pm 2\})$, $D(\{0, \pm 1, \pm\sqrt{2}\})$, *or* $D(\{0, \pm 1, \pm(1+\sqrt{5})/2\})$ *among those arising from matrices in terms of excluded minors.*

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On Strictly Positively Invariant Cones

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ABSTRACT

For a matrix $A \in R^{n\times n}$, it is shown that *strict positive invariance* of a proper con $\mathcal{C} \subset R^n$ (that is, $e^{tA}[\mathcal{C}/\{0\}] \subset \operatorname{int}\mathcal{C}\ \forall t > 0$) implies the existence of a certain direc sum decomposition of R^n into A-invariant subspaces. Our results lead to a characteri zation of the set of initial points which give rise to solution curves that reach $\mathcal{C}$, unde the differential equation $\dot{x} = Ax$. Also given is an application in stability theory.

1. INTRODUCTION

Cones which are invariant under matrix exponentials have received a goo deal of attention in recent years (e.g., Varga [6], Schneider and Vidyasaga [3], Elsner [1], and Stern [4]). In the present work we prove that for a matri $A \in R^{n\times n}$, *strict positive invariance* of a proper cone $\mathcal{C} \subset R^n$ (i.e., $e^{tA}[\mathcal{C}/\{0\}$ $\subset \operatorname{int}\mathcal{C}\ \forall t > 0$) leads to a specific direct-sum decomposition of R^n int A-invariant subspaces. Our results are applied to the characterization o certain asymptotic stability properties of strictly positively invariant prope cones. While the present paper is essentially self-contained, it has in commor with the work in [4] the feature that the main results are obtained via a qualitative differential-equations approach.

We begin in Section 2 by deriving relevant properties of $X_t(\mathcal{C}) := e^{-tA}\mathcal{C}$, the set of initial points which reach $\mathcal{C}$ in time $t \geqslant 0$ under the linear differential equation $\dot{x}(t) = Ax(t)$, and of the set $X(\mathcal{C}) := \cup_{t\geqslant 0} X_t(\mathcal{C})$, unde invariance conditions on $\mathcal{C}$. Then in Section 3 we give our central result o the A-invariant decomposition of R^n, and use it to characterize the set of initial points whose solution curves reach $\mathcal{C}$ under the differential equatio given. Section 4 consists of an application in stability theory.

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