WEIGHTS ON ALMOST SIMPLE POLYTOPES

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JOINT WORK WITH SUE FOEGE

WEIGHTS ON (SIMPLE) POLYTOPES (MCMULLEN)

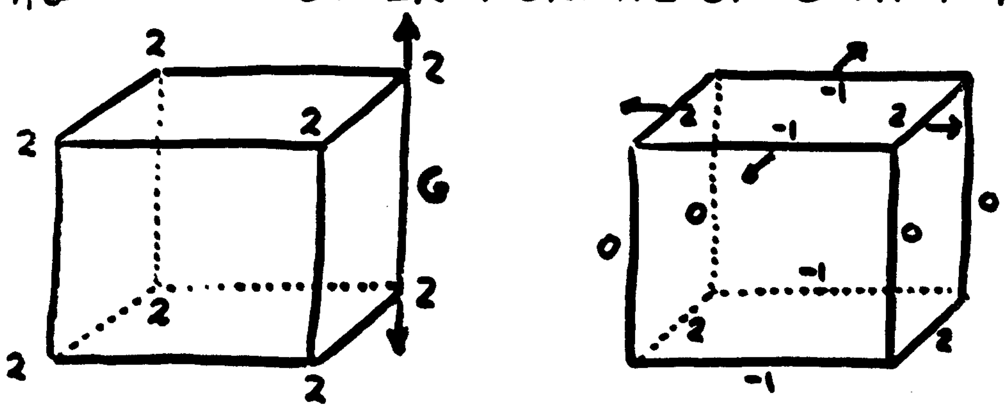
P: SIMPLE d-POLYTOPE

i-WEIGHT: NUMBER $a(F)$ FOR EACH i-FACE F S.T.

$$\sum_{F \subset G} a(F) u_{F,G} = 0$$

FOR EACH $(i+1)$-FACE G.

$u_{F,G}$: UNIT OUTER NORMAL OF G AT F IN AFF G



0-WEIGHT

1-WEIGHT

$\Omega_i(P)$ = SPACE OF i-WEIGHTS

$\Omega(P) = \Omega_0(P) \oplus \cdots \oplus \Omega_d(P)$

PARTICULAR i-WEIGHT: $a(F) = \mathrm{VOL}_i(F)$

PARTICULAR 1-WEIGHT: $p(E)$ = LENGTH OF EDGE E

BASIS FOR $\Omega_i(P)$

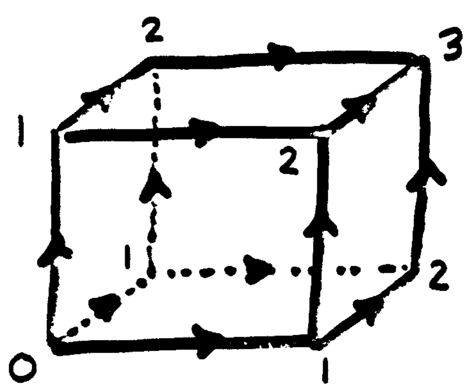
SHELL (THE POLAR) OF P BY A LINE SHELLING
I.E., ORDER THE VERTICES OF P BY A
LINEAR FUNCTIONAL WHICH ASSUMES
DISTINCT VALUES ON ITS VERTICES

ORIENT THE EDGES OF P IN INCREASING
DIRECTION

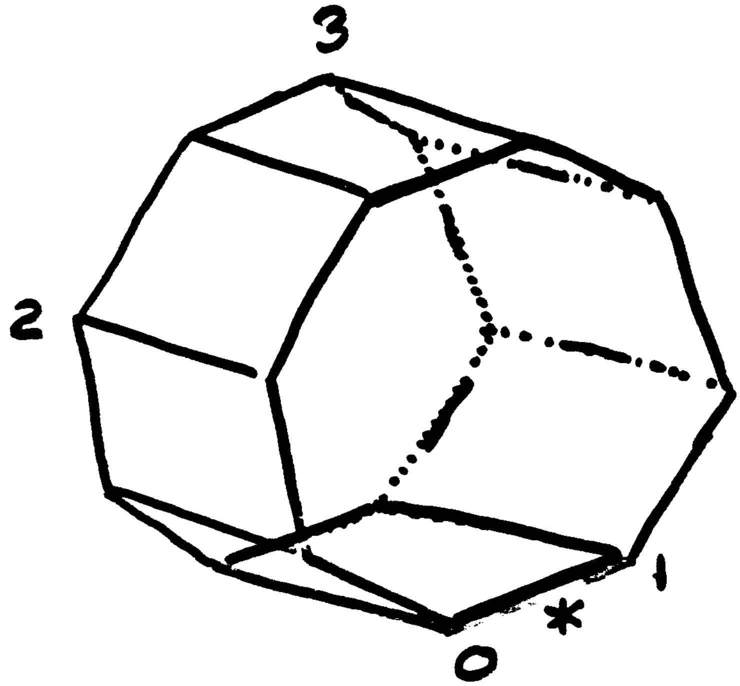
THE TYPE OF A VERTEX = ITS INDEGREE

$h_i(P)$ = NUMBER OF VERTICES OF TYPE i
$h(P) = (h_0(P), \ldots, h_d(P))$ = h-VECTOR

- INDEPENDENT OF CHOICE OF SHELLING
- LINEAR TRANSFORMATION OF f-VECTOR

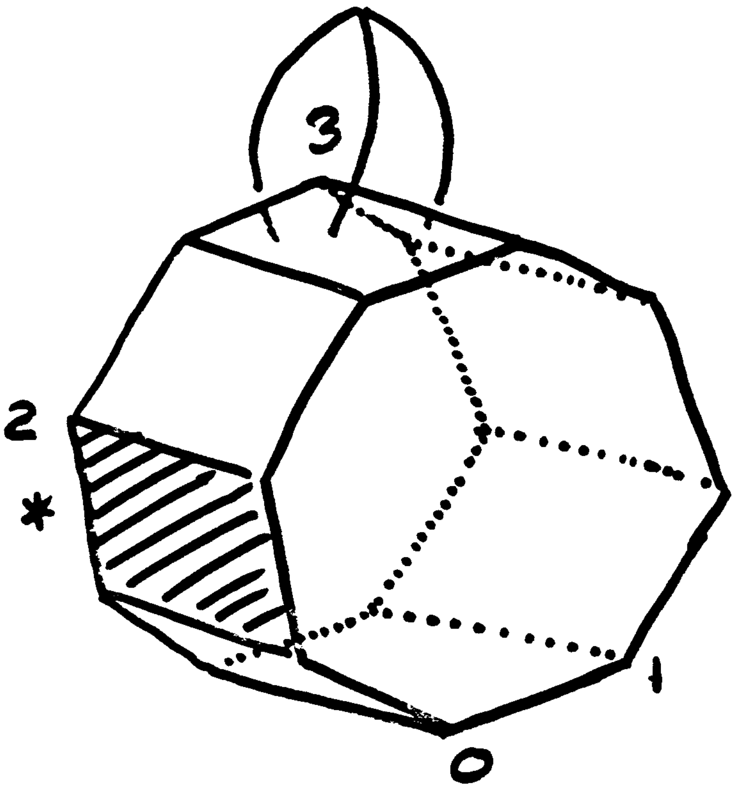


$h(P) = (1, 3, 3, 1)$

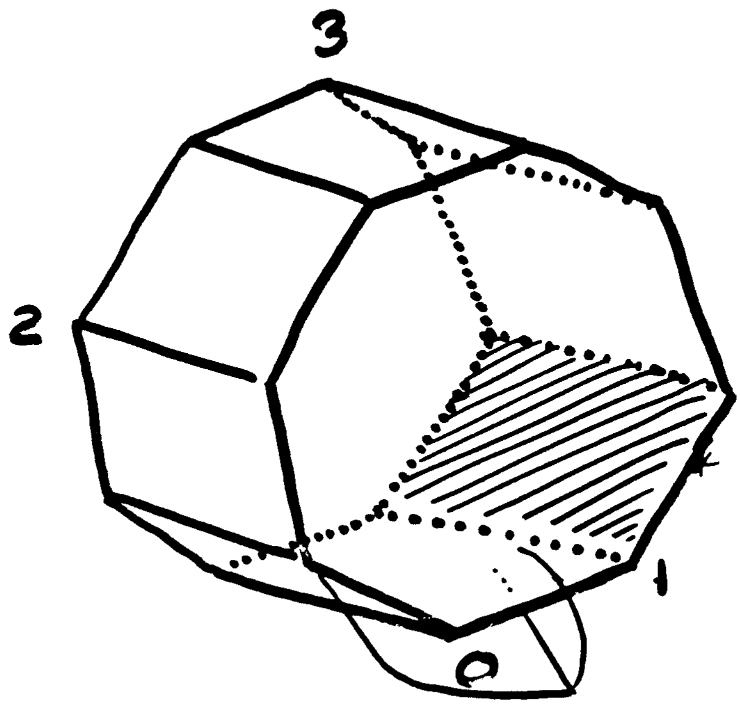


AT VERTEX OF TYPE i ASSIGN ARBITRARY WEIGHT TO NEWLY COMPLETED i-FACE AND "CHASE UPWARD" TO i-FACES BENEATH VERTICES OF TYPE $j > i$.

(MCMULLEN)



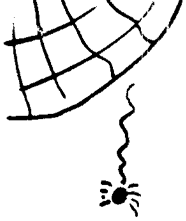
3
2
*
1
0

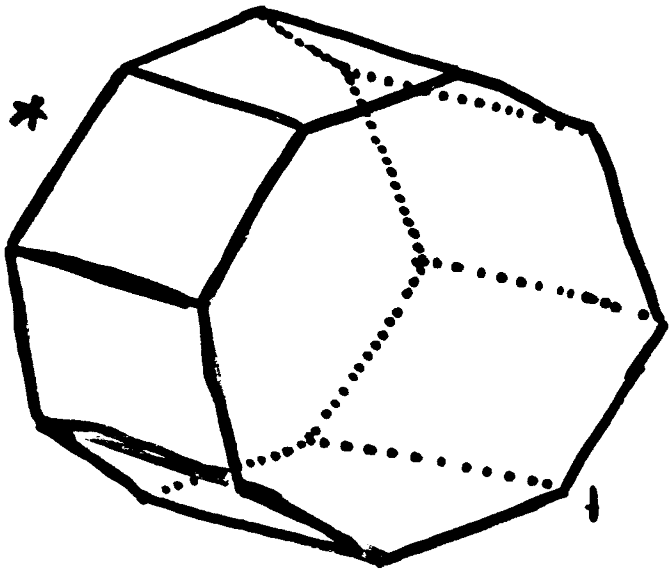


ANOTHER BASIS:

AT VERTEX OF TYPE i ASSIGN ARBITRARY WEIGHT TO NEWLY MET $(d-i)$-FACE AND COMPLETE BY ASSIGNING UNIQUE WEIGHTS TO ALL $(d-i)$-FACES MEETING NEWLY COMPLETED i-FACE

CALL THIS THE $(d-i)$-WEIGHT ASSOCIATED WITH THE i-FACE





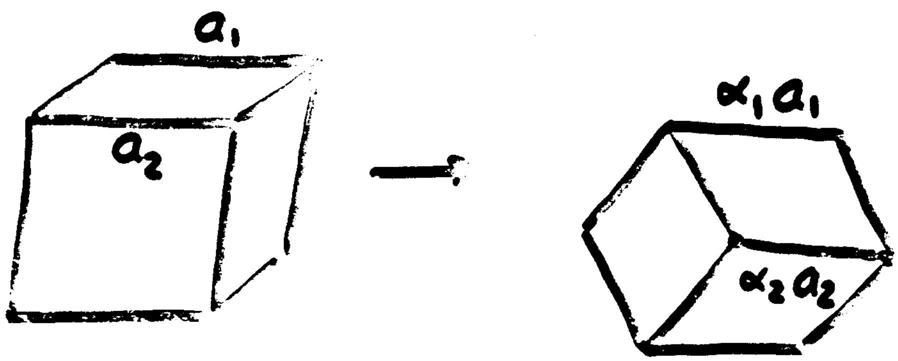
BASIS IS MORE "LOCAL"

HOW TO MULTIPLY WEIGHTS (McMULLEN)

IF a IS AN i-WEIGHT ON P
b IS A j-WEIGHT ON Q
THEN c IS AN $(i+j)$-WEIGHT ON $P \times Q$
WHERE

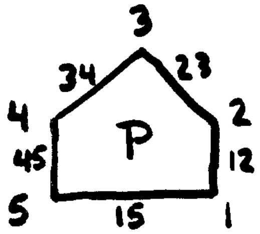
$$c(F \times G) = a(F)\, b(G)$$

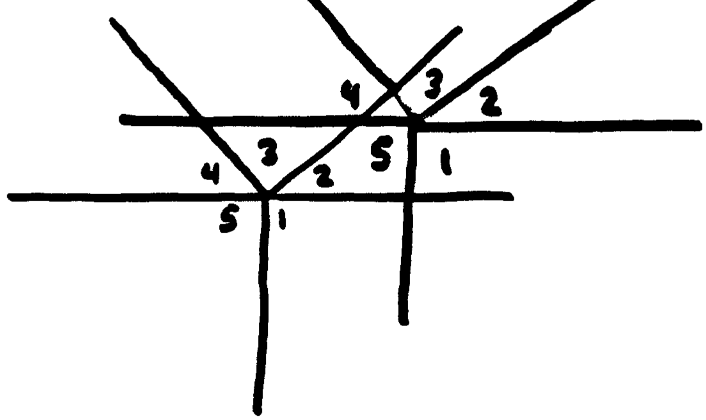
IF a IS AN i-WEIGHT ON P
AND Q IS A PROJECTION (LINEAR IMAGE)
OF P, THEN ONE CAN OBTAIN A
"PROJECTED" i-WEIGHT ON Q

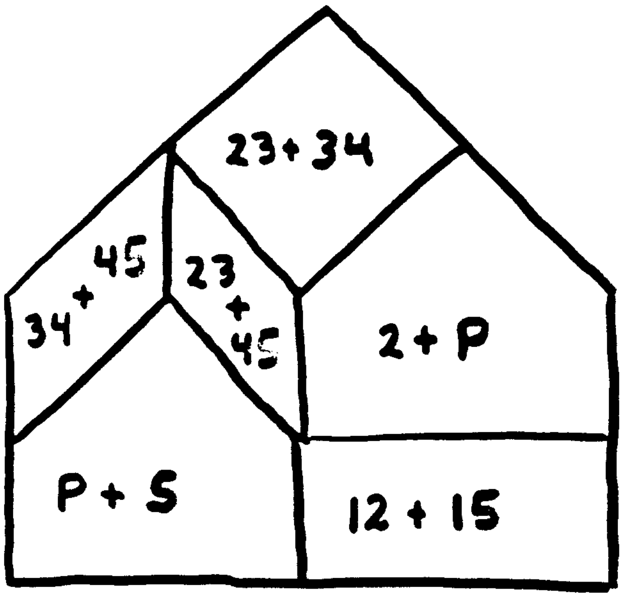


α_j = SAME SCALING FACTOR AS FOR
i-VOLUMES

IF $a \in \Omega_i(P)$, $b \in \Omega_j(P)$
OBTAIN $(i+j)$-WEIGHT ON $P \times P$;
PROJECT TO $P + P$ $(= 2P)$
TO GET $(i+j)$-WEIGHT ON P







ONE DECOMPOSITION INDUCED BY FIBER POLYTOPE FOR $P \times P \longrightarrow P + P$

MCMULLEN'S PROOF OF g-THEOREM

- $\Omega(P)$ IS A GRADED ALGEBRA
- $\dim_{\mathbb{R}} \Omega_i(P) = h_i(P)$
- MULTIPLICATION BY p^{d-2i} IS A BIJECTION :

$$\underset{(i < d/2)}{} \quad \Omega_i(P) \longrightarrow \Omega_{d-i}(P)$$

$$\begin{tikzcd}
\Omega_0 \arrow[r] \arrow[rrrrrr, bend left] & \Omega_1 \arrow[r] \arrow[rrrr, bend left] & \Omega_2 \arrow[r] \arrow[rr, bend left] & \cdots \arrow[r] & \Omega_{d-2} \arrow[r] & \Omega_{d-1} \arrow[r] & \Omega_d
\end{tikzcd}$$

$$h_0 \le h_1 \le h_2 \le \cdots \ge h_{d-2} \ge h_{d-1} \ge h_d$$

h-VECTOR IS SYMMETRIC, NONNEGATIVE, UNIMODAL

"LEFSCHETZ BIJECTION"

UNLIKE FIRST PROOF OF g-THEOREM, NEED NOT REQUIRE P TO BE RATIONAL

WHEN P IS RATIONAL, $\Omega(P)$ CORRESPONDS TO COHOMOLOGY OF ASSOCIATED TORIC VARIETY

ANOTHER PERSPECTIVE ON WEIGHTS

$P = \{ y : a_i^T y \le 1,\ 1 \le i \le n \}$, SIMPLE, $\dim P = d$

$P(x) = \{ y : a_i^T y \le x_i,\ 1 \le i \le n \}$

$V(x) = \mathrm{VOL}(P(x))$

FOR $x_i \approx 1$, $V(x)$ IS A HOMOGENEOUS POLYNOMIAL
OF DEGREE d.

$R = \mathbb{R}\left[\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}\right]$

LET $a \in R$ BE HOMOGENEOUS OF DEGREE i

$a\,V(x)$ IS HOMOGENEOUS OF DEGREE $d-i$

"CORRESPONDS" TO AN i-WEIGHT a'

ALL i-WEIGHTS ARE OBTAINABLE THIS WAY

TO MULTIPLY WEIGHTS a', b'

FIND $a, b \in R$ ST

$a\,V(x)$ CORRESPONDS TO a'

$b\,V(x)$ CORRESPONDS TO b'

THEN $ab\,V(x)$ CORRESPONDS TO $a'b'$

THE SPECIAL 1-WEIGHT p CORRESPONDS
TO $\frac{\partial}{\partial x_1} + \cdots + \frac{\partial}{\partial x_n}$

TORIC h-VECTOR

P: d-POLYTOPE

$$h(P) = (h_0, h_1, \ldots, h_d)$$

$$h(P,t) = \sum_{i=0}^{d} h_i t^{d-i}$$

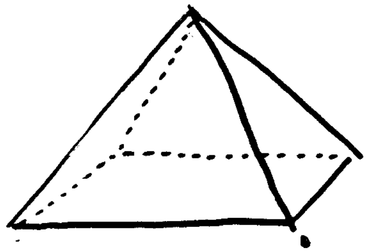
$$g(P) = (g_0, g_1, \ldots, g_{\lfloor d/2 \rfloor})$$

$$g(P,t) = \sum_{i=0}^{\lfloor d/2 \rfloor} g_i t^i$$

$$g_0 = h_0; \quad g_i = h_i - h_{i-1}, \quad 1 \le i \le \lfloor d/2 \rfloor$$

$$g(\emptyset, t) = h(\emptyset, t) = 1$$

$$h(P,t) = \sum_{\substack{G \text{ FACE OF } P \\ G \neq \emptyset}} g(\mathrm{lk}\, G, t)(t-1)^{\dim G}$$



0-FACES	$4 \times (1+0t)(t-1)^0$	4
	$1 \times (1+t)(t-1)^0$	$1+t$
1-FACES	$8 \times (1)(t-1)^1$	$-8+8t$
2-FACES	$5 \times (1)(t-1)^2$	$5-10t+5t^2$
3-FACES	$1 \times (1)(t-1)^3$	$-1+3t-3t^2+t^3$

$$1+2t+2t^2+t^3$$

$(1, 2, 2, 1)$

IF P IS RATIONAL:

COMPONENTS OF TORIC h-VECTOR CORRESPOND TO DIMENSIONS OF INTERSECTION HOMOLOGY

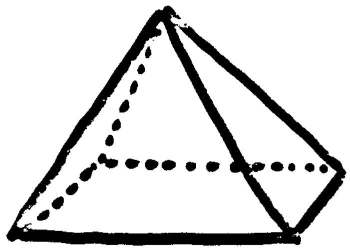
$h(P)$ IS NONNEGATIVE, $\underbrace{\text{SYMMETRIC}}$, UNIMODAL

EASY

THERE ARE COMBINATORIAL TYPES OF POLYTOPES THAT CANNOT BE REALIZED AS RATIONAL POLYTOPES

GUIDED BY WEIGHTS AND INTERSECTION HOMOLOGY, CAN WE DERIVE PROPERTIES OF $h(P)$ FOR NOT-NECESSARILY-RATIONAL REALIZATIONS OF POLYTOPES?

CAUTIONARY EXAMPLE



$$h(P) = (1,2,2,1)$$

$$\dim \Omega(P) = (1,1,2,1)$$

ALMOST-SIMPLE POLYTOPES

P IS ALMOST-SIMPLE IF THE RESULT OF TRUNCATING ALL OF ITS VERTICES IS A SIMPLE POLYTOPE

EXAMPLE: EVERY 3-POLYTOPE IS ALMOST SIMPLE

THEOREM (FOEGE + L): IF P IS ALMOST SIMPLE THEN $h(P)$ IS SYMMETRIC & NONNEGATIVE (& UNIMODAL?)

RESTRICTIONS

IF $a \in \Omega_i(P)$ AND F IS A FACE OF P
THEN $a|_F \in \Omega_i(F)$

IF $a, b \in \Omega(P)$ AND F IS A FACE OF P
THEN

$$(ab)|_F = (a|_F)(b|_F)$$

I & C FORMULAS

IF $Q \cong I \times P$

I ↳ INTERVAL

THEN $h(Q,t) = (1+t)\,h(P,t)$

$$(a,b,c,b,a) \longrightarrow (a,\ a+b,\ b+c,\ b+c,\ a+b,\ a)$$

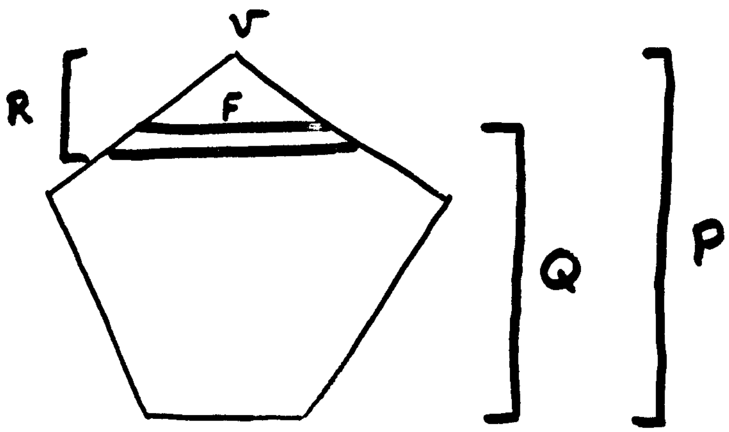
IF $Q \cong CP$

CP ↳ PYRAMID OVER P ("CONE")

THEN $h(Q,t) = g(P,t) + t\,h(P,t)$

$$(a,b,c,b,a) \longrightarrow (a,\ b,\ c,\ c,\ b,\ a)$$

TRUNCATION OF A VERTEX



$$
\begin{aligned}
h(P) &= h(Q) + h(R) - h(Q \cap R) \\
&= h(Q) + h(CF) - h(IF) \\
&= h(Q) + g(F) - h(F)
\end{aligned}
$$

$$
h(Q) = h(P) + \underbrace{h(F) - g(F)}_{\dim\ p|_F\, \Omega(F)}
$$

SUGGESTS TRYING TO FIND SPACES FOR $h(P)$ AS SUBSPACES OR QUOTIENT SPACES OF $\Omega(Q)$

CRUCIAL FACTS

- IF F IS A FACET CREATED BY TRUNCATING A VERTEX THEN

$$\{a|_F : a \text{ IS AN } i\text{-WEIGHT ASSOCIATED WITH A } (d-i)\text{-FACE } G \text{ OF } F\}$$

$$= \bar{p}\,\Omega_{i-1}(F) \qquad 1 \le i \le d-1$$

$$(\bar{p} = p|_F)$$

- IF F_1 AND F_2 ARE EACH FACETS CREATED BY TRUNCATING VERTICES v_1 AND v_2, RESPECTIVELY, AND IF a IS AN i-WEIGHT ASSOCIATED WITH A $(d-i)$-FACE OF F_1, THEN $a|_{F_2} = 0$

THEOREM: SUPPOSE P IS ALMOST SIMPLE WITH VERTICES $v_1, \ldots, v_m$. TRUNCATE THE VERTICES TO OBTAIN SIMPLE Q WITH TRUNCATION FACETS $F_1, \ldots, F_m$.
LET $\overline{\Omega}_i(Q) = \{a \in \Omega_i(Q) : a|_{F_j} = 0 \ \forall j\}$

THEN $\dim_{\mathbb{R}} \overline{\Omega}_i(Q) = h_i(P)$, $\lceil \frac{d}{2} \rceil \le i \le d$

COROLLARY: $\dim_{\mathbb{R}} \Omega_i(P) = h_i(P)$, $\lceil \frac{d}{2} \rceil \le i \le d$.
I.E., NO NEED TO MODIFY DEFINITION OF i-WEIGHTS FOR LARGE i.

QUESTIONS: BASED ON INTUITION FROM INTERSECTION HOMOLOGY,

- ARE THESE THE APPROPRIATE SPACES?
- FOR $0 \le i \le \lfloor \frac{d}{2} \rfloor$ SHOULD WE USE

$$\overline{\Omega}_i(Q) = \left[p^{d-2i} \right]^{-1} \overline{\Omega}_{d-i}(Q) \ ?$$

- IS THERE A LEFSCHETZ ACTION WHICH IMPLIES UNIMODALITY OF $h(F)$?

WHAT ABOUT GENERAL POLYTOPES?

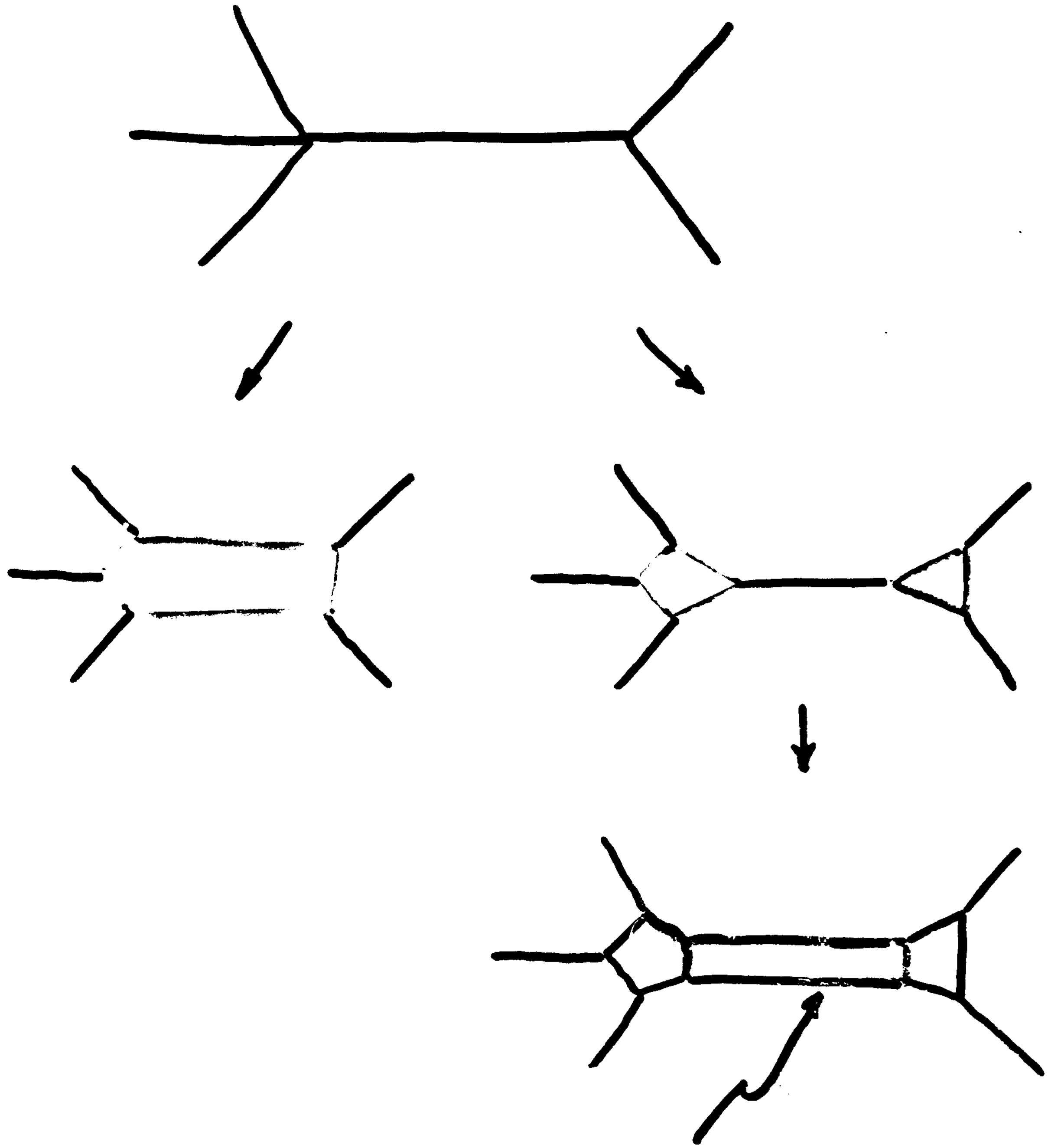
P : d-POLYTOPE

TRUNCATE 0-FACES
1-FACES
2-FACES
⋮
$(d-2)$-FACES

TO OBTAIN SIMPLE $Q = \mathrm{TRUNC}(P)$

$$h(Q) = h(P) + \sum_{\substack{F:\ F \text{ A FACE OF } P \\ F \neq \emptyset \\ F \neq P}} \underbrace{\left[h(\ell k\, F, t) - g(\ell k\, F, t) \right]}_{\substack{\text{NONNEGATIVE IF } h \text{ UNIMODAL} \\ \text{FOR } \ell k\, F}} h(\mathrm{TRUNC}\, F, t)$$

INTERPRETATION VIA WEIGHTS?



A CARTESIAN PRODUCT