The Volume of the Relaxed Boolean Quadric Polytope

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Abstract. The *boolean quadric polytope* P_n is the convex hull in $d := \binom{n+1}{2}$ dimensions of the binary solutions of $x_i x_j = y_{ij}$, for all $i < j$ in $N := \{1, 2, ..., n\}$ $(n \geq 2)$. The polytope is naturally modeled by a somewhat larger polytope; namely, Q_n the solution set of $y_{ij} \leq x_i$, $y_{ij} \leq x_j$, $x_i + x_j \leq 1 + y_{ij}$, $y_{ij} \geq 0$, for all i, j in N. In a first step toward seeing how well Q_n approximates P_n, we establish that the d-dimensional volume of Q_n is $2^{2n-d} n!/(2n)!$.

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1. Introduction. A natural approach to the unconstrained, quadratic-objective, binary program in n (≥ 2) variables

$$\max\left\{\sum_{i\in N} c_i x_i + \sum_{i<j\in N} d_{ij} x_i x_j \ : \ x_i \in \{0,1\} \ \forall\, i \in N\right\}, \tag{1}$$

where $N = \{1, 2, ..., n\}$, is to model the problem as a linearly constrained, linear-objective, binary program, through the use of $\binom{n}{2}$ auxiliary binary variables y_{ij} which model the quadratic terms $x_i x_j$. We obtain the equivalent program

$$\max \sum_{i\in N} c_i x_i + \sum_{i<j\in N} d_{ij} y_{ij}, \tag{2}$$

$$\text{subject to} \quad y_{ij} \leq x_i \quad \forall\, i < j \in N, \tag{3}$$

$$y_{ij} \leq x_j \quad \forall\, i < j \in N, \tag{4}$$

$$y_{ij} \geq 0 \quad \forall\, i < j \in N, \tag{5}$$

$$x_i + x_j \leq 1 + y_{ij} \quad \forall\, i < j \in N, \tag{6}$$

$$x_i \in \{0,1\} \quad \forall\, i \in N, \tag{7}$$

$$y_{ij} \in \{0,1\} \quad \forall\, i < j \in N. \tag{8}$$

The *boolean quadric polytope* P_n is the convex hull (in real $d := \binom{n+1}{2}$ space) of the set of solutions of (3-8). As the problem of solving (2-8) is NP-Hard, it is natural to consider branch-and-cut methods based on (2-6). The relaxed feasible region (3-6) is denoted by Q_n. Padberg (1989) has made a detailed study of P_n and Q_n.

It is natural to consider how good of an approximation Q_n is to P_n. The Chvátal-Gomory rank (see Schrijver (1986)) of P_n with respect to Q_n increases with n, so in a certain combinatorial sense, Q_n is a poor approximation of P_n. In a different combinatorial sense Q_n is quite close to P_n; that is, the 1-skeleton of P_n is a subset of the 1-skeleton of Q_n (the so-called *Trubin Property*) (see Padberg (1989)). Another method has been proposed to study the closeness of pairs of nested polytopes, based on the volumes of the polytopes. Lee and Morris (1992) have suggested the distance function

$$\rho(Q_n, P_n) = \left(\frac{\mathrm{vol}_d(Q_n)}{\mathrm{vol}_d(B^d)}\right)^{1/d} - \left(\frac{\mathrm{vol}_d(P_n)}{\mathrm{vol}_d(B^d)}\right)^{1/d},$$

where B^d is the d-dimensional Euclidean ball, and vol_d denotes d-dimensional Lebesgue measure. For polytope pairs contained in $[0,1]^d$, ρ is at most $O(\sqrt{d})$. In some interesting cases of sets of polytope pairs, ρ may increase more slowly than this upper bound, in other situations the bound is sharp (see Lee and Morris (1992)). In the present paper, as a step toward determining the asymptotic behavior of $\rho(Q_n, P_n)$, we calculate $\mathrm{vol}_d(Q_n)$.

Let $Q'_n := 2Q_n$, that is, the polytope Q_n magnified by a factor of 2. Clearly, $\mathrm{vol}(Q'_n) = 2^d\mathrm{vol}(Q_n)$. Padberg (1989)Clearly, $\mathrm{vol}(Q'_n) = 2^d\mathrm{vol}(Q_n)$. Padberg (1989) demonstrated that Q'_n is a lattice polytope (i.e., its extreme points are lattice points). For simplicity, we state our results for Q'_n, which is defined by the inequalities

$$y_{ij} \leq x_i \quad \forall\, i < j \in N, \tag{9}$$

$$y_{ij} \leq x_j \quad \forall\, i < j \in N, \tag{10}$$

$$y_{ij} \geq 0 \quad \forall\, i < j \in N, \tag{11}$$

$$x_i + x_j \leq 2 + y_{ij} \quad \forall\, i < j \in N. \tag{12}$$

2. The Volume of $\mathbf{Q'_n}$. Our first step in calculating $\mathrm{vol}(Q'_n)$ is to reduce the problem to that of calculating the volume of a subset of Q'_n. Points in Euclidean d-space will be denoted by $(x, y) = (x_1, x_2, \ldots, x_n, y_{12}, y_{13}, \ldots, y_{n-1,n})$. For $a \in \{0,1\}^n$, let

$$C_a = \{(x,y) \in Q'_n \ : \ a \leq x \leq a + \mathbf{1}\},$$

where $\mathbf{1}$ is the n-vector $(1, 1, ..., 1)$. Clearly, Q'_n is the union of all such polytopes C_a. Furthermore, $\mathrm{vol}_d(C_a \cap C_b) = 0$ for $a \neq b$, so $\mathrm{vol}(Q'_n) = \sum_a C_a$.

Proposition 1. $\mathrm{vol}(C_a) = \mathrm{vol}(C_0)$, *for all* $a \in \{0,1\}^n$.

Proof: It suffices to demonstrate that if binary n-vectors a and b differ in precisely one coordinate, then $\mathrm{vol}(C_a) = \mathrm{vol}(C_b)$. Suppose, without loss of generality, that $a_j = b_j$ for $j \neq i$, $a_i = 0$, and $b_i = 1$. We define a map $\Phi_i : C_a \mapsto C_b$ as follows: Φ_i is a composition of coordinate maps $\{\phi_i, \phi_j, \phi_{ki}, \phi_{ij}, \phi_{kj} \ : \ 1 \leq k < i < j \leq n\}$, where $\phi_i(x_i) := 2 - x_i$, $\phi_j(x_j) := x_j$ for $j \neq i$, $\phi_{kj}(y_{kj}) := y_{kj}$, $\phi_{ij}(y_{ij}) = x_j - y_{ij}$, and $\phi_{ki}(y_{ki}) = x_k - y_{ki}$. To see that the range of Φ_i is contained in C_b, we only need to consider ϕ_{ij}; the analysis for ϕ_{ki} is similar. Clearly,

$$\phi_{ij}(y_{ij}) = x_j - y_{ij} \leq x_j = \phi_j(x_j),$$

and

$$\phi_{ij}(y_{ij}) = x_j - y_{ij} \leq x_j - x_i - x_j + 2 = 2 - x_i = \phi_i(x_i).$$

Also,

$$\phi_i(x_i) + \phi_j(x_j) = 2 - x_i + x_j \leq 2 - y_{ij} + x_j = 2 + \phi_{ij}(y_{ij}).$$

Thus, we have shown that Φ_i is, indeed, a map from C_a into C_b.[1] It is trivial to check that Φ_i is an involution. Consequently, Φ_i is bijective and unimodular, and thus measure preserving, so $\text{vol}(C_a) = \text{vol}(C_b)$. Now, given an *arbitrary* binary n-vector a, the composition of the maps in $\{\Phi_i : a_i = 1\}$ gives a measure preserving bijection from C_0 to C_a, so $\text{vol}(C_a) = \text{vol}(C_0)$. ∎

Corollary 2. $\text{vol}(Q'_n) = 2^n \text{vol}(C_0)$. ∎

Let $(S_n, \prec)$ denote the poset (partially ordered set) on $S_n = \{x_i : 1 \leq i \leq n\} \cup \{y_{ij} : 1 \leq i < j \leq n\}$ having $y_{ij} \prec x_i$ and $y_{ij} \prec x_j$. Let $e(S_n, \prec)$ denote the number of *(linear) extensions* of $(S_n, \prec)$, i.e., the number of order-preserving bijections from S_n to $\{1, 2, ..., d\}$.

Proposition 3. $\text{vol}(C_0) = e(S_n, \prec)/d!$
Proof: By definition,

$$C_0 = \{(x, y) \in Q'_n \ : \ 0 \leq x_i \leq 1 \ (1 \leq i \leq n)\} \ .$$

It follows that C_0 is defined by the inequalities $(9 - 11)$ and

$$x_i \leq 1, \ \ 1 \leq i \leq n \ , \tag{13}$$

with (12) rendered vacuous. C_0 is the *order polytope* (see Stanley (1986)) of the poset $(S_n, \prec)$. The result follows by Corollary 4.2 of Stanley. ∎

Theorem 4. $e(S_n, \prec) = n!d!2^n/(2n)!$.
Proof: The proof which we give here is due to R. Stanley and is somewhat more elegant than our original proof. We regard extensions of $(S_n, \prec)$ as permutations of S_n. That is, given a bijection $\pi : S_n \mapsto \{1, 2, ..., d\}$, we represent π by the permutation $\pi^{-1}(1)\pi^{-1}(2)\cdots\pi^{-1}(d)$. Define an *ordered extension* to be an extension such that the x_i's appear in the order $x_1, x_2, .., x_n$. Clearly, the number of extensions is $n!$ times the number of ordered extensions. To count the ordered extensions, we count the number of ways of constructing an ordered extension by successively inserting the y_{ij}'s into the word $x_1 x_2 \ldots x_n$.

We first insert the $(n-1)$ letters y_{1i} with $2 \leq i \leq n$. They can come in any order, which gives a factor of $(n-1)!$, and they have to come before all the x_i's. Next we insert

[1] The map Φ_i is called a "switching" and is a standard tool in the analysis of the "cut polytope" (see Deza and Laurent (1988)).

the $(n-2)$ letters y_{2i} with $3 \le i \le n$. They can come in any order, which gives $(n-2)!$, and they can be inserted anywhere before x_2. There are n letters before x_2, i.e. x_1 and the $(n-1)$ y_{1i}'s. Hence, we have $n+1$ positions (including the one before x_2) in which to place the y_{2i}'s and repetitions are allowed, which gives $\left(\binom{n+1}{n-2}\right) = \binom{2n-2}{n-2}$ (where, as usual, $\left(\binom{m}{p}\right) := \binom{m+p-1}{p}$). Inserting the y_{2i}'s can thus be done in $(n-2)!\binom{2n-2}{n-2} = (2n-2)!/n!$ different ways. The y_{3i}'s with $4 \le i \le n$ can be inserted anywhere before x_3, in $3 + (n-1) + (n-2) = 2n$ positions (1 position before each of x_1, x_2 and x_3, $(n-1)$ position for the y_{1i}'s and $(n-2)$ position for the y_{2i}'s). This gives a factor of $(n-3)!\left(\binom{2n}{n-3}\right) = (n-3)!\binom{3n-4}{n-3} = (3n-4)!/(2n-1)!$. In general, the $(n-k)$ y_{ki}'s can be placed in

$$(n-k)!\left(\binom{(k-1)n - \binom{k-1}{2} + 1}{n-k}\right) = (n-k)!\binom{kn - \binom{k-1}{2} - k}{n-k} = \frac{(kn - \binom{k}{2} - 1)!}{((k-1)n - \binom{k-1}{2})!}$$

different ways.

As k ranges between 1 and $n-1$, the total number of ordered extensions is

$$e(S_n, \prec) = \frac{(n-1)!}{1} \cdot \frac{(2n-2)!}{n!} \cdot \frac{(3n-4)!}{(2n-1)!} \cdots \frac{((n-1)n - \binom{n-1}{2} - 1)!}{((n-2)n - \binom{n-2}{2})!}.$$

Hence,

$$e(S_n, \prec) = \frac{((n-1)n - \binom{n-1}{2} - 1)!}{n \cdot (2n-1) \cdots ((n-2)n - \binom{n-2}{2})} = \frac{(\binom{n+1}{2} - 2)!}{n \cdot (2n-1) \cdots ((n-2)n - \binom{n-2}{2})}.$$

Because $(n-1)n - \binom{n-1}{2} = \binom{n+1}{2} - 1$ and $n^2 - \binom{n}{2} = \binom{n+1}{2}$, we get

$$\begin{aligned} e(S_n, \prec) &= \frac{(\binom{n+1}{2} - 2)!\,(\binom{n+1}{2} - 1)\,\binom{n+1}{2}}{n \cdot (2n-1) \cdots ((n-2)n - \binom{n-2}{2}) \cdot ((n-1)n - \binom{n-1}{2}) \cdot (n^2 - \binom{n}{2})} \\ &= \frac{d!}{\prod_{k=1}^{n} \left(kn - \binom{k}{2}\right)} \end{aligned}$$

It is easy to show (by induction on n) that $\prod_{k=1}^{n}\left(kn - \binom{k}{2}\right) = (2n)!/2^n$.

Therefore, the total number of ordered extensions is $d!2^n/(2n)!$, and the total number of extensions $e(S_n, \prec)$ is $n!d!2^n/(2n)!$. ∎

Corollary 2, Proposition 3, and Theorem 4 now yield

Theorem 5. $\mathrm{vol}(Q'_n) = 2^{2n} n!/(2n)!$. ∎

We note that by Stirling's formula, $\mathrm{vol}(Q'_n) = 2^{-1/2}(e/n)^n(1 + o(1))$. Hence, for example, if it could be shown that $\mathrm{vol}(2P_n) = 2^{-1/2}(e/n)^d(1+o(1))$, then we could conclude that $\rho(Q_n, P_n)$ behaves like $\sqrt{d}$. Finally, we remark that H. Carlsson has pointed out to us that $\mathrm{vol}(C_0)$ can also be calculated (inductively) by integration.

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Geometric Comparison of Combinatorial Polytopes

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Abstract. We survey some analytic methods of volume calculation and introduce a distance function for pairs of polytopes based on their volumes. We study the distance function in the context of three familiar settings of combinatorial optimization: (*i*) Chvátal-Gomory rounding, (*ii*) fixed charge problems, and (*iii*) vertex packing.

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1. Introduction. We assume some familiarity with elementary convexity (see Eggleston (1958)) and with the polyhedral approach to combinatorial optimization problems (see Nemhauser and Wolsey (1988)). A convex d-polyhedron Q, given as the solution set of a finite system of linear inequalities and equations, is often used to model a finite point set X in $\mathbf{R}^d$. If Q is a good approximation of $P := \text{conv}(X)$, then the minimum of a concave function f on Q will be a good lower bound on (or even coincide with) its minimum on X. Ordinarily, the set X is given implicitly by a combinatorial description (e.g., all characteristic vectors of Hamiltonian tours in a finite graph), and it is not computationally feasible to work with an inequality description of P, nor to evaluate f on every element of X. In other situations, we may know a lengthy inequality description of P, but may consider working with an alternative polyhedron $Q \supset P$, if Q has a simpler description than P, and Q is a good approximation to P. In Section 2, we propose a geometric distance function $\rho(Q, P)$ to compare a convex body P with an approximating (i.e., containing) convex body Q. We establish basic properties of ρ. In Section 3, we consider two examples arising from Chvátal-Gomory cutting planes. In Section 4, we evaluate ρ on a class of pairs of idealized polytopes that arise in the consideration of fixed-charge problems. We establish asymptotic properties of ρ as the dimension of the space increases. We also establish an asymptotic result for the simple plant location problem when there are two plants. In Sections 5 and 6, we establish that two well-known relaxations vertex-packing polytopes can be extremely poor geometric models in the worst case.

As many of our results are asymptotic, we set such notation at the outset. Let f and g be functions from $\mathbf{R}_+$ to $\mathbf{R}_+$. We write $f(x) = O(g(x))$ if there exists a constant c such that $f(x) \leq c\ g(x)$, for sufficiently large x. We write $f(x) = \Omega(g(x))$ if $g(x) = O(f(x))$. We write $f(x) = \Theta(g(x))$ if $f(x) = O(g(x))$ and $f(x) = \Omega(g(x))$. We write $f(x) = o(g(x))$ if $\lim_{x\to\infty} \frac{f(x)}{g(x)}$ (exists and) is zero. Finally, we write $f(x) = \omega(g(x))$ if $g(x) = o(f(x))$.

In order to calculate (resp., approximate) $\rho(Q, P)$, we must be able to calculate (approximate) the volume vol_d (d-dimensional Lebesgue measure) of Q and of P. In the worst case this is an extremely difficult problem (see Khachiyan (1990)), but it should be kept in mind that in special situations, it may be possible to develop even analytic expressions for the required volumes. For example, for the d-dimensional unit ball $B^d := \{x \in \mathbf{R}^d\ : \|x\|_2 \leq 1\}$, we have $\text{vol}(B^d) = \pi^{d/2}/\Gamma(d/2+1)$. In the remainder of this section, we survey some analytic results concerning volumes of polytopes, which we employ in the sequel.

Perhaps the most obvious manner in which to attempt to calculate the volume of a polytope Q in $\mathbf{R}^d$ is to develop a triangulation of Q, calculate the volume of each simplex $\text{conv}(\{x^0, x^1, ..., x^d\})$ by dividing the determinant of $[x^1 - x^0, x^2 - x^0, ..., x^d - x^0]$ by $d!$, and summing the terms. Indeed, there are canonical methods for triangulating a polytope, and one can use this to develop a general algorithm (which, practically speaking, can only be used in very low dimension) to calculate volumes (see Cohen and Hickey (1979)). But in special situations, this simple idea of triangulating Q, can yield an analytic result. The best-known example is the "order polytope" $\mathcal{O}(S_<)$ of a poset (partially ordered set) $S_<$ on a d element set S. The *order polytope* $\mathcal{O}(S_<)$ of the poset $S_<$ is the subset of $[0,1]^S$ satisfying $x_i \leq x_j$ if $i \leq j$ in the poset $S_<$. It is well known, and can be established using the idea of triangulating $\mathcal{O}(S_<)$, that $\mathrm{vol}_d(\mathcal{O}(S_<)) = e(S_<)/d!$, where $e(S_<)$ is the number of linear extensions (order preserving bijections) of S (see Stanley (1986), for example). In general, it is not easy to calculate $e(S_<)$ (see Brightwell and Winkler (1990)), but for special posets, we may be able to find analytic formulae or estimates (see Section 4 and 6, for example).

Stanley (1986) associates another polytope with the poset $S_<$. The *chain polytope* $\mathcal{C}(S_<)$ is the subset of $[0,1]^S$ satisfying $x_{i_1} + x_{i_2} + \cdots + x_{i_k} \leq 1$, for every (maximal) chain $i_1 < i_2 < \cdots < i_k$ of $S_<$. Stanley demonstrates that $\mathrm{vol}_d(\mathcal{C}(S_<)) = \mathrm{vol}_d(\mathcal{O}(S_<))$ (despite the fact that $\mathcal{C}(S_<)$ and $\mathcal{O}(S_<)$ are not generally combinatorially equivalent). We make use of Stanley's result in Section 5.

Lawrence (1989) found the following beautiful formula for the volume of the intersection of the *simple* polytope $Q := \{x \in \mathbf{R}^d_+ \ : \ Ax \leq b\}$ with the halfspace $H_{at} := \{x \in \mathbf{R}^d \ : < a, x > \leq t\}$:

$$\mathrm{vol}_d(Q \cap H_{at}) = \frac{1}{d!} \sum_x \frac{(\max\{0, t- < a, x >\})^d}{|\det(B_x)| \overline{a}_{x_1} \cdots \overline{a}_{x_d}},$$

where $< a, x >$ is an arbitrary linear function that is constant on no edge of Q, the sum is over the vertices x of Q, the matrix B_x is the simplex method "basis", associated with vertex x, and $\overline{a}_{x_1}, \ldots, \overline{a}_{x_d}$ are the simplex method "reduced costs" of the d "non-basic variables" at vertex x. All of the required information is available in the usual simplex method tables associated with the vertices (see Schrijver (1986), for example). When $Q = [0,1]^d$, Lawrence observed that his formula reduces to

$$\mathrm{vol}_d([0,1]^d \cap H_{at}) = \frac{1}{d! \prod_{i=1}^d a_i} \sum_x (-1)^{< e, x >} (\max\{0, t- < a, x >\})^d \ ,$$

where the sum is over the set of vertices of $[0,1]^d$, and e is the d-vector of all ones (also see Barrow and Smith (1979)). We make use of this formula in Section 6.

We note that for the special case where $a = e$ and t is an integer, we have

$$\mathrm{vol}_d([0,1]^d \cap H_{et}) = \frac{1}{d!}\sum_{i=1}^{t} A_d^i \ ,$$

where the Eulerian number A_d^i is the number of permutations of $\{1,2,...,d\}$ with i rises (counting one rise at the start) (see Stanley (1977)).

One possible way of obtaining information regarding the volume of a polytope Q, as well as other geometric quantities, is by using certain results from the theory of mixed volumes (see Eggleston (1958) or Burago and Zalgaller (1988), for example). For example, we can appeal to the "Steiner decomposition"

$$\mathrm{vol}_d(Q + tB^d) = \sum_{i=0}^{d} a_i(Q)t^i \ ,$$

where the $a_i(Q)$ $(1 \le i \le d)$ are constants, and $+$ denotes Minkowski sum. The constants in the polynomial can be interpreted. Trivially, $a_0(Q) = \mathrm{vol}_d(Q)$ and $a_d(Q) = \mathrm{vol}_d(B^d)$. It also turns out that $a_{d-1}(Q) = \overline{w}(Q)\ d\ \mathrm{vol}_d(B^d)/2$, where $\overline{w}(Q)$ is the *mean width* of Q (i.e., the average Euclidean distance between the pairs of parallel support hyperplanes of Q). Minkowski established

$$\mathrm{vol}_d(Q) \le \left(\frac{\overline{w}(Q)}{2}\right)^d \mathrm{vol}_d(B^d).$$

Minkowski's inequality implies that $\overline{w}([0,1]^d) = \Theta(\sqrt{d})$. In fact, it turns out that $\overline{w}([0,1]^d) = 2\mathrm{vol}_{d-1}(B^{d-1})/\mathrm{vol}_d(B^d)$.

For a polytope Q, let $L(Q)$ denote the number of points in $Q \cap \mathbf{Z}^d$, where $\mathbf{Z}^d$ is the standard integer lattice. A *lattice polytope* in $\mathbf{R}^d$ is a polytope having all vertices belonging to $\mathbf{Z}^d$. Ehrhart (1962,1967) showed that for a lattice polytope Q, there exist constants $c_i(Q)$ such that

$$L(kQ) = \sum_{i=0}^{d} c_i(Q)k^i \ ,$$

for every positive integer k. This polynomial is known as the *Ehrhart polynomial* of Q. Some of the coefficients can be interpreted. For example, $c_d(Q) = \mathrm{vol}_d(Q)$. Moreover, Macdonald (1963) demonstrated that

$$\mathrm{vol}_d(Q) = \frac{1}{d!}\sum_{i=0}^{d} \binom{d}{i} (-1)^i L((d-i)Q) \ .$$

We make use of Macdonald's formula in Section 5. For a more detailed discussion of $L(Q)$ and the relationship of other "lattice point enumerators" to $\mathrm{vol}_d(Q)$, see Gruber and Lekkerkerker (1987).

2. A Distance Function on Nested Convex Bodies. Let P and Q be convex bodies in $\mathbb{R}^d$ with $P \subset Q$. Let

$$\begin{aligned}\rho(Q,P) :=& \left(\frac{\mathrm{vol}_d(Q)}{\mathrm{vol}_d(B^d)}\right)^{1/d} - \left(\frac{\mathrm{vol}_d(P)}{\mathrm{vol}_d(B^d)}\right)^{1/d} \\ =& \pi^{-1/2}\Gamma(d/2+1)^{1/d}\left(\mathrm{vol}_d(Q)^{1/d} - \mathrm{vol}_d(P)^{1/d}\right) \\ =& \Theta(\sqrt{d})\left(\mathrm{vol}_d(Q)^{1/d} - \mathrm{vol}_d(P)^{1/d}\right) .\end{aligned}$$

In effect, $\rho(Q,P)$ measures the radial distance between concentric balls having the same volumes as Q and P. Hence, we can think of $\rho(Q,P)$ as the *idealized radial distance* between Q and P. We note that $\rho(Q,P) = 0$ if and only if $Q = P$, and that ρ is magnified by λ if space is dilated by a factor of λ. Trivially, for $P \subset Q \subset R$, we have $\rho(R,Q) + \rho(Q,P) = \rho(R,P)$

In the important case where $Q \subset [0,1]^d$, we observe that $\rho = \mathcal{O}(\sqrt{d})$. In Sections 3-6, we will consider certain infinite sets of such polytope pairs and establish asymptotic estimates of ρ as the dimension increases. In particular, we will see when the worst case of $\mathcal{O}(\sqrt{d})$ is attained by ρ , and when it is not.

It is also natural to consider distance functions that are defined relative to a class of objective functions. In particular, we define the *mean height of Q above P* by

$$\bar{h}(Q,P) := \int_{\|c\|_2=1} \left(\max_{x\in Q} cx \;-\; \max_{z\in P} cz\right) d\psi \;,$$

where ψ is $(d-1)$-dimensional Lebesgue measure on the boundary of B^d, normalized so that ψ on the entire boundary is unity. $\bar{h}$ is the average Euclidean distance between a pair of parallel supporting hyperplanes (one for each polytope) having the same orientation. It is clear that $2\bar{h}(Q,P)$ is simply the mean width of Q minus the mean width of P. Hence, we have another interpretation of ρ: by Minkowski's inequality, if Q is a ball, then $\rho(Q,P)$ is an upper bound on $\bar{h}(Q,P)$; alternatively, if P is a ball, then $\rho(Q,P)$ is a lower bound on $\bar{h}(Q,P)$. Indeed, we can think of $\rho(Q,P)$ as a surrogate for $\bar{h}(Q,P)$. The advantage of ρ is that it is more robust – not being defined with respect to a particular class of objective functions.

We remark that there are other possible ways to combine $\mathrm{vol}_d(Q)$ and $\mathrm{vol}_d(P)$ to compare Q with P. For example, a *relative* distance function is $1 - (\mathrm{vol}_d(P)/\mathrm{vol}_d(Q))^{1/d}$. The computational performance of branch-and-bound on an integer linear linear program is related to the *absolute* objective gap associated with a relaxation (for normalized objective functions). For this reason, and the above-mentioned relationship between ρ and $\overline{h}$, we prefer ρ to such relative distance functions.

As was noted for ρ, $\overline{h}(Q,P) = 0$ if and only if $Q = P$, and $\overline{h}$ is magnified by λ if space is dilated by a factor of λ. As $\overline{h}$ is difficult to work with analytically, we may resort to the upper bound

$$h_{\max}(Q,P) := \max_{\|c\|_2=1}\left(\max_{x\in Q} cx \;-\; \max_{x\in P} cx\right).$$

Example 2.1. Let

$$Q := \{x \in [0,1]^2 \;:\; x_1 \le a,\; x_2 \le b\}\;,$$

and

$$P := \{x \in [0,1]^2 \;:\; x_1 \le a/2,\; x_2 \le b\} \quad (a, b \le 1).$$

It can be checked that $h_{\max}(Q,P) = a/2$, $\overline{h}(Q,P) = a/2\pi$, and $\rho(Q,P) = \sqrt{ab}(1-\sqrt{1/2})/\sqrt{\pi}$. Hence, by letting a or b go to zero, we can have $\overline{h}/\rho$ and $h_{\max}/\rho$ go to 0 or infinity. ∎

We see from Example 2.1 that even confining our attention to polytopes contained in the unit square, we can not bound $\overline{h}$ (and $h_{\max}$) from above or below by a positive constant times ρ. This apparent discrepancy between $\overline{h}$ and ρ is caused by the extreme asymmetry of the polytopes in Example 2.1. The polytopes that we consider in the sequel do not suffer from such extreme asymmetry. We remark that for polytope pairs contained in the unit d-cube, it may be possible to develop nontrivial bounds on $\overline{h} - \rho$ (as a function of d).

3. Rounding. For each positive integer d, let K_d be a polytope in $\mathbf{R}^d$, and let

$$(*) \qquad \sum_{i=1}^{d} a_i^{(d)} x_i^{(d)} \le b^{(d)}$$

be an inequality that is satisfied by the integer-valued points of K_d, where all $a_i^{(d)}$ are positive integers. Let $k^{(d)}$ be the greatest common divisor of the $a_i^{(d)}$. It has been observed that $(*)$ can be strengthened to

$$\lfloor * \rfloor \qquad \sum_{i=1}^{d} a_i^{(d)} x_i \le k^{(d)} \lfloor b^{(d)}/k^{(d)} \rfloor \; .$$

In fact, in the sense of "Chvátal-Gomory cutting planes", $\lfloor * \rfloor$ is the strongest inequality that is implied by both integrality and $(*)$, independent of the polytope K_d (see Nemhauser and Wolsey (1988), page 211, for example). Indeed, as observed by Nemhauser and Wolsey, this "shows the limitations of one application of the Chvátal-Gomory rounding method".

Turning to a specific example, which illustrates this point in the geometric sense of Section 2, let $Q_d := \{x \in \mathbf{R}_+^d \; : \; (*)\}$, and let $P_d := \{x \in \mathbf{R}_+^d \; : \; \lfloor * \rfloor\}$. As the two polytopes are both simplices, it is easy to establish

Proposition 3.1.

$$\rho(Q_d, P_d) = \pi^{-1/2} \left(\frac{\Gamma(\frac{d}{2}+1)}{\Gamma(d+1) \prod_{i=1}^{d} a_i^{(d)}} \right)^{1/d} \left(b^{(d)} - k^{(d)} \lfloor b^{(d)}/k^{(d)} \rfloor\right) \; .$$

Corollary 3.2.

$$\rho(Q_d, P_d) = o(1) \; .$$

Proof: Using Proposition 3.1, we have that

$$\begin{aligned} \rho(Q_d, P_d) &\le \pi^{-1/2} \left(\frac{\Gamma(\frac{d}{2}+1)}{\Gamma(d+1) \prod_{i=1}^{d} a_i^{(d)}} \right)^{1/d} k^{(d)} \\ &\le \pi^{-1/2} \left(\frac{\Gamma(\frac{d}{2}+1)}{\Gamma(d+1)} \right)^{1/d} \left(\prod_{i=1}^{d} \left(\frac{k^{(d)}}{\min_{i=1}^{d} a_i^{(d)}} \right) \right)^{1/d} \\ &= O(d^{-1/2}) \; . \end{aligned}$$

■

Hence, we see, geometrically, that in a simple, yet important class of examples, one application of this "inequality tightening" is ineffective, in this asymptotic sense. We give one further example to illustrate the limitations of rounding.

Example 3.3. Let $Q^{(0)}$ be the set of points x in $\mathbf{R}^2$ satisfying

$$tx_1 + x_2 \le t + 1$$
$$-tx_1 + x_2 \le 1$$
$$0 \le x_1 \le 1$$
$$0 \le x_2 \ .$$

Let $P := [0,1]^2$ be the convex hull of the integer-valued points in $Q^{(0)}$ (see Figure 3.4). The polytope P has unbounded "Chvátal rank" with respect to $Q^{(0)}$ (see Nemhauser and Wolsey, pg. 227, for example). For each nonnegative integer p, let $a_1^{(p)}x_1 + a_2^{(p)}x_2 \le b^{(p)}$ be *any* inequality with integer-valued coefficients that is satisfied by all points of $Q^{(p)}$. Let $k^{(p)}$ be the greatest common divisor of $a_1^{(p)}$ and $a_2^{(p)}$. Finally, let $Q^{(p+1)}$ be the subset of points in $Q^{(p)}$ that satisfy $a_1^{(p)}x_1 + a_2^{(p)}x_2 \le k^{(p)}\lfloor b^{(p)}/k^{(p)} \rfloor$. It is easy to check that $\rho(Q^{(0)}, P) = \Theta(\sqrt{t})$, and for any fixed p, $\rho(Q^{(0)}, Q^{(p)}) = \mathcal{O}(1)$ (asymptotically in t), hence $\rho(Q^{(p)}, P) = \Theta(\sqrt{t})$. That is, rounding may do relatively little even when a polytope is a poor approximation of the convex hull of its integer-valued points. ∎

<<< Figure 3.4 about here >>>

4. Fixed Charge Problems. Let $I := \{1, 2, ..., I\}$ and $J := \{1, 2, ..., J\}$. Let P_{IJ} equal the set of $(x \in \mathbf{R}^{I \times J},\ y \in \mathbf{R}^I)$ satisfying

$$x_{ij} \le y_i \quad \forall\ (i,j) \in I \times J,$$
$$0 \le x_{ij} \quad \forall\ (i,j) \in I \times J,$$
$$y_i \le 1 \quad \forall\ i \in I\ ,$$

and let Q_{IJ} equal the set of $(x \in \mathbf{R}^{I \times J},\ y \in \mathbf{R}^I)$ satisfying

$$\sum_{j \in J} x_{ij} \le J y_i \quad \forall\ i \in I,$$
$$0 \le x_{ij} \le 1 \quad \forall\ (i,j) \in I \times J,$$
$$y_i \le 1 \quad \forall\ i \in I\ .$$

Note that P_{IJ} is properly contained in Q_{IJ}, but they contain the same set of integer-valued points. In applications of these models, each y_i is a $\{0,1\}$-valued indicator variable which signals when (the real) x_{ij} is positive for some j (see Nemhauser and Wolsey (1988), for example). In real applications of these models, there are a variety of other inequalities and equations that must also be satisfied by solutions, but we believe that valuable information can already be obtained in this idealized situation.

Proposition 4.1.

$$\rho(Q_{IJ}, P_{IJ}) = \pi^{-1/2}\Gamma(I(J+1)/2+1)^{\frac{1}{I(J+1)}}\left(\left(\frac{1}{2}\right)^{\frac{1}{J+1}} - \left(\frac{1}{J+1}\right)^{\frac{1}{J+1}}\right).$$

Proof: The polytope P_{IJ} is the cross product of I copies of the order polytope P_J equal to the set of $(x \in \mathbf{R}^J,\ y \in \mathbf{R})$ satisfying

$$\begin{aligned} &x_j \leq y \quad \forall\, j \in J, \\ &0 \leq x_j \quad \forall\, j \in J, \\ &y \leq 1\ . \end{aligned}$$

The number of linear extensions of the associated partial order is $J!$, hence $\mathrm{vol}_{J+1}(P_J) = \frac{1}{J+1}$, and $\mathrm{vol}_{I(J+1)}(P_{IJ}) = (\frac{1}{J+1})^I$.

The polytope Q_{IJ} is the cross product of I copies of the polytope Q_J equal to the set of $(x \in \mathbf{R}^J,\ y \in \mathbf{R})$ satisfying

$$\begin{aligned} &\sum_{j\in J} x_j \leq Jy, \\ &0 \leq x_j \leq 1 \quad \forall\, j \in J, \\ &y \leq 1\ . \end{aligned}$$

Define the "box" B to be the set of $(x \in \mathbf{R}^J,\ y \in \mathbf{R})$ satisfying

$$\begin{aligned} &0 \leq x_j \leq 1 \quad \forall\, j \in J, \\ &0 \leq y \leq 1\ . \end{aligned}$$

We have

$$Q_J = B \cap \{(x \in \mathbf{R}^J,\ y \in \mathbf{R})\ :\ \sum_{j\in J} x_j \leq Jy\}\ .$$

The hyperplane $\{(x \in \mathbf{R}^J,\ y \in \mathbf{R})\ :\ \sum_{j\in J} x_j = Jy\}$ divides B into two sets of equal volume. This can be seen by noting that the affine transformation τ taking x_j into $1-x_j$ and y into $1-y$ is unimodular and maps the closure of $B \setminus Q_J$ to Q_J. Hence $\mathrm{vol}_{J+1}(Q_J) = (1/2)\mathrm{vol}_{J+1}(B) = 1/2$, and $\mathrm{vol}_{I(J+1)}(Q_{IJ}) = 1/2^I$. The result follows. ∎

As a basis of comparison, we consider $h_{max}(Q_{IJ}, P_{IJ})$. Consider the objective function

$$(2I)^{-1/2} \max \sum_{i\in I} x_{i1} - \sum_{i\in I} y_i\ .$$

It is easy to check that the origin optimizes the objective over P_{IJ}, while

$$x_{i1} = 1 \;\forall\, i \in I \;,$$
$$x_{ij} = 0 \;\forall\, i \in I \setminus \{1\},\; j \in J \;,$$
$$y_i = 1/J \;\forall\, i \in I \;,$$

giving objective value $\sqrt{I/2}(1 - 1/J)$, optimizes the objective over Q_{IJ}. We conclude that $h_{\max}$ appears to increase quickly with I, but appears to be well behaved in J.

Table 4.2 describes the limiting behavior of ρ and $h_{\max}$ under various rates of increase of I and J satisfying $d = I(J+1)$. In particular, if I does not grow too quickly, ρ is much better behaved than $h_{\max}$. That is, in such situations, Q_{IJ} is a much better geometric model of P_{IJ}, than is predicted by a worst-case analysis of linear objective function discrepancy. One suggestion is that for classes of optimization problems based on these models, branch-and-bound methods based on the weaker model Q_{IJ} may outperform those based on P_{IJ} if J is sufficiently larger than I (i.e., the computational gains realized in optimizing over the simpler polytope Q_{IJ} may more than offset the increased branching related to its weaker objective function bounds).

I	$J+1$	$h_{\max}$	ρ
constant	d/constant	$\Omega(1)$	$o(1)$
$\log d$	$d/\log d$	$\Omega(\sqrt{\log d})$	$o(1)$
$\sqrt{d}$	$\sqrt{d}$	$\Omega(d^{1/4})$	$o(d^{\epsilon})\;\forall\,\epsilon > 0$
d/constant	constant	$\Theta(\sqrt{d})$	$\Theta(\sqrt{d})$

Table 4.2

Problem 4.3. Let $L := \{(x \in \mathbf{R}^{I \times J},\; y \in \mathbf{R}^{I}) \;:\; \sum_{i \in I} x_{ij} = 1 \;\;\forall\;\; j \in J\}$, let $\overline{Q}_{IJ} := Q_{IJ} \cap L$, and let $\overline{P}_{IJ} := P_{IJ} \cap L$. The polytopes $\overline{Q}_{IJ}$ and $\overline{P}_{IJ}$ are the so-called *weak* and *strong formulations* (respectively) of the "simple plant location problem" (see Nemhauser and Wolsey (1988), pp. 384-5, for example). Compute $\rho(\overline{Q}_{IJ}, \overline{P}_{IJ})$, where the associated volumes are to be computed in the $(IJ + I - J)$-dimensional affine set L.

We can solve Problem 4.3, asymptotically in J, for the case $I = 2$.

Proposition 4.4.

$$\rho(\overline{Q}_{2,J}, \overline{P}_{2,J}) = o(1) \ .$$

Proof: The constraints describing $\overline{Q}_{2,J}$ are equivalent to:

$$y_1 \le \sum_{j \in J} x_{1j}/J \le y_2,$$
$$0 \le x_{1j} \le 1 \quad \forall j \in J,$$
$$0 \le y_i \le 1 \quad i = 1,2 \ .$$

Thinking of all variables as independent uniformly distributed random variables, and using the Law of Large Numbers, the volume of this polytope, as J tends to infinity, tends to the volume of the set of $y \in \mathbf{R}^2$ satisfying

$$0 \le y_1 \le 1/2 \le y_2 \le 1 \ ,$$

which is 1/4. The constraints describing $\overline{P}_{2,J}$ are equivalent to:

$$0 \le y_1 \le x_{1j} \le y_2 \le 1 \ \ \forall j \in J \ .$$

These constraints describe an order polytope with volume $1/(J+1)(J+2)$. For both polytopes, for each j, transforming the interval $\{x_{1j} \ : \ 0 \le x_{1j} \le 1\}$ to the interval $\{(x_{1j}, x_{2j}) \ : \ x_{1j} + x_{2j} = 1, \ x_{1j} \ge 0, \ x_{2j} \ge 0\}$ multiplies the volume by $\sqrt{2}$. Hence, the desired limit is the same as

$$\lim_{J \to \infty} \ \pi^{-1/2}\Gamma(J/2+1)^{\frac{1}{(J+2)}} \left(\left(\frac{\sqrt{2}^J}{4} \right)^{\frac{1}{J+2}} - \left(\frac{\sqrt{2}^J}{(J+1)(J+2)} \right)^{\frac{1}{J+2}} \right) .$$

The result follows. ∎

We note that the asymptotic behavior of $\rho(\overline{Q}_{2,J}, \overline{P}_{2,J})$ mimics that of $\rho(Q_{2,J}, P_{2,J})$. We conjecture that $\rho(\overline{Q}_{I,J}, \overline{P}_{I,J}) = o(1)$ when I is held constant.

We can also solve Problem 4.3, asymptotically in I, when J is held constant.

Proposition 4.5.

$$\rho(\overline{Q}_{I,J}, 0) = \Theta(I^{\frac{1}{2} - \frac{1}{J+1}}) \ ,$$

asymptotically in I, with J held constant.

Proof: The $(I-1)$-dimensional volume of the convex hull of the I standard unit vectors in $\mathbb{R}^I$ is $\sqrt{I}/(I-1)!$. Therefore,

$$\rho(\overline{Q}_{I,J},0) = \Theta(\sqrt{I})\left(\frac{\sqrt{I}}{(I-1)!}\right)^{\frac{1}{IJ+I-J}} .$$

The result follows since

$$\lim_{I\to\infty} \sqrt{I}\left(\frac{\sqrt{I}}{(I-1)!}\right)^{\frac{1}{IJ+I-J}} I^{-\frac{1}{2}+\frac{1}{J+1}} = \exp\left\{\frac{1}{J+1}\right\} .$$

∎

Hence, we observe that when J is held constant, $\overline{Q}_{I,J}$ is a better model of $\overline{P}_{I,J}$ than $Q_{I,J}$ is of $P_{I,J}$.

5. Vertex Packing. Let G be a finite simple graph on d vertices. For a graph G, let $V(G)$ denote the vertex set of G. In this section, we demonstrate that, in the worst case, weak fractional vertex-packing polytopes can be extremely bad geometric models of the vertex-packing polytope of a graph. The *weak fractional vertex-packing polytope* of a graph G on d vertices, is the subset of $[0,1]^d$ satisfying $x_i + x_j \le 1$ for all edges $\{i,j\}$ of G. The *vertex-packing polytope* of G is the convex hull of the set of characteristic vectors of stable sets of vertices of G. Let Q_d be the weak fractional vertex-packing polytope of the complete graph K_d. Let P_d be the vertex-packing polytope of K_d.

Proposition 5.1.

$$\rho(Q_d,P_d) = \pi^{-1/2}\Gamma(d/2+1)^{1/d}\left(\left(\frac{1}{2}\right)^{1-1/d} - \left(\frac{1}{d!}\right)^{1/d}\right) .$$

Proof: Since $S \subset V(K_d)$ is a packing if and only if $|S| \in \{0,1\}$, it is easy to see that P_d is a simplex and that $\mathrm{vol}_d(P_d) = 1/d!$. Next, we calculate $\mathrm{vol}_d(Q_d)$ using Macdonald's formula (see Section 1). Recall that for a nonnegative integer k, $L(kQ)$ is the number of lattice points in the polytope kQ. It is easy to check that

$$L(kQ_d) = \left\lceil\frac{k+1}{2}\right\rceil^d + d\sum_{j=1}^{\lceil k/2\rceil} j^{d-1} .$$

Now Q_d is not a lattice polytope. However, it is well known that $Q'_d := 2Q_d$ is a lattice polytope (see Lee (1989), for example). Hence, using Macdonald's formula, we have

$$\begin{aligned}\mathrm{vol}_d(Q'_d) =& \frac{1}{d!}\sum_{i=0}^{d}\binom{d}{i}(-1)^i L(2(d-i)Q_d)\\ =& \frac{1}{d!}\sum_{i=0}^{d}\binom{d}{i}(-1)^i(d-i+1)^d \;+\; \frac{1}{(d-1)!}\sum_{i=0}^{d}\binom{d}{i}(-1)^i\sum_{j=1}^{d-i}j^{d-1} .\end{aligned}$$

The first summand is equal to unity since

$$\begin{aligned}&\frac{1}{d!}\sum_{i=0}^{d}\binom{d}{i}(-1)^i(d-i+1)^d\\ &\qquad=\frac{1}{d!}\sum_{i=0}^{d}\binom{d}{i}(-1)^i\sum_{l=0}^{d}\binom{d}{l}(d-i)^l\\ &\qquad=\sum_{l=0}^{d}\binom{d}{l}\,\frac{1}{d!}\sum_{i=0}^{d}\binom{d}{i}(-1)^i(d-i)^l\\ &\qquad=\sum_{l=0}^{d}\binom{d}{l}S_l^d = 1 ,\end{aligned}$$

where S_l^d is a Stirling number of the second kind (the number of ways of partitioning l (labeled) objects into d nonempty (unlabeled) classes) (see Berge (1971), for example). The last equation holds since S_l^d is zero for $l < d$ and unity for $l = d$.

The second summand is also equal to unity since

$$\begin{aligned}&\frac{1}{(d-1)!}\sum_{i=0}^{d}\binom{d}{i}(-1)^i\sum_{j=1}^{d-i}j^{d-1}\\ &\qquad=\frac{1}{(d-1)!}\sum_{j=0}^{d-1}j^{d-1}(-1)^{d-1-j}\sum_{i=0}^{j}\binom{d}{i}(-1)^{j-i}\\ &\qquad=\frac{1}{(d-1)!}\sum_{j=0}^{d-1}j^{d-1}(-1)^{d-1-j}\binom{d-1}{j}\\ &\qquad=S_{d-1}^{d-1}=1 .\end{aligned}$$

Hence $\mathrm{vol}_d(Q'_d) = 2$, and $\mathrm{vol}_d(Q_d) = 2^{1-d}$. The result follows. ∎

Simple asymptotics reveals that, up to a constant factor, the idealized radial distance between these polytope pairs is as large as possible for polytope pairs contained in the unit cubes:

Corollary 5.2. $\rho(Q_d, P_d) = \Theta(\sqrt{d})$.

We note that Corollary 5.2 can be obtained directly and even generalized. Let $f : \mathbf{Z}_+ \to \mathbf{R}$. The *$f(d)$-weak fractional vertex-packing polytope* of a graph G on d vertices is the subset of $[0,1]^d$ satisfying $\sum_{i \in K_l} x_i \leq 1$ for all complete subgraphs K_l of G having $l \leq f(d)$. In particular, the 2-weak fractional vertex-packing polytope is the weak fractional vertex-packing polytope. Let $Q_d^{f(d)}$ be the $f(d)$-weak fractional vertex-packing polytope of K_d. We observe that $[0, 1/f(d)]^d$ is contained in $Q_d^{f(d)}$. Hence, $\mathrm{vol}_d(Q_d^{f(d)})$ is at least $f(d)^{-d}$. We easily obtain the following.

Proposition 5.3. For $0 < \epsilon \leq 1/2$, and $f(d) = \Theta(d^{1/2-\epsilon})$,

$$\rho(Q_d^{f(d)}, P_d) = \Omega(d^\epsilon) .$$

We note that a by-product of the proof of Proposition 5.1, which is not implied by Corollary 5.2, is that the volume of $[0,1/2]^d$ is only (and exactly) one half that of the containing polytope Q_d. Hence, $\rho(Q_d, [0,1/2]^d) = o(1)$.

A hyperplane with unit normal $(\sqrt{d}, \sqrt{d}, ..., \sqrt{d})$ supports Q_d at $(1/2, 1/2, ..., 1/2)$ and P_d at every standard unit vector. This implies that $h_{\max}(Q_d, P_d) = \Theta(\sqrt{d})$. Hence, in this case, the worst-case behavior of ρ and $h_{\max}$ agree.

6. Vertex Packing on Threshold Graphs. The *fractional vertex-packing polytope* $P(G)$ associated with G is the subset of $[0,1]^d$ satisfying $\sum_{i \in V(K)} x_i \leq 1$, for all cliques (maximal complete subgraphs) K of G. It is well known that for perfect graphs, the vertices of $P(G)$ are integer valued, hence they are precisely the characteristic vectors of the stable sets (of vertices) of G (see Grötschel, Lovasz and Schrijver (1984), for example). A *comparability graph* $G(S_<)$ is a graph defined with respect to a poset $S_<$. The graph has a vertex for every element of S, and an edge between vertices i and j if either $i < j$ or $j < i$ in $S_<$. Comparability graphs are a subclass of the perfect graphs (see Duchet (1984), for example). Notice that the chain polytope $C(S_<)$ is identical to the (fractional) vertex-packing polytope $P(G(S_<))$. A graph G (on d vertices) is a *permutation graph* if there is a bijection τ from $V(G)$ to $\{1, 2, ..., d\}$ and a permutation π such that i and j are adjacent if and only if $(\pi(\tau(i)) - \pi(\tau(j)))(\tau(i) - \tau(j)) < 0$. Clearly, every permutation graph is a comparability graph. The graph G is a *threshold graph* if the characteristic vectors of stable sets of G are precisely the $\{0,1\}$-valued points in $Q(G) := [0,1]^d \cap \{x \ : \ < a, x > \leq b\}$, for some choice of a single inequality $< a, x > \leq b$. We call $Q(G)$ a *threshold polytope* for G. In this section, we endeavour to study the worst-case behavior of $\rho(Q(G), P(G))$

and $h_{\max}(Q(G), P(G))$. In particular, we will demonstrate that for any ϵ satisfying $0 < \epsilon \leq 1/2$, there is a sequence of threshold graphs G_d, indexed by the number of vertices d, so that $\rho(Q(G_d), P(G_d)) = \omega(d^{\frac{1}{2}-\epsilon})$. That is, the idealized radial distance between a binary knapsack polytope (i.e., the convex hull of the vectors in $\{0,1\}^d$ satisfying a single inequality) and its continuous relaxation (i.e., the vectors in $[0,1]^d$ satisfying the single inequality) can behave virtually as badly as it can for any family of polytope pairs in $[0,1]^d$.

The threshold graphs are a very special subclass of the permutation graphs, so they too are perfect. To understand how threshold graphs can be seen to arise, we define a *shuffle product* of $[\sigma_1, \sigma_2, ..., \sigma_p]$ and $[\sigma_{p+1}, \sigma_{p+2}, ..., \sigma_d]$ to be a permutation of $\{\sigma_1, \sigma_2, ..., \sigma_d\}$, such that σ_i appears before σ_j if $i < j \leq p$ or $i > j \geq p+1$.

Proposition 6.1 (Golumbic (1978)). The threshold graphs are precisely those permutation graphs corresponding to a shuffle product of $[1, 2, ..., p]$ and $[d, d-1, ..., p+1]$, where p and d are positive integers.

Threshold graphs have numerous applications, and they are well understood (see Chvátal and Hammer (1977), Duchet (1984), Golumbic (1978), and Orlin(1977), for example). Note that there is some flexibility in the choice of the inequality $< a, x > \leq b$ in the definition of $Q(G)$. The inequality $< a, x > \leq b$ *separates* G *integrally* if:

(1) $a_i \geq 0$, for all $i = 1, 2, ..., d$;
(2) $\sum_{i \in T} a_i \leq b$, for all stable sets T;
(3) $\sum_{i \in N} a_i \geq b + 1$, for all nonstable sets N.

Orlin (1977) gives the Separator Algorithm of Figure 6.2 to find the unique integral separator that minimizes b. For a vertex v, let $\deg(v)$ denote the number of edges incident to v.

<<< Figure 6.2 about here >>>

If G is a threshold graph, then the *minimal separator polytope* of G is $Q^*(G) := [0,1]^d \cap \{x \ : < a^*, x > \leq b^*\}$, where $< a^*, x > \leq b^*$ is the minimal integral separator of G.

Since a threshold graph G is a (special) comparability graph, we can interpret and prove the correctness of the Separator Algorithm directly on the poset $S_<$ associated with G. In doing so, we can see, constructively, how the minimal integral separator is a weighted sum of clique inequalities. We assume that G has arisen as in Proposition 6.1, and we describe the method with respect to the Hasse diagram of $S_<$, interpreting the diagram as a tree T rooted at vertex d.

Note that it can be assumed that vertex p appears after $p+1$ in the shuffle product of Proposition 6.1. Then $\{1,2,...,p\}$ is the set of leaves of $\mathcal{T}$. Let $\kappa(i)$ be the (unique) maximal clique containing leaf i. The map κ is a bijection from the leaves of $\mathcal{T}$ to the maximal cliques of G. We will define nonnegative weights $w(i)$ such that

$$(*) \quad \sum_{i=1}^{p} w(i) \sum_{j\in\kappa(i)} x_j \leq \sum_{i=1}^{p} w(i)$$

is the minimal integral separator of G.

Let the *depth* of vertex i of $\mathcal{T}$ be the length (in edges) of the path from the root d to i, unless i is isolated, in which case its depth is 0. If i has depth 0, then $w(i) = 0$. Then from $j = 1,...,p$, let the leaves of depth j receive weight $w(j)$ equal to the sum of the weights of leaves at lesser depths plus one (hence, all leaves at the same depth receive the same weight), where the empty sum is zero, as usual.

We briefly argue the correctness of this procedure. Clearly, the incidence vector x of every stable set of G satisfies $(*)$, since x satisfies the individual clique inequalities. Let

$$a(j) := \sum_{i:j\in\kappa(i)} w(i) \; .$$

If x is the incidence vector of a nonstable set then $x_j = x_l = 1$ for some element l and some nonleaf j such that $l < j$ in $S_<$ (refer to Figure 6.3). Now $a(j)$ is equal to the sum of the weights of leaves at greater depth than j. Let t be a leaf at the least depth such that $t \leq l$. Clearly $a(l) \geq w(t)$. Now $w(t)$ is greater than the sum of the weights of leaves of lesser depth than t. Then, since $t < l < j$ in $S_<$, we have $a(l) + a(j) \geq w(t) + a(j) > \sum_{i=1}^{p} w(i)$. We leave it to the reader to conclude that the right-hand side of $(*)$ is minimum among all integral separators of G, and that G has no other minimal integral separator.

<<< Figure 6.3 about here >>>

Example 6.4. Consider the threshold graph H^{2k} arising from the "perfect shuffle" (see the Hasse diagram of Figure 6.5):

$$2k, 1, 2k-1, 2, 2k-2, 3, ..., k+2, k-1, k+1, k \; .$$

<<< Figure 6.5 about here >>>

The number of linear extensions of the associated class of posets is at least $k!$, hence $\mathrm{vol}_{2k}(P(H^{2k})) \geq k!/(2k)!$. The minimal integral separator of the graph is

$$\sum_{i=1}^{k} 2^{i-1} x_i + \sum_{i=k+1}^{2k} (2^k - 2^{2k-i}) x_i \leq 2^k - 1 .$$

The volume of the minimal separator polytope $Q^*(H^{2k})$ is no more than the volume of

$$Q'(H^{2k}) := [0,1]^{2k} \cap \{x \in \mathbf{R}^{2k} \ : \ \sum_{i=k+1}^{2k} x_i \leq 2\} .$$

Using the special case of Lawrence's volume formula in which the standard unit cube is intersected with an inequality, we can establish that $\mathrm{vol}_{2k}(Q'(H^{2k})) \leq 2^k/k!$. Using Stirling's approximation to the factorial, it is easy to check that

$$\lim_{k \to \infty} \rho(Q'(H^{2k}), P(H^{2k})) = \frac{1}{\sqrt{\pi}} \left(\sqrt{2} - \frac{1}{2} \right) \approx .5158.$$

Hence, in the limit, $\rho(Q^*(H^{2k}), P(H^{2k}))$ is bounded above by .52 . ∎

Let G_d^p be the threshold graph determined by the "cut" permutation

$$d, d-1, d-2, ..., p+1, 1, 2, 3, ..., p ,$$

in the manner of Proposition 6.1. The graph G_d^p is the comparability graph of the poset $S_<$ depicted by the Hasse diagram of Figure 6.6.

<<< Figure 6.6 about here >>>

Proposition 6.7.

$$\rho(Q^*(G_d^p), P(G_d^p)) = \pi^{-1/2} \Gamma(d/2+1)^{1/d} \left(\frac{p!}{d!} \right)^{1/d} \left(\left(\frac{S_d^p}{p^{d-p}} \right)^{1/d} - 1 \right) ,$$

where S_d^p denote a Stirling number of the second kind.

Proof: It is trivial to observe that $e(S_<) = p!$, hence $\mathrm{vol}_d(P(G_d^p)) = p!/d!$. Using the Separator Algorithm, it can be shown that the minimal integral separator of G_d^p is

$$\sum_{i=1}^{p} x_i + p \sum_{i=p+1}^{d} x_i \leq p .$$

Using the special case of Lawrence's volume formula in which $[0,1]^d$ is intersected with an inequality, we can establish that

$$\text{vol}_d(Q^*(G_d^p)) = \frac{1}{d!p^{d-p}} \sum_{j=0}^{p} \binom{p}{j} (-1)^j (p-j)^d$$
$$= \frac{p!}{d!p^{d-p}} S_d^p \ .$$

The first equation can be seen to hold by considering j to be the number of the variables $x_1, x_2, ..., x_p$ that are equal to one at a vertex of $[0,1]^d$. The last equation follows from a formula of Stirling (see Berge (1971), for example). The result follows. ∎

Next, we wish to examine the worst-case behavior of $\rho(Q^*(G_d^p), P(G_d^p))$, but first we need a couple of lemmas.

Lemma 6.8. For $0 < \epsilon < 1$ and $d = p + p^{1-\epsilon}$,

$$\left(\frac{p!}{d!}\right)^{1/d} = 1 + o(1) \ .$$

Proof: Since $(p!/d!)^{1/d}$ is no more than 1, it suffices to demonstrate that

$$\left(\frac{p!}{(p + p^{1-\epsilon})!}\right)^{1/p} = 1 + o(1) \ .$$

Using Stirling's approximation to the factorial, and letting $t = 1/p$, we consider the asymptotic behavior of

$$\frac{\exp\{t^\epsilon\}\ t^{t^\epsilon}}{(1+t^\epsilon)^{1+t/2+t^\epsilon}} \ .$$

as t vanishes. It is clear that the numerator and denominator of this last expression tend to unity. ∎

Lemma 6.9 (Moser and Wyman (1958); also see Bender (1973), and Knessl and Keller (1991)). As p and d increase such that $0 < p < d$ and $\lim_{d\to\infty} d - p = \infty$, S_d^p is asymptotic to

$$\tilde{S}_d^p := \frac{d!(\exp\{R\} - 1)^p}{2R^d p! \sqrt{\pi p R H}} \ ,$$

where $R \geq 0$ solves

$$\frac{p}{d} = \frac{1 - \exp\{-R\}}{R} \ ,$$

and

$$H := \frac{\exp\{R\}(\exp\{R\} - 1 - R)}{2(\exp\{R\} - 1)^2} .$$

Moser and Wyman noted that $1/4 \le H \le 1/2$, so for the purposes of a lower bound on S_d^p, it suffices to take $H = 1/2$ in the expression for $\widetilde{S}_d^p$.

Corollary 6.10. For all $0 < \epsilon \le 1/2$, there is a sequence of threshold graphs G_d on d vertices, for an infinite set of d, such that $\rho(Q^*(G_d), P(G_d)) = \omega(d^{\frac{1}{2}-\epsilon})$.

Proof: Take d to be approximately $p + p^{1-\epsilon}$. By Lemma 6.8, $(d!/p!)^{1/d}$ tends to unity as p goes to infinity. Since $\Gamma(d/2+1)^{1/d} = \Theta(\sqrt{d})$, by Proposition 6.7, it suffices to demonstrate that

$$(*) \qquad d^{\epsilon}\left(\left(\frac{S_d^p}{p^{d-p}}\right)^{1/d} - 1\right)$$

is unbounded as p goes to infinity. Observe that in this case, we can write p as a function of R (of Lemma 6.9):

$$p = \left(\frac{\exp\{R\} - 1}{(R-1)\exp\{R\} + 1}\right)^{1/\epsilon} .$$

It is easy to check that $R = o(1)$ as p increases, and moreover that p is asymptotic to $R^{-1/\epsilon}$. Thus, we can write $(*)$ (using $\widetilde{S}_d^p$ with H taken to be $1/2$), asymptotically, as a function of $t = 1/p$ alone:

$$(t^{-1} + t^{-1+\epsilon})^{\epsilon}\left(-1 + \frac{2^{\frac{t}{-1-t^{\epsilon}}}(-1+\exp\{t^{\epsilon}\})^{\frac{1}{1+t^{\epsilon}}}\, t^{\frac{2-2\epsilon+2t-\epsilon t+2t^{\epsilon}-2\epsilon t^{\epsilon}}{2+2t^{\epsilon}}}(t^{-1}+t^{-1+\epsilon})^{\frac{2+t+2t^{\epsilon}}{2+2t^{\epsilon}}}}{\exp\{\frac{1}{1+t^{-\epsilon}}\}\pi^{\frac{t}{2(1+t^{\epsilon})}}}\right)$$

and take the limit as t vanishes. It can be checked (with the aid of *Mathematica*, for example) that the resulting limit is infinite. ∎

We note that for any threshold graph G, every extreme point $x(Q)$ of $Q(G)$ has at most one coordinate that is not an integer. Rounding such a coordinate down (to 0), produces a (extreme) point $x(P)$ of $P(G)$. Let c be the unit normal to any supporting hyperplane of $Q(G)$ that supports at $x(Q)$. Let $x'(P)$ be a point on the boundary of $P(G)$ that is supported by a hyperplane with normal c. Clearly, $c(x(Q) - x'(P))$ is less than unity, hence $h_{\max}(Q(G), P(G))$ is less than unity for all threshold graphs G. This implies that for optimization with *linear* objective functions, minimal separator polytopes of threshold graphs are good models of the associated vertex-packing polytopes. Proposition 6.10 suggests that this may not be the case for more general classes of objective functions.

Although, for threshold graphs G, the Euclidean distance between a pair of hyperplanes with the same normal that support $Q(G)$ and $P(G)$ cannot be large, there is another sense (besides the idealized radial distance) in which $Q(G)$ can be thought of as a bad approximation to $P(G)$. We consider the worst case behavior of the Euclidean distance between a point in each of the polytopes that is the (unique) "contact point" of a pair of supporting hyperplanes with the same normal.

Proposition 6.11.

$$\max_{\substack{c \in \mathbf{R}^d \\ G \text{ threshold}}} \left\{ \| \operatorname{argmax}\{cx \; : \; x \in Q^*(G)\} \; - \; \operatorname{argmax}\{cx \; : \; x \in P(G)\} \|_2 \right\} = \Theta(\sqrt{d}) \; .$$

Proof: We consider the graphs G_d^{d-1} (i.e., the threshold graphs of the cut permutation with $p = d - 1$). Consider the hyperplanes in $\mathbf{R}^d$ having (unit) normal

$$n_d := \left(\frac{45}{4} + \frac{1}{d-2} \right)^{-1/2} \cdot \left(\frac{5}{2}, \frac{1}{d-2}, \frac{1}{d-2}, \ldots, \frac{1}{d-2}, 3 \right) .$$

The hyperplane with normal c_d that supports $Q^2(G_d^{d-1})$ does so at

$$x(Q^*, d) := \left(1, 0, 0, \ldots, 0, \frac{d-2}{d-1} \right) ,$$

and the hyperplane with normal c_d that supports $P(G_d^{d-1})$ does so at

$$x(P, d) := (1, 1, \ldots, 1, 0) .$$

The result follows by noting that $c_d(x(Q^*, d) - x(P, d)) = \Theta(\sqrt{d})$. ∎

7. Further Directions. For threshold graphs G, it would be nice to have a better understanding of the asymptotic behavior of $\rho(Q^*(G), P(G))$ in terms of characteristics of the posets underlying the graphs; Example 6.4 and Corollary 6.10 indicate two extremes.

It would interesting to study the behavior of ρ for various pairs of polytopes related to the travelling salesman problem. For example, (1) the Directed Hamiltonian Tour Polytope, and (2) the Diagonal-Free Assignment Polytope relative to the complete (loop-free) digraph on d vertices (see Nemhauser and Wolsey (1988), for example). One possible approach to this problem is through the use of the Ehrhart polynomial (see Section 1). The number of lattice points in these polytopes is $(n-1)!$ and $[n!/e]$, respectively. If one could count the number of lattice points in integral dilations of these polytopes, then expressions for the associated volumes could be developed. Such a study might partially explain the empirical success in using relatively simple facial information to solve travelling salesman problems by cutting-plane methods. Such polytopes have all of the symmetry of the complete digraphs, hence it may be possible (and indeed desirable), in this case, to relate ρ and $\overline{h}$.

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