GENERALIZED STRESS AND MOTIONS

In Memory of Paul Filliman

CARL W. LEE*
Department of Mathematics
University of Kentucky
Lexington, KY 40506, U.S.A.

Abstract. In 1987 Kalai presented a new proof of the Lower Bound Theorem for simplicial convex d-polytopes by linking the problem to results in rigidity and stress. He suggested that if higher-dimensional analogues of stress and rigidity were developed, they might lead to other combinatorial results on polytopes, and in particular another proof of the g-Theorem. Here we discuss such a generalization of stress and its relationship to face rings, h-vectors, shellings, bistellar operations, spheres, and simplicial polytopes. In particular, stress plays a role in McMullen's recent new geometric proof of the g-Theorem using his polytope algebra.

Key words: bistellar operations, convex polytope, face ring, h-vector, infinitesimal motion, p.l.-sphere, rigidity, shelling, stress.

1. Introduction

In 1987 Kalai [8] presented a new proof of the Lower Bound Theorem for simplicial convex d-polytopes by linking the problem to results in rigidity and stress. He suggested that if higher-dimensional analogues of stress and rigidity were developed, they might lead to other combinatorial results on polytopes, and in particular another proof of the g-Theorem. A proposal for such a generalization of stress was introduced in [10]. Here we provide details, discussing the relationship to face rings, h-vectors, shellings, bistellar operations, spheres, and simplicial polytopes. In particular, stress plays a role in McMullen's [11] recent new geometric proof of the g-Theorem using his polytope algebra.

2. Infinitesimal Rigidity and Stress

We first offer some background on rigidity and stress. See, for example, Kalai [8], Roth [14], and Whiteley [19] for more details and references. Begin by considering a graph $G = (V, E)$, where $V = \{1, \ldots, n\}$. Suppose that we make a structure by choosing a point $v_i \in \mathbf{R}^d$ for each vertex of the graph, and placing bars connecting

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pairs of points corresponding to edges. An *infinitesimal motion* of the vertices is a set of vectors $\overline{v}_1, \ldots, \overline{v}_n \in \mathbf{R}^d$ such that $d(\|(v_i + t\overline{v}_i) - (v_j + t\overline{v}_j)\|^2)/dt = 0$ when $t = 0$ for all bars v_iv_j. Equivalently, $(v_i - v_j)^T(\overline{v}_i - \overline{v}_j) = 0$ for all edges, or the projections of $\overline{v}_i$ and $\overline{v}_j$ onto the affine span of $\{v_i, v_j\}$ agree. For example, we could choose a single vector $u \in \mathbf{R}^d$ and set $\overline{v}_i = u$ for all vertices v_i. This would be a trivial motion in the sense that it could be extended to all of $\mathbf{R}^d$. That is to say, we can define an *infinitesimal motion of* $\mathbf{R}^d$ to be a choice of vector $\overline{v}$ for each point $v \in \mathbf{R}^d$ such that $(v - w)^T(\overline{v} - \overline{w}) = 0$ for all pairs v, w of points. Then we say that an infinitesimal motion of a structure is *trivial* if it is the restriction of an infinitesimal motion of $\mathbf{R}^d$. If a structure admits only trivial infinitesimal motions, we say it is ***infinitesimally rigid***.

Now an infinitesimal motion of $\mathbf{R}^d$ is uniquely determined by its restriction to the vertices of any geometric $(d-1)$-simplex, and conversely, any infinitesimal motion of a structure consisting of the vertices and edges of a geometric $(d-1)$-simplex can be extended to $\mathbf{R}^d$. So a geometric $(d-1)$-simplex is infinitesimally rigid, and it is not hard to see that the dimension of the space of infinitesimal motions of such a simplex is $\binom{d+1}{2}$. Hence we conclude that this is also the dimension of the space of trivial infinitesimal motions. Thus a structure is infinitesimally rigid if and only if the dimension of its infinitesimal motion space is $\binom{d+1}{2}$.

The fact that a motion of an infinitesimally rigid structure is determined by the motions on d affinely independent vertices allows us to conclude that the union of two infinitesimally rigid structures in $\mathbf{R}^d$ sharing d affinely independent vertices is infinitesimally rigid.

The space of infinitesimally rigid motions of a structure is the nullspace of a certain ***rigidity matrix*** R. The rows of R are indexed by the edges v_iv_j, and the columns of R occur in n groups of d columns, one group for each vertex of R. The row vector of length d in row v_iv_j, group v_k, will be

$$\begin{cases} O^T & \text{if } k \neq i, j, \\ (v_i - v_j)^T & \text{if } k = i, \\ (v_j - v_i)^T & \text{if } k = j. \end{cases}$$

So a structure is infinitesimally rigid if and only if the dimension of the nullspace of R is $\binom{d+1}{2}$.

It is also useful to consider the left nullspace of R, elements of which are assignment of numbers λ_{ij} to edges v_iv_j such that

$$\sum_{\{j: v_iv_j \in E\}} \lambda_{ij}(v_j - v_i) = O$$

holds for every vertex v_i. Such a vector of numbers is called a *stress*, and the vector space of all stresses of a structure is its *stress space*.

Dehn [2] proved the following:

Theorem 1 (Dehn) *The edge skeleton of a simplicial convex 3-polytope P is infinitesimally rigid.*

This is proved by first showing:

Theorem 2 (Dehn) *A simplicial convex 3-polytope P admits only the trivial stress in which all* $\lambda_{ij} = 0$.

PROOF. The proof we give here is a slight modification of that of Roth [14], which in turn uses some techniques of Cauchy. Suppose there is a non-trivial stress. Label each edge $v_iv_j \in E$ with the sign $(+, -, 0)$ of λ_{ij}. Suppose there is a vertex v such that all edges incident to it are labeled 0. Then delete v and take the convex hull of the remaining vertices. The resulting polytope cannot be two-dimensional, because it is clear that there can be no non-trivial stress on the edges of a single polygon. So the polytope is three-dimensional. If it is not simplicial, triangulate the non-triangular faces arbitrarily, labeling the new edges 0. Repeat this procedure until you have a simplicial 3-polytope Q (possibly with some coplanar faces) such that every vertex is incident to at least one nonzero edge. Note that every nonzero edge of Q is an edge of the original polytope P.

Now in each corner of each face (which is a triangle) of Q place the label 0 if the two edges meeting there are of the same sign, 1 if they are of opposite sign, and 1/2 if one is zero and the other nonzero.

Claim 1. The sum of the corner labels at each vertex v is at least four. First, because v is a vertex of P, the nonzero edges of P incident to v cannot all have the same sign. Consider now the cyclic changes in signs of just the nonzero edges of P incident to v. If there were only two changes in sign, the positive edges could be separated from the negative edges by a plane passing through v, since no three edges incident to v in P are coplanar. So there must be at least four changes in sign. The claim for the corner labels in Q now follows easily.

Claim 2. the sum of the three corner labels for each face is at most two. Just check all the possibilities of the edge and corner labels for a single triangle.

Now consider the sum S of all the corner labels of Q. By Claim 1 the sum is at least $4f_0$, where f_0 is the number of vertices of Q. By Claim 2 the sum is at most $2f_2$, where f_2 is the number of faces of Q. But Euler's relation and $3f_2 = 2f_1$ imply that $f_2 = 2f_0 - 4$. So $4f_0 \leq S \leq 4f_0 - 8$ yields a contradiction. □

PROOF OF THEOREM 1. Because P is simplicial, $f_1 = 3f_0 - 6$. So R has $f_1 = 3f_0 - 6$ rows and $3f_0$ columns. We need to show that the dimension of the nullspace of R is six, so we need to show that R has full row rank. But this is equivalent to there being no nontrivial stresses, which we have done. □

Whiteley [19] extended Theorem 1 to arbitrary $d > 3$:

Theorem 3 (Whiteley) *For* $d > 3$, *the edge skeleton of a simplicial convex d-polytope P is infinitesimally rigid.*

Using induction on d, he explained why the edge skeleton of clstar v, the closed star of v, is infinitesimally rigid for each vertex v of P. Then the rigidity of the entire edge skeleton of P results from the fact that the closed stars of two adjacent vertices share a $(d-1)$-simplex.

Regarding the matrix R for an arbitrary simplicial convex d-polytope, $d \geq 3$, the dimension of its nullspace is $\binom{d+1}{2}$. Hence its rank is $df_0 - \binom{d+1}{2}$. So the dimension of the stress space is $f_1 - df_0 + \binom{d+1}{2}$. In particular, this integer, usually now denoted $g_2(P)$ or g_2, is nonnegative. The Lower Bound Theorem follows from the nonnegativity of g_2, however, and this is Kalai's [8] striking proof. In fact, Kalai used this method to prove:

Theorem 4 *For all convex d-polytopes P, $d \geq 3$,*

$$f_{02} - 3f_2 + f_1 - df_0 + \binom{d+1}{2} \geq 0,$$

where f_{02} is the number of incidences of vertices with 2-faces.

3. McMullen's Conditions

On the other hand, the nonnegativity of g_2 for simplicial convex d-polytopes ($d \geq 3$) is a consequence of McMullen's conditions [17], which we will describe in this section.

Let Δ be a simplicial $(d-1)$-complex on the set $\{1,\ldots,n\}$ (its *vertices*). The *f-vector* of Δ is $f = (f_0, f_1, \ldots, f_{d-1})$, where f_j is the number of faces of Δ of dimension j (cardinality $j+1$). Taking $f_{-1} = 1$, the *h-vector* of Δ is $h = (h_0, \ldots, h_d)$ where

$$h_k = \sum_{j=0}^{k} (-1)^{j-k} \binom{d-j}{d-k} f_{j-1}, \quad k = 0, \ldots, d. \tag{2}$$

These relations are invertible:

$$f_j = \sum_{k=0}^{j+1} \binom{d-k}{d-j-1} h_k, \quad j = -1, \ldots, d-1.$$

Define also $g_0 = h_0 = 1$ and $g_k = h_k - h_{k-1}$, $k = 1, \ldots, \lfloor d/2 \rfloor$.

The *Stanley-Reisner ring* or *face ring* of Δ over $\mathbf{R}$ is $A = \mathbf{R}[x_1, \ldots, x_n]/I_\Delta$, where I_Δ is the ideal generated by all square-free monomials $x_{i_1} \cdots x_{i_s}$ such that $\{i_1, \ldots, i_s\} \notin \Delta$. The ring A inherits the grading by degree, $A = A_0 \oplus A_1 \oplus A_2 \oplus \cdots$. Stanley [15, 16] observed:

Theorem 5 (Stanley) *Let A be the face ring of a simplicial $(d-1)$-complex Δ. Then A is Cohen-Macaulay if and only if there exist $\theta_1, \ldots, \theta_d \in A_1$ such that $\dim B_k = h_k$, $k = 0, \ldots, d$, where $B = B_0 \oplus \cdots \oplus B_d = A/(\theta_1, \ldots, \theta_d)$. In this case the θ_j can be chosen generically (that is, with algebraically independent coefficients over $\mathbf{Q}$).*

If the above situation holds we say that Δ is *Cohen-Macaulay*. Reisner [13] derived a homological characterization of Cohen-Macaulay complexes. In particular, shellable simplicial complexes and simplicial balls and spheres (and hence boundary complexes of simplicial polytopes) are Cohen-Macaulay. To see the effect of this condition on the h-vector, we need another definition. For positive integers a and k, a can be expressed uniquely in the form

$$a = \binom{a_k}{k} + \binom{a_{k-1}}{k-1} + \cdots + \binom{a_\ell}{\ell},$$

where $a_k > a_{k-1} > \cdots > a_\ell \geq \ell \geq 1$. Using this, set

$$a^{\langle k \rangle} = \binom{a_k+1}{k+1} + \binom{a_{k-1}+1}{k} + \cdots + \binom{a_\ell+1}{\ell+1}.$$

Define also $0^{\langle k \rangle} = 0$. Stanley [15] proved:

Theorem 6 (Stanley) *Let Δ be a simplicial $(d-1)$-complex. If Δ is Cohen-Macaulay, then the h-vector is nonnegative and $h_{k+1} \leq h_k^{\langle k \rangle}$, $k = 1, \ldots, d-1$.*

Using a connection between the face ring of a simplicial convex polytope and the cohomology of an associated projective toric variety, Stanley [17] showed that the Hard Lefschetz Theorem implies:

Theorem 7 (Stanley) *Suppose that Δ is the boundary complex of a simplicial d-polytope, and that A is its face ring. Then, for some choice of $\theta_1, \ldots, \theta_d \in A_1$, there exists $\omega \in B_1$ such that multiplication by ω^{d-2k} is a bijection between B_k and B_{d-k}, $k = 0, \ldots, \lfloor d/2 \rfloor$, where $B = B_0 \oplus \cdots \oplus B_d = A/(\theta_1, \ldots, \theta_d)$. In particular, multiplication by ω is an injection from B_k into B_{k+1}, $k = 0, \ldots, \lfloor d/2 \rfloor - 1$. As a consequence, $g_k = \dim C_k$, $k = 0, \ldots, \lfloor d/2 \rfloor$, where $C = C_0 \oplus \cdots \oplus C_{\lfloor d/2 \rfloor} = B/(\omega)$. Hence*

1. $h_k = h_{d-k}$, $k = 0, \ldots, d$ (the Dehn-Sommerville Relations*),*
2. $g_k \geq 0$, $k = 0, \ldots, \lfloor d/2 \rfloor$ (the Generalized Lower-Bound Inequalities*), and*
3. $g_{k+1} \leq g_k^{\langle k \rangle}$, $k = 1, \ldots, \lfloor d/2 \rfloor - 1$.

The above three conditions are *McMullen's conditions* and characterize h-vectors of simplicial convex d-polytopes (the sufficiency was established by Billera and Lee [1]). This characterization is also known as the *g-Theorem*.

4. k-Stress

In this section we offer a generalization to the classical stress space of Section 2 that is motivated by the Stanley-Reisner ring. The original idea arose when contemplating Kalai's algebraic shifting technique [7].

First we give some notation. For $x = (x_1, \ldots, x_n)$, and for $r = (r_1, \ldots, r_n) \in \mathbf{Z}_+^n$, by x^r we mean $x_1^{r_1} \cdots x_n^{r_n}$. Define also $\operatorname{supp} x^r = \{i : r_i \neq 0\}$ (the *support* of x^r), $r! = r_1! \cdots r_n!$, and $|r| = r_1 + \cdots + r_n$.

Let Δ be a simplicial complex (not necessarily of dimension $d-1$) on the set $\{1, \ldots, n\}$, and let $v_1, \ldots, v_n \in \mathbf{R}^d$. Define M to be the $d \times n$ matrix with columns $v_1, \ldots, v_n$, and $\overline{M}$ to be the $(d+1) \times n$ matrix obtained from M by appending a final row of 1's.

For each $k = 0, 1, 2, \ldots$, a *linear k-stress* on Δ (with respect to $v_1, \ldots, v_n$) is a polynomial of the form

$$b(x) = \sum_{r:|r|=k} b_r \frac{x^r}{r!}$$

that satisfies

$$b_r = 0 \text{ if } \operatorname{supp} x^r \notin \Delta, \tag{3}$$

and

$$M\nabla b = O. \tag{4}$$

This last condition is equivalent to

$$\sum_{i=1}^{n}(\frac{\partial b}{\partial x_i})v_i = O,$$

where the left-hand side is to be regarded as a polynomial with vector coefficients, or

$$\sum_{i=1}^{n}(\frac{\partial b}{\partial x_i})v_{ij} = 0,\ j = 1,\ldots,d,$$

where $v_i = (v_{i1},\ldots,v_{id})^T$, or

$$\sum_{i=1}^{n} b_{s+e_i}v_i = O \tag{5}$$

for every $s \in \mathbf{Z}_+^n$ such that $|s| = k-1$, where e_i the ith unit vector in $\mathbf{R}^n$. That is to say, we have a linear relation on the vectors v_i for every such s. The collection of all linear k-stresses forms a vector space, which we will denote S_k^ℓ. (In [10] we used the notation $\overline{B}_k$.)

An *affine k-stress* on Δ (with respect to $v_1,\ldots,v_n$) is a linear k-stress that satisfies the additional condition

$$e^T\nabla b = 0,$$

where e denotes the vector $(1,\ldots,1)^T$. Equivalently,

$$\sum_{i=1}^{n}\frac{\partial b}{\partial x_i} = 0,$$

or

$$\sum_{i=1}^{n} b_{s+e_i} = 0 \tag{6}$$

for every $s \in \mathbf{Z}_+^n$ such that $|s| = k-1$; that is, we have an affine relation on the vectors v_i for every such s. Thus

$$\overline{M}\nabla b = O.$$

Clearly $b(x)$ is an affine k-stress with respect to $v_1,\ldots,v_n$ if and only if it is a linear k-stress with respect to $\overline{v}_1,\ldots,\overline{v}_n$, where

$$\overline{v}_i = \begin{bmatrix} v_i \\ 1 \end{bmatrix},\ i = 1,\ldots,n.$$

The collection of all affine k-stresses forms a subspace of S_k^ℓ, which we will denote S_k^a. (In [10] we used the notation $\overline{C}_k$.)

For $c \in \mathbf{R}^n$, define the function σ_c on the space of linear k-stresses by

$$\sigma_c(b) = c^T\nabla b = \sum_{i=1}^{n} c_i\frac{\partial b}{\partial x_i}$$

for any linear k-stress $b(x)$. In particular, define

$$\omega(b) = \sigma_e(b) = \sum_{i=1}^{n}\frac{\partial b}{\partial x_i}.$$

Theorem 8 *Let Δ be any simplicial complex with n vertices, and let $v_1,\ldots,v_n \in \mathbf{R}^d$. Then for $k = 1,2,3,\ldots$, the function σ_c maps S_k^ℓ into S_{k-1}^ℓ, and for $k = 0,1,2,\ldots$, the kernel of ω restricted to S_k^ℓ is S_k^a.*

Proof. The statement about the kernel of ω follows immediately from the definition of S_k^a. Suppose that $b \in S_k^\ell$ for some $k = 1,2,3,\ldots$. For $r \in \mathbf{Z}_+^n$ such that $|r| = k-1$, the coefficient of $\frac{x^r}{r!}$ in $\sigma_c(b)$ is $\sum_{i=1}^{n} c_i b_{r+e_i}$. Suppose that $\operatorname{supp} x^r \notin \Delta$. Then $\operatorname{supp} x^{r+e_i} \notin \Delta$ for $i = 1,\ldots,n$. Hence $b_{r+e_i} = 0$, $i = 1,\ldots,n$, and so $\sigma_c(b)$ satisfies condition (3). Also, $M\nabla(c^T\nabla b) = M[(\nabla^2 b)c] = [\nabla(M\nabla b)]c = O$ since $M\nabla b = O$, so $\sigma_c(b)$ satisfies condition (4). □

Theorem 9 *Let Δ be any simplicial complex with n vertices, and let $v_1,\ldots,v_n \in \mathbf{R}^d$. Then*

1. *$S_0^\ell = S_0^a = \mathbf{R}$.*
2. *S_1^ℓ is isomorphic to the space of all linear relations on the vectors v_i.*
3. *S_1^a is isomorphic to the space of all affine relations on the vectors v_i.*
4. *S_2^a is isomorphic to the classical stress space, under the correspondence $\lambda_{ij} = b_{e_i+e_j}$.*

Proof. The first three parts are trivial. For the fourth, assume $b \in S_2^a$. Let $\lambda_{ij} = b_{e_i+e_j}$ for all $i,j = 1,\ldots,n$. Note that $\lambda_{ij} = \lambda_{ji}$ and that $\lambda_{ij} = 0$ if $\{i,j\}$ is not an edge. From conditions (5) and (6) we see that for all $j = 1,\ldots,n$,

$$\begin{aligned} O &= \sum_{i=1}^{n}\lambda_{ij}v_i \\ &= \sum_{i:i\neq j}\lambda_{ij}v_i + \lambda_{jj}v_j \\ &= \sum_{i:i\neq j}\lambda_{ij}v_i + \sum_{i:i\neq j}(-\lambda_{ij})v_j \\ &= \sum_{i:\{i,j\}\in E}\lambda_{ij}(v_i - v_j), \end{aligned}$$

where E denotes the edges of Δ. Hence the λ_{ij} satisfy condition (1).

Conversely, suppose we are given λ_{ij} for $\{i,j\} \in E$ that satisfy condition (1). For $j = 1, \ldots, n$ define

$$b_{2e_j} = -\sum_{i:\{i,j\}\in E} \lambda_{ij},$$

and for $i \neq j$ define

$$b_{e_i+e_j} = \begin{cases} \lambda_{ij} & \text{if } \{i,j\} \in E, \\ 0 & \text{otherwise.} \end{cases}$$

The above argument then reverses to show that these coefficients determine an affine 2-stress. □

Example 1 Let Δ be the boundary complex of a d-simplex in $\mathbf{R}^d$ and take $v_1, \ldots, v_{d+1} \in \mathbf{R}^d$ to be the vertices of the simplex. Assume that no proper subset of the vertices is linearly dependent. Then there exist nonzero c_i such that

$$\sum_{i=1}^{d+1} c_i v_i = O,$$

and all linear relations on the v_i are multiples of this one. Then for all $k = 0, \ldots, d$, S_k^ℓ is one-dimensional and is spanned by

$$\sum_{r:|r|=k} c^r \frac{x^r}{r!}.$$

For we can see that

$$\sum_{i=1}^{d+1} c^{s+e_i} v_i = c^s \sum_{i=1}^{d+1} c_i v_i = O$$

for all $s \in \mathbf{Z}_+^{d+1}$ such that $|s| = k-1$. Note that c^r is nonzero for all r. On the other hand, $\dim S_k^\ell = 0$ for all $k > d$, $\dim S_0^a = 1$, and $\dim S_k^a = 0$ for all $k \geq 1$, since the v_i are affinely independent and so $\sum_{i=1}^{d+1} c_i \neq 0$.

Example 2 Let Δ be the boundary complex of the standard octahedron in $\mathbf{R}^3$, with

$$\begin{aligned} v_1 &= (+1,0,0)^T \\ v_2 &= (-1,0,0)^T \\ v_3 &= (0,+1,0)^T \\ v_4 &= (0,-1,0)^T \\ v_5 &= (0,0,+1)^T \\ v_6 &= (0,0,-1)^T \end{aligned}$$

Then it can be checked that

1. $S_0^\ell = \mathbf{R}$.
2. S_1^ℓ is three dimensional and has a basis $\{x_1+x_2, x_3+x_4, x_5+x_6\}$.
3. S_2^ℓ is three dimensional and has a basis $\{x_1x_3+x_1x_4+x_2x_3+x_2x_4, x_1x_5+x_1x_6+x_2x_5+x_2x_6, x_3x_5+x_3x_6+x_4x_5+x_4x_6\}$.
4. S_3^ℓ is one-dimensional and has a basis $\{x_1x_3x_5+x_1x_3x_6+x_1x_4x_5+x_1x_4x_6+x_2x_3x_5+x_2x_3x_6+x_2x_4x_5+x_2x_4x_6\}$.
5. $S_k^\ell = \{0\}$ if $k > 3$.
6. $S_0^a = \mathbf{R}$.
7. S_1^a is two-dimensional and has a basis $\{x_1+x_2-x_3-x_4, x_1+x_2-x_5-x_6\}$.
8. $S_k^a = \{0\}$ if $k > 1$.

5. Relationship to the Face Ring

The definition of generalized stress follows somewhat naturally from the face ring. For suppose that Δ is a simplicial complex (not necessarily of dimension $d-1$) with n vertices $\{1, \ldots, n\}$, and let $R = \mathbf{R}[x_1, \ldots, x_n] = R_0 \oplus R_1 \oplus R_2 \oplus \cdots$ be the ring of polynomials, graded by degree. If we are given $\theta_1, \ldots, \theta_d \in R_1$, we are interested in the dimension of B_k (as a vector space over $\mathbf{R}$), where $B = B_0 \oplus B_1 \oplus B_2 \oplus \cdots$ equals R factored out by the ideal $J = J_0 \oplus J_1 \oplus J_2 \oplus \cdots$ generated by I_Δ and $\theta_1, \ldots, \theta_d$. Using the inner product $\langle \sum_{r:|r|=k} a_r x^r, \sum_{r:|r|=k} b_r x^r \rangle = \sum_{r:|r|=k} a_r b_r$ on R_k, write $R_k = J_k \oplus J_k^\perp$. Now $\sum_{r:|r|=k} b_r x^r$ is in $J_k^\perp$ if and only if it is orthogonal to

1. all monomials of the form $x^s x^q$ where x^q is square-free, $\operatorname{supp} x^q \notin \Delta$, and $|s| + |q| = k$; and
2. all polynomials of the form $x^s \theta_j$, where $|s| = k-1$.

Writing $\theta_j = \sum_{i=1}^n v_{ij} x_i$, $j = 1, \ldots, d$ and defining $v_i = (v_{i1}, \ldots, v_{id})^T$, $i = 1, \ldots, n$, the first condition above is equivalent to condition (3) and the second condition is equivalent to condition (5). Hence $\sum_{r:|r|=k} b_r x^r \in J_k^\perp$ if and only if $\sum_{r:|r|=k} b_r \frac{x^r}{r!} \in S_k^\ell$. Recalling that an affine stress with respect to $v_1, \ldots, v_n$ is a linear stress with respect to $\overline{v}_1, \ldots, \overline{v}_n$, we have:

Theorem 10 *Suppose that Δ is a simplicial complex (not necessarily of dimension $d-1$) with n vertices. Let A be its face ring, and assume that we have $\theta_1, \ldots, \theta_d \in A_1$ and $v_1, \ldots, v_n \in \mathbf{R}^d$ such that $\theta_j = \sum_{i=1}^n v_{ij} x_i$, $j = 1, \ldots, d$ and $v_i = (v_{i1}, \ldots, v_{id})^T$, $i = 1, \ldots, n$. Let $A = A_0 \oplus A_1 \oplus A_2 \oplus \cdots = R/I_\Delta$, $B = B_0 \oplus B_1 \oplus B_2 \oplus \cdots = A/(\theta_1, \ldots, \theta_d)$, and $C = C_0 \oplus C_1 \oplus C_2 \oplus \cdots = B/(x_1 + \cdots + x_n)$. Then*

1. *Regardless of whether or not Δ is Cohen-Macaulay, $\dim B_k = \dim S_k^\ell$, $k = 0, \ldots, d$, and $\dim C_k = \dim S_k^a$, $k = 0, \ldots, d$.*
2. *Let $J = J_0 \oplus J_1 \oplus J_2 \oplus \cdots$ be the ideal of R generated by I_Δ and $\theta_1, \ldots, \theta_d$, and $J' = J_0' \oplus J_1' \oplus J_2' \oplus \cdots$ be the ideal of R generated by J and $x_1 + \cdots + x_n$. Then $b(x) = \sum_{r:|r|=k} b_r \frac{x^r}{r!}$ is a linear (respectively, affine) k-stress if and only if*

$$\sum_{r:|r|=k} a_r b_r = 0$$

for all $a(x) = \sum_{r:|r|=k} a_r x^r$ in J_k (respectively, J_k').

Corollary 11 *Let Δ be any simplicial $(d-1)$-complex with n vertices.*

1. *Δ is Cohen-Macaulay if and only if there exist $v_1, \ldots, v_n \in \mathbf{R}^d$ such that $\dim S_k^\ell = h_k$, $k = 1, \ldots, d$. In this case, the v_i can be chosen generically (that is, with algebraically independent components).*
2. *Suppose that Δ is in fact a simplicial $(d-1)$-sphere. If $\dim S_k^a = g_k$, $k = 0, \ldots, \lfloor d/2 \rfloor$, then its h-vector satisfies McMullen's conditions.*

Proof. This follows from the above result and Theorems 5 and 7. □

6. Formulas for Coefficients

In this section we explain why, under suitable conditions on the v_i, the coefficients of the square-free monomials of a linear stress uniquely determine the coefficients of the non-square-free monomials. We then characterize the former coefficients. For a simplicial complex Δ with n vertices, and for $v_1, \ldots, v_n \in \mathbf{R}^d$, we say that the v_i are in *linearly general position with respect to* Δ if $\{v_{i_1}, \ldots, v_{i_s}\}$ is linearly independent for every face $\{i_1, \ldots, i_s\}$ of Δ.

Theorem 12 *Let Δ be any simplicial complex with n vertices, and let $v_1, \ldots, v_n \in \mathbf{R}^d$ be in linearly general position with respect to Δ. If $b(x)$ is a linear stress, then the coefficients of the non-square-free monomials in $b(x)$ are linear combinations of the coefficients of the square-free monomials and hence are uniquely determined by them.*

Proof. Let $b(x) \in S_k^\ell$. We will use reverse induction on $\ell = \operatorname{card}(\operatorname{supp} x^r)$. The result is trivially true if $\ell = k$, so assume that the result is true for some ℓ such that $1 \le \ell \le k$, and suppose that $\operatorname{card}(\operatorname{supp} x^r) = \ell - 1$ where $\operatorname{supp} x^r \in \Delta$. Choose j such that $r_j > 1$ and let $s = r - e_j$. Condition (5) implies

$$\sum_{i=1}^{n} b_{s+e_i} v_i = O.$$

But, by the induction hypothesis, the coefficients b_{s+e_i} are linear combinations of the coefficients of the square-free monomials when $r_i = 0$, since $\operatorname{card}(\operatorname{supp} x^{s+e_i}) = \ell$ in this case. This leaves the $\ell - 1$ coefficients b_{s+e_i} for $i \in \operatorname{supp} x^r$ to be uniquely determined, since the corresponding v_i are linearly independent by assumption. In particular, $b_{s+e_j} = b_r$ is a linear combination of the coefficients of the square-free monomials. □

Therefore, if you are given the coefficients of the square-free coefficients of a k-stress, you can use conditions (3) and (5) to find the other coefficients systematically.

Corollary 13 *Let Δ be any simplicial complex with n vertices, and let $v_1, \ldots, v_n \in \mathbf{R}^d$ be chosen in linearly general position with respect to Δ. Then $\dim S_k^\ell = 0$ for all $k > \dim \Delta + 1$.*

Proof. If $k > \dim \Delta + 1$ then there are no faces of cardinality k, so all coefficients of square-free monomials of a linear k-stress must be zero. □

The next theorem provides an explicit formula for the coefficients of the non-square-free monomials in terms of the coefficients of the square-free monomials. For $G = \{i_1, \ldots, i_s\} \in \Delta$, define $\operatorname{conv} G$ (with respect to $v_1, \ldots, v_n$) to be $\operatorname{conv}\{v_{i_1}, \ldots, v_{i_s}\}$. We similarly define $\operatorname{aff} G$ and $\operatorname{span} G$. We will sometimes abuse notation and write b_G and x^G for b_r and x^r, respectively, where $r_i = 1$ if $i \in G$ and $r_i = 0$ if $i \notin G$. We will also use the notation $G + i$ for $G \cup \{i\}$ and $G - i$ for $G \setminus \{i\}$. Fixing an ordering of the elements of G and assuming that $s \le d$, define

$$[G] = \det \begin{bmatrix} v_{i_1 1} & v_{i_2 1} & \cdots & v_{i_s 1} \\ v_{i_1 2} & v_{i_2 2} & \cdots & v_{i_s 2} \\ \vdots & \vdots & \ddots & \vdots \\ v_{i_1 s} & v_{i_2 s} & \cdots & v_{i_s s} \end{bmatrix},$$

a subdeterminant of M. Note that only the first s rows of M are used. If $i \in G$, we compute $[G - i]$ using the ordering induced by G and multiply by $+1$ (respectively, -1) if i is in an odd (respectively, even) position with respect to this ordering, and we compute $[G - i + j]$ by replacing the column corresponding to v_i with the column corresponding to v_j.

Theorem 14 *Let Δ be a simplicial complex on n vertices of dimension at most $d-1$, and let $v_1, \ldots, v_n$ be chosen generically in $\mathbf{R}^d$. Suppose that $b(x)$ is a linear k-stress for some $1 \le k \le d$. Suppose that $r \in \mathbf{Z}_+^n$ such that $|r| = k$ and $S = \operatorname{supp} x^r \in \Delta$. Then*

$$b_r = \sum_{(k-1)\text{-faces } F \text{ containing } S} b_F \frac{\prod_{i \in S} [F - i]^{r_i - 1}}{\prod_{i \in F \setminus S} [F - i]}.$$

Proof. We will use reverse induction on $\ell = \operatorname{card} S$. The formula is trivially true when $\ell = k$, so assume that the formula for b_r is true whenever $\operatorname{card}(\operatorname{supp} x^r) = \ell + 1$, for some ℓ such that $1 \le \ell < d$. Suppose that the support S of x^r has cardinality ℓ. Write $S = \{i_1, \ldots, i_\ell\}$ where $i_1 < \cdots < i_\ell$. Since x^r is not square-free, there must be some m for which $r_m > 1$. Let M_ℓ be the submatrix of M consisting of the first ℓ rows of M, and let B be the submatrix of M_ℓ determined by the members of S. Multiplying the mth row of $B^{-1} M_\ell (x_1, \ldots, x_n)^T$ by $x^{r - e_m}$ yields a member of J_k, namely

$$x^r + \sum_{j \in \operatorname{lk} S} \frac{[S - m + j]}{[S]} x^{r - e_m + e_j},$$

Note that each monomial on the right-hand side has support of cardinality $\ell + 1$. By the induction hypothesis, the orthogonality condition (2) of Theorem 10, and some

Grassman-Plücker relations, we compute

$$
\begin{aligned}
b_r &= -\sum_{j\in \mathrm{lk}\, S} \frac{[S-m+j]}{[S]} b_{r-e_m+e_j} \\
&= -\sum_{j\in \mathrm{lk}\, S} \frac{[S-m+j]}{[S]} \sum_{(k-1)\text{-faces } F \text{ containing } S+j} b_F \frac{\prod_{i\in S+j} [F-i]^{(r-e_m+e_j)_i-1}}{\prod_{i\in F\setminus(S+j)} [F-i]} \\
&= -\sum_{(k-1)\text{-faces } F \text{ containing } S} \sum_{j\in F\setminus S} b_F \frac{[S-m+j]\prod_{i\in S}[F-i]^{(r-e_m)_i-1}}{[S]\prod_{i\in F\setminus(S+j)}[F-i]} \\
&= -\sum_{(k-1)\text{-faces } F \text{ containing } S} b_F \frac{\prod_{i\in S}[F-i]^{(r-e_m)_i-1}}{[S]} \sum_{j\in F\setminus S} \frac{[S-m+j]}{\prod_{i\in F\setminus(S+j)}[F-i]} \\
&= -\sum_{(k-1)\text{-faces } F \text{ containing } S} b_F \frac{\prod_{i\in S}[F-i]^{(r-e_m)_i-1}}{[S]\prod_{i\in F\setminus S}[F-i]} \sum_{j\in F\setminus S} [S-m+j][F-j] \\
&= \sum_{(k-1)\text{-faces } F \text{ containing } S} b_F \frac{\prod_{i\in S}[F-i]^{(r-e_m)_i-1}}{[S]\prod_{i\in F\setminus S}[F-i]} [S][F-m] \\
&= \sum_{(k-1)\text{-faces } F \text{ containing } S} b_F \frac{\prod_{i\in S}[F-i]^{r_i-1}}{\prod_{i\in F\setminus S}[F-i]}. \quad \square
\end{aligned}
$$

The formula is not symmetric with respect to permutations of the coordinates of the v_i, but can be made so by averaging over all permutations, for example. Since we know that the coefficients of the square-free monomials determine all of the others, it would be nice to characterize them somehow geometrically.

Theorem 15 *Let Δ be any simplicial complex with n vertices, and let $v_1, \ldots, v_n \in \mathbf{R}^d$. Let $b(x)$ be a linear (respectively, affine) k-stress, $k \geq 1$. Choose any face F of Δ of cardinality $k-1$ and any point v in* span F *(respectively* aff F*). Then*

$$v + \sum_{i\in \mathrm{lk}\, F} b_{F+i}(v_i - v)$$

lies in span F *(respectively,* aff F*). Equivalently, if w_i is the vector joining the projection of v_i onto* span F *(respectively,* aff F*) to v_i, then*

$$\sum_{i\in \mathrm{lk}\, F} b_{F+i} w_i = O.$$

Proof. Suppose that $v \in \operatorname{span} F$. Then, using condition (5),

$$
\begin{aligned}
v + \sum_{i\in \mathrm{lk}\, F} b_{F+i}(v_i - v) &= v + \sum_{i\in \mathrm{lk}\, F} b_{F+i}v_i - \sum_{i\in \mathrm{lk}\, F} b_{F+i}v \\
&= v - \sum_{i\in F} b_{F+i}v_i - \sum_{i\in \mathrm{lk}\, F} b_{F+i}v
\end{aligned}
$$

which is in span F (abusing notation slightly in the penultimate sum). If b is an affine stress, then by condition (6) the sum of the coefficients in the above expression is

$$1 - \sum_{i\in F} b_{F+i} - \sum_{i\in \mathrm{lk}\, F} b_{F+i} = 1.$$

So we have an element of aff F. $\square$

Note that, for a linear k-stress, w_i is the altitude vector for the point v_i in the simplex $\operatorname{conv}(\{O\} \cup F)$, and, for an affine k-stress, w_i is the altitude vector for the point v_i in the simplex conv F. So affine k-stress is a natural generalization of classical stress (affine 2-stress), and is equivalent to the proposed generalization of Kalai (personal communication).

Example 3 Let Δ be the boundary complex of a simplicial d-polytope in $\mathbf{R}^d$, $d \geq 1$, and take the v_i to be its vertices. Then the above theorem implies that $\dim S_d^a = 0$. For take any $b(x) \in S_d^a$ and consider any subfacet F (i.e., of cardinality $d-1$). Then there are precisely two facets containing F, and hence only two altitude vectors w_i with respect to aff F, where $i \in \mathrm{lk}\, F$. By convexity these two vectors are not collinear and we must have

$$\sum_{i\in \mathrm{lk}\, F} b_{F+i} w_i = O,$$

from which it follows that $b_{F+i} = 0$ for $i \in \mathrm{lk}\, F$. Thus all the coefficients of the square-free monomials of $b(x)$ are zero, and hence all of the coefficients of $b(x)$ must also be zero.

The previous theorem provides necessary conditions for the coefficients of the square-free terms. But Filliman [3] and Tay-White-Whiteley [18] have shown that they are also sufficient. So we could just as well define linear or affine k-stress using the conditions provided by the previous theorem, and perhaps this would be more natural.

7. Infinitesimal k-Motions

What is the generalization of infinitesimal motions? Consider an affine k-stress on simplicial complex Δ with respect to $\{v_1,\ldots,v_n\} \subset \mathbf{R}^d$. That is, for each $(k-1)$-fa F, we have a number b_F such that for every $(k-2)$-face G,

$$\sum_{i \in \mathrm{lk}\, G} b_{G+i} w_i = O,$$

where w_i is the altitude vector of v_i in the simplex $\mathrm{conv}(G+i)$. Consider th matrix R with rows indexed by $(k-1)$-faces F and columns occurring in groups d columns, one group for each $(k-2)$-face G. The row vector of length d in row group G, is

$$\begin{cases} O^T & \text{if } G \not\subset F, \\ w^T_{(G,F)} & \text{if } G \subset F, \end{cases}$$

where $w_{(G,F)}$ is the altitude vector of the simplex $\mathrm{conv}\, F$ with respect to conv So the left nullspace of R is S^a_k. The other nullspace of R suggests a definition infinitesimal $(k-1)$-motion, namely, an assignment of a vector $\overline{v}_G \in \mathbf{R}^d$ to eac $(k-2)$-face G, such that for every $(k-1)$-face F,

$$\sum_{i \in F} w_i \cdot \overline{v}_i = 0.$$

In the above expression, we use the notation w_i for $w_{(F\setminus\{i\},F)}$ and $\overline{v}_i$ for $\overline{v}_{F\setminus\{i\}}$

Write G_i for $F\setminus\{i\}$ and let u_i be the unit outer normal vector of $\mathrm{conv}\, G_i$ wi respect to $\mathrm{conv}\, F$ in $\mathrm{aff}\, F$. Then

$$\sum_{i \in F} u_i \|w_i\| \cdot \overline{v}_i = 0.$$

Dividing by $\mathrm{vol}_{k-1}(F)$ implies

$$\sum_{i \in F} u_i \cdot \frac{\overline{v}_i}{\mathrm{vol}_{k-2}(G_i)} = 0$$

and hence

$$\sum_{i \in F} u_i \mathrm{vol}_{k-2}(G_i) \cdot \overline{m}_i = 0,$$

where $\overline{m}_i = \overline{v}_i / \mathrm{vol}^2_{k-2}(G_i)$.

A better definition of infinitesimal $(k-1)$-motion might be a choice of vect $\overline{m}_G \in \mathbf{R}^d$ for each $(k-2)$-face G, such that the above expression is satisfied for eac $(k-1)$-face F. Since $u_i \cdot m_i = u_i \cdot \overline{m}_i$ for the projection m_i of $\overline{m}_i$ onto $\mathrm{aff}\, F$, we als have

$$\sum_{i \in F} u_i \mathrm{vol}_{k-2}(G_i) \cdot m_i = 0.$$

It can be shown that this is equivalent to the existence of a vector $m_F \in \mathbf{R}^d$ fo each $(k-1)$-face F such that $m_F \cdot u_i = m_i \cdot u_i$ for each $i \in F$. Notice in this cas

hat

$$\begin{aligned} &\sum_{i \in F} u_i \mathrm{vol}_{k-2}(G_i) \cdot m_i \\ &= \sum_{i \in F} u_i \mathrm{vol}_{k-2}(G_i) \cdot m_F \\ &= \Big(\sum_{i \in F} u_i \mathrm{vol}_{k-2}(G_i)\Big) \cdot m_F \\ &= O \cdot m_F \\ &= 0 \end{aligned}$$

Minkowski's theorem.

For a real number t, let $F(t)$ be the $(k-1)$ simplex determined by translating G_i by the vector tm_i. Then our definition is also equivalent to

$$\frac{d}{dt}\mathrm{vol}^2_{k-1}(F(t)) = 0$$

en $t=0$. This was also observed by Filliman [3]. Equivalently, $F(1)$ is congruent F.

So generalized stress leads to a fairly natural generalization of infinitesimal mo- ons (2-motions). See Tay, White and Whiteley [18] for a deeper study of the ationship between generalized stress and skeletal rigidity of cell complexes.

Simplicial Spheres

Δ be a simplicial $(d-1)$-sphere (or connected $(d-1)$-pseudo-manifold) with n rtices, and choose $v_1,\ldots,v_n \in \mathbf{R}^d$ in generically. As noted before, Δ is Cohen- acaulay. Also, Euler's relation implies that $h_d = 1$. So $\dim S^a_d = 1$; hence up to lar multiple there is only one linear d-stress $b(x)$. By Theorem 12, it suffices to ermine the square-free coefficients of b. Choose a consistent orientation of all the ets of Δ and use this to induce an ordering of the elements of each facet. Let G be ubfacet of Δ, and F_1, F_2 the two facets containing G. Examining the conditions b_{F_1} and b_{F_2} in terms of their altitudes w_1, w_2 with respect to span$G$, one can adily see that $[F_1]b_{F_1} = [F_2]b_{F_2}$. So we may without loss of generality assume that $= [F]^{-1}$ for every facet F. The coefficients of the non-square-free monomials can en be determined.

In Section 10 we will see the geometrical significance of the canonical linear d- ress in the case that Δ is the boundary complex of a simplicial d-polytope.

For a subset $S = \{i_1,\ldots,i_s\}$ of $\{1,\ldots,n\}$, define the function τ_S on the space of near stresses by

$$\tau_S(b) = \frac{\partial^s b}{\partial x_{i_1}\cdots\partial x_{i_s}}.$$

n particular, write

$$\tau_i(b) = \frac{\partial b}{\partial x_i}.$$

Suppose that Δ is a simplicial $(d-1)$-sphere on $\{1,\ldots,n\}$. Let $G \subseteq \{1,\ldots,n\}$ b a face of Δ of cardinality s. Then clstar G, the closed star of G in Δ, is a simplici $(d-1)$-ball (hence Cohen-Macaulay), that is the join of a simplicial $(d-s-1)$-sphe to G. In such a case it is known that $h_{d-s}(\text{clstar}\, G) = 1$. The next theorem sho how to obtain a canonical linear $(d-s)$-stress for clstar G from the canonical line d-stress of Δ.

Theorem 16 *Let Δ be a simplicial $(d-1)$-complex with n vertices, and let $v_1,\ldots,$ be chosen arbitrarily. Suppose that $b(x)$ is a linear d-stress and that G is a face Δ of cardinality s. Then $\tau_G(b)$ is a linear $(d-s)$-stress supported on clstar G. particular, if Δ is a simplicial $(d-1)$-sphere, $v_1,\ldots,v_n \in \mathbf{R}^d$ are chosen genericall and $b(x)$ is the nonzero canonical linear d-stress, then $\tau_G(b)$ is a nonzero line $(d-s)$-stress supported on clstar G.*

PROOF. That $\tau_G(b)$ is a linear $(d-s)$-stress follows from the fact that $\tau_i = \sigma$ and $\tau_G = \tau_{i_1}\cdots\tau_{i_s}$, where $G = \{i_1,\ldots,i_s\}$. If the coefficient of x^r is nonze in $\tau_G(b)$, then the coefficient of $x_{i_1}\cdots x_{i_s}x^r$ must be nonzero in $b(x)$, and hen supp $x^r \in$ clstar G. Now suppose that Δ is a simplicial sphere, the v_i are cho generically, and $b(x)$ is the canonical linear d-stress. Let F be a facet of Δ th contains G. Then the coefficient of $x^{F\setminus G}$ in $\tau_G(b)$ equals the coefficient of x^F in which is nonzero. So $\tau_G(b)$ is not the zero polynomial. □

Let us consider the case that Δ is a shellable simplicial $(d-1)$-sphere Δ, a $F_1,\ldots,F_m$ constitutes a shelling order of the facets. It is known that as each fa F_k is added, precisely one component, say h_s, of the h-vector increases by one, remaining components remaining unchanged. For generic $v_1,\ldots,v_n$, this impl that the dimension of S_s^ℓ increases by one, while the dimensions of the other lin stress spaces remain unchanged. When F_k is added, the closed star of precisely o face G_k of cardinality $d-s$ is completed. The linear s-stress $\tau_G(b)$ now become member of S_s^ℓ, where $b(x)$ is the canonical linear d-stress. Thus if Δ is shellable, can use the shelling to derive a basis for S_s^ℓ.

Theorem 17 *If Δ is a shellable simplicial $(d-1)$-sphere with n vertices, $v_1,\ldots,v$ $\mathbf{R}^d$ are chosen generically, and $F_1,\ldots,F_m, G_1,\ldots,G_m$ are as above, then $\{\tau_{G_k}(b)$ card $G_k = d-s\}$ is a basis for S_s^ℓ. Hence the collection $\{\tau_G(b) : G$ is a face of Δ cardinality $d-s\}$ spans S_s^ℓ.*

By working directly with the face ring, Kind and Kleinschmidt [9] found a inductive proof that arbitrary shellable simplicial complexes are Cohen-Macaula From the above ideas one can construct another inductive proof of this result usi linear stress.

9. Bistellar Operations and P.L.-Spheres

Suppose that Δ is a simplicial $(d-1)$-complex whose vertices are contained i $\{1,\ldots,n\}$. Assume that F and G are disjoint subsets of $\{1,\ldots,n\}$ of cardinalit k and ℓ, respectively, $(1 \le k, \ell \le d)$ such that $k+\ell = d+1$, $F \in \Delta$, $G \notin \Delta$, an lk $F = \partial G = \{G' : G'$ is a proper subset of $G\}$, the boundary of G. Then we say tha

he simplicial complex $\Delta' = (\Delta \setminus F) \cup (G \cdot \partial F)$ is the result of a *bistellar operation* n Δ. That is, we remove from Δ all faces containing F, and then introduce all sets f the form $G \cup F'$ where $F' \in \partial F$. Notice that the faces of Δ' which are not in Δ re precisely those faces of Δ' which contain G, and the faces of Δ' which are not Δ are precisely the faces of Δ which contain F.

heorem 18 *If Δ and Δ' are as above and $v_1,\ldots,v_n \in \mathbf{R}^d$ are generic, then*

$$\dim S_s^\ell(\Delta') = \begin{cases} \dim S_s^\ell(\Delta)+1 & \text{if } k > \ell \text{ and } \ell \le s \le d-\ell, \\ \dim S_s^\ell(\Delta)-1 & \text{if } k < \ell \text{ and } k \le s \le d-k, \\ \dim S_s^\ell(\Delta) & \text{otherwise.} \end{cases} \qquad (7)$$

ROOF. Let $\Delta'' = \Delta \cup (G \cdot \partial F)$. Note that Δ'' also equals $\Delta' \cup (F \cdot \partial G)$. It suffices o show that

$$\dim S_s^\ell(\Delta'') = \begin{cases} \dim S_s^\ell(\Delta) & \text{if } s = 0,\ldots,\ell-1, \\ \dim S_s^\ell(\Delta)+1 & \text{if } s = \ell,\ldots,d, \end{cases}$$

nd

$$\dim S_s^\ell(\Delta'') = \begin{cases} \dim S_s^\ell(\Delta') & \text{if } s = 0,\ldots,k-1, \\ \dim S_s^\ell(\Delta')+1 & \text{if } s = k,\ldots,d. \end{cases}$$

ince Δ and Δ'' share the same faces of cardinality s when $s = 0,\ldots,\ell-1$, it is ear that $S_s^\ell(\Delta) = S_s^\ell(\Delta'')$ for these values of s. So assume that $\ell \le s \le d$. Define $= F \cup G$, a subset of cardinality $d+1$, all of whose proper faces are in Δ''. Let $c(x)$ the nonzero canonical linear s-stress obtained from the essentially unique linear lation on the set $\{v_i : i \in S\}$, as in Example 1. Since all faces of Δ are also in Δ'', see that every linear s-stress on Δ is also a linear s-stress on Δ''. Now let $b(x)$ any linear s-stress on Δ'' that is not a linear s-stress on Δ. Thus b_r is nonzero r some r such that supp $x^r \in$ openstar G, where by openstar G we mean the set of l faces of Δ'' that contain G. We will show that there is a nonzero $t \in \mathbf{R}$ such at $b_r = tc^r$ whenever supp $x^r \in$ openstar G, and hence that $b(x) - tc(x)$ is a linear tress on Δ. This is clearly true if $s = \ell$ since G is the only face of cardinality ℓ in penstar G, so assume that $\ell+1 \le s \le d$. Choose any r such that b_r is nonzero and pp $x^r \in$ openstar G. Since $s > \ell$, there exists j such that supp $x^{r-e_j} \in$ openstar G. ondition 5 implies that $\sum_{i=1}^n b_{r-e_j+e_i} v_i = O$. But this sum involves only the $d+1$ ctors v_i such that $i \in S$, so the coefficients must be multiples of the coefficients $c(x)$. Repeating this procedure and using induction on the cardinality of supp b_r tablishes the desired result. Therefore $\dim S_s^\ell(\Delta'') = \dim S_s^\ell(\Delta)+1$ for $s = \ell,\cdots,d$. he proof relating $\dim S_s^\ell(\Delta'')$ and $\dim S_s^\ell(\Delta')$ is analogous. □

We remark that the changes in the dimensions of the linear stress spaces under a stellar operation remain valid whether or not Δ is Cohen-Macaulay. But the case which Δ is Cohen-Macaulay, and in particular a p.l.-sphere, is of special interest. achner [12] showed that every (simplicial) p.l.-sphere of dimension $d-1$ can be btained from the boundary of a d-simplex by a sequence of bistellar operations. his fact, together with the above result, can be used to obtain a new proof that l.-spheres are Cohen-Macaulay.

Corollary 19 *If Δ is a simplicial piecewise linear $(d-1)$-sphere, then Δ is Cohen-Macaulay.*

PROOF. Choose $v_1,\dots,v_n \in \mathbf{R}^d$ generically. The boundary of a simplex is Cohe[n-]Macaulay by part (1) of Corollary 11 and Example 1, since the dimensions of t[he] linear stress spaces agree with the components of the h-vector. It is easy to see th[at] for any simplicial complex, the components h_s of the h-vector increase and decrea[se] in exactly the same manner as the dimensions $\dim S_s^\ell$ change in (7). So if Δ [is] obtained from the boundary of a simplex by a sequence of bistellar operations, [it] follows that $h_i(\Delta) = \dim S_s^\ell(\Delta)$ for all s. The result now follows from Corollary [...] and Pachner's theorem. □

10. Simplicial Convex Polytopes

In this section, we will assume that Δ is the boundary complex of some simplic[ial] d-polytope P containing the origin in its interior. In discussing the stress spaces [of] Δ, take $v_1,\dots,v_n \in \mathbf{R}^d$ to be the actual vertices of P. Kind and Kleinschmid[t's] shelling proof shows that this suffices to ensure that $\dim S_i^\ell = h_i$, $i = 0,\dots,d$.

For $x \in \mathbf{R}^n$, consider the polytope $Q(x) = \{y \in \mathbf{R}^d : y^T v_i \le x_i,\ i = 1,\dots,n\}$. Of course, $Q(e)$ is the polar P^* of P. Since P^* is simple, for values of x_i near 1 th[e] combinatorial structure of $Q(x)$ agrees with that of P^*. It is known that the volu[me] of $Q(x)$ as a function of the x_i is a homogeneous polynomial $V(x) = \sum_{r:|r|=d} b_r \dots$ of degree d and $b_r = 0$ whenever $\operatorname{supp} x^r \notin P$.

Theorem 20 *Let P be as above. Then the canonical d-stress $b(x)$ described [in] Theorem 14 is precisely $V(x)$.*

PROOF. For every $u \in \mathbf{R}^d$, $Q(x_1,\dots,x_n) + u = Q(x_1 + u^T v_1, \dots, x_n + u^T v_n)$, [so] $V(x_1,\dots,x_n) - V(x_1+u^T v_1,\dots,x_n+u^T v_n) = 0$. Fix r such that $|r| = d-1$. Th[en]

$$O = \frac{\partial^{d-1}}{(\partial x_1)^{r_1}\cdots(\partial x_n)^{r_n}}[V(x_1,\dots,x_n) - V(x_1+u^T v_1,\dots,x_n+u^T v_n)]$$

$$= \sum_{i=1}^n b_{r+e_i} x_i - \sum_{i=1}^n b_{r+e_i}(x_i + u^T v_i)$$

$$= \sum_{i=1}^n b_{r+e_i} u^T v_i$$

$$= u^T\left(\sum_{i=1}^n b_{r+e_i} v_i\right).$$

But this is true for every u, so that $\sum_{i=1}^n b_{r+e_i} v_i = O$ and $V(x)$ is a linear d-stre[ss]. That $V(x)$ is the same as the canonical linear d-stress follows from the fact th[at] $b_F = \frac{1}{|F|}$ for every facet F of P. □

Note that the above proof is analogous to the proof of Minkowski's Theorem [6] that $\sum_{i=1}^n \operatorname{vol}_{d-1}(F_i)\frac{v_i}{\|v_i\|} = O$ where $\operatorname{vol}_{d-1}(F_i)$ is the $(d-1)$-volume of the facet F_i of P^* corresponding to v_i. In fact, the relationship is much closer, as we shall soo[n] see.

McMullen's conditions would be a consequence of the bijectivity of ω^{d-2i} : $S_{d-i}^\ell \to S_i^\ell$, $i = 0,\dots,\lfloor d/2\rfloor$, or more weakly, of the surjectivity of $\omega : S_i^\ell \to S_{i-1}^\ell$, $i = 1,\dots,\lfloor d/2\rfloor$. We will explore some special cases of these conjectures.

What is the geometrical interpretation of the canonical linear i-stresses $\omega^{d-i}(V(x))$, $i = 1,\dots,d$? Let $W(x) = V(x_1+1,\dots,x_n+1)$. Then for small x, $W(x)$ is the volume of a polytope near P^*. Write $W(x) = \sum_{i=0}^d W_i(x)$, where each $W_i(x)$ is a homogeneous polynomial of degree i. Of course, $W_0(x)$ is the volume of P^*, $W_d(x) = V(x)$, and it is easy to see that $W_1(x) = \sum_{i=1}^n \frac{\operatorname{vol}_{d-1}(F_i)}{\|v_i\|} x_i$.

Theorem 21 *Let P be as above. Then for $i = 0,\dots,d$, $\omega^{d-i}(V(x)) = (d-i)!W_i(x)$.*

PROOF. We calculate the contribution of $b_r\frac{x^r}{r!}$ in $V(x)$ to the coefficient of x^s in $W_i(x)$, where $x^s | x^r$. Expanding

$$b_r\frac{(x_1+1)^{r_1}\cdots(x_n+1)^{r_n}}{r_1!\cdots r_n!}$$

[we] see that the contribution is

$$b_r\frac{\binom{r_1}{s_1}\cdots\binom{r_n}{s_n}}{r_1!\cdots r_n!} = \frac{b_r}{s_1!(r_1-s_1)!\cdots s_n!(r_n-s_n)!}.$$

[On] the other hand, the contribution of

$$b_r\frac{x^r}{r!} = b_r\frac{x_1^{r_1}\cdots x_n^{r_n}}{r_1!\cdots r_n!}$$

[in] $V(x)$ to the coefficient of x^s in $\omega^{d-i}(V(x))$, where $i = d - |s|$, is

$$b_r\frac{\binom{d-i}{r_1-s_1,\dots,r_n-s_n}}{r_1!\cdots r_n!} = \frac{(d-i)!b_r}{s_1!(r_1-s_1)!\cdots s_n!(r_n-s_n)!}. \;\square$$

Corollary 22 *Let P be as above.*

1. *The canonical linear 0-stress $\omega^d(V(x))$ equals $d!\operatorname{vol}(P^*)$.*
2. *The canonical linear 1-stress $\omega^{d-1}(V(x))$ equals $(d-1)!\sum_{i=1}^n \frac{\operatorname{vol}_{d-1}(F_i)}{\|v_i\|}x_i$. That is, the canonical linear combination of the v_i induced by ω is (up to scalar multiple) the same as that induced by Minkowski's Theorem.*

We remark that the coefficient of the square-free term of W_i corresponding to the $(i-1)$-face F of P equals

$$\frac{\operatorname{vol}_{d-i}(F^*)}{i!\operatorname{vol}_i(\operatorname{conv}(\{O\}\cup\{v_i : v_i \in F\}))}, \tag{8}$$

where F^* is the face of P^* corresponding to F. See also Filliman [5].

Theorem 23 *Let P be as above. Then $\omega^d : S_d^\ell \to S_0^\ell$ is a bijection. Further, if $d \ge 3$, then $\omega^{d-2} : S_{d-1}^\ell \to S_1^\ell$ is a bijection.*

PROOF. The first statement is true by part (1) of the previous corollary sim- ply because P^* has positive (and hence nonzero) volume. From the Deh Sommerville Relations we know that $\dim S^\ell_{d-1} = h_{d-1} = h_1 = \dim S^\ell_1$. So suffices to show that $\omega^{d-2} : S^\ell_{d-1} \to S^\ell_1$ is a surjection. From Theorem 1 we know that $\{\tau_1(V(x)), \dots, \tau_n(V(x))\}$ spans S^ℓ_{d-1}. We need to show th $\{\omega^{d-2}\tau_1(V(x)), \dots, \omega^{d-2}\tau_n(V(x))\}$ spans S^ℓ_1. Since $\dim S^\ell_1 = h_1 = n-d$, it sufficient to demonstrate that the given subset of S^ℓ_1 has rank $n-d$. But since and τ_i commute, our set equals $(d-2)!\{\tau_1(W_2(x)), \dots, \tau_n(W_2(x))\}$. It is straigh forward to check that $\sum_{i=1}^n x_i\tau_i(W_2(x)) = 2W_2(x)$, and the result now follows fro the Brunn-Minkowski Theorem [6], which implies that the rank of the Hessian $W(x)$ is $n-d$. □

The above theorem implies that $\omega : S^\ell_2 \to S^\ell_1$ is s surjection when $d \geq 3$. Hen $\dim S^a_2 = h_2 - h_1 = g_2$ and Theorem 3 is a corollary.

Consider the ring of all differential operators with constant coefficients in th variables $x_1, \dots, x_n$. Factor out the ideal of operators that annihilate the polynomi $V(x)$. Khovanskii (personal communication) observed that the resulting ring R isomorphic to the cohomology ring of the projective toric variety associated wi the polytope. This implies that R is isomorphic to $B = A/(\theta_1, \dots, \theta_d)$, where coefficients of the θ_i are given by the coefficients of the vertices of P. Indeed, we c see this latter fact directly. Clearly $\tau_S(V(x))$ equals 0 for any subset $S \notin \Delta$. But t invariance of the polynomial $V(x)$ under translation (see the proof of Theorem 2 implies that

$$\sum_{i=1}^n v_{ij}\frac{\partial V(x)}{\partial x_i} = 0$$

for each $j = 1, \dots, d$. Finally, Theorem 17 implies that the image of $V(x)$ under homogeneous differential operators of degree k spans S^ℓ_{d-k}, and hence has dimens $h_{d-k} = h_k$. Using Theorem 5, this suffices to prove that R is isomorphic to B und the map $\frac{\partial}{\partial x_i} \to x_i$.

11. Simplicial 3-Polytopes

We will summarize the consequences of the previous section in the case that $d =$ Suppose that P is a simplicial convex 3-polytope containing the origin in its inter with vertices $v_1, \dots, v_n$, and that P^* is its polar. Then the canonical linear 0-stre is 3! times the volume of P^*, the canonical linear 1-stress is equivalent to the li ear relation induced by Minkowski's Theorem, the canonical linear 2-stress is t classical Maxwell stress (shown by Filliman [4]), and the canonical linear 3-stre is the volume polynomial $V(x)$. Hence, $S^\ell_0 = \mathbf{R}$, S^ℓ_1 has dimension $n-3$ and isomorphic to the space of all linear relations on the v_i, S^ℓ_2 has dimension $n-3$ spanned by the $\tau_i V(x)$, and is isomorphic to S^ℓ_1, and S^ℓ_3 is spanned by $V(x)$. Al $S^a_0 = \mathbf{R}$, S^a_1 has dimension $n-4$ and is isomorphic to the space of all affine rel tions on the v_i, and both S^a_2 and S^a_3 are trivial. That $\omega^3 : S^\ell_3 \to S^\ell_0$ is a bijecti is equivalent to P^* having nonzero volume, and that $\omega : S^\ell_2 \to S^\ell_1$ is a bijecti

equivalent to Dehn's Theorem (Theorem 2) and is a consequence of the Brunn- inkowski Theorem. So in dimension 3 we see a striking confluence of geometric sults.

2. The Polytope Algebra

cMullen [11] recently found a new proof of the g-Theorem for simplicial con- x d-polytopes using the powerful tool of his polytope algebra. In particular, arguments demonstrate that ω^{d-2i} of the previous section is bijective and ence $\dim S^a_k = g_k(P)$, $k = 0, \dots, \lfloor d/2 \rfloor$. In the process, he discovered new ge- etric inequalities (the Hodge-Riemann-Minkowski inequalities), generalized the ksandrov-Fenchel Inequality, and elucidated the geometric role of the Hard Lef- etz Theorem.

Use a generator $[P]$ for each convex polytope in $\mathbf{R}^d$ to generate an abelian group, d take $0 = [\emptyset]$. Impose the relations $[P \cup Q] + [P \cap Q] = [P] + [Q]$ for all polytopes nd Q whose union is also a polytope, and $[P + t] = [P]$ for all polytopes P and nslation vectors t. Define multiplication by $[P] \cdot [Q] = [P + Q]$, extending by arity. The dilatation operator is defined by $\Delta(\lambda)[P] = [\lambda P]$ for polytopes P and numbers λ. The result is the polytope algebra Π.

Now fix a simple d-polytope $P \subset \mathbf{R}^d$ and consider $\Pi(P)$, the subalgebra generated he classes of the summands of P. McMullen shows that there exists a grading $\Pi(P)$ such that kth weight space has dimension $h_k(P^*)$ and that $\Pi(P)$ admits a chetz decomposition. In so doing he defines a k-weight on P to be a choice of number $\omega(F)$ for each ce F of P, which satisfies the Minkowski relations

$$\sum_{F_j \subset G} \omega(F_j)u_j = O,$$

re the sum runs over all the k-faces of a $(k+1)$-face G of P, with u_j the unit r normal vector to G at F_j with respect to aff G.

Given simplicial convex polytope Q containing O in its interior, with linear sses defined with respect to its vertices $v_1, \dots, v_n$, an isomorphism between linear k)-stresses for Q and the k-weights for Q^* is given by

$$\omega(F^*) = (d-k)!b_F\mathrm{vol}_{d-k}(\mathrm{conv}(\{O\} \cup \{v_i : v_i \in F\})).$$

he above, F^* is the face of Q^* corresponding to F under duality. In particular, canonical linear $(d-k)$-stresses for Q simply become the relative volumes of the faces — see (8).

Refer to Filliman [4] for other connections between stress and the Aleksandrov- chel Inequality.

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