ere $A \approx B$ means the limit as n goes to infinity of A/B is 1. In particular, we have $\approx \sqrt{e}n^{n-2}$.

A score vector $V=\langle v_1, \ldots, v_n \rangle$ is a *score sequence* if $v_1 \geqq v_2 \geqq \ldots \geqq v_n$. Let $_{n;p}$ be the number of distinct p-fold tournament score sequences. The asymptotic valuation of $S_{n;p}$ (and in particular of $S_n = S_{n;1}$) is quite difficult; see for example]. Note however that for fixed n, as p gets large the probability that a vector $=\langle v_1, \ldots, v_n \rangle \in \sigma(K_{n;p})$ has distinct components (that is, $v_i = v_j$ implies $i=j$) gets rger. Using $\approx$ with respect to p here, it therefore follows that

$$S_{n;p} \approx \frac{1}{n!} V_{n;p} \approx \frac{p^{n-1} n^{n-2}}{n!}.$$

Note that the remarks in the last section imply that if V is a 1-fold tournaent score sequence then $M(V)$ is a tree: since player 1 has the highest score it must e in the highest irreducible component. In addition, note that if the sequence is reducible, then the components below 1's component are on the "youngest ranch" of $M(V)$. That is, if P is the component containing 1 and Q is the rest of e vertex set, then there is a path $1=p_1, p_2, \ldots, p_m$ such that p_i is the smallest verx adjacent to p_{i-1} (except possibly for p_{i-2}) and such that all vertices of Q lie elow p_m in $M(V)$. Further observations along these lines may produce a workble set of necessary and sufficient conditions for a tree to be in the image set of f applied to tournament score sequences and perhaps allow an asymptotic determiation of S_n.

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INDEPENDENCE NUMBERS OF GRAPHS AND GENERATORS OF IDEALS

by

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This article investigates the generators of certain homogeneous ideals which are associated with graphs with bounded independence numbers. These ideals first appeared in the theory of t-designs. The main theorem suggests a new approach to the Clique Problem which is $\mathcal{NP}$-complete. This theorem has a more general form in commutative algebra dealing with ideals associated with unions of linear varieties. This general theorem is stated in the article; a corollary to it generalizes Turán's theorem on the maximum graphs with a prescribed clique number.

1. Introduction

Let G be a graph on n vertices $\{1, \ldots, n\}$ and let

$$f_G = \prod \{(x_i - x_j):\ i \text{ and } j \text{ are adjacent and } i < j\}$$

be the associated polynomial. Then G has *independence number* $\tilde{c}(G) < k$ if and only if at least two of any arbitrarily given k vertices are adjacent. Otherwise said, the polynomial f_G vanishes whenever k variables are set equal. Let, for given integers n and k with $1 \leqq k \leqq n$, $I(k, n)$ denote the ideal of the polynomial ring $\mathbf{Z}[x_1, \ldots, x_n]$ consisting of the polynomials which vanish whenever k variables are set equal. Then we have

(1.1) $$\tilde{c}(G) < k \quad \text{if and only if} \quad f_G \in I(k, n).$$

The ideal $I(k, n)$ also arises naturally from the study of block t-designs (cf. [1]). In this paper we prove the following theorem concerning generators of $I(k, n)$, as conjectured in [1]:

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Theorem 1. *The ideal $I(k, n)$ is generated by $\Delta(P)$, where $P=\{P_1, \ldots, P_{k-1}\}$ through all partitions of the set $\{1, \ldots, n\}$ into $k-1$ (possibly empty) subs*

$$\Delta(P) = \prod_{m=1}^{k-1} \prod_{\substack{i, j \in P_m \\ i<j}} (x_i - x_j).$$

Applying this theorem to f_G, we can restate the criterion (1.1) in

Corollary 1.2. *A graph G has independence number $\bar{c}(G) \leqq k$ if and only if*

$$f_G = \sum_H g_H \cdot f_H \tag{1.3}$$

where H is the union of k vertex-disjoint complete graphs and g_H is a polynom

After examining the degrees of the polynomials f_H, a well-known theorem of Turán [3] can be deduced from this corollary. (See section 4). More importa Corollary 1.2 suggests a new way of attacking some of the outstanding problems in graph theory. For instance, one might solve the problem of finding sufficient conditions for $\bar{c}(G) \leqq k$ which can be verified within polynomial time by looking for graphs G such that the number of H's appearing in (1.3) is bounded by a polynomial in $|G|$. There is also an interesting question of finding an infinite fa of graphs G so that the minimal number of H's needed in formula (1.3) is expo nential in $|G|$. Such graphs exist if one assumes the hypothesis that $\mathcal{NP} \neq$

Problems

Dual

NO muy correct

The following dual statement is recently proved by D. Kleitman and L. Lovász using a method similar to our proof of Theorem 1 shown in section 2: A graph G has *chromatic number* $\geqq k$ if and only if f_G lies in the ideal generated by the polynomials f_H where H is a complete k-graph on some subset of vertices of G. It would be quite interesting to study the connection between the representation of f_G in the form (1.3) and its "dual" representation described above.

We remark that the coefficient ring $\mathbf{Z}$ of the polynomials we consider in this paper is irrelevant and can be replaced by any unique factorization domain. A generalization of Theorem 1 in commutative algebra will be proved in a subsequent paper [2]. We only state the result in section 3. Nevertheless, its applications to graph theory concerning the structure of maximal complete k-graph free subgraphs of a given graph, which we call the "Turán property", will be discussed in section 4.

see paper

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2. Proof of Theorem 1

First we introduce some notation. Let X be the set $\{x_1, \ldots, x_n\}$ of indeterminates. For a subset Y of X, let $\mathbf{Z}[Y]$ stand for the polynomial ring over $\mathbf{Z}$ with variables in Y. If $f \in \mathbf{Z}[X]$ and $Z \subset X$, denote by f/Z the polynomial obtained from f by setting the variables in Z equal to the first member in Z. Finally, set, for $Y \subset X$.

$$\Delta(Y) = \prod \{(x_i - x_j)\colon x_i, x_j \in Y \text{ and } i<j\}.$$

Obviously, $I(k, n)=0$ when $k=0$ or 1. Therefore we assume $k\geqq 2$ from now on. We shall prove the following

Theorem 2. *For a sequence $X\supset Y_1\supset \dots \supset Y_{k-1}$, define the ideal*

$$I = \{f\in \mathbf{Z}[X]:\ \Delta(Y_1)\dots\Delta(Y_{k-1}) \text{ divides } f$$

$$\text{and } f/Z=0 \text{ for all } Z\subset X \text{ with } |Z|+\sum_{1<i<k}|Y_i\cap Z|\geqq k\}.$$

Let J be the ideal generated by the polynomials $\Delta(X_1)\dots\Delta(X_{k-1})$, where $X_i\supset Y_i$ and $\bigcup X_i=X$. Then $I=J$. partitions

If we take $Y_1=\dots=Y_{k-1}$ to be the empty set, then I is nothing but the ideal $I(k, n)$ and one sees easily that Theorems 1 and 2 coincide in this case.

Proof. We first check that J is contained in I. For this, it suffices to show that each generator $\Delta(X_1)\dots\Delta(X_{k-1})$ is. The condition that $X_i\supset Y_i$ certainly implies that $\Delta(X_1)\dots\Delta(X_{k-1})$ is divisible by $\Delta(Y_1)\dots\Delta(Y_{k-1})$. Moreover, if $Z\subset X$ satisfies

$$|Z|+\sum_{1<i<k}|Y_i\cap Z|\geqq k,$$

then

$$\sum_{1\leqq i<k}|Z\cap X_i|\geqq |Z-Y_1|+\sum_{1\leqq i<k}|Z\cap Y_i|\geqq k$$

since $X-Y_1$ is covered by the X_i's. Therefore $|Z\cap X_i|\geqq 2$ for some i and $\Delta(X_1)\dots\Delta(X_{k-1})/Z=0$.

Now we show $I=J$ by induction on $|X-Y_1|$. The assertion is trivial if $Y_1=X$. Therefore we assume $Y_1\neq X$. We may further assume that $Y_1=\{x_1, \dots, x_r\}$ and each Y_i is a beginning section of this sequence. Let $y\in X-Y_1$. Given $f\in I$, we want to prove that $f\in J$. Our strategy is to find a sequence $f_0=f, f_1, \dots$ of polynomials in $\mathbf{Z}[X]$ such that

(a) $f-f_i$ belongs to J,

(b) $y-x_j$ divides f_i for $1\leqq j\leqq i$.

If we succeed in finding f_r, then we may replace Y_1 by $Y_1\cup\{y\}$, f by f_r and proceed with the induction.

So suppose that f_{i-1} for some $1\leqq i<r$ is defined. Let $g=f_{i-1}/\{x_i, y\}$, then g is a polynomial in $\mathbf{Z}[X-\{y\}]$. Let t be the largest index such that $x_i\in Y_t$ and $x_i\notin Y_{t+1}$. Set $t=k-1$ if $x_i\in Y_{k-1}$. In the latter case, the set $Z=\{x_i, y\}$ satisfies the condition

$$|Z|+\sum_{1<i<k}|Z\cap Y_i|=k,$$

and hence $f_{i-1}/Z=0$, i.e., $y-x_i$ divides f_{i-1}. We simply let $f_i=f_{i-1}$ in this case. Thus we assume $t<k-1$. Suppose $Y_{t+1}=\{x_1, \dots, x_j\}$. Note that $j<i$.

Since

$$(y-x_1)\dots(y-x_{i-1})\Delta(Y_1)\dots\Delta(Y_{k-1})=$$

$$(y-x_{j+1})\dots(y-x_{i-1})\Delta(Y_1)\dots\Delta(Y_{t+1}\cup\{y\})\dots\Delta(Y_{k-1})$$

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divides f_{i-1}, the polynomial $\frac{g}{h}$ is divisible by

$$\Delta(Y_1)\ldots\Delta(Y_{t+1}\cup\{x_i\})\ldots\Delta(Y_{k-1}),$$

where $h=(x_i-x_{j+1})\ldots(x_i-x_{i-1})$. Moreover, if any set $Z\subset X-\{y\}$ is such that

$$|Z|+\sum_{\substack{1<i<k\\ i\neq t+1}}|Z\cap Y_i|+|Z\cap(Y_{t+1}\cup\{x_i\})|\geqq k;$$

define the set Z_1 to be either $Z\cup\{y\}$ or Z depending on whether or not $x_i\in Z$; then Z_1 satisfies

$$|Z_1|+\sum_{1<i<k}|Z_1\cap Y_i|\geqq k,$$

and consequently, $f_{i-1}/Z_1=g/Z=0$. Since h^2 divides g, we also have $\frac{g}{h}\Big/Z=0$. Thus we may apply induction (on $|X|$) to $\frac{g}{h}$ and conclude that g is generated by the polynomials $h\Delta(X_1)\ldots\Delta(X_{k-1})$, where $X_i\supset Y_i$, $x_i\in X_{t+1}$ and $\bigcup_{1\leqq i<k}X_i=X-\{y\}$. Write

Continue!

$$g=\sum u_{X_1\ldots X_{k-1}}h\Delta(X_1)\ldots\Delta(X_{k-1})$$

with $u_{X_1\ldots X_{k-1}}\in\mathbf{Z}[X-\{y\}]$.

división

Consider the polynomial

$$g_i=\sum u_{X_1\ldots X_{k-1}}(y-x_{j+1})\ldots(y-x_{i-1})\Delta(X_1)\ldots\Delta(\{y\}\cup X_{t+1}-\{x_i\})\ldots\Delta(X_{k-1}).$$

It is clear that $g_i\in J$ and $g_i/\{x_i,y\}=g$. Thus $y-x_i$ divides $f_{i-1}-g_i$. Further, $(y-x_j)\ldots(y-x_{i-1})$ also divides $f_{i-1}-g_i$ by the assumption (b) on f_{i-1} and the construction of g_i. Putting $f_i=f_{i-1}-g_i$, we are done. ∎

3. Consequences and Generalizations

There are relations among the generators $\Delta(P)$ of Theorem 1. Actually, all the $\Delta(P)$'s can be generated by those ones which have the lowest degree, as shown in the following

Proposition 3.1. *Let $A=\{x_1,\ldots,x_m\}$ and $B=\{x_{m+1},\ldots,x_{2m+1}\}$. Then*

$$\Delta(A)\cdot\Delta(B)=\sum_{x_b\in B}(-1)^{b+1}\Delta(A\cup\{x_b\})\cdot\Delta(B-\{x_b\}).$$

Consequently, if $C=\{x_{2m+2},\ldots,x_r\}$, then $\Delta(A)\cdot\Delta(B\cup C)=$

$$\sum_{x_b\in B}\Big[(-1)^{b+1}\Delta(A\cup\{x_b\})\cdot\Delta(C\cup B-\{x_b\})\cdot\sum_{x_c\in C}(x_b-x_c)\Big].$$

Proof. Put

$$F=\sum_{x_b\in B}(-1)^{b+1}\Delta(A\cup\{x_b\})\Delta(B-\{x_b\}).$$

It is obvious that $\Delta(A)$ divides F since it divides $\Delta(A\cup\{x_b\})$. Given $m+1\le i<j\le$ $2m$, we claim that x_i-x_j divides F. Indeed, if $b\neq i$ or j, then x_i-x_j divides $\Delta(B-\{x_b\})$; and the sum of the remaining two terms in F

$$(-1)^{i+1}\Delta(A\cup\{x_i\})\Delta(B-\{x_i\})+(-1)^{j+1}\Delta(A\cup\{x_j\})\Delta(B-\{x_j\})$$

is equal to zero (by the definition of Δ) when we set $x_i=x_j$. Thus F is a polynomial divisible by $\Delta(A)\Delta(B)$. Since the degree of F is at most equal to the degree of $\Delta(A)\Delta(B)$, by comparing the coefficients of both polynomials, we see that $F=\Delta(A)\Delta(B)$. ∎

Note that the $\Delta(P)$'s of the lowest degree correspond to the partitions P of the set $\{1,\dots,n\}$ into $k-1$ subsets of as nearly equal sizes as possible. So combining Theorem 1 and Proposition 3.1 together, we have

Corollary 3.2. *The ideal $I(k,n)$ is generated by the polynomials $\Delta(P)$, where P is a partition of $\{1,\dots,n\}$ into $k-1$ subsets of as nearly equal cardinality as possible.*

The theorem below is a generalization of Theorem 1 mentioned in the Introduction.

Theorem 3. *Let Φ be a homogeneous polynomial in $\mathbf{Z}[x_1,\dots,x_n]$ which factors completely into products of the type x_i-x_j. Assume that Φ satisfies the condition*

if x_j-x_m divides Φ, so does x_i-x_m for every $i<j$.

Then the ideal $I(k,n)\cap\langle\Phi\rangle$ is generated by $\Delta(P)\vee\Phi$, the least common multiple of $\Delta(P)$ and Φ, where $\Delta(P)$ are as in Theorem 1.

The proof of Theorem 3 is similar to the proof of Theorem 1 in spirit but involves a lot more technicalities. This as well as the geometric meaning of Theorem 3 will be given in [2].

4. Applications to Graph Theory

Before discussing the applications of Theorems 1 and 3 to graph theory, we first give a general philosophy on translating problems in graph theory into problems on ideal generators.

All the graphs on the same set of n vertices form a lattice L_n under inclusion. This lattice is isomorphic to the Boolean algebra of the subsets of an $\binom{n}{2}$-element set. A collection $\mathscr{U}$ of graphs is called an *upper ideal* in L_n if whenever a graph G contains a subgraph belonging to $\mathscr{U}$, then G itself belongs to $\mathscr{U}$. Denote by $I_{\mathscr{U}}$ the ideal in $R[x_1,\dots,x_n]$ generated by the associated polynomials f_G, $G\in\mathscr{U}$. Many graph-theoretic problems are concerned with finding the smallest number of edges among all the graphs belonging to an upper ideal $\mathscr{U}$ and also determining all the graphs with this minimum number of edges. In terms of polynomials, this is equivalent to finding the minimum degree among all non-zero polynomials in $I_{\mathscr{U}}$. For this purpose, it suffices to find a set of homogeneous polynomials generating the ideal $I_{\mathscr{U}}$ such that the minimum degree among them is computable.

?sí?
¿y por qué $k-1$?

Lindísimo

The applications we shall see below are examples of this philosophy.

Given a graph G, let $c(G)$ and $\bar{c}(G)$ denote the *clique number* and the *independence number* of G, respectively. The complementary graph $\bar{G}$ of G is the graph on the same set of vertices such that two vertices are adjacent in $\bar{G}$ if and only if they are not adjacent in G. Thus $\bar{c}(G)=c(\bar{G})$. Let K_n denote the complete graph on n vertices.

Fix an integer k with $1<k\leqq n$. Write $n=q(k-1)+r$, where $0\leqq r<k-1$. We know from Corollary 3.2 that every nonzero polynomial in the ideal $I(k,n)$ must have degree $\geqq(k-1)\binom{q}{2}+rq$, which is the common degree of the generators $\Delta(P)$ of $I(k,n)$ described in that corollary. Considering the complementary graphs of the graphs associated with these $\Delta(P)$'s and using the criterion (1.1), we see that a graph with clique number less than k can have at most

$$\binom{n}{2}-(k-1)\binom{q}{2}-rq=\frac{k-2}{2(k-1)}(n^2-r^2)+\binom{r}{2}$$

edges. This is a new proof of a well-known theorem of Turán [3].

Let t be a positive integer. A graph G is said to be *t-partite* (resp. *complete t-partite*) if there is a way of partitioning the vertices into t disjoint subsets $V_1, \ldots, V_t$ such that G is contained in (resp. is) the complement of the t complete graphs on the set of vertices in V_i, $1\leqq i\leqq t$. We have

Corollary 4.1. (Turán) *Given an integer k with $1<k\leqq n$, a graph on n vertices with clique number less than k has at most*

$$\frac{k-2}{2(k-1)}(n^2-r^2)+\binom{r}{2}$$

edges, where $0\leqq r<k-1$ and $r\equiv n \pmod{k-1}$. Moreover, the graph achieving this bound is unique (up to isomorphism); it is a complete $(k-1)$-partite graph.

Proof. It remains to prove the second assertion. Let G be a graph with clique number less than k that has the described maximum number of edges. Thus $f_{\bar{G}}$ is a polynomial in $I(k,n)$ with the minimum possible degree. We want to show that $f_{\bar{G}}$ is one of the $\Delta(P)$'s of Corollary 3.2. For this, it suffices to prove that there is a partition $P'=\{P_1,\ldots\}$ of $\{1,\ldots,n\}$ such that $f_{\bar{G}}=\Delta(P')$, because then there are at most $k-1$ nonempty sets P_i in P' (since $f_{\bar{G}}\in I(k,n)$) and consequently, P' is the desired partition due to the minimality of degree $\Delta(P')$.

Let i be a vertex and G_i be a maximum complete subgraph of $\bar{G}$ containing the vertex i. Let V_i be the set of vertices in G_i. Suppose that j is a vertex outside V_i which is adjacent to say i. Then $f_{\bar{G}}$ is in $I(k,n)\cap\langle\Phi\rangle$, where $\Phi=(x_i-x_j)\Delta(V_i)$. Applying Theorem 3, we see that $f_{\bar{G}}$ is generated by $\Delta(P)\vee\Phi$. Since the degree of $f_{\bar{G}}$ is equal to the minimum degree among all $\Delta(P)$, it follows that $f_{\bar{G}}$ is a linear combination of those $\Delta(P)$ divisible by Φ. Each P being a partition, this means that every $\Delta(P)$, and hence $f_{\bar{G}}$, is divisible by $\Delta(V_i\cup\{j\})$. This contradicts the maximality of G_i. Therefore G_i is unique. Now letting P' consist of the distinct V_i's, we have $f_{\bar{G}}=\Delta(P')$, as desired. ∎

Corollary 4.2. *There exists a graph with n vertices and e edges which has the clique number c if and only if*

$$\binom{c}{2} \leqq e \leqq \frac{c-1}{2c}(n^2-r^2)+\binom{r}{2},$$

where $0 \leqq r < c$ *and* $r \equiv n \pmod{c}$.

Proof. The necessity of the first inequality is obvious. The second inequality follows from Corollary 4.1. On the other hand, given the vertex number n, the edge number e and the clique number c satisfying these inequalities, the construction of a graph with these parameters is straightforward. ∎

In view of Turán's theorem, we shall say that a graph G has the *Turán property* if for every integer k, $1<k\leqq n$, there is, among all K_k-free subgraphs of G, a $(k-1)$-partite subgraph which has maximum number of edges. Thus Corollary 4.1 says that the complete graph K_n has the Turán property. There are graphs, for example, the *pentagon*, which do not have this property. The following theorem gives a wide class of graphs which have the Turán property.

Theorem 4. *Let G be a graph on n vertices, labeled as* 1, ..., *n, satisfying the condition*

(4.3) *If a vertex j is adjacent to a vertex m, so is every vertex i with i>j and i≠m.*

Then G has the Turán property.

Note that if a graph satisfies (4.3), then the complementary graph also satisfies the same condition but with reversed labeling on the vertices.

Proof. Write $\Phi=f_{\tilde{G}}$. Then the condition (4.3) on G in the theorem is equivalent to the condition (3.3) stated in Theorem 3. Fix an integer $1<k\leqq n$. A graph H on the same n vertices is a subgraph of G with clique number $c(H)<k$ if and only if

$$f_{\tilde{H}} \in I(k, n) \cap \langle \varphi \rangle.$$

Hence the largest possible number of edges in such a graph H is $\binom{n}{2}$ minus the minimal degree of nonzero polynomials in the ideal $I(k, n)\cap\langle\Phi\rangle$. From Theorem 3, we see that $I(k, n)\cap\langle\Phi\rangle$ is generated by those $f_{\tilde{H}}$, where $\tilde{H}$ is the union of $\tilde{G}$ with $k-1$ disjoint complete subgraphs of K_n. ∎

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