SOME NOTES ON TRIANGULATING POLYTOPES[1]

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1. INTRODUCTION

Let P be a convex d-polytope with vertex set V. A subdivision of P is a collection of distinct d-polytopes $S = \{P_1,...,P_n\}$ whose union is P, such that the vertex set of each P_i is a subset of V, and that $P_i \cap P_j$ is a (possibly empty) face of both P_i and P_j for all i,j. A subdivision will be called proper if $n>1$. If each P_i is a d-simplex, then we have a triangulation of P. The set of all the faces of all the P_i, including the empty set and each of the P_i themselves, will be called the set of faces of the subdivision. Given two subdivisions $S = \{P_1,...,P_n\}$ and $T = \{Q_1,...,Q_m\}$, we will call T a refinement of S if every P_i is the union of some of the Q_j. Denote by Σ the collection of all subdivisions of P. Place a partial order on Σ by writing $S<T$ if S and T are two different subdivisions for which T is a refinement of S.

2. CONVEX POLYGONS

Consider the complex Σ for a convex n-gon P, i.e. consider all possible ways of choosing a set of noncrossing diagonals of P. This is a simplicial complex of dimension n-4. Perles [14] asked whether Σ is realizable as the boundary complex of some convex (n-3)-polytope Q. As background information, he cited a paper by Huguet and Tamari [6] in which this is claimed to be true. They reference Tamari [15] and Friedman and Tamari [3]. We have not yet had the opportunity to check the first of these two papers, but Perles points out that in the second a totally different complex is shown to be polytopal.

Haiman [5] constructs the dual of the desired Q by obtaining a defining set of inequalities, one for each diagonal of the n-gon. Subsequently, we approached this problem from the point of view of Gale diagrams [9], which led to the following construction: Label the vertices of P from 0 to n-1 consecutively around the perimeter. Also, let Δ be the boundary complex of any (n-3)-dimensional simplex and number its vertices from 1 to n-2. Consider the collection S of all sets of consecutive integers of the form $\{i,i+1,\ldots,j\}$, where $1\le i\le j\le n-2$, excluding the set $\{1,2,\ldots,n-2\}$. On the one hand, each member of S corresponds to a diagonal of P, that joining vertices i-1 and j+1. On the other hand, each member of S corresponds to a face of Δ.

Order the members of S, $F_1,\ldots,F_m$, so that $i<j$ whenever $F_j \subset F_i$. Starting with Δ, perform stellar subdivisions of the faces corresponding to $F_1,\ldots,F_m$ in that order. The resulting complex is isomorphic to Σ and is easily seen to be polytopal. Because the number of triangulations of the n-gon, and hence the number of facets of Q, is a Catalan number, we have chosen to call Q a <u>Catalan polytope</u>. In fact, a more general construction technique allows us to conclude the following:

<u>Theorem 2.1</u>: Let P be a convex polygon. Then Σ is realizable as the boundary complex of some convex simplicial (n-3)-polytope Q. Moreover, for any convex (n-3)-polytope R with at most n vertices, there exists a subdivision of its boundary complex that is isomorphic to Σ. If R is simplicial, the subdivision is achievable by a sequence of stellar subdivisions.

One can in fact construct Q to reflect geometrically the symmetry of a regular n-gon:

Theorem 2.2: There exists a realization of Q such that there is an orthogonal group of transformations acting on Q that is isomorphic to the dihedral group.

The f-vector and the h-vector of Q are also of interest. The f-vector is given by the formula of Kirkman [7] and Cayley [2], but can be derived easily once one has determined the components of the h-vector.

Theorem 2.3: For the polytope Q,

$$h_i = \frac{1}{n-1} \binom{n-3}{i} \binom{n-1}{i+1}, \qquad 0 \le i \le n-3,$$

$$f_{j-1} = \frac{1}{n-1} \binom{n-3}{j} \binom{n+j-1}{j+1}, \quad 0 \le j \le n-3.$$

The components of h(Q) can be interpreted in terms of some of the many problems isomorphic to that of triangulating an n-gon.

Suppose we have a subdivision of some convex d-polytope P and we consider the combinatorial complex obtained by joining every face of the subdivision that lies on the boundary of P to a new point z. If the resulting complex is isomorphic to the boundary complex of some convex (d+1)-polytope P^*, then we will call the subdivision polyhedral. In this case, the complex of faces in the subdivision corresponds to the complex of faces of P^* not containing z.

Theorem 2.4: Every subdivision of a convex n-gon P is polyhedral.

This is trivial to see: Think of P as being made of paper, and imagine creasing it slightly along the diagonals of the subdivision, introducing a new point z above the polygon, and taking the convex hull.

It is also clear that every triangulation of P can be constructed by starting with a triangle, and then successively adding on triangles, each one being glued along one edge to the convex complex of preceding triangles.

This can be generalized to a method of triangulating arbitrary convex d-polytopes described by Billera and Munson [1] in the more general context of oriented matroïds. (See also [10].) The vertices of the polytope P are ordered and successively "placed" into position, and at each stage the convex hull of the currently placed vertices is provided with a triangulation. Intuitively, as each new vertex v_k is "placed," the current triangulation is extended by constructing a simplex with apex v_k and base F for every face F of the present triangulation that is "visible" from v_k. We will say that the resulting triangulation S of P is <u>vertex-placeable</u>. So we have the trivial result:

<u>Theorem 2.5</u>: Every triangulation of a convex polygon is vertex-placeable.

Further, Billera and Munson prove the following:

<u>Theorem 2.6</u>: Every vertex-placeable triangulation of a convex d-polytope is polyhedral. (I think polyhedral = regular!)

3. POLYTOPES WITH FEW VERTICES

By a <u>polytope with few vertices</u> we mean a convex d-polytope with d+1, d+2, or d+3 vertices. Since the case of a convex d-polytope with d+1 vertices is trivial, we assume that we have a d-polytope P with d+2 vertices. Consider any proper subdivision S of P. Then S must be a triangulation. Choose any simplex in the triangulation, which is the convex hull of d+1 vertices. Any facet of this simplex that is not on the boundary of P must be the base of another simplex whose apex is the remaining $(d+2)^{nd}$ vertex of P. This implies that the triangulation is completely determined by the initially given simplex, and that it is vertex-placeable.

meaning??

Choose a point O in the interior of P in affinely general position with respect to the vertices of P, and regard the vertices of P as the Gale diagram [4,13] of a set of d+2 distinct points in R^1. Since the convex hull of these points is 1-dimensional, it has two facets, implying that there are exactly two d-simplices containing O formed from the vertices of P. Hence P admits exactly two triangulations.

Now let P be a convex d-polytope with d+3 vertices, and let S be any subdivision of P. If we join the original faces of P to a new point z, the resulting complex is topologically a d-sphere with d+4 vertices. Hence by Kleinschmidt [8] and Mani [11], this complex is realizable as the boundary complex of some convex (d+1)-polytope P^*, showing (after "pulling" z) that the original subdivision is polyhedral. If we consider the Gale diagram of P^* and remove the point z′ corresponding to z, the resulting configuration will be a Gale diagram for P.

Assume now that S is a proper subdivision that is not a triangulation. Suppose we want to find a refinement of S. Then there must be some P_i of S which is the convex hull of d+2 vertices that is to be further subdivided. But there are only two ways of doing this, and both are triangulations and vertex-placeable. Also, each face in this triangulation of P_i that does not lie on the boundary of P must now become the base of a simplex with apex being the $(d+3)^{rd}$ vertex. This implies that S has exactly two refinements, both vertex-placeable triangulations.

Let V′ be a Gale transform of the vertices of P. Choose a new point z′ on the unit circle centered at the origin and regard the result as the Gale diagram of a convex (d+1)-polytope P^* with d+4 vertices, one of which is the vertex z corresponding to z′. Then the complex Δ of faces of P^* that do not contain z is a topological ball. One can show that Δ is isomorphic to a proper subdivision of P (see McMullen [12]), and that as z moves around the unit circle, all such subdivisions arise in this way.

From the above discussion, we may deduce:

<u>Theorem 3.1</u>: Let P be a convex d-polytope with few vertices. Then all subdivisions of P are polyhedral, and all triangulations of P are vertex-placeable.

We also have the analogue of Catalan polytopes for polytopes with few vertices:

<u>Theorem 3.2</u>: Let P be a convex d-polytope with n vertices, where $d+1 \leq n \leq d+3$. Then Σ is realizable as the boundary complex of some simplicial convex (n−d−1)-polytope Q.

4. GENERAL POLYTOPES

Both of the above results can break down if we have a convex d-polytope P with n vertices where $d>2$ and $n>d+3$. Consider the 3-polytope that is a triangular prism with six vertices C,D,E,F,G,H, triangular faces CDF and HGE, and square faces CDGH, DFEG, and FCHE. Place a new vertex B just above the face CDF and take the convex hull. The resulting polytope P admits a subdivision R into three square-based pyramids BCDGH, BDFEG, BFCHE and one simplex BHGE. Each of the pyramids can be triangulated independently, which allows one to construct a chain $R<S<T<U$ of four distinct proper subdivisions. Thus Σ cannot be isomorphic to the boundary complex of any convex polytope of dimension $n-d-1=3$.

Now take P and twist the triangle CDF slightly so that the three square faces decompose into the six triangular faces CGD, CGH, DEF, DEG, FHC, and FHE. Think of each of the square faces as expanding into a tetrahedron. This new polytope P′ can be triangulated into simplices CDGH, DFEG, FCHE, BDHC, BDHG, BFGD, BFGE, BCEF, BCEH, and BHGE. If the boundary faces are now joined to a new point A, the resulting triangulated 3-sphere is dual to the Brückner sphere, which is not polytopal -- see Grünbaum [4, pp.222-224]. Thus P′ has a nonpolyhedral (and hence non-vertex-placeable) triangulation. This triangulation was discovered in a conversation with Kleinschmidt.

Some ideas of the previous sections do carry over to some extent, however. Let P be a convex d-polytope with n vertices and choose a new point O in the interior of P in affinely general position with respect to the vertices of P. Then treat the versimplicial ??

16 − 12 − 1 = 3
3-dim simplicial polytope.

tices of P as the Gale diagram of some set of n points whose convex hull is a simplicial convex (n-d-1)-polytope Q. One can define a surjection from the collection of subdivisions of P onto the collection of faces of Q such that if S and T are two subdivisions with S<T, then the image of S is contained in (possibly equal to) the image of T. If one first selects any two different subdivisions, by a judicious choice of Q one can obtain a Q for which these two subdivisions are mapped onto different faces.

One can also take a Gale transform V′ of the vertices of P and consider the unit (n-d-2)-sphere centered at the origin. For every choice of a point z′ on this sphere, there is an associated polyhedral subdivision of P exactly as in the case of d-polytopes with d+3 vertices [12]. The sphere can be partitioned into regions corresponding to identical subdivisions, and in this manner one can construct a spherical complex of some (but not necessarily all) subdivisions of P.

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