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A Real Nullstellensatz and Positivstellensatz for the semi-polynomials over an ordered field

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Abstract

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Let $\mathbf{K}$ be an ordered field and $\mathbf{R}$ its real closure. A semipolynomial will be defined as a function from $\mathbf{R}^n$ to $\mathbf{R}$ obtained by composition of polynomial functions and the absolute value. Every semipolynomial can be defined as a straight-line program containing only instructions with the following type: "polynomial", "absolute value", "max" and "min" and such a program will be called a semipolynomial expression. It will be proved, using the ordinary Real Positivstellensatz, a general Real Positivstellensatz concerning the semipolynomial expressions. Using this semipolynomial version for the Real Positivstellensatz we shall get as consequences a continuous and rational solution for the 17th Hilbert problem, rational and continuous versions for several cases in the Real Positivstellensatz and constructive proofs for several theorems concerning the algebra over the real numbers.

1. Introduction

This work can be considered as the natural continuation of [21] and we assume that the reader knows the results contained in such paper. With respect to [21], [19] and

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[22] this work contains an idea really original, to use in an explicit way the difference between a function and the program computing such function. Working with polynomials this difference is not really important but dealing with semipolynomial functions, those obtained composing polynomials with the functions absolute value and/or max and min, becomes crucial. In fact, there are at least three different ways for a semipolynomial ϕ to be null:

(1) the semipolynomial ϕ is the null function over any ordered extension of **K**, but in general this is not easy to be determined (nevertheless for the polynomial case it is enough to write in reduced form the formal polynomial defining the function considered),

(2) the semipolynomial ϕ is defined by a straight-line program such that for all the a priori possible cases concerning the instructions absolute value, max and min provides an identically zero polynomial,

(3) the semipolynomial ϕ is identically zero in a way that will be precised later.

The second way for a semipolynomial to be null which is stronger than the first one and the third one which is stronger than the second one, are applied only to the programs and this will be the key which will allow us to formulate the Real Positivstellensatz for the semipolynomials.

This Positivstellensatz is included in a general research program looking for similar theorems for every first-order formal theory with explicit quantifier elimination. In this setting a weak Nullstellensatz is a theorem saying that every incompatible system of equalities is related with an algebraic identity making this incompatibility evident without using the existential axioms in the theory considered. Moreover, a "general" Nullstellensatz, in this setting, must achieve the same objectives for every incompatible system of "atomic relations" in the theory. If, in the future, this general research program is accomplished we shall have obtained that all the formal proofs of incompatibility between atomic relations (which are universal theorems in the theory considered) can be transformed in an automatic way into proofs without using the existential axioms of the theory and moreover these proofs will be reduced to the construction of algebraic identities.

In our case the theory considered is the one concerning the real closed fields where we shall introduce the symbols for the functions absolute value, max and min. In this context, it is not possible to reduce the equality between two terms, as in the ordinary theory for real closed fields, to the equality of a polynomial to zero, it will be reduced to the equality of a semipolynomial expression to zero.

The search of a Positivstellensatz in the semipolynomial case has been motivated by the rational and continuous solution for the 17th Hilbert problem and has provided a reduced solution (independent of the problem considered here) for this problem that can be founded in [12]. As a by-product of the Positivstellensatz for the semipolynomials we get a parameterized version for the 17th Hilbert problem and for several instances of Positivstellensatz. Namely, the theorems we prove in Sections 4 and 5, in reduced version, are the following ones.

Theorem 4.1. *Let $f_{n,d}$ be the general polynomial of degree d and n variables and $\mathbb{F}_{n,d}$ the semialgebraic set defined by*

$$c \in \mathbb{F}_{n,d} \quad \Leftrightarrow \quad \forall \mathbf{x} \in \mathbf{R}^n \ f_{n,d}(\mathbf{c}, \mathbf{x}) \geq 0.$$

Then $f_{n,d}$ can be written as a weighted sum of squares of rational functions

$$f_{n,d}(\mathbf{c}, \mathbf{X}) = \sum_j p_j(\mathbf{c}) \left(\frac{q_j(\mathbf{c}, \mathbf{X})}{k(\mathbf{c}, \mathbf{X})}\right)^2$$

(for all $\mathbf{c} \in \mathbf{R}^m$), where
- *$k(\mathbf{c}, \mathbf{X})$ and the $q_j(\mathbf{c}, \mathbf{X})$ are polynomials in the variables $\mathbf{X}$ whose coefficients are $\mathbb{Q}$-semipolynomials in the coefficients $\mathbf{c}$. Moreover, if $\mathbf{c} \in \mathbb{F}_{n,d}$, then $k(\mathbf{c}, \mathbf{X})$ vanishes only on the zeros of $f_{n,d}(\mathbf{c}, \mathbf{X})$, and*
- *each $p_j(\mathbf{c})$ is a $\mathbb{Q}$-semipolynomial which is nonnegative on $\mathbb{F}_{n,d}$. Moreover, under the hypothesis $\mathbf{c} \in \mathbb{F}_{n,d}$, the nonnegativity of $p_j(\mathbf{c})$ is "clearly"* evident.

Theorem 5.1. *Let $\mathbb{H}(\mathbf{c}, \mathbf{X})$ be a system of generalized sign conditions on polynomials in $\mathbf{K}[\mathbf{c}, \mathbf{X}]$, where the X_i's are considered as variables and the c_j's as parameters. If $\mathbf{S}_{\mathbb{H}}$ is the semialgebraic set defined by*

$$\mathbf{c} \in \mathbf{S}_{\mathbb{H}} \quad \Leftrightarrow \quad \forall \mathbf{x} \in \mathbf{R}^n \ \mathbb{H}(\mathbf{c}, \mathbf{x}) \text{ is incompatible}$$

and $\mathbf{S}_{\mathbb{H}}$ is locally closed, then (Finiteness Theorem) there exist $H_1(\mathbf{c})$ and $H_2(\mathbf{c})$ $\mathbf{K}$-semipolynomials such that

$$\mathbf{c} \in \mathbf{S}_{\mathbb{H}} \quad \Leftrightarrow \quad [H_1(\mathbf{c}) \geq 0, H_2(\mathbf{c}) > 0].$$

If $\mathbf{c} \in \mathbf{S}_{\mathbb{H}}$ then the incompatibility of $\mathbb{H}(\mathbf{X}) = \mathbb{H}(\mathbf{c}, \mathbf{X})$ inside $\mathbf{R}^n$ is made obvious by an algebraic identity with coefficients given by semipolynomials in $\mathbf{c}$.

This paper has been written with the point of view of a constructive mathematician. Anyway it can be read as a paper in classical mathematics where all the proofs are effective, in particular without using the Axiom of Choice, providing primitive recursive algorithms (in case of discrete primitive recursive real closed fields, see [23]) or uniformly primitive recursive (in case the structure of coefficient field is given by an oracle giving the sign of any polynomial with integer coefficients evaluated in the coefficients of the problem).

In the part devoted to the constructive algebra for the real numbers "à la Cauchy", the proof of a theorem provides a uniformly primitive recursive algorithm, where the uniformity is understood with respect to the oracles giving the rational approximation desired for the real numbers "à la Cauchy" appearing in the hypothesis of the problem.

A brief history of Hilbert's 17th Problem

Hilbert's 17th Problem was introduced by D. Hilbert in 1901 (see [16]) and first solved, in a more general version than the one posed by D. Hilbert, by E. Artin in 1927 (see [1]). Artin's proof was strongly non-constructive (for example, use of Zorn's Lemma). Several attempts were made trying to get a constructive solution to Hilbert's 17th Problem. G. Kreisel in 1957 (see [17]) gave a sketch of a proof which was completed by D.E. Daykin in 1961 (see [4]). Independently, A. Robinson (see [27, 28]) got a constructive solution with (by definition total) general recursive bounds. These authors also expressed the weights and coefficients of the rational functions as $\mathbb{Z}$-piecewise-polynomial functions of $\mathbf{c}$. This kind of proofs work only for the case when the coefficient field has an explicit sign test (which is not the case for $\mathbb{R}$). C.N. Delzell (see [6]) in 1980 solved the problem partially for the case of $\mathbb{R}$. For other commentaries on constructivity of solutions see [11, 12, 20].

Moreover, in [6] was proved that the coefficients of the solution (the $p_j(\mathbf{c})$ and the coefficients of the $q_j(\mathbf{c}, \mathbf{X})$ and $k(\mathbf{c}, \mathbf{X})$ in Theorem 4.1) could be choosen as $\mathbb{Q}$-semialgebraic continuous functions of the parameters of the problem (the $\mathbf{c}$). A natural question arises in this point: can the coefficients of the solution be chosen as polynomials in the parameters, $\mathbf{c}$, of the problem? The negative answer to this question when $d \geq 4$ can be found in [5] and also in [9] or [18] where it is proved that it is impossible to find even an analytically varying representation of the solution.

After all these negative answers, the remaining question is to ask if it is possible to improve in some way the functions appearing in the solution to Hilbert's 17th Problem. The first (and possible the best one) answer to this question was announced in 1988 by Delzell (see [8] or [11]): the coefficients in the solution can be chosen as $\mathbb{Q}$-semipolynomials. This answer provides also a rational solution because the coefficient functions of the solution can be considered as functions from $\mathbf{K}^n$ to $\mathbf{K}$ while in the solution introduced in [6] this was only possible with $\mathbf{K}$ a real closed field.

The authors re-discovered independently the same result in 1991 and this motivated a joint paper [12], concerning Hilbert's 17th Problem where the solution was derived without using the semipolynomial Positivstellensatz. The proof announced by Delzell in [8] can be found in [11]. Such proof is derived from an abstract Positivstellensatz for the real spectrum of a ring.

2. Definitions

Firstly we recall the definitions of strong incompatibility and the general form for the Real Nullstellensatz in the polynomial case (see [21, 22]). We consider an ordered field $\mathbf{K}$, and $\mathbf{X}$ denotes a list of variables $X_1, X_2, \ldots, X_n$. We then denote by $\mathbf{K}[\mathbf{X}]$ the ring $\mathbf{K}[X_1, X_2, \ldots, X_n]$. If F is a finite subset of $\mathbf{K}[\mathbf{X}]$, we let F^{*2} be the set of squares of elements in F, $\mathcal{M}(F)$ be the *multiplicative monoid generated by* $F \cup \{1\}$. $\mathcal{Cp}(F)$ is the *positive cone generated by* F (the additive monoid generated by elements of type $p \cdot P \cdot Q^2$ where p is positive in $\mathbf{K}$, P is in $\mathcal{M}(F)$, Q is in $\mathbf{K}[\mathbf{X}]$). Finally, let $I(F)$ be the ideal generated by F.

Definition 2.1. Consider four finite subsets of $\mathbf{K}[\mathbf{X}]$: $F_>$, $F_\geq$, $F_=$, $F_{\neq}$, containing polynomials for which we want respectively the sign conditions > 0, ≥ 0, $= 0$, $\neq 0$: we say that $\mathbf{F} = [F_>; F_\geq; F_=; F_{\neq}]$ is *strongly incompatible* in $\mathbf{K}$ if we have in $\mathbf{K}[\mathbf{X}]$ an equality of the following type:

$$S + P + Z = 0 \quad \text{with } S \in \mathcal{M}(F_> \cup F_{\neq}^{*2}),\ P \in \mathcal{C}\!p(F_\geq \cup F_>),\ Z \in I(F_=).$$

It is clear that a strong incompatibility is a very strong form of incompatibility. In particular, it implies it is impossible to give the indicated signs to the polynomials, in any ordered extension of $\mathbf{K}$. If one considers the real closure $\mathbf{R}$ or $\mathbf{K}$, the previous impossibility is testable by Hörmander's algorithm, for example.

Notation 2.2. We use the following notation for a strong incompatibility:

$$\downarrow [S_1 > 0, \ldots, S_i > 0, P_1 \geq 0, \ldots, P_j \geq 0,$$
$$Z_1 = 0, \ldots, Z_k = 0, N_1 \neq 0, \ldots, N_h \neq 0] \downarrow$$

or, denoting by $\mathbb{H}(X_1, \ldots, X_n)$ the system of generalized sign conditions considered:

$$\downarrow \mathbb{H}(X_1, \ldots, X_n) \downarrow$$

Remark that we use the same notation as in [22] instead of (as in [21] or [19])

$$\star(\mathbb{H}(X_1, \ldots, X_n) \Rightarrow 1 = 0)\star$$

The different variants of the real Positivstellensatz are consequences of the following general theorem:

Theorem 2.3. *Let* $\mathbf{K}$ *be an ordered field and* $\mathbf{R}$ *a real closed extension of* $\mathbf{K}$. *The three following facts, concerning a generalized sign condition system on polynomials of* $\mathbf{K}[\mathbf{X}]$, *are equivalent:*

- *strong incompatibility in* $\mathbf{K}$,
- *impossibility in* $\mathbf{R}$,
- *impossibility in all the ordered extensions of* $\mathbf{K}$. □

An equivalent form of this Postivstellensatz was first proved in 1974 [31]. Less general variants were given by Krivine [18], Dubois [13], Risler [26], Efroymson [14] and Prestel [25].

Next we generalize the notion of strong incompatibility to the semipolynomial case. Let $\mathbf{K}$ be an ordered discrete field and $\mathbf{R}$ its real closure.

A *semipolynomial function* with coefficients in **K** (a **K**-semipolynomial) from $\mathbf{R}^n$ to $\mathbf{R}$ is a function obtained by a finite repetition of composition of polynomials with coefficients in **K** and the function absolute value. A well-known proposition, not used here, assures that the set of the **K**-semipolynomials agrees with the minimal max–min stable set of functions containing polynomials with coefficients in **K** (see for example [7]).

It could be developed for the **K**-semipolynomials a theory similar to the one introduced in [21] which allows to obtain the constructive version for the Real Positivstellensatz. In fact we shall reduce our problem to the ordinary Real Positivstellensatz (Theorem 2.3).

In order to obtain an explicit Positivstellensatz for the semipolynomials, we shall need firstly a notion for algebraic identities concerning semipolynomials. As the semipolynomials does not have canonical representation, this question is a bit tricky.

To solve this question we consider a new notion, the **K**-*semipolynomial expression* (shortly a **K**-spe, or a spe if **K** is clear in the context). A **K**-spe $F(X_1, \ldots, X_n)$ is a straight-line program with the following structure:

- each instruction is an assignment $z_i \leftarrow \ldots$ with the indexes i ordered in an increasing way (the last z_i is F),
- the instructions can have only the four following types:
  - $z_j \leftarrow P(X_1, \ldots, X_n, z_{i_1}, \ldots, z_{i_k})$ where $P \in \mathbf{K}[X_1, \ldots, X_n, z_{i_1}, \ldots, z_{i_k}]$ and every i_h is smaller than j,
  - $z_j \leftarrow |z_i|$ with $i < j$,
  - $z_j \leftarrow \max\{z_{i_1}, \ldots, z_{i_k}\}$ with every i_h smaller than j,
  - $z_j \leftarrow \min\{z_{i_1}, \ldots, z_{i_k}\}$ with every i_h smaller than j.

It is clear that every **K**-semipolynomial can be obtained from a **K**-spe (we only need to replace every X_i by x_i and to execute the program). Moreover, every **K**-spe can be defined using only one of the three functions, absolute value, max or min.

A *polynomial underlying a* **K**-*spe* is, by definition, a polynomial in $\mathbf{K}[X_1, \ldots, X_n]$ obtained when the straight-line program given by the **K**-spe considered is executed in the following way:

- every instruction $z_j \leftarrow |z_i|$ is replaced by one of the two instructions $z_j \leftarrow z_i$ or $z_j \leftarrow -z_i$,
- every instruction $z_j \leftarrow \max\{z_{i_1}, \ldots, z_{i_k}\}$ is replaced by one of the k instructions $z_j \leftarrow z_{i_h}$,
- every instruction $z_j \leftarrow \min\{z_{i_1}, \ldots, z_{i_k}\}$ is replaced by one of the k instructions $z_j \leftarrow z_{i_h}$.

For example, if our **K**-spe F contains d absolute value instructions (without max or min instructions) then there are a priori 2^d polynomials underlying the **K**-spe F.

Definition 2.4. A **K**-semipolynomial expression F will be said *formally null* when all the polynomials underlying F are null.

For example, the **K**-spe $G^2 - |G|^2$ is formally null but $|G| - |-G|$ and $(1 + X^2) - |1 + X^2|$ are not, which in some sense is disturbing.

A **K**-spe G will be said *interior to* another **K**-spe F if (modulo a renumbering of the variables z_i in G) the straight-line program for G can be obtained from the one for F by deleting some instructions and if the straight-line program for G ends with an instruction absolute value, min or mix.

A **K**-spe H is said *a polynomial inside the context of the* **K**-*spe* F if H is a polynomial in the variables X_i and in the **K**-spe interior to F. More precisely, H must be written as the straight-line program associated to F plus instructions of polynomial type (indeed only one of such instructions would be sufficient). Remark that it is not forbidden to introduce new variables, i.e. not appearing in the F's context.

In some sense, it is not worthy to compute with different **K**-spe outside of a common context. For example if $F = |X|$ and $G = |X|$, without common context, the **K**-spe $H = F - G$ will be computed by the following program,

$$z_1 \leftarrow X, \quad F \leftarrow |z_1|, \quad z_2 \leftarrow X, \quad G \leftarrow |z_2|, \quad H \leftarrow F - G$$

obtaining that H is not formally null. So, it is only in a common context that we can talk about **K**-spe formally equal.

In a fixed context, we have the stronger notion for two **K**-*spe to be identical*, as such **K**-spe defined by the same polynomials in the variables and in the **K**-spe interior to the context[1]. In particular it is clear that the notion of **K**-spe identically null is stronger that the one of **K**-spe formally null.

All what follows will be applied on **K**-*spe which are polynomials inside the context of a* **K**-*spe* F *fixed* (*we shall say, inside a fixed context*).

Let $\mathbb{H}$ be a system of generalized sign conditions on the **K**-spe F_i with $1 \leq i \leq t$. Next, we define in a recursive way which are the **K**-spe "evidently $= 0$, ≥ 0 or > 0 under the hypothesis $\mathbb{H}$".

K-*spe evidently null under the hypothesis* $\mathbb{H}$. The **K**-spe evidently null under the hypothesis $\mathbb{H}$ are:

- the **K**-spe equal to 0 in $\mathbb{H}$,
- the **K**-spe coming from polynomial instructions of the following type,

$$z_j \leftarrow \sum_{h=1}^{k} z_{i_h} P_h(X_1, \ldots, X_n, z_{i_1}, \ldots, z_{i_k}),$$

where the z_{i_h} are yet known as evidently null under $\mathbb{H}$,
- the **K**-spe identical to another **K**-spe yet known as evidentally null under $\mathbb{H}$.

[1] The context notion is not essential. Given $F, G, H, \ldots$ it is always possible to compute a maximal common context (maximal in the sense that it is defined by the maximum of the interior comon **K**-spe) for these **K**-spe, taking first in account the most interior **K**-spe to $F, G, H, \ldots$ (those obtained with only one instruction absolute value, max or min) until the less interior. Anyway the context notion seems to be useful to simplify the understanding of what follows and moreover it is well posed for a future implementation of the algorithms in the proof.

K-*spe evidently nonnegative under the hypothesis* $\mathbb{H}$. The **K**-spe evidently nonnegative under the hypothesis $\mathbb{H}$ are:

- the **K**-spe >0 or ≥ 0 in $\mathbb{H}$,
- every **K**-spe z_j obtained in the context by an absolute value instruction $z_j \leftarrow |z_i|$,
- every **K**-spe of type $z_j - z_i$ where z_j is obtained in the context by a max instruction $z_j \leftarrow \max\{\ldots, z_i, \ldots\}$,
- every **K**-spe of type $z_i - z_j$ where z_j is obtained in the context by an min instruction $z_j \leftarrow \min\{\ldots, z_i, \ldots\}$,
- the square **K**-spe, i.e. the **K**-spe coming from an instruction $z_j \leftarrow z_i^2$,
- the polynomials with positive coefficients in **K** in some **K**-spe $z_{j_1}, \ldots, z_{j_k}$ yet known as evidently ≥ 0 under $\mathbb{H}$,
- the **K**-spe identical to another **K**-spe yet known as evidently ≥ 0 under $\mathbb{H}$.

K-*spe evidently positive under the hypothesis* $\mathbb{H}$. The **K**-spe evidently positive under the hypothesis $\mathbb{H}$ are:

- the **K**-spe >0 in $\mathbb{H}$,
- the positive elements in **K**,
- the square of **K**-spe $\neq 0$ in $\mathbb{H}$,
- the products of **K**-spe yet known as evidently >0 under $\mathbb{H}$,
- the **K**-spe identical to another **K**-spe yet known as evidently >0 under $\mathbb{H}$.

Definition 2.5. A system of generalized sign conditions $\mathbb{H}$ is said *strongly incompatible* (in **K** and with the context fixed) if there exists a **K**-spe formally null, obtained as the sum of a **K**-spe evidently >0, a **K**-spe evidently ≥ 0 and a **K**-spe evidently $=0$ (under the hypothesis $\mathbb{H}$).

We shall use, as in [22], the notation $\downarrow \mathbb{H}(X_1, \ldots, X_n) \downarrow$. Remark here that if all the **K**-spe considered are "true" polynomials then we find the old notions and this allows not to introduce new notations.

Using the notion of strong incompatibility, it is possible to develop the notions of strong implication, the constructions of strong incompatibilities and potential existence as in [19] or [21] and also the notions of dynamic implication and dynamic disjunction as in [22]. Anyway we shall not need these concepts because we shall derive the Real Positivstellensatz for the semipolynomials directly from the ordinary Real Positivstellensatz (Theorem 2.3).

3. The Real Positivstellensatz for the K-semipolynomial expressions

In this section, the **K**-semipolynomial expressions considered will be polynomials inside a fixed context and they will be called **K**-spe. Also the strong incompatibilities will have their coefficients in **K** and they are strong incompatibilities in the fixed context (which implies that the functions absolute value, max and min can appear only as in the context).

Theorem 3.1. *Let* **K** *be a discrete ordered field and* **R** *a real closed field containing* **K**. *Let* $\mathbb{H}$ *be a system of generalized sign conditions defined on a finite family of* **K**-*semipolynomial expressions in the variables* $X_1, \ldots, X_n$ *(these* **K**-*semipolynomial expressions are polynomials inside a fixed context). Then the system* $\mathbb{H}$ *is incompatible in* **R** *if and only if the system* $\mathbb{H}$ *is strongly incompatible in* **K** *(for the fixed context). More precisely,*

if $\downarrow \mathbb{H}(X_1, \ldots, X_n) \downarrow$ *(in* **K**) **then** *the system* $\mathbb{H}$ *is incompatible in any ordered field extension of* **K**,

if *for every* $(x_1, \ldots, x_n) \in \mathbf{R}^n$ *the system* $\mathbb{H}(x_1, \ldots, x_n)$ *is incompatible* **then** $\downarrow \mathbb{H}(X_1, \ldots, X_n) \downarrow$ *(in* **K**).

Proof. Remark, firstly, that the incompatibility of the system $\mathbb{H}$ in **R** can be determined using a decision algorithm for the discrete real closed fields, performing only computations in **K**.

The first part in the statement of the theorem is trivial, it is enough to apply the definition of strong incompatibility introduced in the previous section.

To prove the second part, we shall reduce our problem to the ordinary Real Positivstellensatz. Firstly we introduce a formal variable z_j for every variable z_j in the context. So our system $\mathbb{H}$ can be rewritten as a system $\mathbb{H}'$ containing only polynomials in the variables X_i and z_j.

Now we define a polynomial system of generalized sign conditions $\mathbb{H}_c$ associated to the context in the following way:

- for every polynomial instruction $z_j \leftarrow P(X_1, \ldots, X_n, z_{i_1}, \ldots, z_{i_k})$ we introduce in $\mathbb{H}_c$ the sign condition

$$z_j - P(X_1, \ldots, X_n, z_{i_1}, \ldots, z_{i_k}) = 0,$$

- for every absolute value instruction $z_j \leftarrow |z_i|$ we introduce in $\mathbb{H}_c$ the sign conditions

$$z_j^2 - z_i^2 = 0, \qquad z_j \geq 0,$$

- for every max instruction $z_j \leftarrow \max\{z_{i_1}, \ldots, z_{i_k}\}$, we introduce in $\mathbb{H}_c$ the sign conditions

$$(z_j - z_{i_1})(z_j - z_{i_2}) \ldots (z_j - z_{i_k}) = 0,$$
$$z_j - z_{i_1} \geq 0, \quad z_j - z_{i_2} \geq 0, \quad \ldots, \quad z_j - z_{i_k} \geq 0,$$

- for every min instruction $z_j \leftarrow \min\{z_{i_1}, \ldots, z_{i_k}\}$ we introduce in $\mathbb{H}_c$ the sign conditions

$$(z_j - z_{i_1})(z_j - z_{i_2}) \ldots (z_j - z_{i_k}) = 0,$$
$$z_j - z_{i_1} \leq 0, \quad z_j - z_{i_2} \leq 0, \quad \ldots, \quad z_j - z_{i_k} \leq 0.$$

The system $[\mathbb{H}', \mathbb{H}_c]$ is incompatible in **R** because first, the system $\mathbb{H}$ is incompatible in **R** and second, every solution of the system $[\mathbb{H}', \mathbb{H}_c]$ provides a solution for $\mathbb{H}$. As all the elements involved in the system $[\mathbb{H}', \mathbb{H}_c]$ are polynomials, applying the ordinary Real Positivstellensatz (see Theorem 2.3), we obtain a strong incompatibility

$$\downarrow [\mathbb{H}', \mathbb{H}_c] \downarrow . \tag{1}$$

Now if we replace, in the algebraic identity obtained, every variable z_j by the corresponding **K**-spe then:

- the "positive" part in (1) does not contain any generalized sign condition from $\mathbb{H}_c$ and provides a **K**-spe "evidently positive" under the hypothesis $\mathbb{H}$,
- the "nonnegative" part in (1) provides a **K**-spe "evidently nonnegative" under the hypothesis $\mathbb{H}$ (it is enough to use the definitions),
- the "null" part in (1) can be separated in two pieces:
  - the first one is null under the hypothesis $\mathbb{H}'$ and provides a **K**-spe evidently null under the hypothesis $\mathbb{H}$,
  - the second one is null under the hypothesis $\mathbb{H}_c$ and provides a **K**-spe formally null (in the fixed context), which can be deleted.

So, deleting the last piece in the "null" part we obtain a **K**-spe which is equal to a **K**-spe identically null minus a **K**-spe formally null and so formally null, as we wanted to show. □

Remark 3.2. Theorem 3.1 shows that in particular a straight-line program, as "$G - |G|$" with G everywhere positive, defining a semipolynomial everywhere null, has always an algebraic evidence for its nullity. It is a crucial point that in the definition of a strong incompatibility, the global **K**-spe must be formally null, what is much stronger than "everywhere null".

Remark 3.3. Strong versions for the polynomial Positivstellensatz and Nichtnegativstellensatz can be found in [32] and can be derived easily from Theorem 2.3. In a similar manner we can state the same result for the semipolynomial theorems. For example, assuming that we have an implication

$$\forall x_1, \ldots, x_n \in \mathbf{R}^n \quad (\mathbb{H}(x_1, \ldots, x_n) \;\Rightarrow\; P(x_1, \ldots, x_n) > 0)$$

or, what is the same, the incompatibility of the system

$$\mathbb{H}(x_1, \ldots, x_n), P(x_1, \ldots, x_n) \leq 0$$

Theorem 3.1 gives a corresponding strong incompatibility where we can isolate the role played by the polynomial P:

$$S + Q - PR + Z = 0$$

with S evidently positive, Q and R evidently nonnegative and Z evidently null under the hypothesis $\mathbb{H}$. If we multiply the left-hand side of the last equality by $1 - P$ we get the following formal equality:

$$P(S + R + Q) = (S + Q + RP^2) + Z(1 - P)$$

or, what is the same:

$$P(S + Q_1) = S + Q_2 + Z_1$$

with S evidently positive, Q_1 and Q_2 evidently nonnegative and Z_1 evidently null under the hypothesis $\mathbb{H}$. This is the form of Lam's Positivstellensatz. The same trick works for the Nichtnegativstellensatz.

4. A new rational and continuous solution for Hilbert's 17th problem

Let $f_{n,d}(\mathbf{c}, \mathbf{X})$ be the general polynomial with degree d and n variables ($\mathbf{c}$ denotes the list of coefficients $c_1, \ldots, c_m$ and $\mathbf{X}$ the list of variables $X_1, \ldots, X_n$). It is a standard fact in real algebraic geometry that the set

$$\mathbb{F}_{n,d} = \{\mathbf{c}\colon \forall \mathbf{x} \in \mathbf{R}^n \, f_{n,d}(\mathbf{c}, \mathbf{x}) \geq 0\}$$

is a closed $\mathbb{Q}$-semialgebraic set. So, applying the Finiteness Theorem we have that $\mathbb{F}_{n,d}$ can be described as a finite union of basic closed $\mathbb{Q}$-semialgebraic sets. Looking carefully at the proof of the Finiteness Theorem in [3] (or in other places) we can conclude that such proof is explicit and rational (see [30] for a careful complexity analysis of this theorem), which implies that it is possible to compute in a rational way a finite number of polynomials $R_{n,d,i,j}(\mathbf{c})$ in $\mathbb{Z}[\mathbf{c}]$ such that

$$\mathbb{F}_{n,d} = \bigcup_{i=1}^{k} \bigcap_{j=1}^{n_i} \{\mathbf{c}\colon R_{n,d,i,j}(\mathbf{c}) \geq 0\}.$$

This last equality allows us to describe the set $\mathbb{F}_{n,d}$ in the following way.

$$\mathbb{F}_{n,d} = \left\{\mathbf{c}\colon \left[\max_{i=1,\ldots,k} \{\min\{R_{n,d,i,j}(\mathbf{c})\colon j = 1, \ldots, n_i\}\}\right] \geq 0\right\}.$$

So, if for every i in $\{1, \ldots, k\}$ we define

$$H_{n,d,i}(\mathbf{c}) = \min_{j=1,\ldots,n_i} \{R_{n,d,i,j}(\mathbf{c})\}, \quad H_{n,d}(\mathbf{c}) = \max_{i=1,\ldots,k} \{H_{n,d,i}(\mathbf{c})\},$$

we have obtained the following description for the set $\mathbb{F}_{n,d}$:

$$\mathbb{F}_{n,d} = \{\mathbf{c}\colon H_{n,d}(\mathbf{c}) \geq 0\},$$

where $H_{n,d}(\mathbf{c})$ is a $\mathbb{Q}$-semipolynomial.

We have shown the equivalences

$$\mathbf{c} \in \mathbb{F}_{n,d} \quad \Leftrightarrow \quad H_{n,d}(\mathbf{c}) \geq 0 \quad \Leftrightarrow \quad \forall \mathbf{x} \in \mathbf{R}^n\, f_{n,d}(\mathbf{c}, \mathbf{x}) \geq 0$$

with $f_{n,d}(\mathbf{c}, \mathbf{X})$ a polynomial and $H_{n,d}(\mathbf{c})$ a $\mathbb{Q}$-semipolynomial. Let us consider now $H_{n,d}$ as a $\mathbb{Q}$-spe defined by the straight-line program that translates the definitions of $H_{n,d,1}, \ldots, H_{n,d,k}$ and $H_{n,d}$. So, we can apply the Real Positivstellensatz for the $\mathbb{Q}$-semipolynomial expressions in the context $H_{n,d}$ to the implication

$$\forall \mathbf{c} \in \mathbf{R}^m \;\; \forall \mathbf{x} \in \mathbf{R}^n \quad \{H_{n,d}(\mathbf{c}) \geq 0 \;\Rightarrow\; f_{n,d}(\mathbf{c}, \mathbf{x}) \geq 0\}$$

or, what is the same, to the incompatibility of the system of generalized sign conditions

$$H_{n,d}(\mathbf{c}) \geqslant 0, \qquad f_{n,d}(\mathbf{c}, \mathbf{X}) < 0$$

Applying Theorem 3.1 to this system we obtain a strong incompatibility that can be rewritten as the following formal equality,

$$f_{n,d}(\mathbf{c}, \mathbf{x}) g(\mathbf{c}, \mathbf{x}) = f_{n,d}(\mathbf{c}, \mathbf{x})^{2r} + h(\mathbf{c}, \mathbf{x}), \tag{2}$$

where h and g are $\mathbb{Q}$-spe evidently nonnegative under the hypothesis $H_{n,d}(\mathbf{c}) \geq 0$.

Coming back again to the definitions it is easy to see that g and h are polynomials in $\mathbf{X}$ whose coefficients are $\mathbb{Q}$-spe in $\mathbf{c}$. More precisely, g and h are sum of terms

$$p_j(\mathbf{c}) q_j(\mathbf{c}, \mathbf{X})^2,$$

where the $q_j(\mathbf{c}, \mathbf{X})$ have the same type as g and h and the $p_j(\mathbf{c})$ are $\mathbb{Q}$-spe evidently nonnegative under the hypothesis $H_{n,d}(\mathbf{c}) \geq 0$ and with the context $H_{n,d}(\mathbf{c})$. This allows us to conclude that without loss of generality we can suppose that every $p_j(\mathbf{c})$ is a product whose factors have the following type:

- the $\mathbb{Q}$-spe $H_{n,d}(\mathbf{c})$,
- a $\mathbb{Q}$-spe $H_{n,d}(\mathbf{c}) - H_{n,d,i}(\mathbf{c})$,
- a $\mathbb{Q}$-spe $R_{n,d,i,j}(\mathbf{c}) - H_{n,d,i}(\mathbf{c})$,
- a positive rational or the square of a $\mathbb{Q}$-spe in $\mathbf{c}$.

If we multiply by $f_{n,d}(\mathbf{c}, \mathbf{X})$ every member of equality (2) we get

$$f_{n,d}(\mathbf{c}, \mathbf{X}) = \frac{f_{n,d}(\mathbf{c}, \mathbf{X})^2 g(\mathbf{c}, \mathbf{X})}{f_{n,d}(\mathbf{c}, \mathbf{X})^{2r} + h(\mathbf{c}, \mathbf{X})}$$

and denoting by $k(\mathbf{c}, \mathbf{X})$ the denominator of such fraction we obtain finally

$$f_{n,d}(\mathbf{c}, \mathbf{X}) = \frac{f_{n,d}(\mathbf{c}, \mathbf{X})^2 g(\mathbf{c}, \mathbf{X}) k(\mathbf{c}, \mathbf{X})}{k(\mathbf{c}, \mathbf{X})^2} = \frac{g_1(\mathbf{c}, \mathbf{X})}{k(\mathbf{c}, \mathbf{X})^2},$$

where g_1 has the same type as g and h.

The proof of the following theorem is almost achieved.

Theorem 4.1. *The general polynomial with degree d and n variables can be written as a sum of rational functions*

$$f_{n,d}(\mathbf{c}, \mathbf{X}) = \sum_j p_j(\mathbf{c}) \left(\frac{q_j(\mathbf{c}, \mathbf{X})}{k(\mathbf{c}, \mathbf{X})} \right)^2, \tag{3}$$

where:

- *The $q_j(\mathbf{c}, \mathbf{X})$ and $k(\mathbf{c}, \mathbf{X})$ are polynomials in the variables $\mathbf{X}$ whose coefficients are $\mathbb{Q}$-spe in the variables $\mathbf{c}$. Moreover, if $\mathbf{c} \in \mathbb{F}_{n,d}$ then $k(\mathbf{c}, \mathbf{X})$ only vanishes on the zeros of $f_{n,d}(\mathbf{c}, \mathbf{X})$.*
- *Each $p_j(\mathbf{c})$ is a product whose factors are $H_{n,d}(\mathbf{c})$ or one of the $\mathbb{Q}$-spe $H_{n,d}(\mathbf{c}) - H_{n,d,i}(\mathbf{c})$ or one of the $\mathbb{Q}$-spe $R_{n,d,i,j}(\mathbf{c}) - H_{n,d,i,}(\mathbf{c})$ or a positive rational or the square of a $\mathbb{Q}$-spe in $\mathbf{c}$. So, under the hypothesis $H_{n,d}(\mathbf{c}) \geq 0$ the positivity of $p_j(\mathbf{c})$ is "clearly" evident.*
- *The equality*

$$f_{n,d}(\mathbf{c}, \mathbf{X}) k(\mathbf{c}, \mathbf{X})^2 - \sum_j p_j(\mathbf{c}) q_j(\mathbf{c}, \mathbf{X})^2 = 0$$

is specially evident in the following sense: the first member of the equality, as polynomial in $\mathbf{X}$, has as coefficients $\mathbb{Q}$-spe in $\mathbf{c}$ which are formally null.

Equality (3) provides a rational and continuous solution for Hilbert's 17th problem because

- *all the coefficients (the $p_j(\mathbf{c})$ and the coefficients of the $q_j(\mathbf{c}, \mathbf{X})$ and $k(\mathbf{c}, \mathbf{X})$ considered as polynomials in $\mathbf{X}$) appearing in the equality are rational and continuous functions in $\mathbf{c}$, more precisely they are $\mathbb{Q}$-spe in the variables $\mathbf{c}$,*
- *every term in sum (3),*

$$p_j(\mathbf{c}) \left(\frac{q_j(\mathbf{c}, \mathbf{X})}{k(\mathbf{c}, \mathbf{X})} \right)^2,$$

is a rational function which can be extended by continuity to a semialgebraic continuous function in the semialgebraic closed set $\mathbb{F}_{n,d} \times \mathbf{R}^n$.

Proof. The only statement still not proved is the one concerning the fact that every term in (3) can be extended with continuity to a semialgebraic continuous function on $\mathbb{F}_{n,d} \times \mathbf{R}^n$. For that it is enough to exhibit a modulus of uniform continuity for

$$U(\mathbf{c}, \mathbf{X}) = p_j(\mathbf{c}) \left(\frac{q_j(\mathbf{c}, \mathbf{X})}{k(\mathbf{c}, \mathbf{X})} \right)^2$$

on every bounded set $B \subset \mathbb{F}_{n,d} \times \mathbf{R}^n$.

So, if ε is a positive number then we can choose $\delta > 0$ such that on B we have

$$\|(\mathbf{c}, \mathbf{x}) - (\mathbf{c}', \mathbf{x}')\| < \delta \ \Rightarrow \ |f_{n,d}(\mathbf{c}, \mathbf{x}) - f_{n,d}(\mathbf{c}', \mathbf{x}')| < \tfrac{1}{8}\,\varepsilon$$

and we consider two different cases:

If $f_{n,d}(\mathbf{c}, \mathbf{x}) \leq 3\varepsilon/8$ then $f_{n,d}(\mathbf{c}', \mathbf{x}') \leq \frac{1}{2}\,\varepsilon$, which implies directly that

$$\left.\begin{array}{l} 0 \leq U(\mathbf{c}, \mathbf{x}) \leq \frac{3}{8}\,\varepsilon \\ 0 \leq U(\mathbf{c}', \mathbf{x}') \leq \frac{1}{2}\,\varepsilon \end{array}\right\} \Rightarrow |U(\mathbf{c}, \mathbf{x}) - U(\mathbf{c}', \mathbf{x}')| < \varepsilon.$$

– if $f_{n,d}(\mathbf{c}, \mathbf{x}) \geq \varepsilon/4$ then $f_{n,d}(\mathbf{c}', \mathbf{x}') \geq \frac{1}{8}\,\varepsilon$, which implies

$$k(\mathbf{c}, \mathbf{x}) \geq (\tfrac{1}{8}\,\varepsilon)^{2r}, \qquad k(\mathbf{c}', \mathbf{x}') \geq (\tfrac{1}{8}\,\varepsilon)^{2r},$$

allowing to find $\delta' \leq \delta$ such that

$$\|(\mathbf{c}, \mathbf{x}) - (\mathbf{c}', \mathbf{x}')\| < \delta' \ \Rightarrow \ |U(\mathbf{c}, \mathbf{x}) - U(\mathbf{c}', \mathbf{x}')| < \varepsilon$$

since the minoration of the denominator. □

5. Rational and continuous solution for another cases of the classical Real Positivstellensatz

The solution for Hilbert's 17th problem can be seen as a particular case of the Real Positivstellensatz and for this case we have just proved, in the previous section, the existence of a solution depending on the parameters of the problem in a semipolynomial way. So what we shall do, is to generalize this result for another cases.

Let $\mathbb{H}(\mathbf{c}, \mathbf{X})$ be a system of generalized sign conditions on polynomials in $\mathbf{K}[\mathbf{c}, \mathbf{X}]$ where the X_i's are considered as variables and the c_j's as parameters. We denote by $\mathbf{S}_{\mathbb{H}}$ the semialgebraic set defined by

$$\mathbf{S}_{\mathbb{H}} = \{\mathbf{c}\colon \forall \mathbf{x} \in \mathbf{R}^n \ \ \mathbb{H}(\mathbf{c}, \mathbf{x}) \text{ is incompatible}\}.$$

If $\mathbf{S}_{\mathbb{H}}$ is locally closed (i.e. intersection of a closed and an open semialgebraic set) then, applying the Finiteness Theorem (see [3]) and the strategy followed in Section 4 when dealing with the set $\mathbb{F}_{n,d}$, it is possible to compute two **K**-spe $H_1(\mathbf{c})$ and $H_2(\mathbf{c})$ verifying

$$\mathbf{c} \in \mathbf{S}_{\mathbb{H}} \quad \Leftrightarrow \quad [H_1(\mathbf{c}) \geq 0, \; H_2(\mathbf{c}) > 0]$$

$$\Leftrightarrow \quad \forall \mathbf{x} \in \mathbf{R}^n \quad \mathbb{H}(\mathbf{c}, \mathbf{x}) \text{ is incompatible.}$$

Applying now the Real Positivstellensatz for the **K**-spe in the context defined by H_1 and H_2 to the incompatibility of the system of generalized sign conditions

$$[H_1(\mathbf{c}) \geq 0, \; H_2(\mathbf{c}) > 0, \; \mathbb{H}(\mathbf{c}, \mathbf{X})]$$

one gets a rational and continuous version for the strong incompatibilty of the system $\mathbb{H}(\mathbf{c}, \mathbf{X})$ when the parameters $\mathbf{c}$ vary inside $\mathbf{S}_{\mathbb{H}}$.

In the same way that our rational and continuous solution for Hilbert's 17th problem showed in Section 4, improves Delzell's result (see [6]), which is obtained in this section improves Scowcroft's results (see [29]) in four aspects:

(a) the semialgebraic set $\mathbf{S}_{\mathbb{H}}$ need not be for us, necessarily closed,

(b) the coefficients of our solution are **K**-semipolynomials in the parameters $\mathbf{c}$ for the hypothesis,

(c) the algebraic identity obtained, seen as polynomial in $\mathbf{X}$, has a structure specially simple, its coefficients are **K**-spe in $\mathbf{c}$ formally null,

(d) the positivity or strict positivity of the coefficients (which must verify such condition) in the solution is clearly evident under the hypothesis $H_1(\mathbf{c}) \geq 0$ and $H_2(\mathbf{c}) > 0$.

The next theorem summarizes the results obtained in this section and provides a rational and continuous solution for some cases of Real Positivstellensatz.

Theorem 5.1. *Let* $\mathbb{H}(\mathbf{c}, \mathbf{X})$ *be a system of generalized sign conditions on polynomials in* $\mathbf{K}[\mathbf{c}, \mathbf{X}]$, *where the* X_i*'s are considered as variables and the* c_j*'s as parameters. If* $\mathbf{S}_{\mathbb{H}}$ *is the semialgebraic defined by*

$$\mathbf{c} \in \mathbf{S}_{\mathbb{H}} \quad \Leftrightarrow \quad \forall \mathbf{x} \in \mathbf{R}^n \;\; \mathbb{H}(\mathbf{c}, \mathbf{x}) \textit{ is incompatible}$$

and $\mathbf{S}_{\mathbb{H}}$ *is locally closed, then (Finiteness Theorem) there exist* $H_1(\mathbf{c})$ *and* $H_2(\mathbf{c})$ **K**-*semipolynomial expressions such that*

$$\mathbf{c} \in \mathbf{S}_{\mathbb{H}} \quad \Leftrightarrow \quad [H_1(\mathbf{c}) \geq 0, \; H_2(\mathbf{c}) > 0].$$

If $\mathbf{c} \in \mathbf{S}_{\mathbb{H}}$ *then the incompatibility of* $\mathbb{H}(\mathbf{X}) = \mathbb{H}(\mathbf{c}, \mathbf{X})$ *inside* $\mathbf{R}^n$ *is made obvious by a strong incompatibility* $\downarrow \mathbb{H}(\mathbf{X}) \downarrow$ *with type fixed (independent of* $\mathbf{c}$*) and with coefficients given by* **K**-*semipolynomial expressions in* $\mathbf{c}$ *(which are polynomials inside the context defined by* $H_1(\mathbf{c})$ and $H_2(\mathbf{c})$*). Moreover,*

– *the algebraic identity obtained, seen as polynomial in* $\mathbf{X}$, *has a structure specially simple, more precisely, every coefficient of such identity as polynomial in* $\mathbf{X}$ *is a* $\mathbf{K}$*-semipolynomial expression in* $\mathbf{c}$ *formally null (in particular, this* $\mathbf{K}$*-semipolynomial expression defines the zero function of* $\mathbf{c}$ *without supposing* $H_1(\mathbf{c}) \geq 0$ *and* $H_2(\mathbf{c}) > 0$), *every coefficient* $p(\mathbf{c})$ *in the algebraic identity which must be nonnegative (resp. positive) is given by a* $\mathbf{K}$*-semipolynomial expression evidently nonnegative (resp. positive) under the hypothesis* $H_1(\mathbf{c}) \geq 0$ *and* $H_2(\mathbf{c}) > 0$. □

Remark 5.2. It has been obtained a form of the Real Positivstellensatz where the parameters in a strong incompatibility depends in a rational and continuous way on the parameters in the system considered. The restriction concerning the character locally closed of the semialgebraic $\mathbf{S}_{\mathbb{H}}$ gives a particular significance to the choice of the parameterization. One possibility a priori, is to take as distinct parameters all the coefficients appearing inside the hypothesis, but this is not an obligation. Moreover, since the semi-algebraic set $\mathbf{S}_{\mathbb{H}}$ can be easily described as the projection of a closed semialgebraic set in higher dimension, we always can be placed in the conditions where it is possible to apply Theorem 5.1, merely increasing the number of parameters. Anyway this naive idea does not solve (in a magic way) all the problems provided by the constructive algebra with real numbers given "à la Cauchy".

Example 5.3. *Polynomial positive on a compact and basic semialgebraic set.* Let K be a bounded, closed and basic semialgebraic set in $\mathbf{R}^n$ defined by the system

$$\mathbb{H}_K(\mathbf{X}): \quad q_1(\mathbf{X}) \geq 0, \ldots, q_s(\mathbf{X}) \geq 0$$

with every $q_i(\mathbf{X})$ a polynomial in $\mathbf{K}[\mathbf{X}]$.

Let $f_{n,d}(\mathbf{c}, \mathbf{X})$ be the generic polynomial with degree d and n variables as in Section 4. The semialgebraic set V_K defined by

$$V_K = \{\mathbf{c}\colon \forall x \in K \ f_{n,d}(\mathbf{c}, \mathbf{x}) > 0\}$$

is open. In fact, if the polynomial $f_{n,d}$ is, for a value $\mathbf{c}_0$, positive on K then there is a positive lower bound σ of $f_{n,d}(\mathbf{c}_0, \mathbf{X})$ on K which implies that $\sigma/2$ is a lower bound for $f_{n,d}(\mathbf{c}, \mathbf{X})$ on K with $\mathbf{c}$ enough close to $\mathbf{c}_0$.

So we are in the conditions of Theorem 3.1, the system of generalized sign conditions

$$\mathbb{H}(\mathbf{c}, \mathbf{X}): \quad q_1(\mathbf{X}) \geq 0, \ldots, q_s(\mathbf{X}) \geq 0, \ -f_{n,d}(\mathbf{c}, \mathbf{X}) \geq 0$$

is incompatible in $\mathbf{X}$ if and only if $\mathbf{c} \in V_K$ and, as V_K is an open semialgebraic set, there exists a $\mathbf{K}$-semipolynomial $v(\mathbf{c})$ in $\mathbf{c}$ verifying that the incompatibility of the system $\mathbb{H}(\mathbf{c}, \mathbf{X})$ in $\mathbf{X}$ is equivalent to $v(\mathbf{c}) > 0$.

Applying Theorem 3.1 to the incompatibility of $\mathbb{H}(\mathbf{c}, \mathbf{X})$ with $v(\mathbf{c}) > 0$ we obtain an algebraic identity with the following structure:

$$f_{n,d}(\mathbf{c}, \mathbf{X}) \left(\sum_{i \in I_1} p_i(\mathbf{c}) \left(\prod_{j \in J_i} q_j(\mathbf{X}) \right) r_i(\mathbf{c}, \mathbf{X})^2 \right)$$

$$= v(\mathbf{c})^{2p} + \sum_{i \in I_2} s_i(\mathbf{c}) \left(\prod_{j \in J_i} q_j(\mathbf{X}) \right) t_i(\mathbf{c}, \mathbf{X})^2.$$

This algebraic identity is an identity between polynomials in $\mathbf{X}$ where the coefficients are $\mathbf{K}$-spe formally null (if we equate to zero). All the expressions there appearing are polynomials inside the context defined by $v(\mathbf{c})$ and the $s_i(\mathbf{c})$'s and $p_i(\mathbf{c})$'s are $\mathbf{K}$-spe evidently nonnegative under the hypothesis $v(\mathbf{c}) > 0$.

The structure of the last equality provides us the evidence that, for $\mathbf{c}$ fixed verifying $v(\mathbf{c}) > 0$, there exists a positive lower bound for the polynomial $f_{n,d}(\mathbf{c}, \mathbf{X})$ on the bounded and closed semialgebraic set K,

$$\forall \mathbf{x} \in K \quad f_{n,d}(\mathbf{c}, \mathbf{x}) \geq \frac{v(\mathbf{c})^{2p}}{\sum_{i \in I_1} p_i(\mathbf{c}) \left(\prod_{j \in J_i} q_j(\mathbf{x}) \right) r_i(\mathbf{c}, \mathbf{x})^2} \geq \frac{v(\mathbf{c})^{2p}}{m} > 0,$$

where $m > 0$ is a lower bound of the denominator on K (it is worthy to remark that if $v(\mathbf{c}) > 0$ then the denominator is positive on K).

Example 5.4. *Polynomial positive on a regular family of compact and basic semialgebraic sets.* In the last example when dealing with the question of a polynomial positive on a compact we have parametrized the polynomial, but we can also parameterize the compact. So we will introduce the notion of regular family of compact and basic semialgebraic sets.

Let W be a locally closed semialgebraic set defined by two $\mathbf{K}$-semipolynomials $w_1(\mathbf{u})$ and $w_2(\mathbf{u})$,

$$W = \{\mathbf{u} \in \mathbf{R}^t \colon w_1(\mathbf{u}) \geq 0,\ w_2(\mathbf{u}) > 0\},$$

and we shall consider for every $\mathbf{u} \in W$ a non-empty compact semialgebraic set $K_\mathbf{u}$ defined by

$$K_\mathbf{u} = \{\mathbf{x} \in \mathbf{R}^n \colon q_1(\mathbf{u}, \mathbf{x}) \geq 0, \ldots, q_s(\mathbf{u}, \mathbf{x}) \geq 0,\ \|\mathbf{x}\| \leq p(\mathbf{u})\},$$

where the $q_i(\mathbf{u}, \mathbf{X})$'s are polynomials in $\mathbf{X}$ with coefficients $\mathbf{K}$-semipolynomials in $\mathbf{u}$ and $p(\mathbf{u})$ is a polynomial in $\mathbf{u}$. This family of compacts is said *regular* to mean that the compact set $K_\mathbf{u}$ depends continuously on $\mathbf{u}$ (for the Haussdorf distance between two compacts).

Let V be the semialgebraic set defined by

$$V = \{(\mathbf{c}, \mathbf{u}) \colon \mathbf{u} \in W \text{ and } \forall \mathbf{x} \in K_{\mathbf{u}}\ f_{n,d}(\mathbf{c}, \mathbf{x}) > 0\}$$

and $(\mathbf{c}^0, \mathbf{u}^0) \in V$. The function $f_{n,d}(\mathbf{c}^0, \mathbf{x})$ has a positive lower bound σ on the compact $K_{\mathbf{u}^0}$. Since the family of compacts is regular and $K_{\mathbf{u}}$ is explicitly bounded in terms of $\mathbf{u}$ then for $(\mathbf{c}, \mathbf{u})$ in a neighbourhood of $(\mathbf{c}^0, \mathbf{u}^0)$ in $\mathbf{R}^m \times W$, the function $f_{n,d}(\mathbf{c}, \mathbf{x})$ is bigger than $\sigma/2$ what implies that V is open in $\mathbf{R}^m \times W$ and so locally closed. So there exist two $\mathbf{K}$-semipolynomials $v_1(\mathbf{c}, \mathbf{u})$ and $v_2(\mathbf{c}, \mathbf{u})$ in the variables $(\mathbf{c}, \mathbf{u})$ defining V and giving the following equivalences:

$$(\mathbf{c}, \mathbf{u}) \in V \quad \Leftrightarrow \quad v_1(\mathbf{c}, \mathbf{u}) \geq 0,\ v_2(\mathbf{c}, \mathbf{u}) > 0 \quad \Leftrightarrow \quad \forall \mathbf{x} \in K_{\mathbf{u}}\ f_{n,d}(\mathbf{c}, \mathbf{x}) > 0.$$

These equivalences provide the following incompatible system of generalized sign conditions for the $\mathbf{K}$-semipolynomials:

$$v_1(\mathbf{c}, \mathbf{u}) \geq 0, \quad v_2(\mathbf{c}, \mathbf{u}) > 0, \quad -f_{n,d}(\mathbf{c}, \mathbf{X}) \geq 0.$$

Applying the Real Positivstellensatz for semipolynomials to this system, one gets an algebraic identity in $\mathbf{x}$ parameterized by $\mathbf{K}$-semipolynomials in $(\mathbf{c}, \mathbf{u})$, providing the evidence (in the usual algebraic way) that $f_{n,d}(\mathbf{c}, \mathbf{x}) > 0$ when $\mathbf{u} \in W$, $\mathbf{x} \in K_{\mathbf{u}}$ and $(\mathbf{c}, \mathbf{u}) \in V$.

6. Some consequences for the constructive algebra over the real numbers presented "à la Cauchy"

In constructive mathematics (see [2] or [24]) the theorems introduced in the Sections 3–5 are valid when the parameters belong to the real closure $\mathbf{R}$ of an ordered and discrete field $\mathbf{K}$ (see [23]) because in this setting we have a constructive proof for the Real Positivstellensatz (see [21]).

Every point of view will find its place in the following remark: all our proofs are effective, in particular without using the Axiom of Choice, and more precisely, provide uniformly primitive recursive algorithms if the structure of the field of parameters is given by an oracle showing the sign of every polynomial with integer coefficients in the parameters of the problem considered.

One question still missed is the study of the constructive meaning for these results in the framework of the field $\mathbb{R}$: the field of real numbers for the constructive analysis (see [2]), i.e. the real numbers defined as Cauchy sequences of rational numbers. From the algorithmic point of view, this means that the real parameters $\mathbf{c}$ are given by oracles providing suitable rational approximations for these real numbers and that we are looking for an uniformly primitive recursive algorithm. In [15] we shall provide a study of this question as systematic as possible.

In this section it will be shown how to use the parameterized results obtained concerning Hilbert's 17th problem to derive the same theorem in Constructive Algebra (while the non-parameterized solution does not allow to derive any kind of consequence). The section will be ended showing how, we think, it is necessary to formulate the Positivstellensatz problem when dealing with Cauchy real numbers.

6.1 *Hilbert's 17th problem*

Let **R** be the field of real algebraic numbers. Since the equivalence

$$\forall \mathbf{x} \in \mathbf{R}^n \quad f_{n,d}(\mathbf{c}, \mathbf{x}) \geq 0 \iff H_{n,d}(\mathbf{c}) \geq 0$$

is true for every **c** real algebraic then by continuity we have

$$\forall \mathbf{c} \in \mathbf{R}^n \quad (\forall \mathbf{x} \in \mathbb{R}^n \quad f_{n,d}(\mathbf{c}, \mathbf{x}) \geq 0 \iff H_{n,d}(\mathbf{c}) \geq 0).$$

The answer for Hilbert's 17th problem provided by Theorem 4.1 uses polynomials and semipolynomials with rational coefficients which can be, at least in principle, fully determined. The fact concerning the positivity of the coefficients (which must be positive) is constructively clear when dealing with real numbers "à la Cauchy" under the hypothesis $H_{n,d}(\mathbf{c}) \geq 0$. This implies that if the parameters **c** are in $\mathbb{R}$ and verify $H_{n,d}(\mathbf{c}) \geq 0$ then $f_{n,d}(\mathbf{c}, \mathbf{x}) \geq 0$ for every **x**. We have obtained

$$\forall \mathbf{c} \in \mathbb{R}^m \quad (H_{n,d}(\mathbf{c}) \geq \mathbf{0} \implies \forall \mathbf{x} \in \mathbb{R}^n f_{n,d}(\mathbf{c}, \mathbf{x}) \geq 0)$$

with the evidence of this fact given by an algebraic identity. So, when $H_{n,d}(\mathbf{c}) \geq 0$, Hilbert's 17th problem is solved in a continuous and rational way with respect to its coefficients.

To complete the continuous and rational solution for the field $\mathbb{R}$ we need a constructive proof for the implication

$$\forall \mathbf{x} \in \mathbb{R}^n \quad f_{n,d}(\mathbf{c}, \mathbf{x}) \geq 0 \implies H_{n,d}(\mathbf{c}) \geq 0$$

when **c** is a point with coordinates in $\mathbb{R}$. The simple proof we show here, has been given in [20] for the homogeneous case.

Let $G_{n,d}(\mathbb{R})$ be the subset of $\mathbb{R}^m$ defined by the first member of the implication to be shown and $F_{n,d}(\mathbb{R})$ the second one. We remark that the problem is reduced to the case when d is even and this is assumed in all that follows.

So we have $F_{n,d}(\mathbb{R}) \subseteq G_{n,d}(\mathbb{R})$ and we want to prove the other inclusion. We see that $G_{n,d}(\mathbb{R})$ is a convex and closed cone and as the point **c** corresponding to the polynomial $1 + (\sum_{i=1}^n x_i^2)^{d/2}$ is interior to $G_{n,d}(\mathbb{R})$, we obtain that $G_{n,d}(\mathbb{R})$ is the adherence of its interior.

Moreover, using the equivalence for every rational $\mathbf{c}$

$$\forall \mathbf{x} \in \mathbb{R}^{n} \; [illegible] (\mathbf{c}, \mathbf{x}) \geq 0 \iff H_{n,d}(\mathbf{c}) \geq 0$$

(also true for $\mathbf{c}$ real algebraic) we derive that the sets $F_{n,d}(\mathbb{R})$ and $G_{n,d}(\mathbb{R})$ have the same rational points. So, given a point in $G_{n,d}(\mathbb{R})$ we can express it as the limit of a sequence of rational points in $F_{n,d}(\mathbb{R})$ which implies that it belongs to $F_{n,d}(\mathbb{R})$ because $F_{n,d}(\mathbb{R})$ is closed.

A constructive proof (more delicate) for the equivalence

$$[illegible] \iff H_{n,d}(\mathbf{c}) \geq 0$$

when $\mathbf{c}$ is an arbitrary point with coordinates in $\mathbb{R}$ will be given in [15].

[illegible] which [illegible] the Real Positivstellen- [illegible] the field of real numbers

Let K be a discrete subfield of $\mathbb{R}$ (usually $K = \mathbb{Q}$ but another fields as $\mathbb{Q}(\pi)$ can be considered). Let $S(\mathbf{c}, X)$, as in Section 5, a system of generalized sign conditions on [illegible] where the X's are the true variables and the c's are considered as parameters [illegible] the semialgebraic set defined by

$$[illegible] \mathbb{R}^{m} [illegible] \text{ is incompatible}\}$$

[illegible]

[illegible] applying the Finiteness Theorem we can construct, in [illegible] that

$$[illegible] \mathbb{R}^{m} [illegible]$$

$$[illegible] H_{i}(\mathbf{c}) \geq 0 \text{ and } H_{j}(\mathbf{c}) [illegible]$$

[illegible] algebraic identity [illegible] incompatibility of the system

[illegible]

[illegible] coordinates in [illegible]

[illegible]

[illegible] incompatible

and this implication is made evident by an algebraic identity in $\mathbf{X}$, whose coefficients are $\mathbf{K}$-semipolynomials in $\mathbf{c}$.

So, to prove constructively the corresponding case of the Real Positivstellensatz (continuous and rational) is the same thing that to provide a constructive proof for the implication

$$\forall \mathbf{x} \in \mathbb{R}^r \quad \mathrm{H}(\mathbf{c}, \mathbf{X}) \text{ is incompatible} \Rightarrow (H_1(\mathbf{c}) \geq 0 \text{ and } H_2(\mathbf{c}) > 0)$$

when $\mathbf{c}$ is a point with coordinates in $\mathbb{R}$.

In the particular case of Hilbert's 17th problem the proof was found taking advantage of the particular case we were dealing with. So more general tools to deal with this kind of questions need to be created. A result seems essential, the constructive proof that for any locally closed semialgebraic set $\mathbf{S}$ defined by the conditions $H_1(\mathbf{c}) \geq 0$ and $H_2(\mathbf{c}) > 0$ (with $H_1(\mathbf{c})$ and $H_2(\mathbf{c})$ $\mathbf{K}$-semipolynomials), every point in $\mathbf{S}(\mathbb{R})$ is a limit of points in $\mathbf{S}(\mathbf{R})$.

If this program is fullfilled, Example 5.3 will provide a Positivstellensatz for the case of a polynomial in $\mathbb{R}[\mathbf{X}]$ everywhere positive on a $\mathbb{Q}$-semialgebraic basic compact set, and Example 5.4 will provide a Positivstellensatz for the case of a polynomial in $\mathbb{R}[\mathbf{X}]$ everywhere positive on a $\mathbb{R}$-semialgebraic basic compact set that can be described as a member of a regular family of $\mathbb{Q}$-semialgebraic basic compact sets.

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