Attempt to Find Inverse of Mapping From B_2 to B_1

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Let P be a simplicial d-polytope such that the origin is the sum of its vertices. Let $u^1, \ldots, u^n$ be its vertices and let $\beta_1, \ldots, \beta_n$ be their respective norms.

Let us call a member of B_1 a linear relation on the unit vectors $e^i = u^i/\|u^i\|$, and a member of B_2 numbers S_{ij} such that $S_{ij} = 0$ if $i \neq j$ and $u^i u^j$ is not an edge of P, and such that for all j, $\sum_i S_{ij} e^i = 0$.

Let v^i be positive multiples of the u^i such that $v^1 + \cdots + v^n = 0$. Let us assume that the v^i are very close in norm to the u^i. We want to find a linear 2-stress (element of B_2) S_{ij} such that for all j, $\sum_i \beta_i S_{ij} = \|v^j\|$.

For all λ such that $\sum_i \lambda_i v^i = 0$, $\sum_i \lambda_i = 1$, and $0 \leq \lambda_i \leq 1$ for all i, consider the d-polytope obtained by placing hyperplanes with normals e^i at distances $\lambda_i/\|v^i\|$ from the origin. (Of course, if the v^i are in fact the original vertices of P and $\lambda^i = 1/n$ for all i, this is just a scaling of the polar dual P^* of P.)

Now maximize $\sum_i \beta_i F_i$ over all such λ, where F_i denotes the $(d-1)$-content of facet i. We would like to argue that the optimal polytope Q contains the origin in its interior, but I don't know how to do this yet. But let us suppose that this is the case. In fact, we would like to argue that the combinatorial structure of Q is isomorphic to the combinatorial structure of P^* since Q must be "close to" P^*.

Then for all affine relations a on the v^i we have $\sum_i \beta_i \sum_j S_{ij} a_j/\|v^j\| = 0$, where

$$S_{ij} = \begin{cases} |E_{ij}|/\sin\theta_{ij} & \text{if } j \neq i \text{ are neighboring facets of } Q, \\ -\sum_{\text{facets } k \text{ adjacent to } i} |E_{ik}|\cos\theta_{ik}/\sin\theta_{ik} & \text{if } j = i, \\ 0 & \text{otherwise.} \end{cases}$$

Here θ_{ij} is the dihedral angle between facets i and j and $|E_{ij}|$ is the length of their common edge. This is verified by considering the perturbation $\lambda + ta$. One can check that S_{ij} is a linear 2-stress.

Hence $\sum_j a_j \frac{\sum_i \beta_i S_{ij}}{\|v^j\|} = 0$ for all affine a. Let $L_j = \sum_i \beta_i S_{ij}$. So $\sum_j a_j L_j/\|v^j\| = 0$ for all affine a. Also L is a linear relation on the unit vectors $e^i = u^i/\|u^i\| = v^i/\|v^i\|$, so $\sum_j L_j v^j/\|v^j\| = 0$. Hence $(L_1/\|v^1\|, \ldots, L_n/\|v^n\|)$ must be a multiple of the vector of all 1's. Rescale so that $L'_j = \|v^j\|$.

9 Minkowski's Theorem

Theorem 23 (Minkowski) *Let $P \subset \mathbf{R}^d$ be a convex d-polytope with facets $F_1, \ldots, F_n$. Let $u^1, \ldots, u^n$ be the respective outer unit normals of the facets. Let V_i be the $(d-1)$-dimensional volume of the facet F_i, $i = 1, \ldots, n$. Then*

$$\sum_{i=1}^{n} V_i u^i = 0.$$

Proof. Let c be any point in the interior of P and let d_i be the distance of facet F_i from c, $i = 1, \ldots, n$. Let V be the volume of P. Then

$$V = \frac{1}{d} \sum_{i=1}^{n} d_i V_i.$$

Let the equation of the supporting hyperplane to facet F_i be $u^i \cdot x = b_i$, $i = 1, \ldots, n$. Then

$$d_i = \frac{|u^i \cdot c - b_i|}{\|u^i\|} = b_i - u^i \cdot c$$

so

$$V = \frac{1}{d} \sum_{i=1}^{n} (b_i - u^i \cdot c) V_i.$$

Let $t > 0$ be small enough so that the ball of radius t centered at c lies within the interior of P. Let u be any unit vector and consider the point $c' = c + tu$. Then computing the volume of P from c' we have

$$V = \frac{1}{d} \sum_{i=1}^{n} (b_i - u^i \cdot c') V_i = \frac{1}{d} \sum_{i=1}^{n} (b_i - u^i \cdot c - u^i \cdot tu) V_i.$$

Subtracting the two expressions for V gives

$$0 = \frac{1}{d} \sum_{i=1}^{n} (u^i \cdot tu) V_i$$

so

$$u \cdot \sum_{i=1}^{n} V_i u^i = 0$$

for all unit vectors u. This implies that the sum must itself be the zero vector. □

Theorem 24 (Minkowski) *Let $v^1, \ldots, v^n$ be vectors in $\mathbf{R}^d$ such that*

1. *$v^1, \ldots, v^n$ span $\mathbf{R}^d$,*

2. *No v^i is a positive multiple of any other v^j,*

3. *$\sum_{i=1}^{n} v^i = 0$.*

Then there exists a convex d-polytope P with facets F_i, $i = 1, \ldots, n$, such that the unit outer normals are $u^i = v^i/\|v^i\|$ and the $(d-1)$-dimensional volumes are $V_i = \|v^i\|$, respectively.

Proof. List the vectors v^i as columns of a matrix A. Note that $Ae = 0$, where $e \in \mathbf{R}^n$ is the vector $(1, \ldots, 1)^T$. For $b \in \mathbf{R}^n$ define the polyhedron $P(b) = \{x \in \mathbf{R}^d : A^T x \leq b\}$. Then let $B = \{b \in \mathbf{R}^n : P(b) \neq \varnothing, Ab = 0, \text{ and } e^T b = 1\}$.

Claim 1. B is a convex polyhedron, since it is a projection of $\{(b, x) \in \mathbf{R}^{d+n} : A^T x - b \leq 0, Ab = 0, \text{ and } e^T b = 1\}$.

Claim 2. B is bounded, and hence a convex polytope. For choose any direction $c \neq 0$ such that $Ac = 0$ and $e^T c = 0$. Let $b \in B$ and consider the ray $b + tc$, $t > 0$. We need to show that if t is large enough, then this ray is not in B. Consider the following dual pair of linear programs.

$$\begin{array}{cc} \max 0^T x & \min (b+tc)^T y \\ A^T x \leq b + tc & Ay = 0 \\ & y \geq 0 \\ & \\ (I) & (II) \end{array}$$

We need to show that (I) is not always feasible. But (I) is feasible iff (II) (which is clearly feasible) has bounded objective function value. This is equivalent to the nonexistence of y such that $Ay = 0$, $y \geq 0$, and $(b + tc)^T y < 0$. Choose $\varepsilon > 0$ and let $y = e - \varepsilon c$. Make ε small enough so that $y \geq 0$. Then $Ay = Ae - \varepsilon Ac = 0 - 0 = 0$. But $(b + tc)^T y = b^T e - \varepsilon b^T c + t c^T e - \varepsilon t c^T c = b^T e - \varepsilon b^T c - \varepsilon t \|c\|^2$ which is negative when t is sufficiently large.

Claim 3. $P(b)$ is bounded for all $b \in B$, hence has finite volume. For assume $P(b)$ is not bounded. Then there is some $x \in P(b)$ and some direction $z \neq 0$ such that $A^T(x+tz) \leq b$ for all $t \geq 0$. So $A^T z \leq 0$. If there is strict inequality anywhere, then $0 = z^T(Ae) = (z^T A)e < 0$, a contradiction. So $A^T z = 0$. But the columns of A span $\mathbf{R}^d$, so z, being orthogonal to all of the columns, must itself be the zero vector, a contradiction.

From our claims we now know that $V(P(b))$, the volume of $P(b)$, is well-defined for all $b \in B$. Also, since B is closed and bounded and $V(P(b))$ is a continuous function of b, we can consider the problem $\max\{V(P(b)) : b \in B\}$. Note that e/n is in B and that $P(e/n)$ contains the origin in its interior since $e/n > 0$, so the maximization problem has a positive maximum. The maximum is achieved by some $b^* \in B$. Let $P^* = P(b^*)$. Let $F_1, \ldots, F_n$ be its facets, with outer normals $u_1, \ldots, u_n$, and $(d-1)$-volumes $V_1, \ldots, V_n$, respectively. For $h \in \mathbf{R}^n$ consider the function $V(h) = V(P(b^* + h))$. The gradient of $V(h)$ at $h = 0$ is $(V_1/\|v^1\|, \ldots, V_n/\|v^n\|)$ (remembering that no two v^i are positive multiples of each other). Choose any $c \in \mathbf{R}^n$ such that $Ac = 0$ and $e^T c = 0$ and consider the function $V(t) = V(P(b^* + tc))$. Since P^* is optimal, we have $dV/dt = 0$ so

$$\sum_{i=1}^{n} \frac{V_i}{\|v^i\|} c_i = 0.$$

Also, by Theorem 23,

$$\sum_{i=1}^{n} V_i \frac{v^i}{\|v^i\|} = 0.$$

So the vector $(V_1/\|v^1\|, \ldots, V_n/\|v^n\|)$ is orthogonal to all of the rows of A and is orthogonal to all affine relations on the columns of A. But then this vector must be a multiple of e. So there is some positive number k such that $V_i = k\|v^i\|$, $i = 1, \ldots, n$. Scale P^*, if necessary, to obtain the desired polytope. □

We omit the proof of the following stronger result that states that a polytope is essentially uniquely determined by its unit facet normals and facet volumes.

Theorem 25 (Minkowski) *The polytope which exists by the previous theorem is unique up to translation.*