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Some Recent Results on Convex Polytopes

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ABSTRACT. We sample a few results on the combinatorial structure of convex polytopes, including Lawrence's volume formula, f-vectors and h-vectors, associated algebraic structures, shellability, bistellar operations and p.l.-spheres, connections with stress and rigidity, triangulations, winding numbers, the moment map, and canonical convex combinations.

1. Introduction

The study of polyhedra has enjoyed rapid growth, stimulated partly by the development of mathematical programming in the last few decades, and partly by more recently discovered connections with commutative algebra and algebraic geometry. We informally survey a few results on the combinatorial structure of convex polytopes, beginning with Lawrence's volume formula. This leads naturally into the notions of the f-vector and the h-vector. These, in turn, have algebraic significance in associated algebraic structures. Examining these structures in the context of two inductive methods for constructing polytopes, shellings and bistellar operations, reveals an interplay with stress and rigidity. One consequence is a new proof that p.l.-spheres are Cohen–Macaulay. Gale transforms play a role here and can be used to define a class of triangulations of a convex polytope. They also provide a geometric interpretation of the h-vector in terms of winding numbers. We conclude with a brief discussion of a toric variety associated with a rational simplicial convex polytope. The components of the h-vector appear as the dimensions of its homology groups, and its moment map suggests a canonical way to express a point of the polytope as a convex combination of the vertices.

2. Lawrence's volume formula

Let us start by considering a d-dimensional convex polyhedron P of the form $P = \{x \in \mathbf{R}^d : Ax \leq b, x \geq 0\}$, where A is an $m \times d$ matrix and

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$0 < b \in \mathbf{R}^m$. Further, assume that P is bounded (hence a *polytope*) and *simple* (i.e., nondegenerate). Select any linear function $z = c^T x$ that is nonconstant on every edge of P. Introduce m slack variables, one for each constraint. At every vertex (basic feasible solution) v of P we record the (necessarily nonzero) reduced costs $\bar{c}_{i_1}, \ldots, \bar{c}_{i_d}$ of the d nonbasic variables, the current value $\bar{z}$ of z, and the determinant $|B|$ of the current basis matrix. If we have arrived at v from a sequence of simplex pivots, $|B|$ is the product of the pivot elements. Lawrence [**20**] proves that the volume of P equals

$$\sum_v \frac{\bar{z}^d}{d!|B|\bar{c}_{i_1}\cdots\bar{c}_{i_d}} \tag{1}$$

where the sum is taken over all vertices of P. The formula can be modified to handle polytopes that are not simple.

Thus we can theoretically compute the volume of P without triangulating it first, although as Lawrence points out there may be some difficulties in practice because the sum involves terms of differing sign that can be quite large compared to the volume of P.

As a trivial example, let us calculate the volume of the unit 3-cube $\{x \in \mathbf{R}^3 : 0 \le x_i \le 1,\ i = 1, 2, 3\}$ using the function $z = x_1 + 2x_2 + 4x_3$. With slack variables the description becomes

$$\begin{aligned} x_1 + x_4 &= 1, \\ x_2 + x_5 &= 1, \\ x_3 + x_6 &= 1, \\ x_1, \ldots, x_6 &\ge 0. \end{aligned}$$

The desired numbers are then as follows.

v	reduced costs	$\bar{z}$
$(0, 0, 0)$	$(\bar{c}_4, \bar{c}_5, \bar{c}_6) = (-1, -2, -4)$	0
$(1, 0, 0)$	$(\bar{c}_1, \bar{c}_5, \bar{c}_6) = (+1, -2, -4)$	1
$(0, 1, 0)$	$(\bar{c}_4, \bar{c}_2, \bar{c}_6) = (-1, +2, -4)$	2
$(1, 1, 0)$	$(\bar{c}_1, \bar{c}_2, \bar{c}_6) = (+1, +2, -4)$	3
$(0, 0, 1)$	$(\bar{c}_4, \bar{c}_5, \bar{c}_3) = (-1, -2, +4)$	4
$(1, 0, 1)$	$(\bar{c}_1, \bar{c}_5, \bar{c}_3) = (+1, -2, +4)$	5
$(0, 1, 1)$	$(\bar{c}_4, \bar{c}_2, \bar{c}_3) = (-1, +2, +4)$	6
$(1, 1, 1)$	$(\bar{c}_1, \bar{c}_2, \bar{c}_3) = (+1, +2, +4)$	7

In every case $|B| = 1$, so the volume equals

$$\frac{0^3}{6(-1)(-2)(-4)} + \frac{1^3}{6(1)(-2)(-4)} + \frac{2^3}{6(-1)(2)(-4)} + \frac{3^3}{6(1)(2)(-4)} + \frac{4^3}{6(-1)(-2)(4)} + \frac{5^3}{6(1)(-2)(4)} + \frac{6^3}{6(-1)(2)(4)} + \frac{7^3}{6(1)(2)(4)} = 1.$$

3. *h*-vectors, *f*-vectors, and faces

It may seem surprising enough that the different sign patterns of the reduced costs contribute to the signs of the terms of (1) in just the right way, but in fact the reduced cost signs tell us even more. Let h_i be the number of vertices for which there are exactly i positive reduced costs (and hence exactly $d-i$ negative reduced costs). Define the *h-vector* $h(P)$ to be $(h_0, \dots, h_d)$. So, from the previous example, the h-vector of the 3-cube is $(1, 3, 3, 1)$. From the h-vector it is possible to determine the number of faces of P of all dimensions:

$$f_j = \sum_{i=j}^{d} \binom{i}{j} h_i, \qquad i = 0, \dots, d, \tag{2}$$

where f_j equals the number of j-dimensional faces of P.

The vector $f(P) \equiv (f_0, \dots, f_d)$ is called the *f-vector* of P. For the 3-cube it is $(8, 12, 6, 1)$. Formula (2) implies that the f-vector can be derived from the h-vector by constructing a triangle in a manner similar to Pascal's triangle, but replacing the right-hand side of the triangle by the h-vector. The f-vector emerges in reverse at the bottom. For example,

$$\begin{array}{ccccccc}
 & & & \mathbf{1} & & & \\
 & & 1 & & \mathbf{3} & & \\
 & 1 & & 4 & & \mathbf{3} & \\
1 & & 5 & & 7 & & \mathbf{1} \\
\mathbf{1} & \mathbf{6} & & \mathbf{12} & & \mathbf{8} &
\end{array}$$

One way to verify (2) is to use the one-to-one correspondence between the $d+m$ variables and the $d+m$ constraints. The original d variables correspond to the nonnegativity constraints, and the m slack variables correspond to the m explicit constraints. A constraint is enforced when the corresponding variable is set equal to zero. At a vertex v, let $S(v)$ be the set of nonbasic variables with positive reduced cost. For every subset T of $S(v)$, consider the set of all points in P for which the constraints corresponding to $S(v)\backslash T$ hold with equality. If $\text{card}(T) = j$ then we obtain a face of dimension j. Moreover, if we carry out this process for every vertex of P, we will encounter every face of P once and only once [7, §18]. So, the number of faces of dimension j equals the number of ways of finding an $S(v)$ of cardinality $i \geq j$ and selecting a subset of size j.

The above argument works as long as P is simple and z is nonconstant on every edge. Thus an enumeration of the vertices of P (by the simplex method, for example) provides us with enough information to enumerate *all* of the faces of P efficiently.

I wonder whether this is used by Fukuda?

Inverting (2) yields

$$h_i = \sum_{j=i}^{d} (-1)^{i+j} \binom{j}{i} f_j, \qquad i = 0, \dots, d. \tag{3}$$

This implies that $h(P)$ is independent of the choice of linear function $z = c^T x$. In particular, $z = -c^T x$ would reverse the signs of all reduced costs but yield the same h-vector, so the h-vector must be symmetric; i.e., $h_i = h_{d-i}$, $i = 0, \ldots, d$. This system of equations is equivalent to the *Dehn–Sommerville equations* [7, §17].

4. Algebraic significance of the h-vector

Now let P be a simplicial (rather than simple) d-dimensional polytope and $h(P)$ be the h-vector of a simple polytope dual to P. By replacing f_j by f_{d-j-1} and using the Dehn–Sommerville equations, one can obtain formulas analogous to (2) and (3) directly in terms of the f-vector of P.

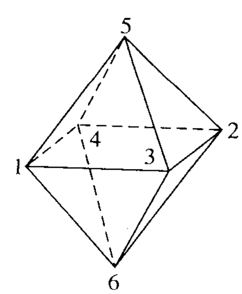
(4) $$h_i = \sum_{j=0}^{i} (-1)^{i-j} \binom{d-j}{d-i} f_{j-1}, \qquad i = 0, \ldots, d$$

(5) $$f_j = \sum_{i=0}^{j+1} \binom{d-i}{d-j-1} h_i, \qquad j = -1, \ldots, d-1$$

So, for example, we say that the h-vector of the octahedron (Figure 1) is $(1, 3, 3, 1)$.

The boundary complex of a simplicial polytope is an example of a simplicial complex. Let V be a finite set, say $V = \{1, \ldots, n\}$. A *simplicial complex* Δ on V is a nonempty collection of subsets of V that is closed under inclusion. For $F \in \Delta$, F is called a *face* of Δ, and its *dimension*, $\dim(F)$, is taken to be $\operatorname{card}(F) - 1$. The *dimension* of Δ itself, $\dim(\Delta)$, is $\max_{F \in \Delta} \dim(F)$.

Given $F \in \Delta$, define $\overline{F}$ to be 2^F, the collection of all subsets of F, and $\partial\overline{F}$ to be $2^F \backslash \{F\}$. The *link* in Δ of F is $\operatorname{lk}_\Delta F \equiv \{G \in \Delta : G \cap F = \emptyset, G \cup F \in \Delta\}$. If F is not the empty set, we also define $\Delta \backslash F \equiv \{G \in \Delta : F \not\subseteq G\}$ to be the *deletion* of F from Δ. Given two simplicial complexes Δ_1 and



$f(P) = (1, 6, 12, 8)$
$h(P) = (1, 3, 3, 1)$

FIGURE 1. The octahedron.

Δ_2 on disjoint sets V_1 and V_2, respectively, the *join* of Δ_1 and Δ_2 is $\Delta_1 \cdot \Delta_2 \equiv \{F_1 \cup F_2 \colon F_1 \in \Delta_1, F_2 \in \Delta_2\}$.

For $(d-1)$-dimensional simplicial complex Δ, its *f-vector* is $f(\Delta) \equiv (f_{-1}, f_0, \dots, f_{d-1})$ where f_j is the number of j-dimensional faces of Δ. The *h-vector* of Δ is then defined by formula (4).

Taking Δ to be any $(d-1)$-dimensional simplicial complex on $V = \{1, \dots, n\}$, form the polynomial ring $R \equiv \mathbf{C}[x_1, \dots, x_n]$, which has a natural grading by degree. For monomial $m = x_{i_1}^{a_{i_1}} \cdots x_{i_k}^{a_{i_k}}$ where each $a_{i_j} > 0$, the *support* of m, supp(m), is the set $\{i_1, \dots, i_k\}$. We now let I be the ideal of all nonfaces; i.e., $I \equiv \langle m \colon \operatorname{supp}(m) \notin \Delta \rangle$. Factoring out I from R yields the ring $A \equiv A_0 \oplus A_1 \oplus A_2 \oplus \cdots$, which inherits the grading by degree. This is known as the *Stanley–Reisner ring* of Δ [**33**], [**38**]. If Δ is the boundary complex of the octahedron given in Figure 1, for example, we take $R = \mathbf{C}[x_1, \dots, x_6]$ and $I = \langle x_1x_2, x_3x_4, x_5x_6 \rangle$.

The ring A is *Cohen–Macaulay* if and only if it contains elements $\theta_1, \dots, \theta_d$ of degree one with the following property: $B \equiv A/\langle \theta_1, \dots, \theta_d \rangle = B_0 \oplus B_1 \oplus \cdots \oplus B_d$ has only finitely many nonzero components, graded by degree, and $\dim(B_i) = h_i$, $i = 0, \dots, d$ as vector spaces over $\mathbf{C}$. When this happens, Δ is called a *Cohen–Macaulay complex* [**38**]. For example, it can be shown for the octahedron that $\theta_1 = x_1 - x_2$, $\theta_2 = x_3 - x_4$, and $\theta_3 = x_5 - x_6$ have the desired property, so the boundary complex of an octahedron is Cohen–Macaulay.

Reisner [**33**] proved that boundary complexes of simplicial polytopes are Cohen–Macaulay, as are the properly larger classes of shellable complexes, p.l.-spheres, and homological spheres. He did this by providing a homological characterization of the class of all Cohen–Macaulay complexes.One corollary of this result is Stanley's new proof of the Upper Bound Theorem for convex polytopes and its extension to homological spheres [**37**].

Further, for simplicial polytopes Stanley [**39**] proved that for suitable θ_i there exists an element $\omega \in B_1$ with the following property: After the ideal generated by ω is factored out of B, the result is an algebra $C \equiv C_0 \oplus \cdots \oplus C_{\lfloor \frac{d}{2} \rfloor}$, graded by degree, such that $\dim(C_i) = g_i \equiv h_i - h_{i-1}$, $i = 1, \dots, \lfloor \frac{d}{2} \rfloor$. His proof exploits a connection between rational convex polytopes and certain complex projective toric varieties. This far-reaching result leads to a complete characterization of f-vectors of simplicial and simple polytopes, as well as tight upper and lower bounds on the numbers of faces of unbounded, simple polyhedra with a given number of bounded and unbounded facets and recession cone of specified dimension [3, 9, 10, 21].

One particular consequence is that the h-vector is unimodal, which was one part of the Generalized Lower-Bound Conjecture [**31**]. The fact that $h_2 - h_1 \geq 0$ provides a new proof of the Lower-Bound Theorem [**1**], [**2**]. Kalai [**17**] has yet another proof of this inequality based upon rigidity considerations. Let E denote the set of edges of a polytope. Kalai considers the *stress space* for a convex polytope that is defined to be all functions $\lambda: E \to \mathbf{R}$ such that

$$\sum_{u\ :\ uv \in E} \lambda_{uv}(u - v) = 0$$

for all vertices v. Since the 1-skeleton of a simplicial d-polytope is generically d-rigid, the stress space is a vector space of dimension $h_2 - h_1$, and hence is nonnegative.

5. Shellability

A $(d-1)$-dimensional simplicial complex is *shellable* if its maximal faces (*facets*) are each of dimension $d-1$ and can be ordered $F_1, \dots, F_m$ such that for $i = 2, \dots, m$ there is a unique minimal face G_i of F_i that is not in the union of $\overline{F}_1, \dots, \overline{F}_{i-1}$. Suppose when adding F_i in the shelling process that the minimal face G_i contains exactly k vertices. Then from formula (4) one readily sees that h_k increases by one, while the other h_j remain unchanged. A shelling of the octahedron (Figure 2) illustrates this.

Bruggesser and Mani **[8]** proved that the boundary complexes of all convex polytopes are shellable. An appropriate ordering of facets can be obtained by translating P so that the origin is in its interior, taking the polar of P, choosing a linear function $c^T x$ as in Section 2 that is nonconstant on every edge of the polar, and ordering the vertices of the polar $v_1, \dots, v_m$ such that $c^T v_1 \le \cdots \le c^T v_m$. The corresponding ordering of the facets of P is a shelling order. Shellings constructed in this manner are called *line shellings*. In terms of P itself, we can take a line in general position through the origin

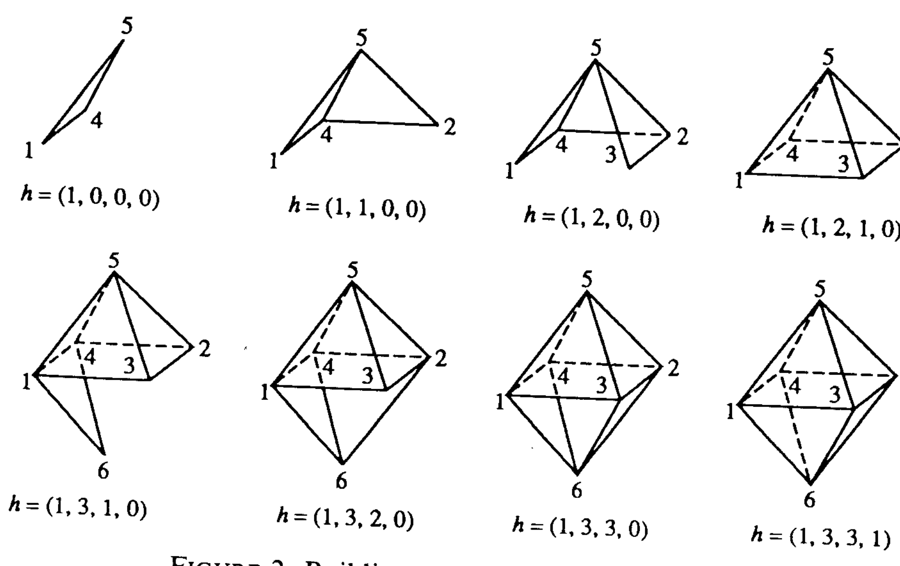


FIGURE 2. Building an octahedron by shelling.

(which is assumed to be in the interior of P) and list the facets according to the order in which their supporting hyperplanes are pierced as we travel along the line from the origin to infinity and back to the origin again from the opposite direction.

One way to prove that A is Cohen–Macaulay for a shellable complex is to show that the dimensions of the B_j change in exactly the same way as the h-vector during the shelling. Kind and Kleinschmidt **[18]** found an inductive proof that shellable complexes are Cohen–Macaulay in this manner.

6. Bistellar operations and p.l.-spheres

So far the methods in the previous section have not led to a more elementary proof of the existence of the element ω. One obstacle might be that the intermediate complexes during the shelling are not themselves polytopes. However, there are other ways to construct a polytope inductively using elementary operations such that one has a polytope at every intermediate stage.

Given a nonempty face F in a simplicial complex Δ and an element $v \notin V$, the *stellar subdivision* of F is $\mathrm{st}(v, F)[\Delta] \equiv (\Delta \backslash F) \cup (\overline{\{v\}} \cdot \partial \overline{F} \cdot \mathrm{lk}_\Delta F)$. The opposite of a stellar subdivision is an *inverse stellar subdivision.* A simplicial complex Δ is a p.l. (*piecewise linear*) $(d-1)$-*sphere* if Δ is obtainable from the boundary of a d-dimensional simplex by a sequence of stellar and inverse stellar subdivisions **[16]**.

Now suppose that F is a nonempty face of a $(d-1)$-dimensional complex Δ and G is a nonface of Δ such that $\mathrm{lk}_\Delta F = \partial \overline{G}$. In this case assume $k + l = d + 1$ where $k = \mathrm{card}(F)$ and $l = \mathrm{card}(G)$. A certain combination of a stellar subdivision of F and an inverse stellar subdivision at the same location is called a *bistellar operation* and results in the simplicial complex $\mathrm{bist}(G, F)[\Delta] \equiv (\Delta \backslash F) \cup (\partial \overline{F} \cdot \overline{G})$.

Formula (4) shows that such a bistellar operation increases $g_l = h_l - h_{l-1}$ by one, decreases $g_k = h_k - h_{k-1}$ by one, and leaves all other differences $g_i = h_i - h_{i-1}$ unchanged. Figure 3 shows how the octahedron can be constructed using three bistellar operations.

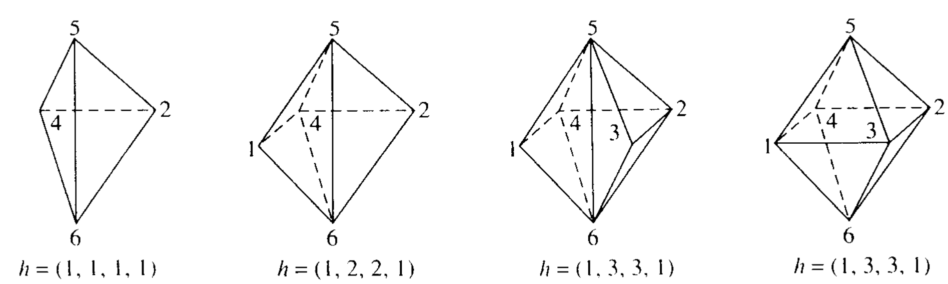


FIGURE 3. Building an octahedron using bistellar operations.

Ewald [12] showed that starting with a d-simplex, any simplicial d-polytope can be obtained by a sequence of bistellar operations such that one has a simplicial d-polytope at each intermediate stage of the sequence. Pachner [32] then proved that every $(d-1)$-dimensional p.l.-sphere can be constructed from the boundary of a d-simplex using bistellar operations.

Kind and Kleinschmidt's proof for shellable complexes suggests trying to do something analogous for p.l.-spheres: Show that the dimensions of B_i change in exactly the same way as the h-vector during a bistellar operation. This is possible, and the result is a new, inductive proof that p.l.-spheres are Cohen–Macaulay [27].

7. Connections with stress

To carry out the above proof, it suffices to work with $\mathbf{R}$ instead of $\mathbf{C}$. Given an element θ_1 in A_1, multiplication by θ_1 is a linear map from the vector space A_i to the vector space A_{i+1}, $i = 0, 1, 2, \ldots$

$$A_0 \xrightarrow{\cdot\theta_1} A_1 \xrightarrow{\cdot\theta_1} A_2 \xrightarrow{\cdot\theta_1} \cdots .$$

Now dualize the vector spaces and the maps:

$$\overline{A}_0 \xleftarrow{\cdot\overline{\theta}_1} \overline{A}_1 \xleftarrow{\cdot\overline{\theta}_1} \overline{A}_2 \xleftarrow{\cdot\overline{\theta}_1} \cdots .$$

Since we are interested in factoring out the images of θ_1, we wish to keep the kernels of $\overline{\theta}_1$. Repeating this process with the other θ_i, we get vector spaces $\overline{B}_0, \overline{B}_1, \overline{B}_2, \ldots$ of common kernels.

To describe these kernels explicitly, let $\theta_i = \sum_{j=1}^n a_{ij}x_j$ and put $V_j = [a_{1j}, \ldots, a_{dj}]^T$. Let M_i be all monomials in the variables $x_1, \ldots, x_n$ of degree i. The space $\overline{B}_i$ is isomorphic to the set of vectors $(c_m)_{m \in M_i}$ indexed by elements of M_i with the property that

$$c_m = 0 \quad \text{if } \operatorname{supp}(m) \notin \Delta, \tag{6}$$

and

$$\sum_{j=1}^{n} c_{x_j m} V_j = 0 \quad \text{for all } m \in M_{i-1}. \tag{7}$$

It is easy to see that $\overline{B}_1$ is the vector space of all linear relations on the V_j, and so has dimension $h_1 = n - d$ if the matrix whose columns are the V_j has full row rank. This would be the case, for example, if Δ is the boundary complex of a simplicial d-polytope containing the origin in its interior and the V_j are chosen to be its vertices.

When the simplicial complex is the boundary complex of the octahedron in Figure 1 and the V_j are taken to be its vertices, elements of $\overline{B}_2$ are of the form $c_{x_1x_3} = c_{x_1x_4} = c_{x_2x_3} = c_{x_2x_4} = p$, $c_{x_1x_5} = c_{x_1x_6} = c_{x_2x_5} = c_{x_2x_6} = q$, $c_{x_3x_5} = c_{x_3x_6} = c_{x_4x_5} = c_{x_4x_6} = r$, and $c_{x^2_i} = 0$, $i = 1, \ldots, 6$.

Now assume that the V_j are chosen in linearly general position, i.e., that every subset of size at most d is linearly independent. Consider a bistellar operation with F, G and k, l as before. Then there is a unique linear relation among $\{V_j : j \in F \cup G\}$, say, $\sum_{j \in F \cup G} a_j V_j = 0$, and $a_j \neq 0$ for all $j \in F \cup G$. Set $a_j = 0$ for all $j \notin F \cup G$. For $1 \leq i \leq d$ define $(c_m)_{m \in M_i}$ by $c_m = m(a)$; i.e., substitute a_j for x_j in m. During the bistellar operation, such elements are gained in $\overline{B}_i$ for $i \geq l$ when G is added, and lost in $\overline{B}_i$ for $i \geq k$ when F is removed. The structure of the links of F and G is enough to guarantee that the dimensions of these $\overline{B}_i$ change by ± 1 in exactly the right way. The proof is completed by demonstrating that $\dim(\overline{B}_i) = 1$, $i = 0, \ldots, d$ for the boundary of a d-simplex.

Now let us consider the special case when Δ is the boundary complex of a simplicial d-polytope P. Translate P, if necessary, so that its vertices v_j are in linearly general position. Then one can choose V_j equal to v_j. There is some evidence to support the proposal that ω can be taken to be $x_1 + \cdots + x_n$, for suppose this ω is factored out of B and this operation is viewed in a dual fashion as we did with the θ_i. The vector spaces $\overline{C}_i$ of kernels are described by

$$c_m = 0 \quad \text{if } \operatorname{supp}(m) \notin \Delta \tag{8}$$

and

$$\sum_{j=1}^{n} c_{x_j m} V_j = 0 \quad \text{and} \quad \sum_{j=1}^{n} c_{x_j m} = 0 \quad \text{for all } m \in M_{i-1}. \tag{9}$$

This is the affine analogue to condition (7). One easily sees that $\overline{C}_1$ is the vector space of all affine relations on the V_j and so its dimension equals $g_1 = n - d - 1$. But $\overline{C}_2$ also is a familiar object: It is isomorphic to the stress space used by Kalai and mentioned in Section 4. The correspondence is given by taking $\lambda_{v_i v_j}$ to be $c_{x_i x_j}$. So the dimension of $\overline{C}_2$ equals $g_2 = h_2 - h_1$ and the dimensions of $\overline{C}_1$ and $\overline{C}_2$ are both correct. It remains to be seen whether the dimensions of the other $\overline{C}_i$ for $i = 3, \ldots, \lfloor d/2 \rfloor$ equal g_i for this choice of ω. The spaces $\overline{C}_i$ suggest a natural way to extend the notion of stress space to higher dimensional faces, which might prove useful [**27**].

8. Triangulations of polytopes

One way to see how a simplicial polytope can be built up by bistellar operations is to look at a *Gale transform*, which can be defined for any convex d-polytope P, whether simplicial or not [**30**]. Let $v_1, \ldots, v_n \in \mathbf{R}^d$ be its vertices and consider the matrix A given by

$$A = \begin{bmatrix} v_1 & \cdots & v_n \\ 1 & \cdots & 1 \end{bmatrix}. \tag{10}$$

Let $y_1, \ldots, y_{n-d-1}$ be a basis for the nullspace of A (the space of all affine relations on the vertices) and list these vectors as the rows of a matrix

$$\begin{bmatrix} y_1^T \\ \vdots \\ y_{n-d-1}^T \end{bmatrix}.$$

The set V' of the n columns of this matrix, $v_1', \ldots, v_n' \in \mathbf{R}^{n-d-1}$, are the points of a Gale transform of P. There is a one-to-one correspondence between the vertices v_i of P and the points v_i' of the transform, and hence between subsets X of V and subsets X' of V'. One key property is that $\text{conv}(X)$ is a face of P if and only if the origin is contained in the relative interior of the convex hull of $V' \backslash X'$ in the Gale transform. This property is maintained even if the points in V' are independently scaled by positive numbers. In this case we say we have a *scaled Gale transform.*

Returning to the example of the octahedron in Figure 1, we see that the matrix A is

$$\begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

The two rows of the following matrix form a basis for the nullspace of A:

$$\begin{bmatrix} 1 & 1 & 0 & 0 & -1 & -1 \\ 0 & 0 & 1 & 1 & -1 & -1 \end{bmatrix}.$$

The result is the Gale transform given in Figure 4. Note that the points of a Gale transform need not be distinct.

Given a Gale transform of any convex d-polytope P, scale the points by positive numbers so that there is no hyperplane missing the origin that contains more than $d' \equiv n - d - 1$ points of V'. Choose any halfline L in $\mathbf{R}^{d'}$ starting at infinity and ending at the origin, but otherwise in general position. As you travel along this line, you will pass through the relative interior of various simplices of the form $\text{conv}(X')$ where $\text{card}(X') = d'$.

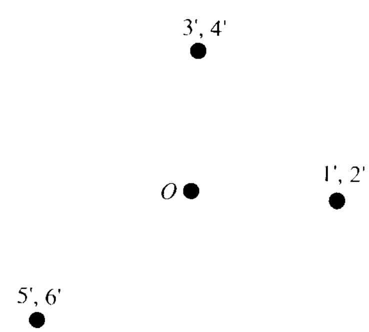


FIGURE 4. A Gale transform of the octahedron.

The complement of each such X' corresponds to a d-simplex in a triangulation of the polytope P, and the order induced by the halfline is a shelling order for the triangulation [**24**], [**25**], [**30**]. (That this procedure resembles line shellings is no coincidence.) Such a triangulation will be called a *Gale triangulation* of P. During the shelling process the boundary of the triangulation changes by bistellar operations, so if P is simplicial this induces a construction of the boundary complex of P by a sequence of bistellar operations.

In Figure 5, the halfline in the scaled Gale transform of the octahedron induces a triangulation of the octahedron into four simplices, each sharing the common interior edge 56. The corresponding bistellar operations are precisely those depicted in Figure 3.

If $d = 2$ or $n \leq d + 3$ then every triangulation of P is a Gale triangulation. In fact, Gale transforms can be used to prove that in these cases the collection of all subdivisions of P, ordered by refinement, is isomorphic to the face lattice of some $(n - d - 1)$-dimensional polytope [**23**], [**24**], [**25**].

In general, not all triangulations of convex polytopes are Gale triangulations. For example, there exist 3-polytopes with seven vertices that have triangulations unobtainable in this way [**24**]. However, any triangulation induced by *pulling* or *placing* the vertices in any order is a Gale triangulation. In fact, if a triangulation T is determined by pulling the vertices in the order $v_1, \dots, v_n$ and the triangulation T' is determined by placing the vertices in the opposite order, then in the Gale transform there exists an oppositely directed pair of halflines L and L' inducing T and T', respectively [**25**].

There are several equivalent ways of defining Gale triangulations. One way is to take the vertices $v_1, \dots, v_n$, lift them into general position in $\mathbf{R}^{d+1}$ yielding $(v_1, t_1), \dots, (v_n, t_n)$, determine their convex hull, and project the facets of the "lower half" of the resulting $(d+1)$-polytope back into $\mathbf{R}^d$. See, for example, [**14**]. Another way is to take the matrix A given in (10), choose a vector $c \equiv (c_1, \dots, c_n)$ in general position, and form the polyhedron

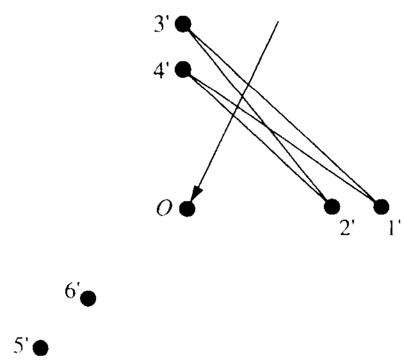


FIGURE 5. Inducing a Gale triangulation of the octahedron.

$Q = \{x : x^T A \leq c^T\}$. Then the vertices of Q are in one-to-one correspondence with simplices in a triangulation of P [15].

This latter perspective can be used to prove that any simple d-polytope P with n facets can be realized as a facet of a simple $(d+1)$-polytope P' with $n+1$ facets that has an edge-path diameter not exceeding $2n-2d$. If P is the feasible region for a linear program, this implies that we can solve the linear program from any starting point with at most $2n-2d$ pivots, if we somehow know which pivots should be made [25]. It has been conjectured that the edge-path diameter of any simple d-polytope with n facets is at most $n-d$, but this has not been settled and related examples have suggested that it might be false. The establishment of a good upper bound is one of the significant open problems in the theory of convex polytopes [19].

One interesting d-polytope to try to triangulate efficiently is the d-cube. The smallest triangulation of the 4-cube has 16 four-dimensional simplices, but for higher dimensions the minimum number is not known [11], [22], [34], [35], [36]. There is a close connection between Hadamard matrices and Gale transforms of the d-cube that might be exploited to shed some light on this problem and the more general task of finding new, interesting triangulations of the d-cube.

9. Winding numbers

The relationship between bistellar operations, Gale transforms, and triangulations leads fairly easily to the following result [28], also known to Lawrence. Let W be a collection of at least $e+1$ points in $\mathbf{R}^e$ such that no hyperplane contains more than e points of $W \cup \{O\}$, where O is the origin. Choose an integer $0 \leq k < (n-e)/2$. For X a subset of W of cardinality e, let us say that X (or $\text{conv}(X)$) is of *type* k if the hyperplane $H = \text{aff}(X)$ partitions the remaining $n-e$ points into two sets, one of which, say F, has cardinality k. Figure 6 shows the subsets of type 0, 1, and 2 in a set of seven points. (The origin is not marked.)

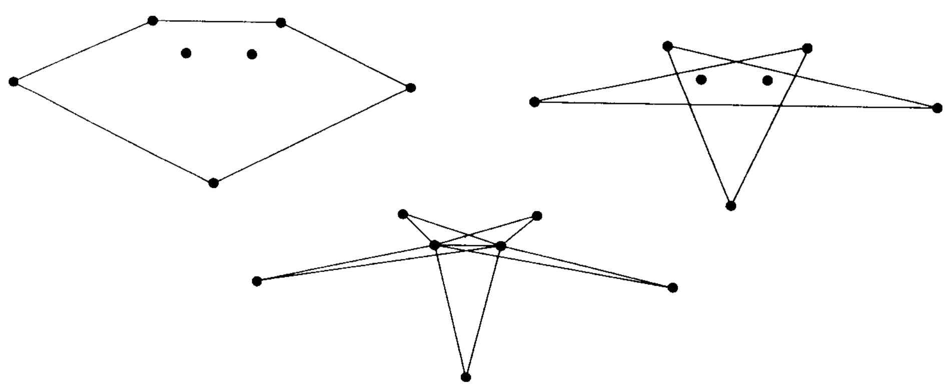


FIGURE 6. Subsets of type 0, 1, and 2.

For such a subset X, define the *sign* of X, $\mathrm{sg}(X)$, to be *positive* if F and O lie on opposite sides of H, and *negative* if F and O lie on the same side of H. Finally, define $\alpha(X)$ to be the measure of the solid angle with vertex O determined by X. The value $\alpha(X)$ is normalized to equal the fraction of the surface area of a unit sphere centered at the origin that is intersected by the cone determined by X. Set

$$w_k = \sum_X \mathrm{sg}(X)\alpha(X) \tag{11}$$

where the sum is taken over all X of type k. Then one can prove that this kth *winding number* w_k is in fact a nonnegative integer.

In the case that W is the scaled Gale transform of some simplicial $n - e - 1$ polytope P, the result follows by proving that $w_k = h_k - h_{k-1}$. The result for general W then follows readily. This suggests trying to prove the nonnegativity of w_k directly, which would yield a new proof of the unimodality of the h-vector. Such a direct proof has already been found for $e \leq 2$ **[28]**.

The unimodality of the h-vector of simplicial d-polytopes with n vertices was first conjectured as a part of the Generalized Lower-Bound Conjecture. The second part of the conjecture is that $h_k = h_{k+1}$ for some $k < \lfloor d/2 \rfloor$ if and only if P admits a triangulation with no simplex of dimension less than $d - k$ in the interior. This part of the conjecture is still unresolved in the general case, but has been confirmed when $n \leq d + 3$ as a part of the winding number proof for $e \leq 2$.

10. The moment map

We conclude with a brief discussion of a connection between convex polytopes and algebraic varieties. See also **[41]**. Let Q be a rational convex d-polytope with vertices $v_1, \dots, v_n$. Consider all nontrivial affine relations $a = (a_1, \dots, a_n)$ on the vertices: $D = \{a : \sum_{i=1}^n a_i v_i = 0, \sum_{i=1}^n a_i = 0, a_i \in \mathbf{R}, \text{not all zero}\}$. For any a, define $A_+ = \{i : a_i > 0\}$, $A_- = \{i : a_i < 0\}$, and $A_0 = \{i : a_i = 0\}$. We say that *a conforms* to a' if $A_+ \subseteq A'_+$ and $A_- \subseteq A'_-$. It is not difficult to see that there exists a finite set $\{a^1, \dots, a^m\}$ of integer $a^j \in D$ such that every integer $a \in D$ is a nonnegative integer combination of a^j conforming to a.

Fine **[13]** constructs a variety in the following way. Let $u_1, \dots, u_n$ be indeterminates. For each integer $a = (a_1, \dots, a_n) \in D$ associate the relation

$$\prod_{i \in A_+} u_i^{a_i} = \prod_{i \in A_-} u_i^{-a_i}. \tag{12}$$

Let $A_Q = \{u \in \mathbf{C}^n : u \text{ satisfies (12) for all integer } a \in D\}$. One can show that $A_Q = \{u \in \mathbf{C}^n : u \text{ satisfies (12) for all } a \in \{a^1, \dots, a^m\}\}$. So, A_Q is a variety in $\mathbf{C}^n$.

Note that $u \in A_Q$, $k \in \mathbf{C}$ implies that $ku \in A_Q$ since $\sum_{i \in A_+} a_i = \sum_{i \in A_-} (-a_i)$ for each $a \in D$. For $u, v \in A_Q$, let $u \sim v$ if $u = kv$ for some $0 \neq k \in \mathbf{C}$. Let $P_Q = (A_Q \setminus \{0\})/\sim$, a projective variety in $\mathbf{CP}^{n-1}$. Note that $u, v \in P_Q$ implies that $uv \in P_Q$ under component-wise multiplication; $u, v \in P_Q$ satisfying $v_i \neq 0$ whenever $u_i \neq 0$ implies $u/v \in P_Q$ under component-wise division (taking $0/0 = 0$); and $u \in P_Q$ satisfying $u \in \mathbf{R}^n_+$ implies $\sqrt{u} \in P_Q$, where $\sqrt{u} = (\sqrt{u_1}, \ldots, \sqrt{u_n})$.

For example, let Q be a d-simplex. Then there are no affine relations among the vertices, so $A_Q = \mathbf{C}^{d+1}$ and $P_Q = \mathbf{CP}^d$.

As a second example, take Q to be a square with vertices labeled v_1, v_2, v_3, and v_4, consecutively around the perimeter. Then the unique affine relation $v_1 + v_3 = v_2 + v_4$ yields the relation $u_1 u_3 = u_2 u_4$.

For F a nonempty face of Q ($F = Q$ allowed), define $\operatorname{supp}(F) = \{i : v_i \in F\}$. For $u \in P_Q$, define $\operatorname{supp}(u) = \{i : u_i \neq 0\}$. For any $u \in P_Q$ it can be shown that there is a face F of Q such that $\operatorname{supp}(u) = \operatorname{supp}(F)$, so P_Q is the disjoint union of sets of the form B_F, where $B_F = \{u \in P_Q : \operatorname{supp}(u) = \operatorname{supp}(F)\}$ for F a face of Q.

In fact there is at least one "canonical" element of P_Q associated with each face F of Q: Take $u = (u_1, \ldots, u_n)$ where

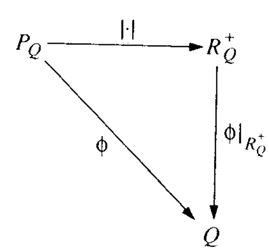
$$u_i = \begin{cases} 1 & \text{if } i \in \operatorname{supp}(F); \\ 0 & \text{otherwise.} \end{cases}$$

The above results can be used to show a direct connection between the homology groups of the variety and the graded components of the ring B, and hence that the dimensions of the homology groups are equal to the h_i when Q is simple.

Define the *moment map* $\phi : P_Q \to Q$ by

$$\phi(u) = \frac{\sum_{i=1}^n |u_i|^2 v_i}{\sum_{i=1}^n |u_i|^2}.$$

Note that if $u \in P_Q$ then $|u| = (|u_1|, \ldots, |u_n|) \in P_Q$. Let $R_Q^+ = \{u \in P_Q : u_i \in \mathbf{R},\ u_i \geq 0 \text{ for all } i\}$. We have the commutative diagram:



For example, let F be a face of Q and consider $u = (u_1, \dots, u_n)$ such that

$$u_i = \begin{cases} 1 & i \in \text{supp}(F); \\ 0 & \text{otherwise}. \end{cases}$$

Then

$$\phi(u) = \sum_{i \in \text{supp}(F)} \frac{1}{\text{card}(\text{supp}(F))} v_i,$$

which is the centroid of the vertices of F.

As another example, suppose Q is a d-simplex. For $u \in R_Q^+$, scale u so that $\sum_{i=1}^n u_i^2 = 1$. Then $\phi(u) = \sum_{i=1}^n u_i^2 v_i = \sum_{i=1}^n \lambda_i v_i$ where $u_i = \sqrt{\lambda_i}$. Hence there is a one-to-one correspondence between the elements of R_Q^+ and the points of Q.

Fine **[13]** proved that for any Q, $\phi|_{R_Q^+}$ is a bijection between R_Q^+ and Q. See also **[41]**. Another way to show this is to describe $\phi|_{R_Q^+}^{-1}$ explicitly **[26]**. For any point $x \in Q$, consider all $\lambda = (\lambda_1, \dots, \lambda_n)$ such that

$$\sum_{i=1}^n \lambda_i x^i = x, \qquad \sum_{i=1}^n \lambda_i = 1, \qquad 0 \le \lambda_i \le 1.$$

Call this set L. There is a unique λ^* that minimizes $\sum_{i=1}^n \lambda_i \log \lambda_i$ over L. It turns out that $\sqrt{\lambda^*} = (\sqrt{\lambda_1^*}, \dots, \sqrt{\lambda_n^*}) \in R_Q^+$ equals $\phi|_{R_Q^+}^{-1}(x)$.

The inverse of the moment map offers a canonical way of expressing any point $x \in Q$ as a convex combination of the vertices that is a natural generalization of barycentric coordinates for a simplex. The function $-\sum_{i=1}^n \lambda_i \log \lambda_i$ is the familiar entropy function and suggests the following whimsical interpretation of the canonical convex expression: Suppose two individuals A and B are playing a game on a polytope Q. A referee chooses a point x in Q, which is known to both A and B, and A chooses a way of expressing x as a convex combination $\sum_{i=1}^n \lambda_i v_i$ of the vertices of Q. Interpreting the λ_i as probabilities assigned to the vertices, A then randomly chooses a vertex of Q using this probability distribution. B now attempts to guess the vertex A has chosen by asking questions of the form "Is the vertex in the set S?" where S is a subset of the vertices. The object of B is to guess the vertex using as few questions as possible, so the object of A is to choose the λ_i that keep B guessing as long as possible, even if B should happen to discover which λ_i his opponent has chosen. The inverse of the moment map provides the best choice of λ_i.

11. Nonsimplicial polytopes

There is considerable interest in extending some of the results for simplicial polytopes to general convex polytopes. The discovery of the generalized Dehn–Sommerville equations by Bayer and Billera **[4]**, **[5]**, **[6]**, the notion of the generalized h-vector, and the connections with intersection homology

[40] are very encouraging and suggest that there is still much to be done to understand fully the interplay between geometry and algebra that has evolved with the study of convex polyhedra.

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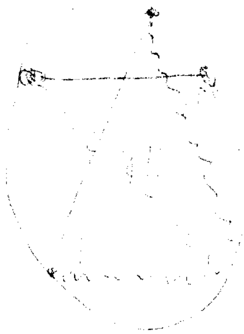

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