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TRIANGULATING THE *d*-CUBE

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INTRODUCTION

Let $I = [0, 1]$ be the unit interval. By a *triangulation* of the d-cube we mean a triangulation of I^d into (closed) simplices whose vertices are vertices of the cube itself. Mara [4, 5] described a triangulation of the 4-cube having 16 4-simplices and a triangulation of the 5-cube having 68 5-simplices, conjecturing that these triangulations are minimal and suggesting a formula for the minimum number of d-simplices in any triangulation of the d-cube. Sallee [9] disproved Mara's conjecture by exhibiting a new triangulation of I^d having fewer d-simplices than Mara's conjectured formula for $d \geq 5$. Both Sallee [8, 9] and Cottle [1] have demonstrated that Mara's triangulation of the 4-cube is minimal. In this paper we give an alternate proof of the minimality of Mara's 4-cube triangulation, rederive Sallee's method of triangulation by showing its relationship to pulling vertices of a polytope, and briefly mention another technique of triangulating polytopes.

1. MINIMAL TRIANGULATIONS OF THE 4-CUBE

We first give an alternate proof of the fact that any triangulation of I^4 requires at least 16 4-simplices.

Let Δ be any finite collection of simplices. For $F \in \Delta$ we call F a *face* of Δ and write dim F for the dimension of F. If dim $F = j$ we will call F a *j-face* of Δ. Faces of Δ of dimension 0 and 1 will be called *vertices* and *edges* of Δ, respectively. Taking $d = \max\{\dim F\colon F \in \Delta\}$, define

$$f(\Delta) = (f_{-1}(\Delta), f_0(\Delta), \ldots, f_d(\Delta)),$$

where

$$f_j(\Delta) = \text{card}\,\{F \in \Delta\colon \dim F = j\}, \qquad -1 \leq j \leq d$$

(regarding the empty set as a simplex of dimension -1), and

$$h(\Delta) = (h_0(\Delta), h_1(\Delta), \ldots, h_{d+1}(\Delta)),$$

where

$$h_i(\Delta) = \sum_{j=0}^{i} (-1)^{i-j} \binom{d-j+1}{d-i+1} f_{j-1}(\Delta), \qquad 0 \leq i \leq d+1$$

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Set $h_i(\Delta) = 0$ if $i < 0$ or $i > d + 1$. It is known [6] that $f(\Delta)$ can be recovered from $h(\Delta)$ using the relations

$$f_j(\Delta) = \sum_{i=0}^{j+1} \binom{d-i+1}{d-j} h_i(\Delta), \qquad -1 \le j \le d.$$

Proposition 1.1. *Suppose Δ is a triangulation of the d-cube, $\partial\Delta$ is the induced triangulation of ∂I^d, and Δ° is the set of those faces of Δ that are not in $\partial\Delta$. Then (with the convention that $\phi \in \partial\Delta$):*

(i) $h_i(\partial\Delta) = h_{d-i}(\partial\Delta)$, $0 \le i \le d$,
(ii) $h_i(\Delta^\circ) = h_{d-i+1}(\Delta)$, $0 \le i \le d + 1$,
(iii) $h_i(\partial\Delta) - h_{i-1}(\partial\Delta) = h_i(\Delta) - h_{d-i+1}(\Delta)$, $0 \le i \le d + 1$.

Proof. See [6]. ∎

We remark that Proposition 1.1 holds more generally for all simplicial complexes Δ that are triangulations of homology d-balls.

Example. Let Δ be the standard triangulation of the 4-cube having $4! = 24$ 4-simplices [5, 11]. Then,

$$f(\Delta) = (1, 16, 65, 110, 84, 24), \qquad h(\Delta) = (1, 11, 11, 1, 0, 0),$$
$$f(\Delta^\circ) = (0, 0, 1, 14, 36, 24), \qquad h(\Delta^\circ) = (0, 0, 1, 11, 11, 1),$$
$$f(\partial\Delta) = (1, 16, 64, 96, 48), \qquad h(\partial\Delta) = (1, 12, 22, 12, 1).$$

For any two 0-1 vectors v, w of the same length, we define $d(v, w)$ to be i if exactly i of the coordinates of v and w are different. For example, if $v = (0, 1, 0, 0)$ and $w = (0, 0, 1, 1)$, then $d(v, w) = 3$. Further, if $d(v, w) = 1$, we will say that v and w are *neighbors*.

Let Δ be a triangulation of I^4, and v, w be vertices of I^4. An edge of Δ joining v and w will be said to be of *type i* if $d(v, w) = i$.

Lemma 1.2. *Suppose Δ is a triangulation of I^4. Let F be a 4-simplex of Δ and $V(F)$ be the set of the five vertices of I^4 contained in F. Assume that $d(v, w) \le 2$ for all v, $w \in F$. Then $V(F)$ consists of a vertex of I^4 together with its four neighbors in I^4.*

Proof. Assume $d(v, w) = 2$ for all v, $w \in V(F)$. Then all of the vertices of F must be of the same parity. But then card $V(F) = 5$ forces $V(F)$ to contain an antipodal pair (v, w) of vertices, i.e., such that $d(v, w) = 4$, contradicting our assumption. Thus there must exist $u^1, u^2, u^3 \in V(F)$ such that $d(u^1, u^2) = 1$ and $d(u^2, u^3) = 2$. Without loss of generality, let us say

$$u^1 = (0, 0, 0, 0),$$
$$u^2 = (1, 0, 0, 0),$$
$$u^3 = (0, 1, 0, 0).$$

Now there must be a $u \in V(F)$ of the form $(*, *, 1, *)$. Without loss of generality, we may say

$$u^4 = (0, 0, 1, 0).$$

Again, there must be a $u \in V(F)$ of the form $(*, *, *, 1)$, necessarily

$$u^5 = (0, 0, 0, 1).$$

Thus, $V(F)$ consists of the vertex $(0, 0, 0, 0)$ together with its four neighbors. ∎

Lemma 1.3. *Suppose Δ is a triangulation of I^4. Then Δ must contain at least one type 4 edge or three type 3 edges.*

Proof. The 4-simplices of Δ cannot include more than eight 4-simplices of the kind described in Lemma 1.2, since each such simplex contains exactly 4 of the 32 original edges of I^4, and in no triangulation of I^4 may any of these original edges be contained in more than one such 4-simplex. Removing from Δ all such 4-simplices yields Σ, a simplicial complex with at least eight vertices. Every 4-simplex of Σ must contain at least one type 3 edge or one type 4 edge. If any 4-simplex of Σ contains a type 4 edge we are done, so assume that Σ has no type 4 edge. Let F be any 4-simplex of Σ. Without loss of generality, suppose $u^1, u^2 \in V(F)$, where

$$u^1 = (0, 0, 0, 0),$$
$$u^2 = (1, 1, 1, 0).$$

Now there must be a $u \in V(F)$ of the form $(*, *, *, 1)$, and it will be impossible to satisfy both $d(u, u^1) \le 2$ and $d(u, u^2) \le 2$. Thus we may, without loss of generality, take $u^3 = (1, 1, 0, 1)$, leaving $(0, 1, 0, 0)$, $(1, 0, 0, 0)$, and $(1, 1, 0, 0)$ as candidates for the remaining two vertices of F, assuming that F has no other type 3 edges. If (u^1, u^2) and (u^1, u^3) are the only two type 3 edges in Σ, then all 4-simplices of Σ must contain u^1, u^2, and u^3, forcing Σ to have at most six vertices, which is a contradiction. Therefore Δ must have at least three edges of type 3. ∎

Theorem 1.4. *Suppose Δ is a triangulation of I^4. Then $f_4(\Delta) \ge 16$.*

Proof. First note that

$$f_{-1}(\partial\Delta) = 1 = f_{-1}(\Delta),$$
$$f_0(\partial\Delta) = 16 = f_0(\Delta),$$
$$f_{-1}(\Delta^\circ) = 0 = f_0(\Delta^\circ).$$

Now I^4 contains 32 edges (which are type 1 edges in Δ), and each of the 24 squares (2-faces) of I^4 must be triangulated by adding a diagonal (type 2 edge). Thus,

$$f_1(\partial\Delta) \ge 56.$$

We may then use Proposition 1.1 to conclude

$$f(\partial\Delta) = (1, 16, a, *, *),$$

where $f_1(\partial\Delta) = a \ge 56$,

$$h(\partial\Delta) = (1, 12, b, 12, 1),$$

where $b = a - 42$,

$$h(\Delta) = (1, 11, c, d, 0, 0),$$

where $d = f_1(\Delta^\circ) \geq 0$ and $c - d = b - 12$, and

$$\begin{aligned} f_4(\Delta) &= \sum_{i=0}^{5} h_i(\Delta) \\ &= 12 + c + d \\ &= b + 2d \\ &= f_1(\partial\Delta) + 2f_1(\Delta^\circ) - 42. \end{aligned}$$

If Δ contains three type 3 edges, then $f_1(\partial\Delta) \geq 59$ and hence $f_4(\Delta) > 17$. If, on the other hand, Δ contains a type 4 edge, then $f_1(\Delta^\circ) \geq 1$ and $f_4(\Delta) \geq 16$. ▮

COROLLARY 1.5. *Every minimal triangulation of I^4 must contain a type 4 edge (i.e., an edge joining an antipodal pair of vertices of the 4-cube).*

We have as a consequence of arguments in [1] and the above corollary that there is no triangulation of the 4-cube having exactly nine 4-simplices of volume 1/24, six 4-simplices of volume 2/24, and one 4-simplex of volume 3/24.

2. TRIANGULATING ARBITRARY POLYTOPES

When a d-polytope P is a facet (d-face) of some $(d + 1)$-polytope Q, making Q simplicial by *pulling its vertices* [2, Section 5.2] will induce a triangulation of P. It is not necessary, however, to regard P as a facet of any $(d + 1)$-polytope to describe this triangulation of P directly: Let $V(P) = \{v^0, v^1, \ldots, v^n\}$ be the set of vertices of P. We obtain successive subdivisions $S_{-1}, S_0, \ldots, S_n$ of P by letting S_{-1} be P itself, and for $0 \leq k \leq n$, $0 \leq j \leq d$, letting the j-faces in S_k be

(1) the j-faces in S_{k-1} which do not contain v^k, and

(2) the j-faces of the form conv $(\{v^k\} \cup F)$, where F is a $(j - 1)$-face not containing v^k of a d-face in S_{k-1} which does contain v^k.

We denote S_n by $S(P)$, or by $S(P; v^0, v^1, \ldots, v^n)$ when we want to acknowledge the dependence of the subdivision upon the particular ordering of the vertices of P explicitly. Exactly as in the case of pulling vertices of a polytope, every j-face of $S(P)$ is a j-simplex, $0 \leq j \leq d$.

PROPOSITION 2.1. *Every d-simplex in $S(P;\ v^0, \ldots, v^n)$ is of the form conv $(\{v^0\} \cup G)$, where G is a $(d - 1)$-simplex in $S(F; v^{i_0}, \ldots, v^{i_m})$, for some $(d - 1)$-face F of P that does not contain v^0.*

Proof. It is immediate from the definition that for every j-face F of P, the triangulation of F induced by $S(P; v^0, \ldots, v^n)$ is precisely $S(F; v^{i_0}, \ldots, v^{i_m})$, regarding F itself as a j-polytope with vertex set $V(F) = \{v^{i_0}, \ldots, v^{i_m}\} \subset V(P)$, $i_0 < i_1 < \cdots < i_m$. Now, every d-face in S_0 is a pyramid with apex v^0 over some $(d - 1)$-face of P. Also, every d-dimensional face in any subdivision of such a pyramid must contain the apex of the pyramid. Therefore, every d-simplex in S_n contains v^0. ▮

For every j-face of P, $0 \leq j \leq d$, let $v(F)$ denote the vertex of P of smallest index that is in F. Let $P = F_d \supset F_{d-1} \supset \cdots \supset F_1 \supset F_0$ be a chain of faces of P such that F_j is a j-face of P and $v(F_j) \neq v(F_{j+1})$, $0 \leq j \leq d - 1$. Let $\mathscr{C}$ be the set of all such chains.

PROPOSITION 2.2. *W is the set of vertices of some d-simplex in $S(P; v^0, \ldots, v^n)$ if and only if $W = \{v(F_d), v(F_{d-1}), \ldots, v(F_0)\}$ for some chain $F_d \supset F_{d-1} \supset \cdots \supset F_0$ in $\mathscr{C}$.*

Proof. Use PROPOSITION 2.1 and induction on $d = \dim P$. ▮

PROPOSITION 2.2 implies that $S(P)$ is the same as the *full-flag triangulation* of P discussed by both Stanley [10] and Sallee, and its equivalence with the triangulation of Von Hohenbalken [12] is clear.

EXAMPLE. Consider the case $P = I^d$ with the vertices of the d-cube indexed according to the binary representation of the index; e.g., if $d = 4$, then $v^{11} = (1, 0, 1, 1)$. It is an easy exercise to see that $S(I^d)$ is the standard triangulation of the d-cube into $d!$ d-simplices [5, 11]. In fact, the number of d-simplices obtained is independent of the order in which the vertices of the d-cube are pulled.

3. SALLEE'S TRIANGULATION OF THE d-CUBE

We now turn to the d-cube and describe a triangulation due to Sallee and independently studied by the author.

For any 0-1 vector $v = (v_1, v_2, \ldots, v_d)$ we define the *parity* of v to be *odd* if v has an odd number of components equal to 1 and *even* if v has an even number of components equal to 1. Choose some $u \in V(I^d)$. The simplex whose vertex set is $\{u\} \cup \{v \in V(I^d)\colon v \text{ is a neighbor of } u\}$ will be called a simplex of *type I* with *apex* u. Let $d \geq 3$. We label the vertices of the d-cube according to the binary representation of the index exactly as in the example of Section 2. We begin our triangulation of I^d by "slicing off" from the d-cube each of the type I simplices with odd parity apex, leaving a d-polytope which we will denote Q_d; i.e., $Q_d = \text{conv}\ \{v\colon v \text{ is an even vertex of } I^d\}$. (If $d = 3$, then Q_d is itself a 3-simplex.) Maintaining the original labeling of the vertices, we now obtain the triangulation $S(Q_d)$ of Q_d. The d-simplices of $S(Q_d)$ will be said to be of *type II*. Together the odd apex type I d-simplices and the type II d-simplices in $S(Q_d)$ constitute the d-simplices of a triangulation T_d of I^d.

Note that every $(d - 1)$-face of Q_d is either (a) the convex hull of the set of neighbors of some odd vertex of I^d, or (b) congruent to Q_{d-1}, case (b) being possible only if $d \geq 4$.

Let $T(d)$ be the number of d-simplices in T_d and $\tilde{T}(d)$ be the number of d-simplices in $S(Q_d)$. Because T_d contains 2^{d-1} d-simplices of type I, it is clear that $T(d) = \tilde{T}(d) + 2^{d-1}$. Observe that $T(3) = 5$. Using PROPOSITION 2.1 and the above facts, we can determine a recursive formula for $T(d)$. Suppose $d \leq 4$. To begin, we have 2^{d-1} d-simplices of type I. Now Q_d has 2^{d-1} $(d - 1)$-faces of type (a), each of which is a $(d - 1)$-simplex, d of which contain v^0. This gives us $2^{d-1} - d$ d-simplices of $S(Q_d)$. On the other hand, there are $2d$ $(d - 1)$-faces of type (b) of Q_d, half of which contain v^0. This gives us the remaining $d\tilde{T}(d - 1)$ d-simplices of $S(Q_d)$. Therefore $T(d) = 2^{d-1} + 2^{d-1} - d + d\tilde{T}(d - 1)$, implying:

THEOREM 3.1.

(i) $T(d) = dT(d - 1) - d2^{d-2} + 2^d - d$,

(ii) $T(d) = d!\,p_d(2)/2 - d!\,p_d(1) + 2^{d-1} - d!/2 + 1$,

where $p_d(x) = \sum_{i=0}^{d} (x^i/i!)$.

Proof. Part (i) is immediate from the preceding comments, and part (ii) follows by induction on $d \geq 3$. ∎

Mara [5] conjectured that any triangulation of the d-cube has at least $M(d) = (2^{d-1} + d!)/2$ d-simplices, having constructed triangulations for $d = 4$ and $d = 5$ with these numbers of d-simplices; however, $T(d)$ is strictly smaller than $M(d)$ for $d \geq 5$, a fact that can be verified, for example, by comparing THEOREM 3.1(i) with the recursive formula $M(d) = dM(d-1) + 2^{d-2} - d2^{d-3}$. In particular, $T(5) = 67$. Todd [11] suggests that one comparison of a triangulation of the d-cube with the standard $d!$ triangulation (with respect to pivoting algorithms) is the extent to which the quantity $R(d) = (T(d)/d!)^{1/d}$ is less than 1. From THEOREM 3.1(ii), however, we see that $\lim_{d\to\infty} T(d)/d! = e^2/2 - e - \frac{1}{2} \simeq 0.476$ and so $\lim_{d\to\infty} R(d) = 1$.

Below are the values of $T(d)$ and $R(d)$ for $3 \leq d \leq 10$.

d	$d!$	$T(d)$	$R(d)$
3	6	5	0.941
4	24	16	0.904
5	120	67	0.890
6	720	364	0.893
7	5,040	2,445	0.902
8	40,320	19,296	0.912
9	362,880	173,015	0.921
10	3,628,800	1,728,604	0.929

An explicit description of the vertex sets of the d-simplices in T_d and a procedure for pivoting from simplex to simplex are informally discussed in [3].

4. ANOTHER TRIANGULATION METHOD: "PLACING" VERTICES

Another way to triangulate polytopes is described by Munson [7] in a more general context. The vertices of the polytope are successively "placed" into position, and at each stage the convex hull of the currently placed vertices is provided with a triangulation.

Again, let $V(P) = \{v^0, v^1, \ldots, v^n\}$. We obtain a sequence of polytopes $P_0, P_1, \ldots, P_n = P$ with respective dimensions $d_0, d_1, \ldots, d_n$ and respective triangulations $S'_0, S'_1, \ldots, S'_n$ as follows: Let $P_0 = \{v^0\}$, itself a simplex of dimension 0. For $0 < k \leq n$, let $P_k = \text{conv}(P_{k-1} \cup \{v^k\})$ and $d_k = \dim P_k$. If $d_k > d_{k-1}$ (i.e., $d_k = d_{k-1} + 1$), then for $0 \leq j \leq d_k$ the j-faces in S'_k are

(1) the j-faces in S'_{k-1}, and

(2) the j-faces of the form conv $(F \cup \{v^k\})$, where F is a $(j-1)$-face of S'_{k-1}.

If, on the other hand, we have $d_k = d_{k-1}$, then for $0 \leq j \leq d_k$ the j-faces in S'_k are

(1) the j-faces in S'_{k-1}, and

(2) the j-faces of the form conv $(F \cup \{v^k\})$, where F is a $(j-1)$-face of S'_{k-1} that is contained in a facet of P_{k-1} that v^k is "beyond" [relative to aff (P_{k-1})] [2, Section 5.2].

Intuitively, as each new vertex v^k is "placed," the triangulation S'_{k-1} is extended by constructing a pyramid with apex v^k and base F for every face $F \in S'_{k-1}$ that is "visible" from v^k. As with pulling vertices there is an explicit dependence upon the order in which the vertices are placed so we may sometimes wish to write $S'(P) = S'(P; v^0, \ldots, v^n)$. S' shares with S the property that for every j-face F of P, the triangulation of F induced by $S'(P; v^0, \ldots, v^n)$ is $S'(F; v^{i_0}, \ldots, v^{i_m})$, regarding F as a j-polytope with vertices $\{v^{i_0}, \ldots, v^{i_m}\}$, $i_0 < i_1 < \cdots < i_m$. Unlike S, however, the number of d-simplices in $S'(I^d)$ changes with the order in which the vertices of the d-cube are placed. For example, if the even parity vertices of the 3-cube are placed before the odd parity vertices, a minimal triangulation is obtained. Similarly, minimal triangulations of I^4 can be obtained directly by an appropriate placement order, without recourse to "slicing off" the odd parity vertices first. Whether the same holds for higher dimensions remains unexplored.

5. REMARKS

That T_d is not a minimal triangulation of the d-cube in general has been shown by Sallee [9] with his triangulation of I^6 requiring 344 simplices. The nonminimality of T_d is not surprising in view of the fact that the only aspect of the triangulation that specifically depends upon the special structure of the cube is the "slicing off" of all of the odd parity vertices of I^d. The triangulation $S(P)$ can be applied to any d-polytope P, but of course the combinatorial properties of the d-cube enable us to use induction and easily compute the number of and describe the vertex sets of the d-simplices of T_d. So the size of a minimum triangulation of the d-cube remains unknown.

A more modest question to ask at this point is whether or not the 5-cube can be triangulated into fewer than 67 5-simplices. Using techniques similar to those of Section 1, one can see that it would be sufficient to prove that any minimal triangulation of I^5 must contain in its interior either no edge and at least 15 triangles (2-faces), or else 1 edge and at least 17 triangles.

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