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On Approximation Behavior of the Greedy Triangulation for Convex Polygons

Christos Levcopoulos[1] and Andrzej Lingas[1]

Abstract. We prove that the greedy triangulation heuristic for minimum weight triangulation of convex polygons yields solutions within a constant factor from the optimum. For interesting classes of convex polygons, we derive small upper bounds on the constant approximation factor. Our results contrast with Kirkpatrick's $\Omega(n)$ bound on the approximation factor of the Delaunay triangulation heuristic for minimum weight triangulation of convex n-vertex polygons. On the other hand, we present a straightforward implementation of the greedy triangulation heuristic for an n-vertex convex point set or a convex polygon taking $O(n^2)$ time and $O(n)$ space. To derive the latter result, we show that given a convex polygon P, one can find for all vertices v of P a shortest diagonal of P incident to v in linear time. Finally, we observe that the greedy triangulation for convex polygons having so-called semicircular property can be constructed in time $O(n \log n)$.

Key Words. Triangulation, Polygon, Approximation heuristic, Time and space complexity.

1. Introduction and Preliminaries. Given a planar point set S, a *diagonal* of S is a straight-line segment whose endpoints are in S and whose inside does not contain any point in S. A *triangulation* of S is a maximal set of nonintersecting diagonals of S. For a set T of straight-line segments in the plane, the term $|T|$ denotes the total length of the segments in T. A *minimum weight triangulation* (MWT for short) of S is any triangulation T of S which achieves the smallest possible value of $|T|$. The length of an MWT of S will be denoted by $\mathrm{M}(S)$.

The MWT problem was raised in numerical analysis many years ago but its complexity status has not yet been resolved [13], [9]. There exist two known heuristics for MWT, the *greedy triangulation heuristic* and the *Delaunay triangulation heuristic*. The former inserts a diagonal of S into the plane if it is the smallest among all diagonals of S neither intersecting nor overlapping with those already in the plane. The latter constructs the dual of the Voronoi diagram of S and completes it to a full triangulation of S. Let $\mathrm{GT}(S)$ and $\mathrm{DT}(S)$ denote the outcome of the greedy heuristic and the Delaunay heuristic run on S, respectively. For arbitrarily large n, Manacher and Zobrist [10] construct sets of n points in the plane, S', S'', such that

$$|\mathrm{GT}(S')|/\mathrm{M}(S') = \Omega(n^{1/3}),$$
$$|\mathrm{DT}(S'')|/\mathrm{M}(S'') = \Omega(n/\log n).$$

[1] The Department of Computer and Information Science, Linköping University, 581 83 Linköping, Sweden.

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Kirkpatrick [3] strengthens the latter result by exhibiting n-point planar sets S''' for which

$$|\mathrm{DT}(S''')|/\mathrm{M}(S''') = \Omega(n).$$

Interestingly, the sets S''' are *convex*, i.e., the points in S''' lie on their *convex hull* and no three of them are collinear. Lingas [8] shows that an analogous result cannot hold for the greedy heuristic in the convex case by constructing a positive real $\varepsilon < 1$ such that for any convex planar point set P with n vertices,

$$|\mathrm{GT}(P)|/\mathrm{M}(P) = O(n^{\varepsilon}).$$

holds.

By a *convex polygon* we shall mean a (simple) polygon P such that all vertices of P lie on their convex hull and no three vertices of P are collinear. Observe that a triangulation of the set S of the vertices of a convex polygon consists of the edges of the convex hull of S and diagonals of S internal to the hull. Hence, a triangulation of a convex polygon P can be defined as a triangulation of the set of vertices of P minus the set of edges of P. More generally, a triangulation of a nonnecessarily convex polygon can be defined as a maximal set of properly nonintersecting internal diagonals of the polygon. Via the above definition, the notions of MWT, $\mathrm{M}(\cdot)$, the greedy triangulation heuristic, $\mathrm{GT}(\cdot)$, the Delaunay triangulation heuristic, and $\mathrm{DT}(\cdot)$ can be easily generalized to include polygons or convex polygons, respectively.

In [7] it has been observed that the counterexample sets S' constructed by Manacher and Zobrist [10] induce nonconvex polygons P' such that

$$|\mathrm{GT}(P')|/\mathrm{M}(P') = \Omega(n^{1/3}).$$

It has been also noted in [7] that Kirkpatrick [3] has actually proved a stronger result which can be expressed as follows. The convex hulls P''' of the counter example Kirkpatrick's sets S''' are convex polygons such that

$$|\mathrm{DT}(P''')|/\mathrm{M}(P''') = \Omega(n).$$

In contrast, in [7] and [8] it has been conjectured that for any convex polygon P,

$$|\mathrm{GT}(P)|/\mathrm{M}(P) = O(1)$$

holds. The proof of this conjecture is the main result of our paper.[2] Our result supports the hypothesis of Manacher and Zobrist [11] stating that there is $\varepsilon < 1$ such that for any planar n-point set S, $|\mathrm{GT}(S)|$ is within an $O(n^{\varepsilon})$ factor of $\mathrm{M}(S)$. We believe that several lemmas derived by us to prove the $O(1)$ bound for convex polygons could also be helpful in proving the conjecture of Manacher and Zobrist. For this reason, some of the lemmas are proved for nonnecessarily convex polygons.

[2] The authors have proved this result independently at the same time. Recently, the result has been generalized by showing that for any polygon P with r reflex angles, $|\mathrm{GT}(P)|/\mathrm{M}(P) = O(r)$ [6].

We prove more refined results on the approximation behavior of the greedy triangulation heuristic for so-called *semicircular* polygons. Following [5], a polygon Q is semicircular (has the semicircle property, originally) if it is convex and satisfies the two following conditions:

(i) The two farthest vertices of Q are the endpoints of an edge (v_i, v_{i+1}) of Q.
(ii) All vertices of Q lie inside the circle whose diameter is equal to the length of the edge (v_i, v_{i+1}).

The longest edge of a semicircular polygon Q will be called the *base* of Q. Given a positive real number $q < 60$, a polygon Q is called *q-bent* if and only if Q is semicircular and the sum of degrees of the two interior angles of P at the endpoints of its base is not greater than $2 \times q$ degrees. We prove that for any q-bent polygon Q, where $q < 60$,

$$|\mathrm{GT}(Q)| \le \frac{1}{\cos(q) - 0.5} \times \mathrm{M}(Q)$$

holds. Thus, for example, if q is 5, then $|\mathrm{GT}(P)| \le 2.01 \times \mathrm{M}(P)$.

Our third result presents a straightforward, recursive implementation of the greedy heuristic for MWT of convex planar point sets or convex polygons running in time $O(n^2)$ and space $O(n)$. It improves Gilbert's simultaneous $O(n^2 \log n)$-time and $O(n^2)$-space bound on the implementation of the greedy heuristic for MWT of planar point sets [1] in the particular convex case. In our implementation, we employ Lee and Preparata's [5] linear-time algorithm for the all nearest-neighbors problem for convex polygons to solve the corresponding, all shortest-diagonals problem for convex polygons in linear time. We also show that the greedy heuristic for MWT of semicircular polygons can be implemented in time $O(n \log n)$.

The structure of the paper is as follows. In Section 2 we derive several properties of greedy and MWT triangulations of polygons and convex polygons, and the $O(1)$ bound on the approximation factor of the greedy heuristic for convex polygons. In Section 3 we derive the upper bound on the approximation ratio of the greedy heuristic for q-bent polygons. In Section 4 we present the solution to the shortest-diagonal problem for convex polygons and the straightforward implementations of the greedy heuristic for convex polygons and semicircular polygons.

2. The Greedy Triangulation for Convex Polygons Approximates the Optimum.

By the counterexample of Lloyd [9] to the greedy triangulation heuristic, it does not necessarily produce an MWT even when applied to a convex polygon. However, there are some similarities between a greedy triangulation and an MWT of a polygon or especially a convex polygon. The first similarity is *local optimality* in the sense of the following definition.

Definition 2.1. A triangulation of a nonnecessarily convex polygon P is said to be *locally optimal* if, for any two triangular faces (v_1, v_2, v_3) and (v_1, v_2, v_4) in $P \cup T$, the segment (v_3, v_4) is an internal diagonal of P, then the edge (v_1, v_2) of T is not longer than (v_3, v_4).

In the first lemma we look at locally optimal triangulation T of a polygon P with r reflex angles from the point of edges of T incident to a given vertex v of P. We show that the sequence of edges of T incident to v in clockwise order can be decomposed into $O(r)$ subsequences such that the lengths of edges of T in the subsequences monotonously increase or decrease at least as like in geometric progressions.

Lemma 2.1. *Let $P = (v_0, v_1, \ldots, v_{m-1})$ be a polygon with r reflex angles such that there is a locally optimal triangulation T of P composed of the segments (v_0, v_j), $j = 2, 3, \ldots, m-2$. Next, let D be a function defined on the set of vertices of P such that for a vertex w of P, D(w) is the distance between v_0 and w. We can decompose the sequence $D(v_1), D(v_2), \ldots, D(v_{m-1})$ into $O(r)$ disjoint subsequences of the form $S_1, \ldots, S_k$ such that either $\frac{3}{2}S_j \le S_{j+1}$ for $j = 1, \ldots, k-1$ or $S_j \ge \frac{3}{2}S_{j+1}$ for $j = 1, \ldots, k-1$.*

Proof. First, let us assume that P is convex. Partition the sequence $D(v_1), D(v_2), \ldots, D(v_{m-1})$ into maximal, continuous subsequences $D(v_{q_1}), \ldots, D(v_{q_r})$ such that for $j = 1, \ldots, r-1$ the angle $(v_{q_j}, v_0, v_{q_{j+1}})$ is of no more than 15°. It is easily seen that the number of the subsequences does not exceed $\lfloor 180/\alpha \rfloor$, where $\alpha > 15$, i.e., 11, and for $j = 1, \ldots, r-2$ the angle $(v_{q_j}, v_0, v_{q_{j+2}})$ is no more than 30°.

Let $1 \le j \le r-2$. First, suppose that $D(v_{q_j}) \le D(v_{q_{j+2}})$. Since $v_{q_{j+1}}$ is not inside the triangle $(v_{q_j}, v_0, v_{q_{j+2}})$ and the angle $(v_{q_j}, v_0, v_{q_{j+2}})$ is no more than 30° we have $D(v_{q_j}) \le D(v_{q_{j+1}})/\cos 15$. By the properties of T, we also have $|(v_{q_{j+2}}, v_{q_j})| \ge D(v_{q_{j+1}})$. It follows that $\frac{3}{2}D(v_{q_j}) \le D(v_{q_{j+2}})$ since otherwise the angle $(v_{q_j}, v_0, v_{q_{j+2}})$ is more than 30°. Analogously, if $D(v_{q_j}) > D(v_{q_{j+2}})$, we can conclude that $D(v_{q_j}) \ge \frac{3}{2}D(v_{q_{j+2}})$. Suppose that j satisfies the following, breaking monotonous decrease, condition:

$$\text{(A)} \qquad 1 \le j \le r-4, \qquad D(v_{q_j}) \ge \tfrac{3}{2}D(v_{q_{j+2}}) \quad \text{and} \quad D(v_{q_{j+4}}) \ge \tfrac{3}{2}D(v_{q_{j+2}}).$$

Then the angle $(v_{q_j}, v_0, v_{q_{j+4}})$ is more than 90° since $v_{q_{j+2}}$ is outside the triangle $(v_0, v_{q_j}, v_{q_{j+4}})$. Hence, there are no more than four odd j and four even j satisfying (A). In turn, suppose that j satisfies the following, breaking monotonous increase, symmetric condition:

$$\text{(B)} \qquad 1 \le j \le r-4, \qquad \tfrac{3}{2}D(v_{q_j}) \le D(v_{q_{j+2}}) \quad \text{and} \quad \tfrac{3}{2}D(v_{q_{j+4}}) \le D(v_{q_{j+2}}).$$

We shall show that then the angle $(v_{q_j}, v_{q_{j+2}}, v_{q_{j+4}})$ is no more than 60°. It is easily seen that $D(v_{q_j}) \le D(v_{q_{j+1}})$ and $D(v_{q_{j+3}}) \ge D(v_{q_{j+4}})$ since otherwise the angle $(v_{q_j}, v_{q_{j+1}}, v_{q_{j+2}})$ or the angle $(v_{q_{j+2}}, v_{q_{j+3}}, v_{q_{j+4}})$ would be more than 30°. Moreover, we have $|(v_{q_{j+2}}, v_{q_j})| \ge D(v_{q_{j+1}})$ and $|(v_{q_{j+2}}, v_{q_{j+4}})| \ge D(v_{q_{j+3}})$ by the definition of the greedy triangulation. Hence, the angle $(v_{q_j}, v_{q_{j+2}}, v_0)$ is smallest among the angles of the triangle $(v_{q_j}, v_{q_{j+2}}, v_0)$ and the angle $(v_0, v_{q_{j+2}}, v_{q_{j+4}})$ is smallest among the angles of the triangle $(v_0, v_{q_{j+2}}, v_{q_{j+4}})$. Thus, the sum of the two angles is less than 30°+30°. Since any convex polygon can have at most two angles of less than 60°, we conclude that there are at most eight j satisfying (B). By the above two estimations on the number of j satisfying (A) and (B), there is a partition of $D(v_{q_1}), \ldots, D(v_{q_r})$ into $O(1)$ subsequences satisfying the requirements from the thesis of the lemma. This proves the lemma if P is convex. If P is nonconvex then we can trivially decompose it into at most $r+1$ convex polygons by using at most $2r$ edges of T. The decomposition induces a partition of $D(v_{q_1}), \ldots, D(v_{q_r})$ into at most $r+1$ continuous sequences each of which can be decomposed into $O(1)$ subsequences satisfying the thesis of the lemma by the convex case. This proves the lemma in the general case. □

The following corollary from Lemma 2.1 is quite obvious.

Corollary 2.1. *Let P be a polygon with r reflex angles. The total length of all edges of GT(P) incident to a vertex v of P is $O(r|e|)$ where e is a longest edge in GT(P) incident to v.*

Proof. The endpoints of the edges incident to v induce a polygon P' with at most r reflex angles. We may assume that $P' = \{v_0, v_1, \ldots, v_{m-1}\}$, where $v_0 = v$. By Lemma 2.1, the total length of the edges of GT(P) incident to v is $O(r \cdot \sum_{l=0}^{\infty} \cdot (\frac{2}{3})^l |e|)$ which is $O(r|e|)$. □

In the next key lemma, we show that if a diagonal d of a convex polygon P intersects an edge e of GT(P) then at least one endpoint of e is within an $O(|d|)$ distance from d.

Lemma 2.2. *Let P be a convex polygon. Next, let e be an edge of GT(P) and let d be a diagonal of P such that d intersects e inside. Finally, let e′, e″ be the two segments resulting from intersecting e by d. We have $|e'| \le 3.5|d|$ or $|e''| \le 3.5|d|$.*

Proof. Let $G(d)$ be the set of all edges of GT(P) that properly intersect d. Next, let $e_1, e_2, \ldots, e_m$ be the sequence of all edges in $G(d)$, in the order they are inserted in the plane.

Case 1. The edges in $G(d)$ share a common endpoint on one side of d. By the definition of GT(P), the shortest edge in $G(d)$ is of length not greater than $|d|$. Hence, all pieces of the edges in $G(d)$ on the side of the common endpoint are of length not greater than $2|d|$ by triangle inequalities.

Case 2. There are j, k, l such that $j > k > l$, the edge e_j lies between the edges e_k and e_l, and each of the endpoints of e_j can be connected by a diagonal with each of the endpoints of e_k and e_l without intersecting any of the edges $e_1, e_2, \ldots, e_{j-1}$. It follows in particular that the edge e_j does not share an endpoint with e_k or e_l. Consider j, k, and l satisfying the above condition, where j is as large as possible. Let Q be the quadrilateral whose vertices are the four endpoints of e_k and e_l. Each of the diagonals of Q and each of the two sides of Q not intersecting d is of length not less than $|e_j|$ by the definition of GT(P). Let g be an edge of Q not intersected by d such that the straight lines induced by the edges of Q incident to g intersect in the half-plane induced by g and not containing Q or they are parallel. By the relationships betweeen the lengths of diagonals and edges of Q, none of the diagonals of Q together with g form an angle of more than 60° within Q. If d and g are not parallel, let ζ denote the angle whose arms include d and g, respectively. We may assume without loss of generality that ζ lies on the e_k's side of Q (the other possibility would be the e_l's side of Q). Clearly, the angle ζ is of no more degrees than any of the angles between a diagonal of Q and g within Q. Therefore, it is of no more than 60°. Draw the straight line L parallel to e_j and passing through the endpoint of e_k incident to g. By the definition of g, d intersects L. Since it also intersects e_j, it is of length at least $\cos\zeta|g|$. This implies $|d| \ge \frac{1}{2}|e_j|$. Hence, each point in the set of endpoints of the edges e_1 through e_j is in the distance of at most $2|d|$ from d.

Let EN be the set of endpoints of edges in $G(d)$ that are in the distance of at most $2.5|d|$ from d. By triangle inequality, the lemma holds for every edge in $G(d)$ that has an endpoint in EN. Thus, in particular, the lemma holds for the edges e_1 through e_j.

Consider an edge e_l that has no endpoint in EN where l is as small as possible. Delete the edges e_{l+1} through e_m. Since $l > j$, the edge e_l cannot be between two edges in the remainder of $G(d)$ without sharing any endpoint with them.

First, suppose that e_l lies between a couple of edges in the remainder sharing with exactly one of them an endpoint, say q. We shall show that such a configuration is impossible. Consider the quadrilateral R formed by the endpoints of the couple of edges. Each diagonal of R is of length not less than $|e_l|$ by the definiton of GT(P). For the same reason, the edge f of R in the half-plane induced by d and not containing q is of length not less than $|e_l|$. Note that $|e_l| > 5|d|$. If the straight lines induced by the couple of edges intersect in the half-plane induced by d and containing f or they are parallel then we can prove $|d| \ge \frac{1}{2}|f|$ by repeating the proof of the inequality $|d| \ge \frac{1}{2}|g|$ with f, R substituted for g, Q, respectively. Since $|d| \ge \frac{1}{2}|f|$ contradicts $|f| > 5|d|$, we may assume without loss of generality that the straight lines induced by the couple of edges intersect in the half-plane induced by d and containing q. Consider the triangle t formed by these straight lines together with f. Let γ be the angle of t between f and the remaining edge of t not including q (see Figure 2.1). First, we shall prove that $|f| \le 9.5|d|$ and the angle γ is no less than $\arctan(2.5/(9.5-\sqrt{5^2-(2.5)^2}))$. Remember that by the definition of l, each edge in the couple has at least one endpoint in EN. Suppose that both endpoints of the couple of edges in the half-plane induced by d and containing f are in EN. Then, we have $|f| \le (2\cdot 2.5+1)|d|$ by triangle inequalities. To estimate the angle γ in this subcase, note that it is greater than the angle β between the diagonal h of R crossing it and f (see Figure 2.1). Further, let e_i denote the edge in the couple incident to q and let x denote the length of the part of e_i between q and d. We have $|h| \le 2.5|d| + |d| + x$

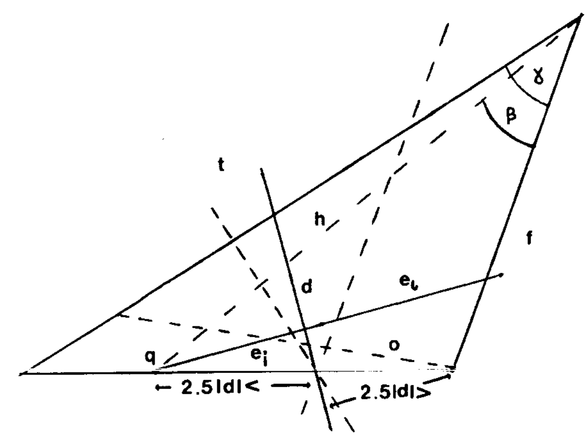

Fig. 2.1. An example of the placement of the diagonal d and the triangle t.

by triangle inequalities. Since $x > 2.5|d|$, $|e_i| > x + \text{abs}(5|d| - x)$, $|f| > 5|d|$, the angle β is no less than $\arccos(\frac{3}{5})$ by a straightforward trigonometric argumentation. Note that $\arccos(\frac{3}{5}) > \arctan(2.5/(9.5-\sqrt{5^2-(2.5)^2}))$ holds.

Since q is not in EN, the subcase where the edges in the couple have endpoints in EN on different sides of d have to be considered. Then, the diagonal o of R between such a pair of endpoints is of length not greater than $6|d|$ by triangle inequalities. Since the diagonal intersects e_l, we have $|e_l| \le 6|d|$. (Here, one can notice that the lemma holds for e_l. However, this does not guarantee that the lemma holds for edges of $G(d)$ inserted after e_l between the couple of edges. Recall that we want to prove that e_l cannot share only one endpoint with one of the edges in the couple.) By $|e_l| \le 6|d|$, each of the edges in the couple is of length not greater than $6|d|$. On the other hand, the endpoint of f incident to e_i is in EN by the definition of l. Putting everything together, we conclude that f is of length not greater than $(6+1+2.5)|d|$ by triangle inequalities. We estimate the angle γ in this subcase as follows. Let e_k be the edge in the couple not incident to q. Note that the perpendicular projection of the endpoint of e_k not incident to f on the straight line induced by f lies inside f by $|e_k| > 5|d|$, $|f| > 5|d|$ and $o \le 6|d|$. Further, since the diagonal o and the edge e_k have one endpoint in the distance greater than $2.5|d|$ from d, their common endpoint lies in the distance greater than $2.5|d|$ from f. Hence, the angle γ is no less than $\arctan(2.5/(9.5-\sqrt{5^2-(2.5)^2}))$ by the bounds on the length of edges and diagonals of R (see Figure 2.2).

Now, recall that the endpoint of e_i different from q is in EN. Hence, since q is not in EN, the edge d is of length not less than $[2.5/(\sqrt{(2.5)^2+1}+2.5)]|f|\sin(\gamma)$ (see Figure 2.1). We obtain a contradiction by $|f| > 5|d|$ and

$$\sin(\arctan(2.5/(9.5-\sqrt{5^2-(2.5)^2}))) > (\sqrt{(2.5)^2+1}+2.5)/12.5.$$

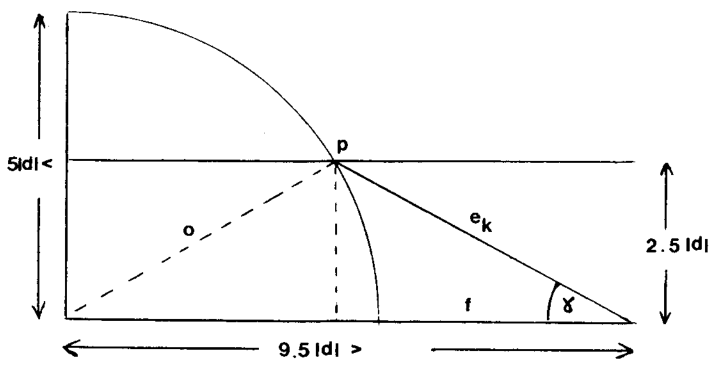

Fig. 2.2. The angle γ is smallest when the endpoint e_k not incident to f is located at the intersection point p.

This shows that e_l cannot share only one endpoint with one of the edges in the couple.

Next, consider the case where the edge e_l lies between a couple of edges in the remainder of $G(d)$ sharing with each of the edges a different endpoint. Then, the two remaining endpoints of the couple of edges are in EN by the definition of l. Hence, the diagonal connecting these two endpoints is of length less than $(2 \cdot 2.5+1)|d|$ by triangle inequalities. Since the above diagonal intersects e_l, it is no shorter than e_l by the definition of $G(d)$. Thus, we also have $|e_l|<6|d|$. In consequence, each of the endpoints of e_l is in the distance not greater than $(6-2.5)|d|$ from d. For this reason, the edge e_l and all edges of $G(d)$ between the couple of edges satisfy the lemma by triangle inequalities.

We proceed similarly in the case when the edge e_l lies between a couple of edges in the remainder of $G(d)$ with a common endpoint. Then, by the choice of l, the two other endpoints of these two edges are in EN. Therefore, they are in the distance of at most $(2 \cdot 2.5+1)|d|$ from each other. Hence, by the definition of GT(P), the edge e_l is of length not greater than $6|d|$. In consequence, the common endpoint is in the distance not greater than $(6-2.5)|d|$ from d and all edges of $G(d)$ lying between these couple of edges satisfy the lemma.

Finally, suppose that e_l lies between an edge in the remainder of $G(d)$ and an endpoint q of d. By the definition of l, at least one endpoint, say p, of the edge is in EN and in consequence is within the distance not greater than $(2.5+1)|d|$ from q. Naturally, the edge e_l intersects the diagonal (p, q) and in consequence it is of length not greater than $3.5|d|$ by the definition of GT(P). Thus, at least one of the endpoints of e_l is in EN. We obtain a contradiction. Summarizing the above four possible cases of the location of e_l with respect to the remainder of $G(d)$, we conclude that e_l can only lie between a couple of edges in the remainder, sharing with each of them either a different endpoint or the same endpoint. We can iterate our proof by deleting e_l and all other edges

between the couple of edges from the remainder of $G(d)$ since, as has been proved, they satisfy the lemma any way.

Case 3. Neither Case 1 nor Case 2 holds. We proceed analogously as in Case 2 under the convention that $j=0$. The set of edges of GT(P) with endpoints in EN is nonempty since at least one edge of GT(P) not longer than d intersects d. □

For convenience, MT(P) will denote a given, arbitrary MWT of a polygon P further. In the context of tracing similarities between MT(P) and GT(P), it would be interesting to know whether the statement resulting from substituting MT(P) for GT(P) in Lemma 2.2 holds. However, for our purposes, it will be sufficient to prove the following, slightly weaker statement.

LEMMA 2.3. *Given a convex polygon P, let e be a diagonal of P and let d be an edge of* MT(P) *intersected by e. Next, let d′, d″ be the two segments resulting from intersecting d by e. We have* $|d'|=O(|e|)$ or $|d''|=O(|e|)$.

PROOF. For the purpose of the proof, we introduce the following notation. Given a vertex v of P, its Euclidean distance from the edge e is denoted by $d(v)$. Next, the set of edges of MT(P) intersecting e is denoted by E. Given an edge d in E, by a neighbor of d, we mean another edge of MT(P) incident to d that is either in E or is incident to e and no other edge in E lies between it and e_1.

Let M be the set of all edges in E that have at least one endpoint in the distance not greater than $5|e|$ from e. Next, let w_{max} be a vertex of P incident to an edge in $E-M$ that maximizes the distance to e. Suppose that $d(w_{max})$ is greater than $6|e|$. We derive a contradiction as follows.

Case 1. w_{max} is incident to only one edge in E. Let w_1 be the other endpoint of the edge. Since the edge (w_{max}, w_1) is not in M, we have $d(w_1)>5|e|$. Next, let (w_1, w_2) and (w_1, w_3) be the two neighbors of (w_{max}, w_1) incident to w_1. If the edge (w_1, w_2) is not in M then we have $d(w_2) \le d(w_{max})$. Otherwise, since (w_{max}, w_1) is not in M, we have $d(w_2) \le 5|e|$. Thus, in both cases, we have $d(w_2) \le d(w_{max})$. Analogously, we derive the inequality $d(w_3) \le d(w_{max})$. Note that the angle (w_2, w_1, w_3) intersected by e is greatest when $d(w_1)$ tends to $5|e|$ and the perpendicular projection of w_1 on the straight line induced by e lies in the middle point of e. Therefore, the angle (w_2, w_1, w_3) is not greater than $2\arctan(0.1)$. Now, to estimate the length of (w_2, w_3), draw the half-line extensions H_2, H_3 of (w_1, w_2) and (w_1, w_3), respectively, starting at w_1. Next, draw the straight lines L_2, L_3 passing through w_2 or w_3, respectively, such that for $i=2, 3$, L_i intersects H_2 and H_3 in the same distance from w_1. Finally, draw the straight line L_1 parallel to L_2 and L_3, passing through the point of intersection between e and H_2 or H_3 closest to w_1. By the symmetry between w_2 and w_3, we may assume without loss of generality that L_2 lies between L_3 and L_1. Let s_2, s_3 denote the segment of H_3 between L_1 and L_2, and between L_2 and L_3, respectively. Next, let s_1 be the segment of L_1 between H_2 and H_3. By triangle inequalities and the fact that the angle (w_2, w_1, w_3) is, in particular, of no more than 60°, it is easy to see that the inequality $|(w_2, w_3)|<|s_1|+|s_2|+|s_3|$ holds. Clearly, by the definition of L_1, we have $|s_1|<|e|$. Further, if the edge e does not intersect s_2 then we have $|s_2|+|s_3|<d(w_{max})+|e|$ by $d(w_3)\le d(w_{max})$. Otherwise, we have to add the length of the segment of s_2 between L_1 and e to the above upper bound. It is easy to see that the latter segment is shorter than e by the definition of L_1. Putting everything together, we obtain $|(w_2, w_3)|\le d(w_{max})+3|e|$. By $|(w_{max}, w_1)|\ge d(w_{max})+d(w_1)$ and $d(w_1)>5|e|$, we obtain a contradiction with the optimality of MT(P).

Case 2. w_{max} is incident to two edges in E. Let w_1 and w_2 be the other endpoints of these edges. By the definition of w_{max}, at least one of the edges is not in M. Hence, we may assume without loss of generality that the edge (w_{max}, w_2) is not in M, $d(w_1)\le d(w_2)$ and $d(w_2)>5|e|$. Let (w_2, w_3) be the edge neighboring (w_{max}, w_2) incident to w_2. If we had $d(w_3)<d(w_{max})-|e|$ then the diagonal (w_1, w_3) would be of length less than $d(w_1)+d(w_{max})$ by triangle inequalities. Since we have $|(w_{max}, w_2)|\ge d(w_{max})+d(w_2)$ and $d(w_2)\ge d(w_1)$, the above diagonal would be shorter than (w_{max}, w_2) which contradicts the optimality of MT(P). We conclude that $d(w_3)>d(w_{max})-|e|$. It follows in particular that w_3 is not an endpoint of e. Let e_1 be the edge neighboring (w_2, w_3) different from (w_{max}, w_2).

Subcase A. The edge e_1 is of the form (w_2, w_4). Note that $d(w_4)\le d(w_{max})$ by $d(w_2)>5|e|$. Arguing as in Case 1 (with w_{max}, w_3, w_4 substituted for w_2, w_{max}, w_3, respectively), we show that the diagonal (w_{max}, w_4) is of length less than $d(w_{max})+3|e|$. Since the edge (w_2, w_3) is of length greater than $d(w_{max})-|e|+d(w_2)$ and $d(w_2)>5|e|$, we obtain a contradiction.

Subcase B. The edge e_1 is of the form (w_3, w_4). Recall that $d(w_3)>d(w_{max})-|e|$. Consequently, if the edge (w_3, w_4) is in M, we have $d(w_4)\le 5|e|$. Thus, we have $d(w_4)\le d(w_{max})$ independently of whether the edge (w_3, w_4) is in M or not. Naturally, we also have $d(w_1)\le d(w_{max})$ by our starting assumptions. Since, in particular, $d(w_3)>5|e|$, holds, the angle (w_2, w_3, w_4) intersected by e is less than 2 arctan$(\frac{1}{10})$ by an argumentation analogous to that in Case 1 (for the angle (w_2, w_{max}, w_3)). Similarly, the angle (w_1, w_{max}, w_2) intersected by e is less than 2 arctan$(\frac{1}{10})$. Thus, if the extensions of (w_1, w_{max}) and (w_3, w_4) intersect on the side of e where w_{max} lies then the angle formed by the extensions and intersected by e is less than 4 arctan$(\frac{1}{10})$ (which is less than 60°). Therefore, we have $(w_1, w_4)\le d(w_{max})+3|e|$ by an argumentation analogous to that in Case 1 (for the diagonal (w_2, w_3)). On the other hand, the diagonal (w_1, w_3) is of length not greater than $d(w_1)+d(w_{max})+|e|$ by triangle inequalities. Transform MT(P) as follows. Remove the edges (w_2, w_3), (w_{max}, w_2) and insert the diagonals (w_1, w_3), (w_1, w_4). The length of the resulting triangulation is not greater than

$$|\mathrm{MT}(P)|-(d(w_3)+d(w_2))-(d(w_{max})+d(w_2)) \\ +(d(w_{max})+3|e|)+(d(w_1)+d(w_{max})+|e|).$$

By $d(w_3)>d(w_{max})-|e|$, $d(w_2)\ge d(w_1)$ and $d(w_2)>5|e|$, we obtain a contradiction.

Case 3. The vertex w_{max} is incident to at least three edges in E. Let p and q be, respectively, the other endpoint of the lowest edge e_l and the highest e_h incident to w_{max}. By the definition of w_{max}, the endpoints p and q, even if e_l or e_h is in M, are in the distance not greater than $d(w_{max})$ from e. Hence, the diagonal (p, q) is of length less than $d(w_{max})+3|e|$ by an argumentation analogous to that in Case 1 (for the diagonal (w_2, w_3)). We may assume without loss of generality that the straight line L parallel to e and passing through p intersects e_h. Let s denote the segment of L between e_l and e_h. By the definition of w_{max}, we obtain $|e|/d(w_{max})\ge|s|/|e_l|$. On the other hand, we have $|e_l|<2d(w_{max})+|e|$ by triangle inequalities. Combining the two inequalities with $d(w_{max})>5|e|$, we conclude that $2.2|e|>|s|$. Now, transform MT(P) as follows. Move the left endpoints of the edges f incident to w_{max} and different from e_l and e_h from w_{max} to p. Note that the length of the edge resulting from f is less than $(|f|-d(w_{max}))+|s|$ by the definitions of w_{max}, s, and triangle inequality. This moving of endpoints cancels the next to lowest edge incident to w_{max} and intersecting e, decreasing the number of edges by one. Draw the edge (p, q). Let k be the number of edges in E incident to w_{max}. Recall that $|(p, q)|<d(w_{max})+3|e|$. Putting everything together, we conclude that the resulting triangulation of P is of length not greater than $|\mathrm{MT}(P)|-d(w_{max})+(d(w_{max})+3|e|)+(k-2)(2.2|e|-d(w_{max}))$. We obtain a contradiction by $k\ge 3$ and $d(w_{max})>6|e|$. □

The next, technical lemma provides an upper bound on the number of edges in GT(P) with lengths within given bounds, intersecting a given segment of a diagonal of the convex polygon P.

LEMMA 2.4. *Let P be a convex polygon and let k be a natural number. For all segments s of diagonals of P, and all positive reals $o\le k|s|$, there are $O(k|s|/o)$ edges e in* GT(P) *such that $o\le|e|\le 2o$ and e properly intersects s.*

PROOF. Consider the straight-line graph $G(s, o)$ induced by the edges of GT(P) that are of length not less than o and not greater than $2o$ and properly intersect s. By Lemma 2.1, for any vertex v of P, there are at most $O(1)$ edges in $G(s, o)$ incident to v. Therefore, it is sufficient to show that the number of edges in a maximum cardinality *matching* [2] MA of $G(s, o)$ is not greater than $c_2k|s|/o$ for some constant c_2. Suppose otherwise. Assume that s is placed horizontally. By the convexity of P and $o\le k|s|$, the total sum of absolute values of the differences between the X- and Y-coordinates, respectively, of the upper endpoints of consecutive edges of MA in the order from the left to the right is not greater than $(2k+1)|s|$. The analogous estimation holds for the bottom endpoints. Hence, by a straightforward geometric-counting argument, there are three edges (f_1, f_2), (g_1, g_2), and (h_1, h_2) in MA such that (g_1, g_2) lies between (f_1, f_2) and (h_1, h_2) and it holds abs$(X(f_i)-X(h_i))=O(c_2^{-1}o)$, abs$(Y(f_i)-Y(h_i))=O(c_2^{-1}o)$, where for a point p, $X(p)$ and $Y(p)$ denote its X- and Y-coordinate, respectively. We may assume without loss of generality that c_2 is small enough to make the values of abs$(X(f_i)-X(h_i))$ and abs$(Y(f_i)-Y(h_i))$ less than $o/16$. Then, the segments (f_1, h_1) and (f_2, h_2) are of length less than $o/8$. Since these two segments are not in GT(P), there are shorter edges e' in GT(P), intersecting them. However, such an edge e' has to intersect or touch the two segments simultaneously and therefore it is of length no less than the minimum distance between these segments. On the other hand, the distance between (f_1, h_1) and (f_2, h_2) cannot be less than $|(f_1, f_2)| - |(f_1, h_1)| - |(f_2, h_2)|$. Thus, such an edge e' is no shorter than $o - 2 \cdot \frac{1}{8}o$. We obtain a contradiction. □

To derive more tight bounds on the number of edges in GT(P) of lengths within given bounds intersecting a given, longer diagonal of the convex polygon P, it will be convenient to assume the following definition.

DEFINITION 2.2. Given a convex polygon P, for all edges d of MT(P) and all natural i, let $G(i, d)$ be the set of all edges e of GT(P) such that $|d|/2^{i+1} < |e|$, $|e| < |d|/2^i$ and d properly intersects e. □

By combining Lemma 2.3 and 2.4, we obtain the following upper bound on the cardinality of $G(d, i)$.

LEMMA 2.5. *For all edges d of* MT(P) *and all natural i, the set $G(i, d)$ is of cardinality $O(1)$.*

PROOF. Let e be in $G(d, i)$. By Lemma 2.3, the edge e intersects d in the distance not greater than $c_1|e|$ from an endpoint of d, where c_1 is the constant from the thesis of Lemma 2.3. By Lemma 2.4, the number of edges of GT(P) that are of length greater that $2^{-\log c_1 - 1}$ and intersect d is $O(1)$. Therefore, we may assume without loss of generality that i satisfies the inequality $c_1/2^i < \frac{1}{2}$. Let $(v, u]$ be one of the two, disjoint initial fragments of d of length $c_1|d|/2^i$. Since we may assume without loss of generality that at least half of the edges in $G(d, i)$ intersects $(v, u]$, it is sufficient to observe that the number of the edges in $G(d, i)$ intersecting $(v, u]$ is $O(1)$ by Lemma 2.4. □

THEOREM 2.1. *For any convex polygon $P = (v_0, \ldots, v_m)$, it holds $|\mathrm{GT}(P)| = O(M(P))$.*

PROOF. For $i = 0, \ldots, m$, let e_i be the longest edge in GT(P) incident to v_i and let LGT(P) be the set of all these longest edges e_i. By Corollary 2.1, it is sufficient to show that $|\mathrm{LGT}(P)| = O(|\mathrm{MT}(P)|)$. We shall do it by assigning to each edge e_i in LGT(P) an edge in MT(P) and observing that for any edge d of MT(P), the set $A(d)$ of edges in LGT(P) to which d is assigned is of length $O(|d|)$. The assignment procedure for an edge e_i in LGT(P) is as follows.

Case 1. No edge in MT(P) intersects e_i. In other words, the edge e_i occurs in MT(P). We assign e_i to itself.

To define the remaining cases, we assume that $e_i = (v_1, v_2)$ and q is the middle point of e_i.

Case 2. There are two edges d_1, d_2 in MT(P) such that $(v_1, q) \cap d_1 \neq \emptyset$ and $[q, v_2) \cap d_2 \neq \emptyset$ or $(v_1, q] \cap d_1 \neq \emptyset$ and $(q, v_2) \cap d_2 \neq \emptyset$. We may assume without loss of generality that d_1 and d_2 minimize the distance between their intersections with e_i among the pairs of edges of MT(P) satisfying the above condition. It is easy to see that such d_1 and d_2 share an endpoint. Hence, the edge e_i is of length $O(|d_1| + |d_2|)$ by Lemma 2.2 and triangle inequality. It follows that if d is a longest edge in MT(P) that intersects e_i then $|e_i| = O(|d|)$. Therefore, we assign such an edge d to e_i in this case.

Case 3. Neither Case 1 nor Case 2 holds. Let d be an edge in MT(P) intersecting e_i in the minimum distance from q. Since Case 2 does not hold, either v_1 or v_2 together with the endpoints of d form a triangle that is a triangular face in $P \cup \mathrm{MT}(P)$ covering at least half of e_i.

Subcase A. We have $|d| \geq \frac{1}{3}|e_i|$. Here, a longest edge d' in MT(P) intersecting e_i is of length not less than $\frac{1}{3}|e_i|$. We assign such an edge d' to e_i.

Subcase B. We have $|d| < \frac{1}{3}|e_i|$. It follows from the properties of the triangle that at least one of its remaining edges not intersecting e_i is of length not less than $|e_i|/3$ and no more than $\frac{4}{3}|e_i|$. We assign such an edge of the triangle to e_i in this case.

We need show that for every edge d in MT(P), we have $|A(d)| = O(|d|)$. We shall show it by examining the above cases of an assignment of d to an edge in LGT(P).

(i) Case 1 yields at most one assignment of d charging d with $|d|$.
(ii) Suppose that d has been assigned to some edges e_i in LGT(P) in Case 2 or Case 3A. Thus, the edge d is a longest edge in MT(P) intersecting the edges e_i. Each of the edges e_i is of length not greater than $c|d|$, where c is a positive constant, by the definition of the assignment. First, let us consider the edges e_i of length greater than $|d|/2$. By applying Lemma 2.4 about $1 + \log_2 c$ times, we conclude that the number of such edges e_i is $O(1)$. Hence, they charge d with $O(|d|)$. In turn, let us consider the edges e_i of length not greater than $|d|/2$, i.e., the edges in $\bigcup_{j=1}^{\infty} G(d, j)$. By Lemma 2.5, they charge d with $O(\sum_{j=1}^{\infty} |d|/2^j)$ which is $O(|d|)$.
(iii) Finally, suppose that d has been assigned to some edges e_i in Case 3B (remember that d now denotes another edge than that in the assignment procedure). By the definition of the assignment, each of the edges e_i is incident to an endpoint of d and is of length no less than $\frac{3}{4}|d|$ and no greater than $3|d|$. Hence, their number is $O(1)$ by Lemma 2.1. Thus, they charge $|d|$ with $O(|d|)$. □

3. The Greedy Triangulation for Sharp Semicircular Polygons Closely Approximates the Optimum. Although we have proved in Section 2 that the greedy triangulation heuristic for convex polygons approximates the optimum, we have not derived any small constant upper bound on the approximation factor. Interestingly, we can derive such an upper bound for the special case of convex polygons which are q-bent polygons where q is not greater than, say, 50, using a quite different approach than in Section 2. Our new approach relies on the following fact:

FACT 3.1 (Lemma 2 in [5]). If v is a vertex of a semicircular polygon P then each nearest neighbor (see [13]) of v in the set of vertices of P is incident to one of the two edges of P incident to v.

The result of this section can be precisely stated as follows.

THEOREM 3.1. *For any q-bent polygon P, $0<q<60$, it holds that* $|\mathrm{GT}(P)| \le \mathrm{M}(P)/(\cos(q)-0.5)$.

PROOF. To start with, we need the following definitions. The horizontal distance between any two points, say A and B, denoted by $h(A, B)$ is the absolute value of the difference between their X-coordinates. Analogously, given a straight-line segment s, the horizontal length $h(s)$ of s is the horizontal distance between its endpoints. Consequently, given a set of straight-line segments S, the term $h(S)$ denotes the horizontal length of S, i.e., the total sum of the horizontal lengths of the segments in S. Finally, given a polygon P', the term $\mathrm{HMT}(P')$ denotes a triangulation of P' achieving the minimum horizontal length.

Let P be any q-bent polygon. We may assume without loss of generality an orientation of P such that every edge and diagonal of P has slope $\le \tan(q)$ and $\ge -\tan(q)$. (A way to achieve this is to turn P until the base of P lies below all other edges, and the leftmost edge of P which is adjacent to the base has slope $\tan(q)$.) By the assumed orientation of P, we have:

(i) If e' is an edge or a diagonal of P, then the inequalities $|e'| \times \cos(q) \le h(e') \le |e'|$ are satisfied.

Since $h(\mathrm{HMT}(P)) \le h(\mathrm{MT}(P))$ and $h(\mathrm{MT}(P)) \le \mathrm{M}(P)$, to prove the theorem it is sufficient to prove the following proposition:

PROPOSITION 3.1. *For any q-bounded polygon P, $0<q<60$, placed as above,* $|\mathrm{GT}(P)| \le h(\mathrm{HMT}(P))/(\cos(q)-0.5)$ *holds.*

PROOF. The proof is by induction on the number of vertices of P. If P has three vertices then the proposition trivially holds. Suppose that P has at least four vertices. Let e be the shortest edge in $\mathrm{GT}(P)$. By the definition of the greedy triangulation, we have:

(ii) e is a shortest diagonal of P.

Let P' and P'' be the subpolygons of P in the partition of P induced by e, such that the base of P is an edge of P'. There is at least one vertex v of P incident to e that is not incident to the base of P. Note that the convex hull of the set of vertices of P decreased by the vertices adjacent to v different from the endpoints of the base, is also semicircular. Hence, by (ii) and Fact 3.1, we conclude that P'' is a triangle. It follows that $|\mathrm{GT}(P)| = |e| + |\mathrm{GT}(P')|$. Clearly, the polygon P' is also q-bent, and its placement satisfies the requirement of Propositon 3.1. Hence, by the induction hypothesis, we have $|\mathrm{GT}(P')| \le h(\mathrm{HMT}(P'))/(\cos(q)-0.5)$. Consequently, we obtain

$$|\mathrm{GT}(P)| \le |e| + \frac{1}{\cos(q)-0.5} \times h(\mathrm{HMT}(P')).$$

Thus, to complete the induction step, it is sufficient to show the inequality

$$|e| + \frac{1}{\cos(q)-0.5} \times h(\mathrm{HMT}(P')) \le \frac{1}{\cos(q)-0.5} \times h(\mathrm{HMT}(P))$$

or the following inequality:

$$(*) \qquad h(\mathrm{HMT}(P')) \le h(\mathrm{HMT}(P)) - |e| \times (\cos(q)-0.5).$$

To prove the latter inequality, we consider the two following cases.

Case 1. e is in $\mathrm{HMT}(P)$. In this case, $\mathrm{HMT}(P)-\{e\}$ is a triangulation of P', and it has total horizontal length $h(\mathrm{HMT}(P)) - h(e)$. Hence, by (i), we obtain the inequality $h(\mathrm{HMT}(P')) \le h(\mathrm{HMT}(P)) - |e| \times \cos(q)$ which proves the inequality $(*)$ in this case.

Case 2. e is not in $\mathrm{HMT}(P)$. Let v_1, v_2, and v_3 be the vertices of P'' from the left to the right (thus $e=(v_1, v_3)$). Since (v_1, v_2) is not in $\mathrm{HMT}(P)$, there are one or more edges in $\mathrm{HMT}(P)$ which intersect (v_1, v_3) and, hence, they touch v_2. Let E be the set of all these edges. Combining (i) with the fact that e is a shortest diagonal of P, we conclude that:

(iii) For every edge e' in E, the inequalities $h(e) \le |e| \le |e'| \le h(e')/\cos(q)$ hold.

Let E_1, respectively E_2, be the set consisting of all edges in E whose right endpoint, respectively left endpoint, is v_2. Next, for $i=1,2$, let $\mathrm{card}(E_i)$ be the cardinality of E_i. We may assume without loss of generality that $\mathrm{card}(E_1) \ge \mathrm{card}(E_2)$ and if $\mathrm{card}(E_1) = \mathrm{card}(E_2)$ then $h(v_1, v_2) \le h(v_2, v_3)$. Let v_4 be the left endpoint of the shortest edge in E_1. Employing again Fact 3.1 in the known way, we conclude that the triangle (v_1, v_4, v_2) is in the partition induced by $\mathrm{HMT}(P)$. Next, let E' be the set of all diagonals of P whose one endpoint is v_1 and whose other endpoint is an endpoint of some edge in E disjoint from the vertices v_2 and v_4. Intuitively, E' can be obtained from E by deleting the edge (v_2, v_4) and by turning all other diagonals in E to the left, to touch v_1 instead of v_2. Let T be the set $(\mathrm{HMT}(P) - E) \cup E'$. It is easily seen that T is a triangulation of P'. To show the inequality $(*)$, it is sufficient to show the inequality $h(T) \le h(\mathrm{HMT}(P)) - |e| \times (\cos(q)-0.5)$, or, equivalently, the inequality $h(E') \le h(E) - |e| \times (\cos(q)-0.5)$. By the definitions of E, E_1, and E_2, we obtain the following equalities:

(iv) $h(E') = h(E) - h(v_2, v_4) - h(v_1, v_2) \times (\mathrm{card}(E_1)-1) + h(v_1, v_2) \times \mathrm{card}(E_2)$

$$= h(E) - h(v_2, v_4) - h(v_1, v_2) \times (\mathrm{card}(E_1) - \mathrm{card}(E_2) - 1).$$

In the context of the intuitive description of E', the term $h(v_2, v_4)$ in (iv) corresponds to the deletion of the edge (v_2, v_4) from E, the term $h(v_1, v_2)\times(\text{card}(E_1)-1)$ corresponds to the fact that the horizontal lengths of all other diagonals in E_1 become smaller during the turn, etc. If card(E_1) is greater than card(E_2), then we obtain the inequality $h(E')\le h(E)-h(v_2, v_4)$ by (iv), which combined with (iii) yields $h(E')\le h(E)-|e|\times\cos(q)$. Thus, the inequality (*) holds in this situation. It remains only to consider the situation where card(E_1) = card(E_2) and $h(v_1, v_2)\le h(v_2, v_3)$. Then, the equality $h(E')=h(E)-h(v_2, v_4)+h(v_1, v_2)$ holds by (iv). This yields the equality $h(E')=h(E)-h(v_1, v_4)$ by $h(v_2, v_4)=h(v_1, v_4)+h(v_1, v_2)$. Therefore, to prove the inequality (*), it is sufficient to show the inequality $h(v_4, v_1)\ge|e|\times(\cos(q)-0.5)$. By the definition of the considered situation, the following chain of inequalities holds: $h(v_1, v_2)\le h(v_1, v_3)/2\le h(e)/2\le|e|/2$. Combining these inequalities with (i), we obtain the inequality $|(v_1, v_2)|\le|e|/(2\times\cos(q))$. On the other hand, by triangle inequality, the inequality $|(v_2, v_4)|<|(v_1, v_4)|+|(v_1, v_2)|$ or, equivalently, the inequality $|(v_1, v_4)|>|(v_2, v_4)|-|(v_1, v_2)|$ holds. Moreover, the inequality $|(v_4, v_2)|\ge|e|$ holds by (iii). Combining the three latter inequalities, we obtain the following chain of inequalities:

$$|(v_1, v_4)|>|e|-\frac{|e|}{2\times\cos(q)}\ge|e|\times\left(1-\frac{1}{2\times\cos(q)}\right).$$

Hence, by (i), we obtain $h(v_1, v_4)>|e|\times(\cos(q)-0.5)$. This completes the proofs of the inequality (*), Proposition 3.1, and Theorem 3.1. □

4. The Implementation of the Greedy Triangulation for Convex Polygons and Convex Point Sets. Here, we present a recursive implementation of the greedy heuristic in the convex case. After finding a shortest diagonal of the current polygon, the polygon is split along the diagonal and our procedure is recursively applied to the two resulting subpolygons. To find a shortest diagonal of a convex polygon efficiently, we actually solve a more general so-called *all-shortest-diagonals problem for convex polygons* in linear time. The more general problem can be defined as follows: given a convex polygon P, find for every vertex v of P a shortest internal diagonal of P incident to v.

We solve this problem by using the linear-time algorithm of Lee and Preparata [5] for finding all nearest neighbors of a convex polygon, and its refinement due to Yang and Lee [14].

THEOREM 4.1. *Given a convex polygon $P=(v_0, v_1, \dots, v_{n-1})$, the all shortest diagonal problem for P can be solved in time $O(n)$.*

PROOF. Let v_{i_1}, v_{i_2}, v_{i_3}, and v_{i_4} be the vertices with the smallest X-coordinate, smallest Y-coordinate, largest X-coordinate, and largest Y-coordinate, respectively. We may assume without loss of generality that $i_1=0$ and $i_1\le i_2\le i_3\le i_4$.

It was shown in [14] that the polygons $P_1=(v_{i_1}, v_{i_1+1}, \dots, v_{i_2})$, $P_2=(v_{i_2}, v_{i_2+1}, \dots, v_{i_3})$, $P_3=(v_{i_3}, v_{i_3+1}, \dots, v_{i_4})$, and $P_4=(v_{i_4}, v_{i_4+1 \bmod n}, \dots, v_{i_1})$ are semicircular. (In the degenerate case P_l may consist of a single vertex or an edge.) Let $1\le l\le 4$. For $i_l+1<j<i_{l+1}-1$, let $P(j)$ be the subpolygon $(v_{i_l}, v_{i_l+1}, \dots, v_{j-2}, v_j, v_{j+2}, \dots, v_{i_{l+1}})$. Note that the problem of finding a shortest diagonal of P incident to v_j is equivalent to the problem of finding the nearest neighbor of v in the convex polygon that can be expressed as $P_1\cup P_2\cup P_3\cup P_4-P_l\cup P(j)$. By Fact 3.1, the nearest neighbor of v_j among the vertices of $P(j)$ is either v_{j-2} or v_{j+2}. For $\alpha=1, 2$, let $k(l, \alpha)=\text{MAX}(\{i_l+\alpha+2k \mid k\in N\ \&\ i_l+\alpha+2k\le i_{l+1}-1 \bmod n\})$ and let P_l^α be the subpolygon $(v_{i_l}, v_{i_l+\alpha}, v_{i_l+\alpha+2}, v_{i_l+\alpha+4}, \dots, v_{k(l,\alpha)}, v_{i_{l+1}})$. Note that P_l^α is also semicircular. Clearly, there exists $\beta\in\{1, 2\}$ such that v_j is a vertex of P_l^β. Now, since the nearest neighbor of v_j in $P(j)$ is either v_{j-2} or v_{j+2}, by the definition of P_l^β it is sufficient to find a nearest neighbor of v_j in the convex polygon $P_1\cup P_2\cup P_3\cup P_4-P_l\cup P_l^\beta$ to obtain the other endpoint of a shortest diagonal of P incident to v_j. Since the all-nearest-neighbor problem for convex polygons is solvable in linear time [5], [14], we can find, for every vertex v_j of P_l^β where $i_l+1<j<i_{l+1}-1 \bmod n$, a shortest diagonal of P incident to v_j in time $O(n)$. As for the four extreme vertices v_{i_l}, $v_{i_l+1 \bmod n}$, $v_{i_{l+1}-1 \bmod n}$, and $v_{i_{l+1}}$, we can find shortest diagonals of P incident to them by a brute-force method of scanning the vertices of P in clockwise order, which also takes $O(n)$ time. Since there are eight possible valuations of l and β, the all shortest diagonal problem can be solved in time $O(n)$. □

By Theorem 4.1, our recursive implementation of the greedy heuristic for MWT of convex polygons and convex planar point sets becomes more efficient than those known for planar point sets.

THEOREM 4.2. *Given a convex polygon P with n vertices,* GT(P) *can be constructed in time $O(n^2)$ and space $O(n)$.*

PROOF. Consider the following recursive procedure for constructing GT(P):

```
procedure Greedy(P)
  if n ≤ 3 then
  begin
    return the empty set;
    halt
  end
  e ← a shortest diagonal of P;
  split P along e into the polygons P1 and P2;
  return {e} ∪ Greedy(P1) ∪ Greedy(P2)
```

Since the two first instructions for $n>3$ are performed $O(n)$ times, the first in time $O(n)$ by Theorem 4.2, the second obviously in time $O(n)$, the total time performance of Greedy(P) is $O(n^2)$. Next, at any stage of recursively splitting the original polygon P, all the subpolygons to be maintained can be stored in space $O(n)$ since they form a partition of P. Also, finding a shortest diagonal of any one of these polygons takes $O(n)$ space. Hence, Greedy(P) can be implemented in space $O(n)$. □

Since the convex hull of a planar n-point set can be found in time $O(n \log n)$ and space $O(n)$ [13], we immediately obtain the following corollary from Theorem 4.1:

Corollary 4.1. *Given a convex set S of n points in the plane,* GT(S) *can be constructed in time $O(n^2)$ and space $O(n)$.*

Fact 3.1 also yields a straightforward $O(n \log n)$-time implementation of the greedy triangulation heuristic for semicircular polygons.

Theorem 4.3. *Given a semicircular polygon $P = (v_0, \ldots, v_{n-1})$,* GT($P$) *can be constructed in time $O(n \log n)$ and space $O(n)$.*

Proof. We may assume without loss of generality that (v_0, v_{n-1}) is the base of P. For $1 < j < n-2$, consider the subpolygon $P(j) = (v_0, \ldots, v^*_{j-2}, v_j, v^*_{j+2}, v_{n-1})$, where $v^*_{j-2} = v_{\max(0,j-2)}$ and $v^*_{j+2} = v_{\min(j+2,n-1)}$. By Fact 3.1, the nearest neighbor of v_j among the vertices of $P(j)$ is v^*_{j-2} or v^*_{j+2}. Hence, the shortest diagonal of P incident to v_j is (v_j, v^*_{j-2}) or (v_j, v^*_{j+2}). Thus, we can find, for $j = 0, 1, \ldots, n-1$, a shortest diagonal of P incident to v_j in total linear time.

Consider the following procedure.

Insert, for each vertex v_j of P, a shortest diagonal incident to v_j into a *heap* (see [12]). Pick a shortest diagonal $e = (w_1, w_3)$ from the heap, draw it in the plane and delete it from the heap. By Fact 3.1, the diagonal e cuts off a single vertex w_2 of P. Delete from the heap the diagonals incident to w_2. Let w_4 and w_5, respectively, be the vertex of P different from w_2 that is incident to the same edge of P as w_1 and w_3, respectively. Insert (w_3, w_4), respectively (w_1, w_5), in the heap if it is a diagonal of P.

Note that now for each vertex of the polygon P' resulting from P by cutting off the vertex w_2, the heap contains a shortest diagonal of P' incident to it. Therefore, it is sufficient to iterate the above procedure $n-3$ times to produce GT(P). An insertion of a diagonal into the heap and a deletion of the topmost diagonal from the heap can be easily implemented in time $O(\log n)$ [12]. As for the deletions of diagonals incident to any vertex w of P from the heap, they can be done in constant time per deletion by keeping pointers between the vertex w and the diagonals in the heap incident to w. Since the total number of insertions and deletions needed to build the heap and iterate the procedure is $O(n)$, the total time performance is $O(n \log n)$. The construction and maintenance of the heap takes $O(n)$ space. □

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