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Regular Triangulations of Convex Polytopes

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ABSTRACT. Let V be a finite set of points in $\mathbf{R}^d$. We are interested in subdivisions of the convex hull of V into polytopes whose vertices lie in V. We begin with a general introduction to lexicographic subdivisions, and refinements and subdivisions by pulling and placing, and present some implications for edge-path diameters of simple convex polytopes and linear programming. In particular, we show that if P is any simple d-polytope with n facets, then there exists a simple $(d+1)$-polytope Q with $n+1$ facets, one of which is congruent to P, such that the edge-path diameter of Q is at most $2(n-d)$. After reviewing the notions of Gale transforms and diagrams we discuss several equivalent definitions for the properly larger class of regular or Gale subdivisions. There is a simple characterization of such subdivisions, and shellings can be obtained from Gale transforms. If V is a set of at most $n \leq d+3$ points in $\mathbf{R}^d$ with d-dimensional convex hull, we prove that all subdivisions are regular and all triangulations are placeable. Examples show that these results break down when $n > d+3$. We conclude with some comments on the secondary polytope of all regular subdivisions of a given set V.

is $Q_A = P_A$ in this case?

1. Faces, subdivisions and triangulations

Let $V = \{v_1, \ldots, v_n\}$ be a finite set of points in $\mathbf{R}^d$ such that $[V]$ is a d-dimensional convex polytope, where $[\cdot]$ denotes convex hull. We do not insist that each point of V be a vertex of $[V]$ and in fact will not even require that they all be distinct. We say a subset F of V is a *face* of V if F is the intersection of V with some supporting hyperplane of $[V]$. In addition, the empty set and V itself will also be called faces. All faces except V are *proper*. Note, in particular, that what we call a face depends not only on the geometrical structure of $[V]$ but also on the defining set V. The *dimension* of a face F, $\dim(F)$, equals the dimension of $[F]$. A j-dimensional face

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will be called a *j-face* for brevity. Faces of dimension 0, 1, and $d-1$ are *vertices, edges*, and *facets* of V, respectively.

The polytope itself is a subdivision

A *subdivision* of $[V]$ is a collection $\mathcal{S} = \{S_1, \dots, S_m\}$ of subsets of V such that (1) $\dim([S_i]) = d$ for all i, (2) $\bigcup_{i=1}^{m}[S_i] = [V]$, and (3) for $1 \le i < j \le m$ we have $[S_i] \cap [S_j] = [F]$ for some common proper face F of S_i and S_j. If in additional all S_i have cardinality $d+1$ then the subdivision is called a *triangulation*. A subset F is a face of $\mathcal{S}$ if it is a face of at least one of the S_i.

For two subdivisions $\mathcal{S} = \{S_1, \dots, S_m\}$ and $\mathcal{T} = \{T_1, \dots, T_p\}$ we call $\mathcal{T}$ a *refinement* of $\mathcal{S}$ if for each S_i, there exist $T_{i_1}, \dots, T_{i_k} \in \mathcal{T}$ such that $\{T_{i_1}, \dots, T_{i_k}\}$ is a subdivision of S_i. In this case we write $\mathcal{T} \le \mathcal{S}$, with strict inequality if the two subdivisions are different.

In §2 we give a general introduction to lexicographic subdivisions, and refinements and subdivisions by pulling and placing, and discuss some implications for edge-path diameters of simple convex polytopes and linear programming (Theorem 1). §3 reviews the notions of Gale transforms and diagrams. Several equivalent definitions for the properly larger class of regular or Gale subdivisions are presented in §4. Theorem 2 provides a simple characterization and Theorem 3 addresses shellability. In §5 we discuss subdivisions of small sets of points, proving that in such cases all subdivisions are regular and all triangulations are placeable (Theorem 4). The last section, §6, contains some examples of nonregular triangulations and some notes on the secondary polytope. Some of the results in this paper were announced in [9].

2. Pulling, placing and lexicographic triangulations

Let $\mathcal{S}$ be a subdivision of V and let $v \in V$. One refinement $p_v^-(\mathcal{S})$ of $\mathcal{S}$ can be obtained as follows. Consider each $S_i \in \mathcal{S}$.

1. If $v \notin S_i$ then $S_i \in p_v^-(\mathcal{S})$.
2. If $v \in S_i$ then $p_v^-(\mathcal{S})$ contains all sets of the form $F \cup \{v\}$, where F is a facet of S_i which does not contain v.

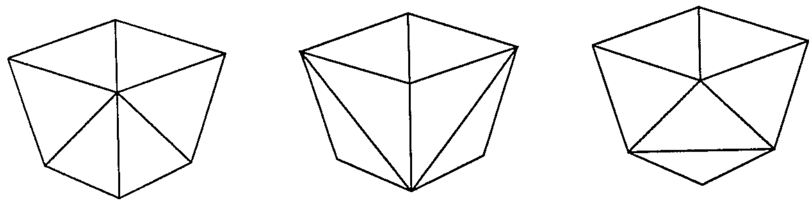
We define another refinement $p_v^+(\mathcal{S})$ of $\mathcal{S}$ by again considering each $S_i \in \mathcal{S}$.

1. If $v \notin S_i$ then $S_i \in p_v^+(\mathcal{S})$.
2. If $v \in S_i$ and $\dim([S_i \backslash \{v\}]) = d-1$ then $S_i \in p_v^+(\mathcal{S})$.
3. If $v \in S_i$ and $\dim([S_i \backslash \{v\}]) = d$, let $S_i' = S_i \backslash \{v\}$. Then $p_v^+(\mathcal{S})$ contains S_i' together with all sets of the form $F \cup \{v\}$, where F is a facet of S_i' which v is *beyond* (with respect to $[S_i']$ [**4**, §**5.2**].

Let the points of V be given in some order $v_1, \dots, v_n$, let $(\varepsilon_1, \dots, \varepsilon_n) \in \{\pm\}^n$, and consider any subdivision $\mathcal{S}$ of V. Then it is easy to check that $p_{v_n}^{\varepsilon_n} \cdots p_{v_1}^{\varepsilon_1}(\mathcal{S})$ is a triangulation of V. Triangulations that result from choosing $\mathcal{S}$ to be the trivial subdivision $\{V\}$ are *lexicographic* [**19**].

That it is a subdivision is corroborated at each step. For triangulation we need to check sizes of the pieces at the last step.

If $\mathcal{S}$ is $\{V\}$ and $\varepsilon_i = -$ for all i, the resulting triangulation is said to be obtained by *pulling* the points of V in the order $v_1, \ldots, v_n$. On the other hand, if $\mathcal{S}$ is $\{V\}$ and $\varepsilon_i = +$ for all i, the resulting triangulation is said to be obtained by *placing* the points of V in the order $v_n, \ldots, v_1$ (although perhaps the term *pushing* would be more appropriate). In Figure 1, (a) is achievable by pulling but not by placing, (b) is achievable by placing but not by pulling, and (c) is lexicographic, but not achievable by either pulling or placing. (only ! you mean)



(a) (b) (c)

FIGURE 1. Some lexicographic triangulations

Pulling is described in Hudson [7, **Lemma 1.4**] and has been considered by others in several equivalent ways. Billera and Munson [2] defined triangulations by placing in the more general context of oriented matroids. Its designation is suggested by an equivalent formulation. One "places" the points of V into position in the order $v_n, \ldots, v_1$, successively updating the resulting convex hulls and their triangulations. This inductive method for constructing the convex hull of V appears in Grünbaum [**4, §5.2**] and recent work of Seidel [**16**] shows that it can be implemented as a particularly simple and effective algorithm.

what is the other guy in Minneso ta doing differently.

Now suppose that $\mathcal{T}$ is the placing triangulation $p^+_{v_n} \cdots p^+_{v_1}(\{V\})$. For $k = 1, \ldots, d+1$, let v_{i_k} be chosen recursively such that

$$i_k = \max\{j\colon \dim([\{v_{i_1}, \ldots, v_{i_{k-1}}\} \cup \{v_j\}]) = k-1\}.$$

It is not difficult to show that if one reorders $v_1, \ldots, v_n$ by bringing the points $v_{i_1}, \ldots, v_{i_{d+1}}$ to the end of the list (without reordering any of the other points), then the resulting placing triangulation remains unchanged. So without loss of generality we may assume that $\dim([\{v_{n-d}, \ldots, v_n\}]) = d$. In this case it is immediate that $\mathcal{T}$ is *weakly vertex decomposable* (see [**15**]) with decomposition order $v_1, \ldots, v_n$. As a consequence, the diameter of $\mathcal{T}$ is no more than $2(n-d-1)$. That is, for any two d-simplices S', S'' of $\mathcal{T}$ there is a chain of d-simplices $S' = S_0, S_1, \ldots, S_{k-1}, S_k = S''$ such that $k \le 2(n-d-1)$ and $S_{i-1} \cap S_i$ has dimension $d-1$ for all $1 \le i \le k$.

Diameter in nonregular triangulations ??

This has a nice dual interpretation. Let $P \subset \mathbf{R}^d$ be a simple d-polytope containing the origin that has n facets $F_1, \ldots, F_n$. Let $V = \{v_1, \ldots, v_n\}$

be the corresponding vertices of the simplicial polytope polar to P. Regard P as sitting in $\mathbf{R}^{d+1}$ by appending a zero $(d+1)$st coordinate. Construct a pyramid Q' over P with apex $z = (\mathbf{O}, 1)$. There will be n facets $F'_1, \ldots, F'_n$ of Q' containing z such that $F'_i \cap P = F_i$. By "rotating" $F'_1, \ldots, F'_n$ away from z about the F_i by successively smaller amounts, one obtains a simple $(d+1)$-polytope Q, one of whose facets is P. The vertices of Q that do not lie in P correspond to the d-simplices of $\mathcal{T} = p^+_{v_n} \cdots p^+_{v_1}(\{V\})$. Similarly, the edges of Q that do not lie in P correspond to the $(d-1)$-simplices of $\mathcal{T}$. Each vertex in P is joined by an edge to a unique vertex of Q not in P. Hence we can conclude that the edge-path diameter of Q is no more than $2(n-d-1)+2 = 2(n-d)$. Therefore we have proved the following theorem.

no entiendo

THEOREM 1. *Let P be any simple d-polytope with n facets. Then there exists a simple $(d+1)$-polytope Q with $n+1$ facets, one of which is congruent to P, such that the edge-path diameter of Q is at most $2(n-d)$.*

The significance of this result is due to the unresolved *Hirsch conjecture* that the edge-path diameter of any simple d-polytope with n facets is not greater than $n-d$.

Suppose we start with the linear program

$$\begin{aligned} &\max c \cdot x, \\ &a_i \cdot x \le b_i, \quad 1 \le i \le n \end{aligned} \tag{1}$$

where $x \in \mathbf{R}^d$. Assume that $b_i > 0$ for all i. From Theorem 1 and the accompanying discussion it is not hard to verify that if the feasible region is bounded (even if it is not simple) one can consider the augmented linear program

$$\begin{aligned} &\max c \cdot x + 0x_{d+1} \\ &a_i \cdot x + (b_i - \varepsilon_i)x_{d+1} \le b_i, \ 1 \le i \le n, \\ &x_{d+1} \ge 0. \end{aligned} \tag{2}$$

Here the ε_i are indeterminates with lexicographic order $0 < \varepsilon_n \ll \cdots \ll \varepsilon_1 \ll 1$. The feasible region of (2) is obtained by making a pyramid over the feasible region of (1) with apex $(\mathbf{O}, 1)$, and then rotating the hyperplanes away from the apex in the order $i = 1, \ldots, n$. Because $\mathbf{O}$ is in the interior of the feasible region of (1) and the perturbations are small, one can conclude that $(x, 0)$ is optimal for (2) if and only if x is optimal for (1). From any basic feasible solution to (2) there exists a sequence of at most $2(n-d)$ pivots to an optimal solution. So in principle the linear program can be solved in a linear number of pivots, if one only knew which pivots to make. Perhaps there is a way that the objective function can be used to suggest a favorable ordering of the constraints.

3. Gale transforms and diagrams

In this section we summarize some useful facts about Gale transforms and diagrams. More detailed expositions can be found in **[4, §§5.4, 6.3]** and **[14]**. Let us return to a finite set of (not necessarily distinct) points V in $\mathbf{R}^d$ such that $\dim([V]) = d$. List the points v_i as columns of a matrix and append a row of ones, obtaining the $(d+1) \times n$ matrix

$$A = \begin{bmatrix} v_1 & \cdots & v_n \\ 1 & \cdots & 1 \end{bmatrix}.$$

This matrix has full row rank because the points of V do not lie in a common hyperplane. The nullspace $\mathscr{N}(A)$ of A is the space of all affine relations on V. Choose a basis for $\mathscr{N}(A)$ and list these vectors as the rows of an $(n-d-1) \times n$ matrix $\overline{A}$. The columns $\overline{v}_1, \dots, \overline{v}_n$ of $\overline{A}$ are in a natural one-to-one correspondence with the original points $v_1, \dots, v_n$ of V, which extends to a bijection between subsets F of V and subsets $\overline{F}$ of $\overline{V}$. The set $\overline{V} = \{\overline{v}_1, \dots, \overline{v}_n\}$ is called a *Gale transform* of V. Note, in particular, that $\overline{v}_1 + \cdots + \overline{v}_n = \mathbf{O}$. The key property of Gale transforms is the following well-known characterization of faces.

PROPOSITION 1. *A subset* $F \subsetneq V$ *is a face of* V *if and only if* $\mathbf{O} \in \operatorname{relint}([\overline{V} \backslash \overline{F}])$.

A set of n points $\overline{V}'$ is a *Gale diagram* of V if it is *isomorphic* to $\overline{V}$; i.e., $\mathbf{O} \in \operatorname{relint}([\overline{V} \backslash \overline{F}])$ if and only if $\mathbf{O} \in \operatorname{relint}([\overline{V}' \backslash \overline{F}])$ for all $F \subsetneq V$. For example, one can scale the points in a Gale transform independently by positive amounts—such a Gale diagram will be called a *scaled Gale transform*. If a Gale transform (respectively, diagram) is scaled so that the nonzero points lie on the unit $(n-d-2)$-sphere centered at $\mathbf{O}$, then we say we have a *normalized* Gale transform (respectively, diagram).

4. Regular subdivisions and triangulations

Again let V be a finite set of (not necessarily distinct) points in $\mathbf{R}^d$ such that $\dim([V]) = d$. There are several equivalent ways to define the notion of a *regular* or *Gale* subdivision of V.

Gale transform construction

DEFINITION 1. Let $\overline{V}$ be a Gale transform of V and $\overline{z} \in \mathbf{R}^{n-d-1}$. Put $\overline{V}' = \overline{V} \cup \{\overline{z}\}$. If $\mathbf{O} \in \operatorname{relint}([\overline{V}' \backslash \overline{S}])$ for $S \subseteq V$ and S is maximal with respect to this property then S is a d-face of the subdivision $\mathscr{S}$.

DEFINITION 2. Choose numbers $\lambda_1, \dots, \lambda_n$ and let $W = \{(v_1, \lambda_1), \dots, (v_n, \lambda_n)\} \subset \mathrm{R}^{d+1}$. If $S' = \{(v_{i_1}, \lambda_{i_1}), \dots, (v_{i_k}, \lambda_{i_k})\}$ is a facet of W in the *upper hull* of W (i.e., the last component of the outward normal of its supporting hyperplane is positive) then $S = \{v_{i_1}, \dots, v_{i_k}\}$ is a d-face of the subdivision $\mathscr{S}$. (If $\dim([W]) = d$ then the subdivision is taken to be the trivial one.)

Very intriguing

DEFINITION 3. Choose numbers $\mu_1, \ldots, \mu_n$ and let $Q = \{(x, x_{d+1}) \in \mathbf{R}^{d+1} : x \cdot v_i + x_{d+1} \leq \mu_i,\ i = 1, \ldots, n\}$. Let (w, w_{d+1}) be a vertex of Q and let $S = \{v_i : w \cdot v_i + w_{d+1} = \mu_i\}$. Then S is a d-face of the subdivision $\mathcal{S}$.

Definition 2 appears, for example, in Gel′fand, Kapranov, and Zelevinskii [3] who call such subdivisions regular because of their connection to solutions of systems of differential equations. McMullen [14] discusses Definition 1, and Definition 3 is due to Cottle and Hoffman [6].

To show the equivalences, assume first that $\mathcal{S}$ is given via Definition 1. Scale $\overline{z}$ by a positive amount, if necessary, so that we can assume it lies in the interior of $[\overline{V}]$. Find numbers $\lambda_1, \ldots, \lambda_n$ such that $\overline{z} = \lambda_1\overline{v}_1 + \cdots + \lambda_n\overline{v}_n$, $\lambda_1 + \cdots + \lambda_n = 1$, and $0 < \lambda_i < 1$ for all i. One can show that $\overline{V}'$ is a Gale diagram of $V' = \{(1-\lambda_1)^{-1}(v_1, \lambda_1), \ldots, (1-\lambda_n)^{-1}(v_n, \lambda_n), (\mathbf{O}, -1)\}$. So the facets of V' not containing $z = (\mathbf{O}, -1)$ correspond to the proposed elements of the subdivision of V. Noting that $(1-\lambda_i)^{-1}\lambda_i > 0$, apply the projective transformation $f(x, x_{d+1}) = (x, x_{d+1})/(x_{d+1}+1)$ to $[V']$. This sends the point z onto the hyperplane at infinity. The result is the unbounded polyhedron $[W] + [\mathbf{O}, -\infty)$, where $W = \{(v_1, \lambda_1), \ldots, (v_n, \lambda_n)\}$. The facets of V' not containing z correspond to the bounded facets of $[W]$, which in turn correspond to the facets in the upper hull of W. So $\mathcal{S}$ falls under Definition 2.

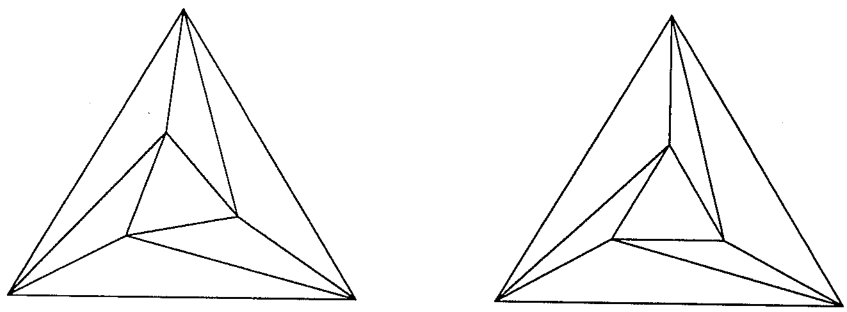
Conversely let $\mathcal{S}$ be given by Definition 2 and let $\overline{V}$ be a Gale transform of V. By translation and positive scaling of the last coordinate one can obtain $\lambda_1', \ldots, \lambda_n'$ that yield the same subdivision, such that $0 < \lambda_i' < 1$ and $\lambda_1' + \cdots + \lambda_n' = 1$. Set $\overline{z} = \lambda_1'\overline{v}_1 + \cdots + \lambda_n'\overline{v}_n$. Then with this $\overline{z}$ we get $\mathcal{S}$ using Definition 1.

The equivalence of Definitions 2 and 3 via $\lambda_i = -\mu_i$ is a straightforward consequence of duality.

EXAMPLE 1. The twisted triangle example of Figure 2(a) shows that there are regular triangulations that are not lexicographic.

EXAMPLE 2. Let $\lambda_k \neq 0$ and $\lambda_i = 0$ for $i \neq k$. Then the corresponding subdivision of V is $p_{v_k}^{-sg(\lambda_k)}(\{V\})$. More generally, if $0 \ll \lambda_n' \ll \cdots \ll \lambda_1'$, $\varepsilon_i \in \{\pm\}$, and $\lambda_i = -\varepsilon_i\lambda_i'$ for $i = 1, \ldots, n$, then we get the lexicographic triangulation $p_{v_n}^{\varepsilon_n} \cdots p_{v_1}^{\varepsilon_1}(\{V\})$. In particular, there are points $\overline{z}$ and $-\overline{z}$ such that $\overline{z}$ corresponds to the triangulation by pulling the points in the order $v_1, \ldots, v_n$ and $-\overline{z}$ corresponds to placing the points in the order $v_n, \ldots, v_1$. So these two triangulations are in a sense opposite or *complementary.*

EXAMPLE 3. Let $\lambda_i = -\|v_i\|^2$ for $i = 1, \ldots, n$. Then we get the (nearest point) Delaunay subdivision of V. Of course this is invariant under rigid motion, and one can verify that $\overline{z} = \lambda_1\overline{v}_1 + \cdots + \lambda_n\overline{v}_n$ also remains unchanged when V is moved rigidly. Similarly the furthest point Delaunay subdivision



(a) (b)

FIGURE 2. A regular and a nonregular triangulation

is constructed by choosing $\lambda_i = \|v_i\|^2$, $i = 1, \ldots, n$. So these two subdivisions are also complementary. For more on complementary subdivisions see [12].

Analogous to Shephard's theorem [17] we have the following test for regularity.

Shepar

THEOREM 2. *Suppose $\mathcal{S} = \{S_1, \ldots, S_m\}$ is a subdivision of V. Let $\overline{V}$ be a Gale transform of V. Then $\mathcal{S}$ is a regular subdivision if and only if $\bigcap_{i=1}^m \operatorname{relint}(\operatorname{cone}(\overline{V}\backslash\overline{S}_i)) \neq \varnothing$.*

What can you say about Gale transforms

PROOF. The point $\overline{z}$ must be placed in $-\bigcap_{i=1}^m \operatorname{relint}(\operatorname{cone}(\overline{V}\backslash\overline{S}_i))$. □

The equivalence of Definitions 1 and 2 shows immediately that any regular subdivision is shellable. Let us make the connection more explicit when we have a regular triangulation. Suppose $\overline{V}$ is a Gale transform of V, where $\operatorname{card}(V) > d+1$. By applying an appropriate projective transformation to V, if necessary, we can assume that no hyperplane missing $\mathbf{O}$ contains an affinely dependent set of points of $\overline{V}$. Hence every set of $n-d-1$ linearly independent points of $\overline{V}$ determines a unique hyperplane. By translating V we may also assume that $v_1 + \cdots + v_n = \mathbf{O}$. Let $\mathcal{T}$ be a regular triangulation induced by $\overline{z} = \lambda_1\overline{v}_1 + \cdots + \lambda_n\overline{v}_n$, where $\lambda_1 + \cdots + \lambda_n = 1$ and $0 < \lambda_i < 1$ for all i. It is easy to see that a set S_i of cardinality $d+1$ is a d-simplex of $\mathcal{T}$ if and only if $[\overline{T}_i \cup \{\overline{z}\}]$ is an $(n-d-1)$-simplex containing $\mathbf{O}$ in its interior, where $\overline{T}_i = \overline{V}\backslash\overline{S}_i$. This happens if and only if the ray $-t\overline{z}$, $t > 0$, intersects the relative interior of $[\overline{T}_i]$ for some positive value t_i of t. Because $\mathcal{T}$ is a triangulation, perturbing $\overline{z}$ by a small amount will not alter $\mathcal{T}$, so we may assume that the numbers t_i are all distinct. Order such sets $\overline{T}_1, \ldots, \overline{T}_m$ in order of decreasing t_i. Consider the corresponding order $S_1, \ldots, S_m$ of the d-simplices of $\mathcal{T}$.

THEOREM 3. *The above ordering of the d-simplices of $\mathcal{T}$ constitutes a shelling order and in fact corresponds to the line shelling of the upper hull of $W = \{(v_1, \lambda_1), \ldots, (v_n, \lambda_n)\}$ induced by the ray $(\mathbf{O}, u)$, $u > 0$.*

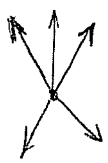
PROOF. Straightforward. □

In [11] we discuss a connection between shellings, winding numbers in Gale transforms, and f-vectors of simplicial convex polytopes, and show how this can be used to understand some specific cases of the generalized lower-bound conjecture.

5. Subdivisions of sets with few points

Again let V be a finite set of (not necessarily distinct) points in R^d such that $\dim([V]) = d$. If $\mathrm{card}(V) \leq d+3$ we say that V has *few points*. The goal of this section is the following result.

THEOREM 4. *If V has few points then every subdivision is regular and in fact every triangulation is placeable.*



PROOF. If $\mathrm{card}(V) = d+1$ then the only subdivision of V is the trivial one. Suppose $\mathrm{card}(V) = d+2$. Consider any nontrivial subdivision $\mathcal{S}$ of V. Then $\mathcal{S}$ must be a triangulation. Choose any simplex in the triangulation, which is the convex hull of $d+1$ points, say, $v_1, \ldots, v_{d+1}$. Any facet of this simplex that is not on the boundary of V must be the base of another simplex whose apex is v_{d+2}. This implies that the triangulation is completely determined by the initially given simplex, and that it is $p^+_{v_{d+2}}(\{V\})$. Assume that the origin $\mathbf{O}$ is in the interior of $[V]$ in affinely general position with respect to the points in V. Then V is the Gale diagram of some set of $d+2$ points in $\mathbf{R}^1$, not all identical. Since the convex hull of these points is one-dimensional it has two facets, implying that there are exactly two d-simplices containing $\mathbf{O}$ formed from the points in V. Hence V admits exactly two triangulations.

Consider a Gale transform $\overline{V}$ of V. It will be one-dimensional, with $\overline{V}$ partitioned as $\overline{V} = \overline{W} \cup \overline{X} \cup \overline{Y}$, where $\overline{W}, \overline{X}, \overline{Y}$ are the points in $\overline{V}$ to the left of, at, and to the right of $\mathbf{O}$, respectively. It is well known and easy to see that if S is a subset of V that lies in no face of V and is minimal with respect to this property, then either $S = W \cup X$ or $S = X \cup Y$. The two triangulations of V consist of either (1) the simplices of the form $(W \cup X \cup Y)\backslash\{u\}$, where $u \in W$, or (2) the simplices of the form $(W \cup X \cup Y)\backslash\{u\}$, where $u \in Y$.

Now suppose $\mathrm{card}(V) = d+3$ and let $\mathcal{S}$ be any nontrivial subdivision of V. Assume first that $\mathcal{S}$ is not a triangulation. Then there must be some $S_k \in \mathcal{S}$ which consists of $d+2$ points, say, $v_1, \ldots v_{d+2}$. Any facet of S_k that is not on the boundary of V must be the base of a pyramidal d-face whose apex is v_{d+3}. Hence $\mathcal{S} = p^+_{v_{d+3}}(\{V\})$.

Assume next that $\mathcal{S}$ is a triangulation. If we join the original boundary faces of V to a new point z, then together with the faces of $\mathcal{S}$ the resulting complex is topologically a d-sphere with $d+4$ vertices. Hence by

 How??

Kleinschmidt [8] and Mani [13] this complex is realizable as the boundary complex of some $(d+1)$-polytope P. Consider a Gale transform $\overline{V}'$ of P (which will be two-dimensional). Normalize $\overline{V}'$ by scaling the nonzero points of $\overline{V}'$ by positive amounts so that they all lie on the unit circle centered at $\mathbf{O}$. Note that $\overline{z} \neq \mathbf{O}$ since $\mathcal{S}$ is not trivial. ← Why? (by the faces - cones interplay in the Gale transform)

By appropriately "rotating diagonals" (see [4, §6.3] for the precise statement of the allowable operations), one can construct normalized Gale diagrams isomorphic to $\overline{V}'$. What we need to show is that there is a normalized Gale diagram $\overline{U}'$ such that (1) it is isomorphic to $\overline{V}'$, and (2) $\overline{U} = \overline{U}'\backslash\{\overline{z}\}$ is *strongly isomorphic* to a normalized Gale transform $\overline{V}$ of V. By strongly isomorphic we mean that the corresponding nonzero points of $\overline{U}$ and $\overline{V}$ occur in the same order around the perimeter of the circle, with $\overline{u}, \overline{v}$, coinciding in $\overline{U}$ if and only if $\overline{u}$, $\overline{v}$ coincide in $\overline{V}$. This will prove that $\mathcal{S}$ is regular. ? why

Suppose there is no way to construct such a $\overline{U}'$ from $\overline{V}'$ by rotating diagonals. Then one of two types of obstacles must be encountered. The first possibility is that there are two nonzero points $\overline{u}$, $\overline{v}$ which coincide in $\overline{V}$ but cannot be merged in $\overline{V}'$ because of the presence of $-\overline{z}$ between them (see Figure 3). Then $V\backslash\{u, v\}$ is a d-simplex in $\mathcal{S}$ since $\mathbf{O}$ is in the interior of the triangle $\{\overline{u}, \overline{v}, \overline{z}\}$. On the other hand, if we begin with $\overline{V}$ (in which $\overline{u}$ and $\overline{v}$ coincide) and project orthogonally onto the linear span of $\{\overline{u}, \overline{v}\}$, then we obtain a one-dimensional Gale diagram for $V\backslash\{u, v\}$. Hence the dimension of $V\backslash\{u, v\}$ is $(d+1)-1-1 = d-1 < d$, which is a contradiction.

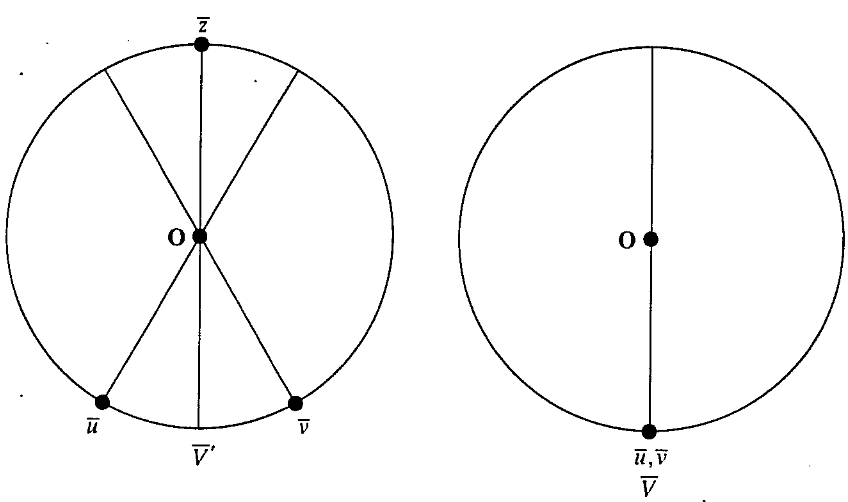


FIGURE 3. Obstacle 1

Assume the first obstacle does not occur. The second possibility is that everything can be ordered properly by rotating diagonals, except for a group of points which ought to appear in the order $\overline{v}_1, \ldots, \overline{v}_m$, but cannot be so ordered in $\overline{V}'$ because $-\overline{z}$ falls among them. This is depicted in Figure 4.

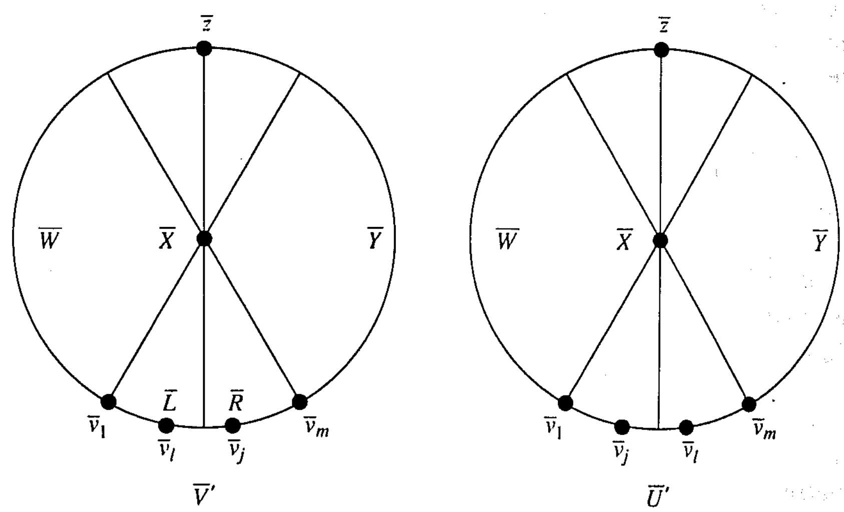


FIGURE 4. Obstacle 2

The only point in the upper closed arc of the circle in $\overline{z}$. There are nonempty sets $\overline{W}$, $\overline{Y}$ in the left and right open arcs, respectively. The (possibly empty) set of points at $\mathbf{O}$ is denoted $\overline{X}$. The points $\overline{v}_1, \ldots, \overline{v}_m$ fall in the lower closed arc. Of these, $\overline{L}$ and $\overline{R}$ are those points lying to the left and right of $-\overline{z}$, respectively. Assume that $\overline{v}_j$ is the point of smallest index in $\overline{R}$ and $\overline{v}_l$ is the point of largest index lying in $\overline{L}$. Let $\overline{v}_{j+1}, \ldots, \overline{v}_k$ be the points, if any, which coincide with $\overline{v}_j$ in $\overline{V}'$ (and $\overline{V}$ since we are assuming that the first obstacle does not occur). Our assumption implies that $j \leq k < l$.

Consider the set $S = X \cup Y \cup (R \setminus \{v_j\})$. First, it is a face of $\mathscr{S}$ since $[\overline{V}' \setminus \overline{S}]$ contains $\mathbf{O}$ in its interior. Second, it is not in the boundary of $\mathscr{S}$. If it were, then $S \cup \{z\}$ would be a face of P and so $[\overline{V}' \setminus (\overline{S} \cup \{\overline{z}\})]$ should contain $\mathbf{O}$ in its interior, which is not the case. Third, $S \cup \{v_j\}$ is not a face of $\mathscr{S}$ because $\mathbf{O}$ is not in the interior of $[\overline{V}' \setminus (\overline{S} \cup \{\overline{v}_j\})]$. Therefore, S cannot lie in any face of $V \setminus \{v_j\}$. But a Gale diagram for $V \setminus \{v_j\}$ is obtained by projecting $\overline{V}$ (in which the points $\overline{v}_1, \ldots, \overline{v}_m$ appear in the correct order) onto the orthogonal complement of the linear span of $\overline{v}_j$. The result is a one-dimensional Gale diagram. The sets projecting to the left of, on, and to the right of $\mathbf{O}$ are, respectively, $\overline{S}_1 = \overline{W} \cup \{\overline{v}_1, \ldots, \overline{v}_{j-1}\}$, $\overline{X} \cup \{\overline{v}_{j+1}, \ldots, \overline{v}_k\}$, and $\overline{S}_2 = \overline{Y} \cup \{\overline{v}_{k+1}, \ldots, \overline{v}_m\}$. Our contradiction now comes from the fact that S contains neither $S_1 \cup X$ nor $S_2 \cup X$ (the minimal nonboundary sets of $V \backslash \{v_j\}$), since it does not contain W or v_l. Therefore, $\mathscr{S}$ is regular and the appropriate $\overline{U}'$ can be constructed.

To show that $\mathscr{S}$ is placeable consider the point $\overline{v}$ nearest to $-\overline{z}$ in $\overline{U}'$ counterclockwise around the perimeter of the circle (see Figure 5). It will not coincide with $-\overline{z}$ because $\mathscr{S}$ is a triangulation. Let $\overline{W}$, $\overline{X}$, $\overline{Y}$ be the points to the left of, on, and to the right of the diagonal joining $\overline{z}$ and $-\overline{z}$, respectively. The triangles containing **O** which have $\overline{z}$ and $\overline{v}$ as two vertices are those of the form $[\overline{z}, \overline{v}, \overline{u}]$, where $\overline{u} \in \overline{W}$. So the d-simplices of $\mathscr{S}$ not containing v are of the form $(W \cup X \cup Y) \backslash (\{u, v\})$, where $u \in W$. But by projecting $\overline{U}' \backslash \{\overline{z}\}$ onto the orthogonal complement of the linear span of $\overline{v}$ we see that these simplices are precisely those of one of the two triangulations of $V \backslash \{v\}$, which is vertex placeable. So $\mathscr{S}$ is vertex placeable. □

(*)

Note: you thought that he had proved placea-bility before! when he argued that a (non-triangulation) subdivision for a d-dim point conf. with d+2 vertices had a placing vertex. THAT IS NOT enough, because you can have convex pieces forming non-convex bodies that don't coarsening to a non triangulation subdivision

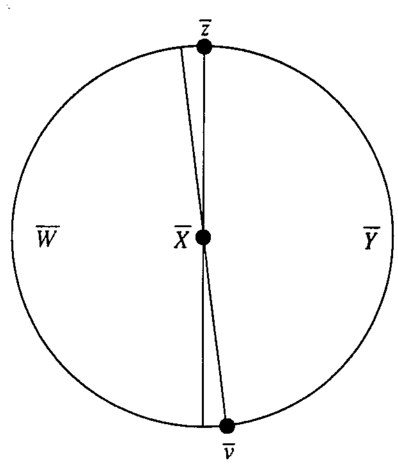
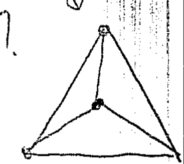


FIGURE 5. $\mathscr{S}$ is vertex placeable.

6. Subdivisions of larger sets

Why?



Consider again any subdivision $\mathscr{S}$ of a d-dimensional set of n points V. If $d = 2$ and all the elements of V are vertices of V then it is easy to see that $\mathscr{S}$ is vertex placeable. However, if some points of V are not vertices, the well-known example of Figure 2(b) (in which the three lines joining corresponding vertices of the outer and inner triangles meet at a common point when extended) shows that there can be nonregular subdivisions even when $n = 6$. We say that $\mathscr{S}$ is *weakly regular* if there exists a d-dimensional set V' having a regular subdivision $\mathscr{S}'$ that is combinatorially isomorphic to $\mathscr{S}$. Equivalently, if one joins the original boundary faces of V to a new point z, then together with the faces of $\mathscr{S}$ the result must be combinatorially equivalent to the boundary complex of some $(d+1)$-polytope. If $d = 2$ then $\mathscr{S}$ is always weakly regular by Steinitz's theorem.

you DO need argument (*)

Que bonito !

In three dimensions a subdivision can be nonregular even if all of the points of V are vertices of V. Consider the capped prism of Figure 6(a). The two triangulations {2368, 2568, 2347, 2378, 2456, 2457, 2578} and {2356, 2358, 2348, 2478, 2467, 2567, 2578} (using an abbreviated notation) are both nonregular, though they are weakly regular. On the other hand, if the upper capped triangle of the prism is twisted slightly the resulting polytope shown in Figure 6(b) has a triangulation {2356, 2358, 2348, 2478, 2467, 2567, 2578, 3568, 3478, 4567} that is not even weakly regular. For, joining the boundary faces to a new point 1 results in a triangulated 3-sphere that is dual to the Brückner sphere, which is not polytopal (see Grünbaum [4, pp. **222–224**] where the points $A, \ldots, H$ correspond to our points $1, \ldots, 8$, respectively). This example was constructed with Peter Kleinschmidt.

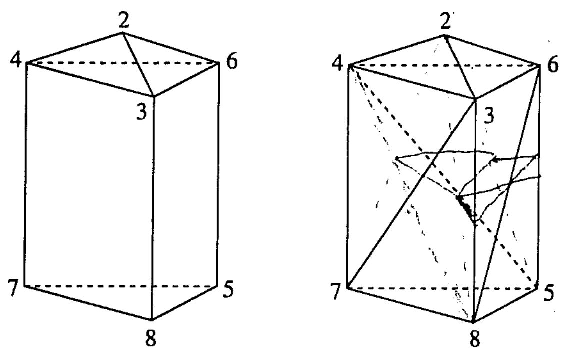


FIGURE 6. Two polytopes with nonregular triangulations

In general, one can take a Gale transform $\overline{V}$ of V and consider the unit $(n-d-2)$-sphere centered at the origin. For every choice of a point $\overline{z}$ on this sphere there is an associated regular subdivision of V. The sphere can be partitioned into regions corresponding to identical subdivisions and in this manner one can construct a spherical complex of all the nontrivial regular subdivisions of V. The partial ordering of the faces of this complex by reverse inclusion coincides with the corresponding partial ordering of the regular subdivisions by refinement as defined in the first section. Gel′fand, Kapranov, and Zelevinskiĭ [3] proved that there is a convex $(n-d-1)$-polytope, known as the *secondary polytope*, that is dual to this spherical complex, and hence the complex itself is also polytopal. Billera, Filliman, and Sturmfels [1] present several proofs of this and they discuss the secondary polytope in greater detail.

When V is the set of vertices of a two-dimensional convex n-gon the dual of the secondary polytope is known as the *associahedron* [5, 10]. Gel′fand, Kapranov, and Zelevinskiĭ have pointed out that the secondary polytope of the n-gon already appears in work of Stasheff [**18**].

We conclude with a sketch of the construction of the secondary polytope for a general set of points V. Assume that $\mathbf{O}$ is the centroid of V and let $\overline{V}$ be a Gale transform of V. Then we may just as well regard V as a Gale transform of $\overline{V}$. Let z be any point in the interior of $[V]$, in affinely general position with respect to V. It is known that the translation $V - z$ is the Gale transform of some set of points $\overline{V}'$ which is a scaling of $\overline{V}$. Let $P(z)$ be the simplicial $(n-d-1)$-polytope $[\overline{V}']$. The facets of $P(z)$ are in one-to-one correspondence with the d-simplices in V which contain z. We can put an equivalence relation on the regular triangulations of V by saying $\mathcal{T}$ is equivalent to $\mathcal{T}'$ if the unique d-simplex containing z is the same in both $\mathcal{T}$ and $\mathcal{T}'$. Then the facets of $P(z)$ correspond to the equivalence classes.

For any two different nontrivial regular triangulations $\mathcal{T}$ and $\mathcal{T}'$ there is always some point z such that $\mathcal{T}$ and $\mathcal{T}'$ fall into two different equivalence classes relative to z. From here it is not too hard to see that the cones determined by the facets of the dual of the secondary polytope are precisely full-dimensional intersections of facet cones of the various possible $P(z)$ for different choices of z.

The set of all d-simplices in V naturally induces a finite collection of open subsets U_i of $[V]$, where x and y are in a common subset U_i if and only if, for all d-simplices S, $x \in \text{int}([S]) \Leftrightarrow y \in \text{int}([S])$. Equivalently, x and y are in the same subset U_i if and only if $P(x)$ and $P(y)$ are combinatorially equivalent simplicial polytopes. Finally (see [**1**]), one can show that the secondary polytope is combinatorially equivalent to the Minkowski sum of the polars of the $P(z_i)$, with one z_i chosen from each subset U_i.

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