14 SUBDIVISIONS AND TRIANGULATIONS OF POLYTOPES

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INTRODUCTION

Starting from a given finite set of points V, we consider subdivisions of the convex hull of V into polytopes $\{P_1, \ldots, P_m\}$. A subdivision is a triangulation if each P_i is a simplex. We start with definitions and properties, then turn to methods of constructing subdivisions and triangulations, face-counting results, some particular triangulations, and secondary and fiber polytopes. We confine ourselves to triangulations of convex structures for the most part.

14.1 BASIC CONCEPTS

GLOSSARY

Affine span: The affine span of a set V is the smallest affine space, or flat, containing V. It is denoted by aff(V).

Convex hull: The convex hull of a set V is the smallest convex set containing V. It is denoted by conv(V).

Polytope: A polytope P is the convex hull of a finite set of points. If it is d-dimensional, its boundary consists of faces of dimension -1 (the empty set), 0 (vertices), 1 (edges), 2, ..., and $d-1$ (facets). Its set of vertices will be denoted by vert(P).

Subdivision: Suppose V is a finite set of points such that $P = \text{conv}(V)$ is d-dimensional (a d-polytope). A subdivision of V is a finite collection $S = \{P_1, \ldots, P_m\}$ of d-polytopes such that:

- The vertices of each P_i are drawn from V (though it is not required that every point in V be used as a vertex of some P_i);
- P is the union of $P_1, \ldots, P_m$; and
- If $i \neq j$ then $P_i \cap P_j$ is a common (possibly empty) face of the boundaries of P_i and P_j.

In this case we will also say that S is a subdivision of the polytope P.

Trivial subdivision: The trivial subdivision of V is the subdivision $\{\text{conv}(V)\}$.

Simplex: A d-dimensional simplex is a d-polytope with exactly $d+1$ vertices.

Triangulation: A subdivision in which each P_i is a simplex.

Faces: The faces of a subdivision $\{P_1, \ldots, P_m\}$ consist of $P_1, \ldots, P_m$ together with their faces.

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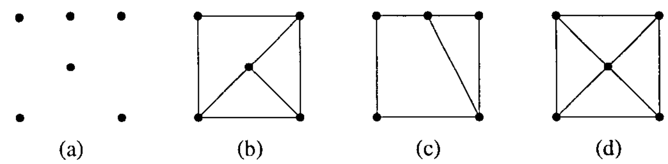
EXAMPLES

In Figure 14.1.1, (a) shows a set of points. The collection of three polygons in (b) is not a subdivision of that set since not every pair of polygons meets along a common edge or vertex; (c) shows a subdivision that is not a triangulation; and (d) gives a triangulation.

FIGURE 14.1.1
(a) *A set of points.*
(b) *A nonsubdivision.*
(c) *A subdivision.*
(d) *A triangulation.*



14.2 SEQUENTIAL CONSTRUCTION PROCEDURES

The convex hull of a finite set of points $V = \{v_1, \ldots, v_n\}$ can be constructed sequentially by successively constructing $R_1 = \text{conv}(\{v_1\})$, $R_2 = \text{conv}(R_1 \cup \{v_2\})$, $R_3 = \text{conv}(R_2 \cup \{v_3\}), \ldots, R_n = \text{conv}(R_{n-1} \cup \{v_n\})$. With little additional effort, a triangulation of each R_i can also be constructed, resulting finally in a triangulation of V. Another method of constructing a triangulation of V is to begin with the trivial subdivision of V, and then obtain a sequence of refinements. See [Lee91, Zie95].

GLOSSARY

Refinement of a subdivision: Suppose $S = \{P_1, \ldots, P_l\}$ and $T = \{Q_1, \ldots, Q_m\}$ are two subdivisions of V. Then T is a refinement of S if for each j, $1 \le j \le m$, there exists i, $1 \le i \le l$, such that $Q_j \subseteq P_i$. In this case we will write $T \le S$.

Visible facet: Suppose P is a d-polytope in $\mathbb{R}^d$, F is a facet of P, and v is a point in $\mathbb{R}^d$. There is a unique hyperplane H (affine set of dimension $d-1$) containing F. The polytope P is contained in exactly one of the closed halfspaces determined by H. If v is contained in the opposite open halfspace, then F is said to be visible from v. If P is a k-polytope in $\mathbb{R}^d$ with $k < d$ and $v \in \text{aff}(P)$, then the above definition is modified in the obvious way so that everything is considered relative to the ambient space $\text{aff}(P)$.

Placing a vertex: Suppose $S = \{P_1, \ldots, P_m\}$ is a subdivision of V and $v \notin V$. The subdivision T of $V \cup \{v\}$ that results from placing v is obtained as follows:

- If $v \notin \text{aff}(V)$, then for each $P_i \in S$, include $\text{conv}(P_i \cup \{v\})$ in T.
- If $v \in \text{aff}(V)$, then for each $P_i \in S$, $P_i \in T$; and if F is a facet of P_i that is contained in a facet of $\text{conv}(V)$ visible from v, then $\text{conv}(F \cup \{v\}) \in T$.
- Note: if $v \in \text{conv}(V)$, then $S = T$.

Pulling a vertex: Suppose $S = \{P_1, \ldots, P_m\}$ is a subdivision of V and $v \in V$. The result of pulling v is the subdivision T of V obtained by modifying each $P_i \in S$ as follows:

- If $v \notin P_i$, then $P_i \in T$.
- If $v \in P_i$, then for every facet F of P_i not containing v, $\text{conv}\,(F \cup \{v\}) \in T$.

Note that T is a refinement of S.

Pushing a vertex: Suppose $S = \{P_1, \ldots, P_m\}$ is a subdivision of V (where $\dim(\text{conv}\,(V)) = d$) and $v \in V$. The result of pushing v is the subdivision T of V obtained by modifying each $P_i \in S$ as follows:

- If $v \notin P_i$, then $P_i \in T$.
- If $v \in P_i$ and $\text{conv}\,(\text{vert}\,(P_i)\backslash\{v\})$ is $(d-1)$-dimensional (i.e., P_i is a pyramid with apex v), then $P_i \in T$.
- If $v \in P_i$ and $P_i' = \text{conv}\,(\text{vert}\,(P_i) \setminus \{v\})$ is d-dimensional, then $P_i' \in T$. Also, if F is any facet of P_i' that is visible from v, then $\text{conv}\,(F \cup \{v\}) \in T$.

Note that T is a refinement of S.

Lexicographic subdivisions: If T is any subdivision of V constructed by starting with the trivial subdivision of V and then pushing and/or pulling some/all of the points in V in some order, then T is a lexicographic subdivision.

Diameter of a subdivision: Suppose $\{P_1, \ldots, P_m\}$ is a subdivision. Polytopes $P_i \neq P_j$ are ***adjacent*** if they share a common facet. A sequence $P_{i_0}, \ldots, P_{i_k}$ is a ***path*** if P_{i_j} and $P_{i_{j-1}}$ are adjacent for each $1 \leq j \leq k$. The ***length*** of such a path is k. The ***distance*** between polytopes P_i and P_j is the length of the shortest path connecting them. The diameter of the subdivision is the maximum distance occurring between pairs of polytopes P_i, P_j.

MAIN RESULTS

1. If the points of V are ordered $\{v_1, \ldots, v_n\}$ and T is the subdivision obtained by placing the points of V in that order, then
   (a) T is a triangulation of V.
   (b) The same triangulation is obtained by starting with the trivial subdivision of V and pushing the points of V in the opposite order $v_n, \ldots, v_1$.
   (c) The diameter of T does not exceed $2(n-d-1)$, where $d = \dim(\text{conv}\,(V))$ [Lee91].
2. If S is any subdivision of $V = \{v_1, \ldots, v_n\}$, then S can be refined to a triangulation by sequentially pushing and/or pulling all the vertices in some order.
3. For any specified point $v_k \in V = \{v_1, \ldots, v_n\}$, there is a triangulation of V in which every simplex of maximum dimension contains v_k as a vertex—begin with the trivial subdivision $S = \{\text{conv}\,(V)\}$, pull v_k first, then pull the remaining points in any order.
4. For any specified simplex F with vertices in $V = \{v_1, \ldots, v_n\}$, there is a triangulation of V in which F is a face—first place the vertices of F, then place the remaining vertices in any order.

5. If $\dim(\text{conv}(V)) = d$ and $\text{card}(V) \leq d+3$, then every triangulation of V can be obtained by placing (respectively, pushing) the points of V in some order [Lee91].

6. If V is the set of vertices of a convex n-gon in $\mathbb{R}^2$, then every triangulation of V can be obtained by placing (respectively, pushing) the points of V in some order.

7. Suppose $V = \{v_1, \ldots, v_n\}$ is the set of vertices of some d-polytope P. For a face F of P, define $v(F) = v_k$, where $k = \min\{i \mid v_i \in F\}$. A ***full flag*** of P is a chain C of faces $F_0 \subset F_1 \subset F_2 \subset \cdots \subset F_{d-1} \subset F_d = P$ such that $\dim(F_i) = i$, $0 \leq i \leq d$, and $v(F_i) \neq v(F_{i-1})$, $1 \leq i \leq d$. For a full flag C, write $v(C) = \{v(F_0), \ldots, v(F_d)\}$. Then the simplices of the triangulation of P determined by pulling the vertices of P in the order $v_1, \ldots, v_n$ are $\{\text{conv}(v(C)) \mid C \text{ is a full flag of } P\}$.

EXAMPLES

Figure 14.2.1 gives three triangulations of a set of seven points that can be obtained from the trivial subdivision by pulling and pushing [Lee91]. The triangulation in (a) is obtained by pulling point 1, but cannot be obtained by pushing alone. The triangulation in (b) is obtained by pushing the points in the indicated order, or placing them in the opposite order, but cannot be obtained by pulling points alone. The lexicographic triangulation in (c) is obtained by pushing point 1 and then pulling point 2, but cannot be obtained by pulling points alone or by pushing points alone.

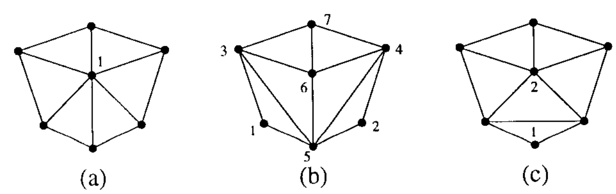


FIGURE 14.2.1
(a) *A pulling triangulation.*
(b) *A pushing triangulation.*
(c) *A lexicographic triangulation.*

14.3 REGULAR TRIANGULATIONS AND SUBDIVISIONS

GLOSSARY

Regular subdivision: Any convex hull algorithm for points in $\mathbb{R}^{d+1}$ can be used to compute subdivisions of sets of points $V = \{v_1, \ldots, v_n\}$ in $\mathbb{R}^d$ (see Chapter 19 of this Handbook). Such subdivisions are called regular and are obtained in the following way:

(i) Regard V as sitting naturally in $(\mathbb{R}^d, 0)$.

(ii) Choose arbitrary real numbers $\alpha_1, \ldots, \alpha_n$.

(iii) Determine $Q = \text{conv}(\{(v_1, \alpha_1), \ldots, (v_n, \alpha_n)\})$.
(iv) Project the lower facets of Q onto $(\mathbb{R}^d, 0)$.

Here, a lower facet is a facet of Q that is visible from the point $(0, -\alpha)$ for α sufficiently large. See [GKZ94, Lee91, Zie95]. Some algorithmic aspects of computing regular triangulations can be found in [ES96].

Weakly regular subdivision: A subdivision S of a set V is weakly regular if there exists a set V' having a regular subdivision S' such that (V', S') is combinatorially isomorphic to (V, S). That is, there is a one-to-one correspondence between the points of V and the points of V' such that for every subset $F \subseteq V$ and corresponding subset $F' \subseteq V'$, $\text{conv}(F)$ is a face of S if and only if $\text{conv}(F')$ is a face of S'.

Polytopal complex: A polytopal complex is a finite, nonempty collection S of polytopes in $\mathbb{R}^d$ that contains all the faces of its polytopes, and such that the intersection of any two of its polytopes is a common face of each of them. The dimension of S, $\dim(S)$, is the largest dimension of a polytope in S, and S is ***pure*** if every polytope in S is contained in one of dimension $\dim(S)$ [Zie95]. (Thus every subdivision is a pure polytopal complex.)

Shellable: A pure polytopal complex S is shellable if it is 0-dimensional (i.e., a finite set of points) or else $\dim(S) = k > 0$ and S has a ***shelling***, i.e., an ordering of its maximal faces $P_1, \ldots, P_m$ such that for $2 \leq j \leq m$ the intersection of P_j with $P_1 \cup \cdots \cup P_{j-1}$ is nonempty and is the beginning segment of a shelling of the $(k-1)$-dimensional boundary complex of P_j [Zie95].

MAIN RESULTS

1. All regular subdivisions are shellable. On the other hand, there exist nonshellable subdivisions, starting in dimension 3 (see [Zie95]).

2. All lexicographic triangulations are regular. In particular, if $v_1, \ldots, v_n$ are pushed/pulled in that order, then the corresponding triangulation is obtained by choosing $|\alpha_1| \gg |\alpha_2| \gg \cdots \gg |\alpha_n| \gg 0$, where $\alpha_i > 0$ if v_i is pushed and $\alpha_i < 0$ if v_i is pulled [Lee91].

3. If $\text{card}(V) = \dim(\text{conv}(V)) + 2$, then there are exactly two triangulations of V, and both are regular.

4. If $\text{card}(V) = \dim(\text{conv}(V)) + 3$, then all subdivisions of V are regular [Lee91].

5. If V is the set of vertices of a convex n-gon in $\mathbb{R}^2$, then all subdivisions of V are regular.

6. If $V \subset \mathbb{R}^2$, then all subdivisions of V are weakly regular as a consequence of Steinitz's Theorem. However, there exists a set V of 6 points in $\mathbb{R}^2$ having a nonregular triangulation [Lee91] (see Figure 14.3.2(b)).

7. There exists a set V of 7 points that are the vertices of a 3-polytope having a nonregular triangulation that is not even weakly regular [Lee91] (see Figure 14.3.3(b)).

8. If V is the vertex set of $C_{4n-4}(4n)$, the cyclic polytope of dimension $4n-4$ with $4n$ vertices, then V has at least 2^n triangulations, of which only $O(n^4)$ are regular [dHSS96]. (See Chapter 13 of this Handbook for the definition of the cyclic polytope.)

9. If $\alpha_i = \|v_i\|^2$, then the resulting subdivision is the Delaunay subdivision. If $\alpha_i = -\|v_i\|^2$, then the resulting subdivision is the "farthest site" Delaunay subdivision. (See Chapters 20 and 22 of this Handbook.)

10. Given a subdivision of V, one can test its regularity by using linear programming to check the existence of appropriate α_i, $1 \le i \le n$. On the other hand, checking weak regularity is quite hard, perhaps as difficult as determining solutions to systems of real polynomial inequalities (see comments on the Universality Theorem in Chapter 13 and in [Zie95]).

EXAMPLES

Figure 14.3.1 shows the two triangulations (both regular) of the vertices of a 3-dimensional bipyramid over a triangle. In (a) there are two tetrahedra in the triangulation, sharing a common internal triangle; in (b) there are three, sharing a common internal edge.

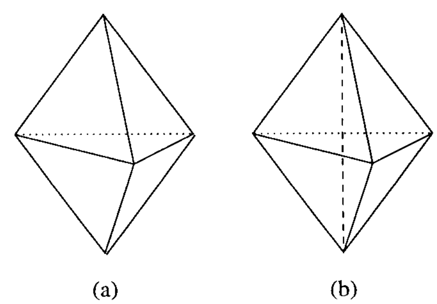


FIGURE 14.3.1
The two triangulations of a set of 5 points in $\mathbb{R}^3$.

Figure 14.3.2 shows triangulations of two different sets of 6 points in $\mathbb{R}^2$. The first triangulation is regular, the second is not. But by virtue of the first triangulation, the second is weakly regular.

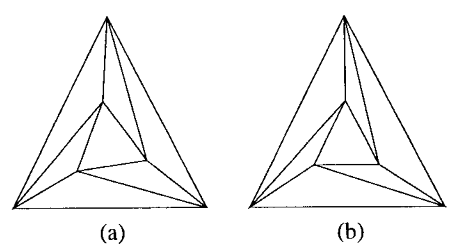


FIGURE 14.3.2
A regular and a nonregular (but weakly regular) triangulation.

Figure 14.3.3 shows two 3-polytopes, each with 7 vertices. The polytope in (a) is a "capped triangular prism" and its vertex set admits two nonregular triangulations. Denoting the simplices by their vertex sets, these are: {1257, 1457, 1236, 1267, 1345, 1346, 1467} and {1245, 1247, 1237, 1367, 1356, 1456, 1467}. Both triangulations are, however, weakly regular. The polytope in (b) is obtained from the capped triangular prism by rotating the top triangle by a small amount. Its vertex set has one nonregular triangulation, which is not even weakly regular: {1245, 1247, 1237, 1367, 1356, 1456, 1467, 2457, 2367, 2345}. See [Lee91].

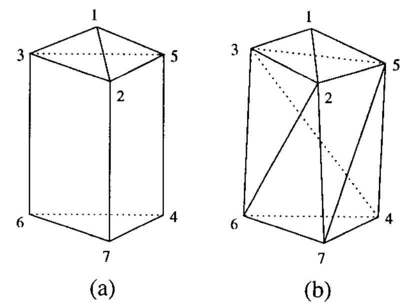


FIGURE 14.3.3
Two polytopes with nonregular triangulations.

14.3.1 TRIANGULATING REGIONS BETWEEN POLYTOPES

Suppose P and Q are two d-polytopes in $\mathbb{R}^d$ with disjoint vertex sets V and W, respectively, and Q is contained in P. One can triangulate the region inside of P and outside of Q by the following procedure [GP88]:

1. Construct the regular subdivision of $V \cup W$ by setting $\alpha_i = 1$ for the $v_i \in V$ and $\alpha_i = 0$ for $v_i \in W$.
2. Refine this subdivision to a triangulation by pushing and/or pulling each point in $V \cup W$.
3. Ignore the portion of the triangulation within Q.

Now suppose P and Q are two d-polytopes in $\mathbb{R}^d$ with disjoint vertex sets V and W, respectively, and that there is a hyperplane H for which P and Q are contained in opposite open halfspaces. One can triangulate the region in $\text{conv}(P \cup Q)$ that is exterior to P and Q by the following procedure [GP88]:

1. Construct the regular subdivision of $V \cup W$ by setting α equal to the distance of v_i to H for each $v_i \in V \cup W$. For example, if $H = \{x \mid a \cdot x = \beta\}$, then α_i can be taken to equal $|a \cdot v_i - \beta|$. (It would also suffice to use these values of α_i for $v_i \in V$ and to set $\alpha_i = 0$ for $v_i \in W$.)
2. Refine this subdivision to a triangulation by pushing and/or pulling each point in $V \cup W$.
3. Ignore the portion of the triangulation within P or Q.

14.4 SUBDIVISIONS, TRIANGULATIONS, AND FACE VECTORS

Suppose $S = \{P_1, \ldots, P_m\}$ is a subdivision of V with $\dim(\text{conv}(V)) = d$. In this section we examine some of the properties of its face numbers. See [Bay94, BL93].

GLOSSARY

Boundary: Suppose S is as above. The boundary complex ∂S of S is the set of those faces of S given by $\{F \in S \mid F \subseteq G$ for some face G of dimension $d-1$ contained in exactly one $P_i\}$. In particular, the empty set is a member of ∂S.

Interior: Suppose S is as above. The interior $\operatorname{int} S$ is the set of those faces of S that are not in the boundary.

f-vector: Suppose S is as above. Let $f_j(S)$ denote the number of j-dimensional faces of S, $-1 \leq j \leq d$. Note that $f_{-1}(S) = 1$ since the empty set is the unique face of S of dimension -1. The f-vector of S is $f(S) = (f_0(S), \ldots, f_d(S))$. In an analogous way we define $f(\partial S)$ and $f(\operatorname{int} S)$. Note that $f_{-1}(\partial S) = 1$ and $f_{-1}(\operatorname{int} S) = 0$.

Simplicial polytope: A simplicial polytope is one for which every facet (and hence every face) is a simplex.

14.4.1 h-VECTORS and g-VECTORS

Suppose S is any polytopal complex of dimension d. For example, S might be the boundary complex of a polytope of dimension $d+1$ or a subdivision of a finite set V such that $\operatorname{conv}(V)$ has dimension d.

We define the h-vector $h(S) = (h_0(S), \ldots, h_{d+1}(S))$ with generating function $h(S,x) = \sum_{i=0}^{d+1} h_i x^{d+1-i}$, and the g-vector $g(S) = (g_0(S), \ldots, g_{\lfloor (d+1)/2 \rfloor}(S))$ with generating function $g(S,x) = \sum_{i=0}^{\lfloor (d+1)/2 \rfloor} g_i x^i$ in the following recursive way:

1. $g_0(S) = h_0(S)$.
2. $g_i(S) = h_i(S) - h_{i-1}(S)$, $1 \leq i \leq \lfloor (d+1)/2 \rfloor$.
3. $g(\emptyset, x) = h(\emptyset, x) = 1$.
4. $h(S,x) = \displaystyle\sum_{G \text{ face of } S} g(\partial G, x)(t-1)^{d-\dim(G)}$.

For more information on f-vectors, g-vectors, and h-vectors, refer to Chapter 15. The formulas are simpler when all of the faces of S are simplices.

MAIN RESULTS

1. Assume that T is a triangulation of a d-polytope [BL93].
   (a) The number of d-simplices in T equals the sum of the components of the h-vector.
   (b) The h-vector is nonnegative.
   (c) The h-vectors of T, ∂T, and $\operatorname{int} T$ are related in the following ways:

$$h_i(T) - h_{d+1-i}(T) = h_i(\partial T) - h_{i-1}(\partial T), \quad 0 \leq i \leq d+1.$$

$$h_i(T) = h_{d+1-i}(\operatorname{int} T), \quad 0 \leq i \leq d+1.$$

In particular, the h-vectors and the f-vectors of ∂T and $\operatorname{int} T$ are completely determined by the h-vector (and hence the f-vector) of T.

(d) Assume further that T is shellable and that $P_1, \ldots, P_m$ is a shelling order of the d-dimensional simplices in T. In particular, each P_j meets $\bigcup_{i=1}^{j-1} P_i$ in some positive number s_j of facets of P_j, $2 \le j \le m$. Define also $s_1 = 0$. Then $h_i(T)$ equals card $\{j \mid s_j = i\}$, $0 \le i \le d+1$.

2. If S is the trivial subdivision of a convex d-polytope P consisting of P itself, then

$$h_i(S) = \begin{cases} h_i(\partial P) - h_{i-1}(\partial P), & 1 \le i \le \lfloor d/2 \rfloor, \\ 0, & \lfloor d/2 \rfloor < i \le d. \end{cases}$$

See [Bay94].

3. Suppose V is a finite set of points with rational coordinates, S is a subdivision of V, and $P = \operatorname{conv}(V)$. Then for all i, $h_i(S) \ge h_i(P)$ and $h_i(\partial S) \ge h_i(\partial P)$. Further, if P is simplicial and S is a triangulation, the result holds even without the rationality assumption. In either case, $f_d(S) \ge h_{\lfloor d/2 \rfloor}(\partial S) \ge h_{\lfloor d/2 \rfloor}(\partial P)$ [Bay94, Sta92].

14.4.2 SHALLOW TRIANGULATIONS

The concept of shallow triangulation is motivated by an attempt to understand the case of equality in the last result mentioned above. See [Bay94, BL93].

GLOSSARY

The following definitions concern triangulations T of a finite set V of vertices of a convex d-polytope P.

Carrier: If F is a face of T, the carrier $C(F)$ of F is the smallest face of P containing F.

Shallow: If $\dim(C(F)) \le 2\dim(F)$ for every face F of T, then T is a shallow triangulation.

Weakly neighborly: If all triangulations of V are shallow, then P is weakly neighborly.

Equidecomposable: If all triangulations of V have the same f-vector, then P is equidecomposable.

Stacked: If P has a triangulation in which there are no interior faces of dimension less than $d-1$, then P is stacked.

k-stacked: If P is a simplicial d-polytope that has a triangulation in which there are no interior faces of dimension less than $d-k$, then P is k-stacked. In particular, a simplicial polytope is stacked if and only if it is 1-stacked.

MAIN RESULTS

1. A polytope P is weakly neighborly if and only if every set of $k+1$ vertices is contained in a face of dimension at most $2k$ for all k [Bay94].

2. If P is weakly neighborly, then P is equidecomposable.

3. If T is a shallow triangulation of a polytope P, then $h(T) = h(P)$ and $h(\partial T) = h(\partial P)$ [Bay94].

4. If T is a triangulation of a polytope P with rational vertices and $h(T) = h(P)$, then T is shallow. Hence, if P is a rational polytope and $h(T) = h(P)$ for all triangulations T of P, then P is weakly neighborly [Bay94].

5. If P is a simplicial d-polytope, then it has a shallow triangulation if and only if it is k-stacked for some $1 \leq k \leq d/2$. In this case there is exactly one triangulation T of P having no interior faces of dimension less than $d - k$ (and this triangulation is the unique shallow one) [Bay94].

6. Suppose P is a d-polytope where $d > 3$. Then P is 1-stacked if and only if $g_2(\partial P) = 0$. See [BL93].

7. **Suppose P is a simplicial d-polytope such that $g_k(\partial P) = 0$ for some k with** $3 \leq k \leq \lfloor d/2 \rfloor$. Then there is another simplicial d-polytope that has the same f-vector and is $(k-1)$-stacked. **It is an open problem whether P itself is always $(k-1)$-stacked under this hypothesis;** this is known to be true if $f_0(P) \leq d+3$ or $k < f_0(P)/(f_0(P) - d)$. (See [BL93], but note that there are places where "k" appears instead of the correct "$k - 1$.")

Some classes of weakly neighborly polytopes are given below [Bay94]:

- In dimension less than 3, all polytopes are weakly neighborly.
- In dimension 3, the only weakly neighborly polytopes are pyramids (over polygons) and the triangular prism.
- The product of two simplices of any dimensions is weakly neighborly. (See Section 14.5.1 for the definition of product.)
- The only simplicial weakly neighborly polytopes are simplices and even-dimensional neighborly polytopes (those for which every subset of $d/2$ vertices determines a face of the polytope).
- Lawrence polytopes are weakly neighborly. (See [Bay94, Zie95] for the definition of Lawrence polytopes.)
- Pyramids over weakly neighborly polytopes are weakly neighborly.
- Subpolytopes of weakly neighborly polytopes are weakly neighborly.

14.4.3 RELATIONSHIPS TO COUNTING LATTICE POINTS

Triangulations of polytopes can be used to enumerate lattice points in polytopes. See [BL93].

GLOSSARY

Integral: A polytope is integral if every vertex has integer coordinates.

i (P, n): For an integral polytope P and a nonnegative integer n, $i(P,n)$ is the number of points $x \in P$ for which nx has integer coordinates. Equivalently, it is the number of integer points in nP.

Compressed ordering: An ordering of the vertices of an integral polytope P is compressed if every d-dimensional simplex of the triangulation obtained by pulling the vertices of P in that order has volume $1/d!$. P itself is ***compressed*** if every ordering of its vertices is compressed. (For example, the standard d-dimensional unit cube is compressed.)

MAIN RESULTS

1. $i(P,n)$ is a polynomial in n of degree d, called the Ehrhart polynomial of P (see Chapter 7).
2. For integral d-polytope P write $J(P,t) = 1 + \sum_{n=1}^{\infty} i(P,n)t^n$. Then $J(P,t) = W(P,t)/(1-t)^{d+1}$, where $W(P,t)$ is a polynomial of degree at most d with nonnegative integer coefficients.
3. If P is an integral d-polytope with compressed order σ, then

$$i(P,n) = \sum_{i=0}^{d} \binom{n-1}{i} f_i(T),$$

and $W(P,t) = h_0(T) + h_1(T)t + \cdots + h_d(T)t^d$, where T is the pulling triangulation induced by σ.

4. If P is a compressed integral d-polytope and σ is an ordering of its vertices, then the f-vector of the triangulation induced by σ depends only on P, not on σ.

For example, if P is the standard unit 3-cube, then any ordering σ produces a compressed triangulation T with h-vector $h(T) = (1,4,1,0,0)$. Thus $J(P,t) = (1+4t+t^2)/(1-t)^4 = (1+4t+t^2)(1+4t+10t^2+20t^3+35t^4+\cdots) = 1+8t+27t^2+64t^3+125t^4+\cdots$.

14.5 SOME PARTICULAR TRIANGULATIONS

We gather together some information on some particular triangulations, including triangulations of the product of two simplices, the d-dimensional cube, the convex n-gon, and complete barycentric subdivisions.

14.5.1 PRODUCT OF TWO SIMPLICES

Consider the $(k+l)$-polytope $P = \Delta_k \times \Delta_l$, the product of a k-dimensional simplex Δ_k and an l-dimensional simplex Δ_l. We consider triangulations of P using the points in its vertex set V. See [BCS88, deL96, GKZ94, Hai91].

GLOSSARY

Product: If P is a subset of $\mathbb{R}^k$ and Q is a subset of $\mathbb{R}^l$, then the product of P and Q is the subset of $\mathbb{R}^{k+l}$ given by $\{(v,w) \mid v \in P,\ w \in Q\}$.

MAIN RESULTS

1. As mentioned before, $P = \Delta_k \times \Delta_l$ is weakly neighborly, and so every triangulation has the same f-vector and h-vector. In particular, if T is a triangulation of P, then $f_{k+l}(T) = (k+l)!/(k!l!)$, and $h_i(T) = \binom{k}{i}\binom{l}{i}$ for $0 \le i \le k+l$ (with $h_i(t)$ taken to be zero if $i > \min\{k,l\}$) [BCS88].

2. Given a triangulation $\{P_1,\dots,P_s\}$ of a k-polytope P and a triangulation $\{Q_1,\dots,Q_t\}$ of an l-polytope Q, then there is a triangulation of $P \times Q$ using $s \cdot t \cdot (k+l)!/(k!l!)$ simplices of dimension $k+l$. To see this, observe that $\{P_i \times Q_j \mid 1 \le i \le s,\ 1 \le j \le t\}$ is a subdivision of $P \times Q$. Now refine this subdivision to a triangulation by, for example, pulling the vertices of $P \times Q$. Each $P_i \times Q_j$ will thereby be refined into $(k+l)!/(k!l!)$ simplices [Hai91].

3. All triangulations of $\Delta_2 \times \Delta_3$ and $\Delta_2 \times \Delta_4$ are regular. On the other hand, if $k, l \ge 3$, then there exist nonregular triangulations of $\Delta_k \times \Delta_l$ [deL96].

To describe one triangulation of $\Delta_k \times \Delta_l$ explicitly [BCS88, GKZ94], assume that Δ_k has vertex set $\{v_0,\dots,v_k\}$ and that Δ_l has vertex set $\{w_0,\dots,w_l\}$. Then $P = \Delta_k \times \Delta_l$ has vertex set $\{(v_i,w_j) \mid 0 \le i \le k,\ 0 \le j \le l\}$.

Consider paths from the vertex (v_0,w_0) to the vertex (v_k,w_l) in which each step involves increasing either the index of v or the index of w by one. Each such path selects a subset of $k+l+1$ vertices of P, which determines a $(k+l)$-dimensional simplex. The collection of simplices associated with all such paths constitutes a triangulation of P. This is the same triangulation of P as the one obtained by starting with the trivial subdivision of P and pulling the vertices in the order

$$\begin{array}{c}(v_0,w_0),(v_0,w_1),\dots,(v_0,w_l),\\ (v_1,w_0),(v_1,w_1),\dots,(v_1,w_l),\\ \vdots \\ (v_k,w_0),(v_k,w_1),\dots,(v_k,w_l).\end{array}$$

Figure 14.5.1 shows this triangulation for $\Delta_2 \times \Delta_1$, a prism. The label ij on a vertex is an abbreviation for (v_i,w_j).

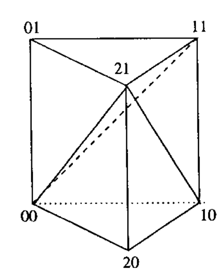


FIGURE 14.5.1
A triangulation of $\Delta_2 \times \Delta_1$.

14.5.2 d-CUBES

Here we consider triangulations of a d-dimensional cube using only the set V of its vertices. See [Hai91].

GLOSSARY

d-cube: The unit d-dimensional cube I^d is the d-fold product of the unit interval $I = [0,1]$ with itself.

Index: A vertex of the d-dimensional cube is a point of the form $(a_1, \ldots, a_d) \in \{0,1\}^d$. Define the index of the vertex to be $\sum_{i=0}^{d-1} a_{i+1} 2^i$.

Size: The size of a triangulation T is the number $f_d(T)$ of d-simplices in T.

$\varphi(d)$: The size of the smallest triangulation of I^d. That is, $\varphi(d) = \min\{f_d(T) \mid T$ is a triangulation of $I^d\}$.

MAIN RESULTS

1. The maximum size of a triangulation of I^d is $d!$ (since the minimum volume of a d-simplex using the vertices of I^d is $1/d!$), and this is achievable for every d by pulling the vertices in any order.

2. $\varphi(d) \geq 2^d(d+1)^{-(d+1)/2} d!$. This bound is derived by observing that I^d can be inscribed in a sphere of diameter $\sqrt{d}$, and that the maximum volume of a simplex contained in this sphere is $(d+1)^{(d+1)/2}/(2^d d!)$ (the volume of a regular simplex) [Hai91].

3. There are precisely 74 triangulations of the 3-cube, and these fall into 6 classes of combinatorially different types [Big91, deL95]. All are regular. On the other hand, if $d \geq 4$, then not all triangulations of the d-cube are regular [deL96].

4. If I^d can be triangulated into $T(d)$ simplices, then I^{kd} can be triangulated into $[(kd)!/(d!)^k]T(d)^k = \rho^{kd}(kd)!$ simplices, where $\rho = (T(d)/d!)^{1/d}$. One measure of the efficiency of a triangulation is $\rho = (T(d)/d!)^{1/d}$. This result shows that any value of ρ achievable for one triangulation is achievable asymptotically. The smallest value of ρ obtainable from triangulations to date is $\rho = (13,248/40,320)^{1/8} \approx 0.870$ [Hai91].

Table 14.5.1 lists the known values of $\varphi(d)$ [Hug93].

TABLE 14.5.1 Minimal triangulations of d-cubes.

d	1	2	3	4	5
$\varphi(d)$	1	2	5	16	67

It is also known that the smallest size of a triangulation of I^6 that slices off alternate vertices of I^6 is 324 [Hug93].

Figure 14.5.2 shows a triangulation of the 3-cube of size 5.

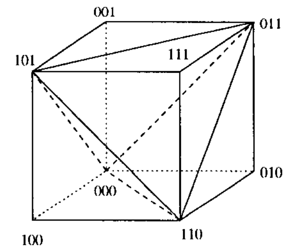


FIGURE 14.5.2
A minimum size triangulation of the 3-cube.

SOME SPECIFIC TRIANGULATIONS OF I^d

Pushing vertices: Start from the trivial subdivision of I^d and push the vertices in order of decreasing index. The resulting triangulation will have $d!$ simplices [Big91].

Pulling vertices: Start from the trivial subdivision of I^d and pull the vertices in order of increasing index to obtain the triangulation T. Pulling the vertices in any order yields a triangulation with $d!$ simplices, so that $f_d(T) = d!$. $h_d(T) = h_{d+1}(T) = 0$ and $h_i(T) = A(d,i)$, $0 \leq i \leq d-1$, where $A(d,i)$ is the Eulerian number (it equals the number of permutations of $\{1,\ldots,d\}$ having exactly i descents). There is a one-to-one correspondence between the simplices in T and the permutations of $\{1,\ldots,d\}$, given in the following way: For a given permutation σ, the corresponding simplex has vertices $(0,\ldots,0)+e_{\sigma(1)}+e_{\sigma(2)}+\cdots+e_{\sigma(k)}$, $0 \leq k \leq d$, where e_i denotes the standard ith unit vector. This is also known as Kuhn's triangulation [Big91, Tod76].

Sallee's corner slicing triangulation: Assume $d \geq 3$. For each vertex with an odd number of coordinates equaling 1, construct the simplex consisting of this vertex and its d neighbors (those joined to this vertex by an edge). These simplices, together with the central polytope remaining when these simplices are removed, constitute a subdivision of I^d. Refine this subdivision to a triangulation by pulling the vertices in order of increasing index. This triangulation has size $O(d!)$ [Hai91, Sal82].

Sallee's middle cut triangulation: Assume $d \geq 2$. Slice the cube into two polytopes by the hyperplane $x_1+\cdots+x_d = \lfloor d/2 \rfloor$. Refine this subdivision to a triangulation by pulling the vertices in order of increasing index. This triangulation has size $O(d!/d^2)$ [Sal84].

Haiman's triangulation: This triangulation method, which bootstraps a triangulation of I^d to a triangulation of I^{kd} as described in Section 14.5.1, Main Result 2, has size $O(\rho^d d!)$, where $\rho < 1$ [Hai91].

EXAMPLES

Figure 14.5.3 shows two triangulations of the 3-cube: (a) the one resulting from pulling the vertices in order of increasing index, and (b) the one resulting from pushing the vertices in order of decreasing index.

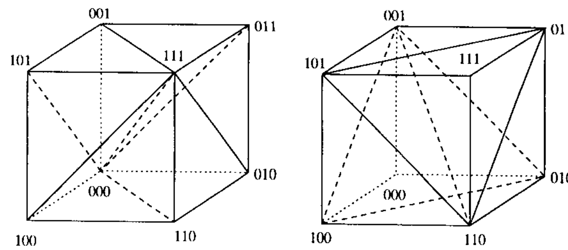


FIGURE 14.5.3
(a) *The pulling triangulation of the 3-cube.*
(b) *The pushing triangulation of the 3-cube.*

14.5.3 CONVEX n-GONS

There is no difficulty in finding subdivisions and triangulations of a convex n-gon using its set V of vertices. All subdivisions are regular, and all triangulations are constructible by pushing (or placing). Any subdivision is determined by a collection of mutually noncrossing internal diagonals. The set of all triangulations of the n-gon is isomorphic to many other combinatorial structures, including the set of all ways to parenthesize a string of $n-1$ symbols and the set of all rooted binary trees with $n-2$ nodes. See [Lee89, Zie95].

MAIN RESULTS

1. There are $\frac{1}{n-1}\binom{n-3}{j}\binom{n+j-1}{j+1}$ subdivisions of a convex n-gon having exactly j diagonals, $0 \leq j \leq n-3$. In particular, the number of triangulations is the ***Catalan number*** $\frac{1}{n-1}\binom{2n-4}{n-2}$.
2. Two triangulations are ***adjacent*** if and only if they share all but one diagonal. The ***distance*** between two triangulations T and T' is the length k of the shortest path $T = T_0, T_1, T_2, \ldots, T_k = T'$ of triangulations in which T_i and T_{i-1} are adjacent for all $1 \leq i \leq k$. The distance between two triangulations of a convex n-gon does not exceed $2n-6$ [Luc89]. This bound is achievable for infinitely many values of n [STT88].
3. The set of all triangulations of a convex n-gon is connected by a Hamiltonian cycle—a closed path $T_0, T_1, T_2, \ldots, T_m = T_0$ containing each triangulation exactly once (except for T_0, which starts and ends the path), in which T_i and T_{i-1} are adjacent for all $1 \leq i \leq m$ [Luc87].

14.5.4 COMPLETE BARYCENTRIC SUBDIVISIONS

For a given d-polytope P, let V be the collection of the centroids of the nonempty faces. Give the centroid of each k-dimensional face the label k, $0 \leq k \leq d$. Note that points labeled 0 are the vertices of P. Triangulate P by pulling the points of V in order of nonincreasing label. The resulting triangulation is the ***complete barycentric subdivision*** of P. The procedure can be extended in the obvious way to be applied to any polytopal complex. See [Bay88].

Figure 14.5.4 shows the complete barycentric subdivision of a 3-cube, a triangulation of size 48—there are eight pyramids into the center of the cube from each of the six original facets.

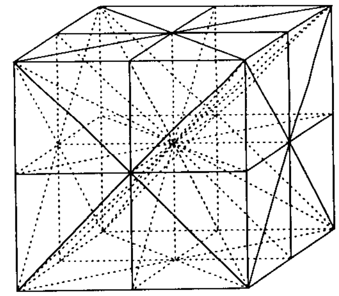


FIGURE 14.5.4
The complete barycentric subdivision of a 3-cube.

MAIN RESULTS

1. For every polytope P, there is a ***dual polytope*** (or polar polytope: see Chapter 13) P^* of the same dimension, whose face lattice is anti-isomorphic to that of P. The complete barycentric subdivisions T and T^* of P and P^* are combinatorially isomorphic. That is to say, there is a bijection between the vertices of T and of T^* such that a subset of vertices of T determines a simplex in T precisely when the corresponding subset of vertices of T^* determines a simplex in T^*.
2. If T is the complete barycentric subdivision of a polytope P, then the combinatorial structure of the face lattice of P (up to lattice reversal by the previous result) can be recovered from the combinatorial structure of T, even if one is not given the specific geometric realization or the labels of the points [Bay88].
3. Suppose T is the complete barycentric subdivision of a d-dimensional simplex. Then $f_d(T) = d!$, $h_{d+1}(T) = 0$, and $h_i(T) = A(d+1, i)$, $0 \le i \le d$. These are the Eulerian numbers encountered in Kuhn's triangulation of I^{d+1}. In fact, Kuhn's triangulation is combinatorially isomorphic to the join of T to a new point (make a pyramid with this new point over every d-simplex in T) [Big91].
4. If T is the complete barycentric subdivision of I^d, then $f_d(T) = d!2^d$. Also, $h_{d+1}(T) = 0$, and $h_i(T)$ equals the number of signed permutations of $\{1, \ldots, d\}$ with exactly i descents [Bre94].

14.6 SECONDARY AND FIBER POLYTOPES

This section concerns itself with the structure of the collection of all regular subdivisions of a given finite set of points $V = \{v_1, \ldots, v_n\} \subset \mathbb{R}^d$. See [GKZ94, Lee91, Zie95]. Assume that $\dim(\text{conv}(V)) = d$.

GLOSSARY

z-vector: Suppose T is a regular triangulation of V. Define the z-vector $z(T) = (z_1, \ldots, z_n) \in \mathbb{R}^n$ by $z_i = \sum \text{vol}(F)$, where the sum is taken over all d-simplices F in T having v_i as a vertex.

Secondary polytope: The secondary polytope $\Sigma(V)$ is the convex hull of the z-vectors of all regular triangulations of V.

Link: If F is a face of a triangulation T, then the link of F is the set $\{G \mid G$ is a face of T, $\text{conv}(F \cup G)$ is a face of T of dimension $\dim F + \dim G + 1$, and $F \cap G = \emptyset\}$.

Adjacent triangulations: Suppose T is a triangulation of V (not necessarily regular). Suppose there is a subset W of $k+2$ points in V such that $\dim(\text{aff}(W)) = k$, T contains one of the (only) two triangulations of W, and the links with respect to T of all the k-dimensional faces F in the triangulation of W are identical. Then it is possible to interchange the triangulations of W, giving the new k-simplices the same links with respect to T, and thereby obtain a new triangulation of V. This operation is called a ***flip***, and the resulting triangulation is said to be adjacent to T.

Connected: Two triangulations (not necessarily regular) are said to be connected if one can be obtained from the other by a sequence of flips.

The secondary polytope plays an important role in the study of Gröbner bases [Stu96] and generalized discriminants and determinants [GKZ94].

MAIN RESULTS

1. The collection of all regular subdivisions of the set V, partially ordered by refinement, is combinatorially equivalent to the boundary complex of the polytope $\Sigma(V)$, which has dimension $n-d-1$ [GKZ94].

2. The vertices of $\Sigma(V)$ are precisely the z-vectors. In particular, no two regular triangulations have the same z-vector. The edges of $\Sigma(V)$ correspond to adjacent regular triangulations [GKZ94].

3. As an immediate consequence of the existence of $\Sigma(V)$, every regular triangulation has at least $n-d-1$ adjacent triangulations, and every pair of regular triangulations is connected. There are examples of nonregular triangulations with fewer than $n-d-1$ adjacent triangulations [deL95]. It is an open problem to determine in general whether or not every pair of triangulations, regular or not, is connected, although this is true of point sets in $\mathbb{R}^2$ and of vertex sets of cyclic polytopes [Ram96]. In particular, it is unknown if a (necessarily nonregular) triangulation exists with no adjacent triangulations.

4. $\Sigma(V)$ can also be expressed as a discrete or continuous Minkowski sum of polytopes coming from a representation of V as a projection of the vertices of an $(n-1)$-dimensional simplex. See [Zie95].

5. In the special case that $n = d+2$, there are precisely two nontrivial subdivisions of V (both regular), so that $\Sigma(V)$ is a line segment.

6. In the special case that $n = d+3$, all subdivisions are regular, and $\Sigma(V)$ is a convex polygon.

7. In the special case that V is the set of vertices of a convex n-gon, $\Sigma(V)$ is called the ***associahedron*** [Lee89]. Its dual is a simplicial polytope Q of

dimension $n-3$ having the following f-vector and h-vector:

$$f_{j-1}(Q) = \frac{1}{n-1}\binom{n-3}{j}\binom{n+j-1}{j+1}, \quad 0 \le j \le n-3,$$

$$h_i(Q) = \frac{1}{n-1}\binom{n-3}{i}\binom{n-1}{i+1}, \quad 0 \le i \le n-3.$$

From the discussion in Section 14.5.3, $f_{j-1}(Q)$ is the number of subdivisions of the n-gon having exactly j diagonals. There are various combinatorial interpretations of the h-vector. Explicit coordinates and inequalities for $\Sigma(V)$ can be found in [Zie95].

Figure 14.6.1 shows the five regular triangulations of a set of 5 points in $\mathbb{R}^2$, marking which pairs of triangulations are adjacent.

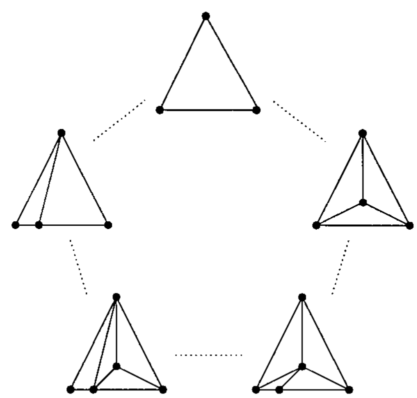


FIGURE 14.6.1
A polygon of regular triangulations.

The secondary polytope of the product of two simplices is discussed, for example, in [deL96, GKZ94]. See [dHSS96] for properties of the polytope that is the convex hull of the $(0,1)$ incidence vectors of all triangulations of V, and for the relationship of this polytope to $\Sigma(V)$. The special case when V is the set of vertices of a convex n-gon was first described in [DHH85].

14.6.1 FIBER POLYTOPES

A secondary polytope is a special case of a fiber polytope, which is associated with an affine map $\pi : P \to Q$ from a polytope P in $\mathbb{R}^p$ onto a polytope Q in $\mathbb{R}^q$. Such a map induces certain regular subdivisions of Q (called *π-coherent subdivisions*). The fiber polytope $\Sigma(P,Q)$ has dimension $\dim(P) - \dim(Q)$, and its nonempty faces correspond to these π-coherent subdivisions.

A ***section*** is a continuous map $\gamma : Q \to P$ with $\pi(\gamma(x)) = x$ for all $x \in Q$. The ***fiber polytope*** is defined to be the set of all average values of the sections of π:

$$\Sigma(P,Q) = \left\{ \frac{1}{\text{vol}\,(Q)} \int_Q \gamma(x)dx \mid \gamma \text{ is a section of } \pi \right\}.$$

The associahedron and the permutohedron (see Chapter 13) are examples of fiber polytopes, and there are applications to zonotopal subdivisions and oriented matroids. For more details, see [Zie95].

14.7 SOURCES AND RELATED MATERIAL

FURTHER READING

Chapter 22 discusses triangulations of more general (e.g., nonconvex) objects. Chapter 20 provides details on Delaunay triangulations and Voronoi diagrams. Refer also to Chapter 13, on basic properties of convex polytopes.

A section on triangulations and subdivisions of convex polytopes can be found in the survey article [BL93]. The book [Zie95] and the article [Lee91] contain information on regular subdivisions and triangulations; for their important role in generalized discriminants and determinants see the book [GKZ94], and for their significance in computational algebra see the book [Stu96]. Additional references can be found in the above-mentioned sources, as well as the citations given in this chapter.

RELATED CHAPTERS

Chapter 3: Tilings
Chapter 7: Lattice points and lattice polytopes
Chapter 13: Basic properties of convex polytopes
Chapter 15: Face numbers of polytopes and complexes
Chapter 19: Convex hull computations
Chapter 20: Voronoi diagrams and Delaunay triangulations
Chapter 22: Triangulations
Chapter 27: Computational convexity

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