# ON GENERATORS OF IDEALS ASSOCIATED WITH UNIONS OF LINEAR VARIETIES

ideales que son en su conjunto de zeros la union de espacios lineales

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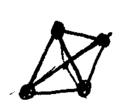
ABSTRACT

Consider the polynomial ring $R[x_1, \dots, x_n]$ over a unique factorization domain $R$. A form (i.e., a homogeneous polynomial) is said to *split* if it is a product of linear forms. When a homogeneous ideal is generated by splitting forms, the associated projective algebraic set is a finite union of linear subvarieties of $P^{n-1}(R)$. But conversely, when a projective algebraic set decomposes into linear subvarieties, its associated radical ideal may not be generated by splitting forms. In this paper we construct a recursive algorithm for establishing sufficient conditions for an ideal to be generated by a prescribed set of splitting forms and apply this algorithm to a family of ideals that have arisen in the study of block designs. Our results on ideal generators have very interesting applications to graph theory, which are discussed elsewhere.

## 1. *Introduction*

Let $R$ be a unique factorization domain and let $R[x_1, \dots, x_n]$ denote the polynomial ring with $n$ variables. The ideal of a polynomial ring generated by the elements $f_1, \dots, f_r$ will be noted by $\langle f_1, \dots, f_r \rangle$. **In this paper we investigate generators of the ideals of $R[x_1, \dots, x_n]$ associated with unions of *linear varieties*.** In what follows, all the ideals are homogeneous, algebraic sets are projective, and varieties are irreducible algebraic sets. A form is said to *split* if it is a product of linear forms. One can refer to [1] for undefined algebraic terminology in this paper.

**When an ideal is generated by splitting forms, its associated algebraic set is a finite union of linear subvarieties in $P^{n-1}(R)$. But the converse is not true.** Shown below is a radical ideal not generated by splitting forms, although its associated algebraic set is the union of three 1-dimensional linear subvarieties of $P^3(R)$.



*Example.* Let $I$ denote the ideal

$$\langle x, z \rangle \cap \langle y, w \rangle \cap \langle x+y, z+w \rangle$$

in the polynomial ring $R[x, y, z, w]$. This ideal contains the polynomial $xw - yz = x(z+w) - z(x+y)$. We claim that $xw - yz$ cannot be generated by splitting forms in $I$. Since the ideal $I$ clearly does not contain any linear form, we need only to show that it does not contain any splitting form of degree 2 either. Assuming $f \cdot g \in I$, where $f$ and $g$ are linear forms, we want to derive a contradiction. Because the ideals $\langle x, z \rangle$, $\langle y, w \rangle$ and $\langle x+y, z+w \rangle$ are all prime, each of them must contain either $f$ or $g$. Therefore two of these three ideals must contain a linear form in common. This is obviously impossible.

In view of the above counterexample, we are interested in sufficient conditions for an ideal $I$ to be generated by a prescribed finite set $\mathcal{G}$ of splitting forms. For technical reasons, we shall consider ideals of the form $I \cap \langle \phi \rangle$, where $\phi$ is a splitting form. The following proposition provides a recursive algorithm for establishing a sufficient condition for the ideal $I \cap \langle \phi \rangle$ to be generated by $\{g \vee \phi : g \in \mathscr{G}\}$. Here the notation "$\vee$" stands for least common multiple (unique up to multiplications by units of $R$). We shall write $\mathscr{G} \vee \phi$ for $\{g \vee \phi : g \in \mathscr{G}\}$.

PROPOSITION 1.1. *Let $\mathscr{P}$ be the set of all quadruples $(Y, I, \phi, \mathscr{G})$ such that $I \cap \langle \phi \rangle$ is generated by $\mathscr{G} \vee \phi$, where $Y = \{y_1, y_2, \ldots\}$ is a finite set of indeterminates, $I$ is an ideal of the polynomial ring $R[Y]$, $\phi$ is a splitting form in $R[Y]$, and $\mathscr{G}$ is a finite set of nonzero splitting forms in $I$. Then a quadruple $(Y, I, \phi, \mathscr{G})$ is in $\mathscr{P}$ if there exists a linear form*

$$\lambda = y_v - \sum_{i \neq v} r_i y_i \in R[Y]$$

*that satisfies the following three requirements.*

(1.2) *$(Y, I, \phi\lambda, \mathscr{G}') \in \mathscr{P}$ for some $\mathscr{G}' \subset \mathscr{G}$.*

(1.3) *$(Y - \{y_v\}, I'', \alpha(\phi), \mathscr{G}'') \in \mathscr{P}$ for some $I'' \supset \alpha(I)$ and for some $\mathscr{G}''$, where $\alpha$ is the ring homomorphism over $R$ from $R[Y]$ to $R[Y - \{y_v\}]$ defined by*

$$\alpha(y_i) = \begin{cases} y_i & \text{if } i \neq v, \\ \sum_{i \neq v} r_i y_i & \text{if } i = v. \end{cases}$$

(1.4) *For the same set $\mathscr{G}''$ as in (1.3), $\mathscr{G}'' \vee \alpha(\phi) \subset \langle \alpha(\mathscr{G} \vee \phi) \rangle$.*

*Proof.* Let $(Y, I, \phi, \mathscr{G})$ be a quadruple such that there exists a linear form $\lambda$ satisfying the requirements (1.2)–(1.4). We know from (1.2) that $I \cap \langle \phi\lambda \rangle$ is generated by $\mathscr{G}' \vee \phi\lambda$ for some $\mathscr{G}' \subset \mathscr{G}$. The condition (1.3) says that the ideal $I'' \cap \langle \alpha(\phi) \rangle$ is generated by $\mathscr{G}'' \vee \alpha(\phi)$ for some $I'' \supset \alpha(I)$ and some $\mathscr{G}''$. Thus the image of $I \cap \langle \phi \rangle$ under $\alpha$ is contained in $I'' \cap \langle \alpha(\phi) \rangle$. The condition (1.4) implies that

$$I'' \cap \langle \alpha(\phi) \rangle = \alpha(I) \cap \langle \alpha(\phi) \rangle = \alpha(I \cap \langle \phi \rangle).$$

Therefore we have the following commutative diagram of exact sequences:

$$\begin{array}{ccccccccc} & & \langle \mathscr{G}' \vee \phi\lambda \rangle & \longrightarrow & \langle \mathscr{G} \vee \phi \rangle & \xrightarrow{\alpha} & \langle \mathscr{G}'' \vee \alpha(\phi) \rangle & \longrightarrow & 0 \\ & & \| & & \downarrow\cap & & \| & & \\ 0 & \longrightarrow & I \cap \langle \phi\lambda \rangle & \longrightarrow & I \cap \langle \phi \rangle & \xrightarrow{\alpha} & I'' \cap \langle \alpha(\phi) \rangle & \longrightarrow & 0. \end{array}$$

By diagram chasing we see that the inclusion map from $\langle \mathscr{G} \vee \phi \rangle$ to $I \cap \langle \phi \rangle$ is surjective. This shows that $(Y, I, \phi, \mathscr{G})$ is in $\mathscr{P}$.

We shall apply this algorithm to investigate generators of a family of ideals arising from the study of block $t$-designs. (See [2].) We will show that these ideal admit generators which are the products of the forms $x_i - x_j$ (see Theorem 2.1 below). The proof will, in particular, describe a way of selecting an appropriate linear form $\lambda$ at each intermediate stage. A consequence, Corollary 2.3, verifies a conjecture raised in [2]. Our results on ideal generators also have quite interesting applications in graph theory. These are discussed in [3].

Since linear varieties are very simple objects, one would hope to find a criterion for a radical ideal whose associated algebraic set is a finite union of linear varieties to be generated by splitting forms. As the associated radical ideal $A_i$ of a linear variety $V_i$ is generated by linear forms, there is an ideal, namely, the product of $A_i$, $i = 1, \dots, r$, which is generated by splitting forms and whose associated algebraic set is the union of $V_i$, $i = 1, \dots, r$. So the problem is equivalent to finding a criterion for the radical of an ideal generated by splitting forms to be generated also by splitting forms. A lot remains to be explored. For instance, the counterexample above certainly describes a type of radical ideals which are not generated by splitting forms, but one does not know if this is the only kind of exception.

## 2. *Main results*

Let $R$ be as before. We fix integers $n \geqslant k \geqslant 0$. Let $X$ stand for the set $\{x_1, \dots, x_n\}$ of indeterminates. For $f \in R[X]$ and $Z \subset X$, denote by $f/Z$ the polynomial obtained from $f$ by setting the variables in $Z$ equal to the first member in $Z$. Let $J = J(k, n)$ be the ideal of $R[X]$ consisting of polynomials $f$ such that $f/Z = 0$ for all subsets $Z$ of $X$ with $|Z| \geqslant k$. **This ideal has arisen from the study of *block t-designs*** (cf. [2]). We are interested in generators of ideals of the form $J \cap \langle \phi \rangle$.

It is clear that $J = 0$ when $k = 0$ or 1. Therefore we assume that $k \geqslant 2$ from now on.

For a subset $Y$ of $X$, put

$$\Delta(Y) = \prod_{\substack{x_i, x_j \in Y \\ i < j}} (x_i - x_j)\,.$$

By an *X-partition* we shall mean a partition $P$ of $X$ into $k-1$ (possibly empty) subsets $X_1, \dots, X_{k-1}$, and we shall write $\Delta(P)$ for $\prod_{i=1}^{k-1} \Delta(X_i)$. One checks easily that $\Delta(P)$ lies in $J$. A form in $R[X]$ is said to be *diagonal* if it is a product of polynomials of the type $x_i - x_j$, with repeated factors allowed. Thus **diagonal forms are splitting forms of a special kind.** The polynomials $\Delta(P)$ defined above are clearly diagonal forms, and for every diagonal form $\phi$, $\Delta(P) \vee \phi$ is a diagonal form in the ideal $J \cap \langle \phi \rangle$. The following theorem provides a sufficient condition on $\phi$ for these forms $\Delta(P) \vee \phi$ to generate $J \cap \langle \phi \rangle$.

THEOREM 2.1. *Let $\phi$ be a diagonal form satisfying the condition*

(2.2) *if* $x_j - x_m$ *divides* $\phi$, *so does* $x_i - x_m$ *for every* $i < j$.

*Then the ideal $J \cap \langle \phi \rangle$ is generated by the diagonal forms $\Delta(P) \vee \phi$, where $P$ runs through all X-partitions.*

*Remark.* The condition (2.2) in the theorem is not superfluous, as one can see from the example when $n = 5$, $k = 3$, and

$$\phi(x_1, \ldots, x_5) = (x_1 - x_2)(x_2 - x_3)(x_3 - x_4)(x_4 - x_5)(x_1 - x_5).$$

It is easily seen that $\phi$ lies in $J$, hence $J \cap \langle\phi\rangle = \langle\phi\rangle$. On the other hand, since $\phi$ is not equal to $\Delta(P)$ for any $X$-partition $P$, the polynomials $\Delta(P) \vee \phi$ must all be of degree 6 or higher. Therefore $\phi$ cannot be generated by the polynomials $\Delta(P) \vee \phi$.

The following special case is proved in [3] using a simpler method.

COROLLARY 2.3. *The ideal $J(k, n)$ is generated by the forms $\Delta(P)$. In fact, it is generated by those $\Delta(P)$, where $P$ is a partition of $X$ into $k-1$ subsets of as nearly equal cardinality as possible.*

## 3. *Generalizations and proofs*

In order to prove Theorem 2.1, we shall first adapt the condition (2.2) and the theorem to a more general form suitable for applying Proposition 1.1.

Let $Y$ be a finite set of indeterminates. We shall consider diagonal forms $\phi$ in $R[Y]$ satisfying the condition

(3.1) *there exists a linear order $\prec$ on $Y$ such that if $x - y$ divides $\phi$, then so does $z - y$ for every $z \prec x$.*

Given such a $\phi$, we say that a sequence

$$\mathrm{T}: Y \supset T_1 \supset T_2 \supset \ldots \supset T_{k-1}$$

of subsets of $Y$ of length $k-1$ is *$\phi$-admissible* if

(3.2) $T_1$ is the largest subset of $Y$ such that $\Delta(T_1)$ divides $\phi$ ($T_1$ is unique by the condition (3.1));

(3.3) $\Delta(T_1) \ldots \Delta(T_{k-1})$ divides $\phi$.

For a $\phi$-admissible sequence T as above, define

(3.4) $$I(\mathrm{T}) = \left\{ f \in R[Y] : f/Z = 0 \text{ for all subsets } Z \text{ of } Y \text{ with } |Z| + \sum_{i=2}^{k-1} |Z \cap T_i| \geqslant k \right\}.$$

Thus $I(\mathrm{T})$ is an ideal of $R[Y]$. Obviously, if we let $T_1$ be defined by (3.2) and $T_i$, for $i = 2, \ldots, k-1$, be the empty set, then the sequence T is $\phi$-admissible and $I(\mathrm{T})$ is equal to $J(k, n)$ provided $Y = X$.

In this general setting, the role of an $X$-partition will be played by an *irredundant* T-*covering of* $Y$, that is, a covering $\{Y_1, \ldots, Y_{k-1}\}$ of $Y$ such that $Y_i \supset T_i$ for $i = 1, \ldots, k-1$ and it has no proper subcovering which has the same property.

It is easily seen that Theorem 2.1 is an immediate consequence of

THEOREM 3.5. *Let $\phi$ be a diagonal form in* $R[Y]$ *satisfying* (3.1) *and* T *be a $\phi$-admissible sequence. Then the ideal $I(\mathrm{T}) \cap \langle\phi\rangle$ is generated by the diagonal forms*

$\Delta(Y_1)\ldots\Delta(Y_{k-1})\vee\phi$, *where* $\{Y_1,\ldots,Y_{k-1}\}$ *runs through all irredundant* T*-coverings of* $Y$.

*Proof.* For a $\phi$-admissible sequence T, put

(3.6) $\mathscr{G}(\mathrm{T}) = \{\Delta(Y_1)\ldots\Delta(Y_{k-1}) : \{Y_1,\ldots,Y_{k-1}\}$ is an irredundant T-covering of $Y\}$.

Let $\mathscr{P}'$ be the set consisting of quadruples $(Y, I, \phi, \mathscr{G})$, where $Y$ is a finite set of indeterminates, $\phi$ is a diagonal form in $R[Y]$ satisfying (3.1), $I = I(\mathrm{T})$ and $\mathscr{G} = \mathscr{G}(\mathrm{T})$ are defined by (3.4) and (3.6) respectively for some $\phi$-admissible sequence T. We want to show that $\mathscr{P}'$ is a subset of $\mathscr{P}$ defined in Proposition 1.1. Note that $\mathscr{G}(\mathrm{T}) \subset I(\mathrm{T})$. Indeed, if $\{Y_1,\ldots,Y_{k-1}\}$ is an irredundant T-covering of $Y$ and $Z \subset Y$ with

$$|Z| + \sum_{i=2}^{k-1} |Z \cap T_i| \geqslant k\,,$$

then

$$\sum_{i=1}^{k-1} |Z \cap Y_i| \geqslant |Z - T_1| + \sum_{i=1}^{k-1} |Z \cap T_i| \geqslant k\,.$$

Therefore $|Z \cap Y_i| \geqslant 2$ for some $i$ and $\Delta(Y_1)\ldots\Delta(Y_{k-1})/Z = 0$.

Let $(Y, I, \phi, \mathscr{G})$ be an element of $\mathscr{P}'$ with $I = I(\mathrm{T})$ and $\mathscr{G} = \mathscr{G}(\mathrm{T})$ for some $\phi$-admissible sequence

$$\mathrm{T} : Y \supset T_1 \supset \ldots \supset T_{k-1}\,.$$

If $T_1 = Y$, then $\phi \in I$ and $I \cap \langle\phi\rangle = \langle\phi\rangle$. On the other hand, $\Delta(T_1)\ldots\Delta(T_{k-1})$ lies in $\mathscr{G}$ and divides $\phi$. This shows

$$\langle \mathscr{G} \vee \phi\rangle = \langle\phi\rangle = I \cap \langle\phi\rangle\,,$$

as desired. So assume $T_1 \neq Y$. The proof will be by induction on $|Y - T_1|$ and, for a fixed $|Y - T_1|$, by induction on $s(\phi, \mathscr{G})$, the number of distinct linear factors of $\bigvee_{g\in\mathscr{G}} g$ not dividing $\phi$. We shall show the existence of a linear form $\lambda$ which meets the requirements (1.2)–(1.4) and thus conclude from Proposition 1.1 that $(Y, I, \phi, \mathscr{G})$ is in $\mathscr{P}$.

Let $y$ be the smallest element in $Y - T_1$ under the order $\prec$ given in (3.1). Let $x$ be the smallest element in $Y$ such that $x - y$ does not divide $\phi$. Then we have $x \in T_1$ and $z \prec y$ for all $z \in T_1$ by conditions (3.1) and (3.2). Further, if $z \in T_1$ and $x \prec z$, then $z - y$ does not divide $\phi$. Reordering the elements in $T_1$ which are not less than $x$ if necessary, we may assume

(3.7) *if* $x \notin T_i$, *then* $z \notin T_i$ *for every* $z \in T_1$ *with* $x \prec z$.

Choose $\lambda$ to be the linear form $y - x$. We verify (1.2)–(1.4) as follows.

*Proof of* (1.2). Put $\phi' = \phi\lambda$. By the choice of $y$ and $x$, $\phi'$ clearly satisfies (3.1) with the same order $\prec$. Define the sequence

$$\mathrm{T}' : Y \supset T_1' \supset T_2' \supset \ldots \supset T_{k-1}'$$

by
$$T'_1 = \begin{cases} T_1 \cup \{y\} & \text{if } x \text{ is the largest element in } T_1, \\ T_1 & \text{otherwise,} \end{cases}$$
and
$$T'_i = T_i \qquad \text{for } i = 2, \dots, k-1.$$

Then T′ is $\phi'$-admissible. Moreover, $I = I(\mathrm{T}) = I(\mathrm{T}')$ since it is independent of $T_1$ and $T'_1$. Finally let $\mathscr{G}' = \mathscr{G}(\mathrm{T}')$, which is a subset of $\mathscr{G} = \mathscr{G}(\mathrm{T})$. Thus $(Y, I, \phi', \mathscr{G}')$ lies in $\mathscr{P}'$ and hence in $\mathscr{P}$ by induction on $s(\phi', \mathscr{G}')$.

*Proof of* (1.3). Put $\phi'' = \phi/\{x, y\}$ and $Y'' = Y - \{y\}$. We want to find $I'' \supset I/\{x, y\}$ and $\mathscr{G}''$ such that $(Y'', I'', \phi'', \mathscr{G}'') \in \mathscr{P}$. One checks easily that $\phi''$ satisfies (3.1) with the induced order on $Y''$.

Let $t$ be the index such that $x \in T_t$ and $x \notin T_{t+1}$. We let $t = k-1$ if $x \in T_{k-1}$. In the latter case, we have
$$I \subset \langle \psi \rangle \quad \text{and} \quad I \cap \langle \phi \rangle = I \cap \langle \phi \vee \psi \rangle$$
where
$$\psi = \prod \{x - z : z \in Y \quad \text{and} \quad z \neq x\}.$$

The form $\phi \vee \psi$ satisfies (3.1) with respect to the new order which assigns $x$ to be the smallest element and preserves the orders of the elements in $Y - \{x\}$. Moreover, the sequence
$$\mathrm{T}^\circ : Y \supset T_1^\circ \supset T_2 \supset \dots \supset T_{k-1},$$
where $T_1^\circ \supset T_1$ is given by (3.2) with $\phi \vee \psi$ replacing $\phi$, is clearly $\phi \vee \psi$-admissible and $I = I(\mathrm{T}^\circ)$. Therefore $(Y, I, \phi \vee \psi, \mathscr{G}(\mathrm{T}^\circ))$ is in $\mathscr{P}'$ and hence in $\mathscr{P}$ by induction. Since $I \cap \langle \phi \rangle = \langle \mathscr{G}(\mathrm{T}^\circ) \vee \phi \vee \psi \rangle$ is contained in $\langle \mathscr{G}(\mathrm{T}) \vee \phi \rangle$, we have $(Y, I, \phi, \mathscr{G}) \in \mathscr{P}$. So we assume $t < k-1$.

The conditions (3.2), (3.7) and the minimality of $x$ imply that the sequence
$$\mathrm{T}'' : Y'' \supset T''_1 \supset T''_2 \supset \dots \supset T''_{k-1},$$
where
$$T''_i = T_i \qquad \text{if } 1 \leqslant i \leqslant k-1 \text{ and } i \neq t+1,$$
$$T''_{t+1} = T_{t+1} \cup \{x\},$$
is $\phi''$-admissible. Then the quadruple $(Y'', I'', \phi'', \mathscr{G}'')$ with $I'' = I(\mathrm{T}'')$ and $\mathscr{G}'' = \mathscr{G}(\mathrm{T}'')$ lies in $\mathscr{P}'$ and consequently in $\mathscr{P}$ by induction on $|Y'' - T''_1|$.

It remains to show that $I/\{x, y\} \subset I''$. Let $Z'' \subset Y''$ be such that
$$|Z''| + \sum_{i=2}^{k-1} |Z'' \cap T''_i| \geqslant k.$$
Then $Z = Z'' \cup \{x, y\} \subset Y$ satisfies
$$|Z| + \sum_{i=2}^{k-1} |Z \cap T_i| \geqslant k.$$
The desired inclusion is now obvious. This completes the proof of (1.3).

*Proof of* (1.4). We have to show that $\langle \mathscr{G}'' \vee \phi'' \rangle \subset \langle (\mathscr{G} \vee \phi)/\{x, y\} \rangle$, where $\mathscr{G}'' = \mathscr{G}(T'')$. Let $g'' = \Delta(Y_1'') \ldots \Delta(Y_{k-1}'')$ be an element of $\mathscr{G}''$. Put $g = \Delta(Y_1) \ldots \Delta(Y_{k-1})$, where

$$Y_i = Y_i'' \qquad \text{if } i \neq t+1,$$

and

$$Y_{t+1} = \{y\} \cup (Y_{t+1}'' - \{x\}).$$

Then $\{Y_1, \ldots, Y_{k-1}\}$ is an irredundant T-covering of $Y$ and $g/\{x, y\} = \pm g''$. We claim that $(g \vee \phi)/\{x, y\} = \pm g'' \vee \phi''$.

Clearly, $g'' \vee \phi''$ divides $(g \vee \phi)/\{x, y\}$ and a factor $y_1 - y_2$ with $y_1, y_2 \notin \{x, y\}$ appears in both forms with the same multiplicity. So it suffices to check the order of $x - z$ in both forms.

*Case* I. $x \prec z$. If $z \notin T_1$, then both $y - z$ and $x - z$ do not divide $\phi$, and consequently, $x - z$ does not divide $\phi''$. Thus the orders of $x - z$ in both forms are equal to the order of $x - z$ in $g''$. *If* $z \in T_1$, then $z \notin T_{t+1}$ by construction. In this case, $y - z$ divides neither $\phi$ nor $g$ since $\{Y_1, \ldots, Y_{k-1}\}$ is irredundant. Thus the orders of $x - z$ in both forms are equal to the order of $x - z$ in $g \vee \phi$.

*Case* II. $z \prec x$. First, from condition (3.3) and the irredundancy of $\{Y_1, \ldots, Y_{k-1}\}$, we know that the multiplicity of $x - z$ in $g$ does not exceed that in $\phi$. Moreover, since $y - z$ divides $\phi$ and it appears in $g$ at most of order 1, we conclude that the multiplicity of $x - z$ in $(g \vee \phi)/\{x, y\}$ is equal to the multiplicity of $x - z$ in $\phi''$. On the other hand, by the same reasoning as above, we know that the order of $x - z$ in $g''$ is dominated by that in $\phi''$.

In both cases we have obtained the desired equality. This completes the proof of (1.4). Theorem 3.5 now follows.

## *References*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