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On Two Minimax Theorems in Graph Theory

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This paper contains new proofs of, and certain remarks on, two minimax theorems recently found by Edmonds [2] and Lucchesi and Younger [6], respectively. An interesting feature of both theorems is that they do not seem to follow from the known minimax theorems of matroid theory (it is of course a different question whether they have generalizations to matroids: Section 1 below seems to indicate some connection). The reason why we include these proofs here, in one paper, is not only the similar character of the two theorems but also that we formulate a problem on hypergraphs which could serve as a common generalization of both.

1. Let G be a digraph with a root a. A *branching* (rooted at a) is a spanning tree which is directed in such a way that each $x \neq a$ has one edge coming in. A *a-cut* of G determined by a set $S \subset V(G)$ is the set of edges going from S to $V(G) - S$. It will be denoted by $\Delta_G(S)$. We also set $\delta_G(S) = |\Delta_G(S)|$.

THEOREM 1 (Edmonds). *The maximum number of edge-disjoint branchings (rooted at a) equals the minimum number of edges in a-cuts.*

Proof. The nontrivial part of the theorem says that if $\delta_G(S) \geq k$ for every $S \subset V(G)$, $a \in S$ then there are k edge-disjoint branchings. We use induction on k.

Let F be a set of edges such that

(i) F is an arborescence rooted at a (i.e., a tree such that $a \in V(F)$ and it is directed in such a way that exactly one edge enters each point $x \neq a$ of F);

(ii) $\delta_{G-F}(S) \geq k - 1$ for every $S \subset V(G)$, $a \in S$.

If F covers all points, i.e., it is a branching we are finished: $G - F$ contains $k - 1$ edge-disjoint branchings and F is in the kth one.

Suppose F only covers a set $T \subset V(G)$. We show we can add an edge $e \in \Delta_G(T)$ to F so that the arising arborescence $F + e$ still satisfies (i) and (ii).

Consider a maximal set $A \subset V(G)$ such that

(a) $a \in A$;

(b) $A \cup T \neq V(G)$;

(c) $\delta_{G-F}(A) = k - 1$.

If no such A exists any edge of $\Delta_G(T)$ can be added to F.

Since

$$\delta_{G-F}(A \cup T) = \delta_G(A \cup T) \geq k,$$

we have $A \cup T \neq A$, $T \not\subseteq A$. Also,

$$\delta_{G-F}(A \cup T) > \delta_{G-F}(A)$$

and so, there must be an edge $e = (x, y)$ which belongs to $\Delta_{G-F}(A \cup T) - \Delta_{G-F}(A)$. Hence $x \in T - A$ and $y \in V(G) - T - A$. We claim e can be added to F, i.e., $F + e$ satisfies (i) and (ii). (i) is trivial.

Let $S \subset V(G)$, $a \in S$. If $e \notin \Delta_G(S)$ then

$$\delta_{G-F-e}(S) = \delta_{G-F}(S) \geq k - 1.$$

If $e \in \Delta_G(S)$ then $x \in S$, $y \in V(G) - S$. We use now the inequality

$$\delta_{G-F}(S \cup A) + \delta_{G-F}(S \cap A) \leq \delta_{G-F}(S) + \delta_{G-F}(A), \tag{1}$$

which follows by an easy counting. Here

$$\delta_{G-F}(A) = k - 1, \qquad \delta_{G-F}(S \cap A) \geq k - 1$$

and, by the maximality of A,

$$\delta_{G-F}(S \cup A) \geq k,$$

since $S \cup A \neq A$ as $x \in S - A$ and $S \cup A \neq V(G)$ as $y \notin S \cup A$. Thus (1) implies

$$\delta_{G-F}(S) \geq k$$

and so,

$$\delta_{G-F-e}(S) \geq k - 1.$$

Thus, we can increase F till finally it will satisfy (i), (ii), and $V(F) = V(G)$. Then apply the induction hypothesis on $G - F$. This completes the proof.

Remark. The above proof yields an efficient algorithm to construct a maximum set of edge-disjoint branching. We will need a good algorithm to determine

$$K(G) = \min_{\substack{S \subset V(G) \\ a \in S}} \delta_G(S).$$

This requires the computation of $n - 1$ flow values ($n = |V(G)|$, $m = |E(G)|$ and hence, it can be done in p steps, where p is a polynomial in n, m. Now we start defining F. At any stage, we can increase it by checking at most m edges e whether or not

$$K(G - F - e) \geqslant k - 1.$$

In fact, we do not need to check more than m edges altogether; since if an edge is checked then it is either put into F or it cannot come into consideration as an element of F anymore. This shows that we obtain F in mp steps and thus, we obtain k edge-disjoint branchings in kmp steps.

2. Let G be a weakly connected digraph.[1] By a *directed cut* of G we mean the set $D = \Delta_G(S)$ ($S \subset V(G)$, $S \neq \varnothing$) provided $\Delta_G(V(G) - S) = \varnothing$. Note that a directed cut D uniquely determines a set S_D with $D = \Delta_G(S_D)$.

THEOREM 2 (Lucchesi and Younger). *The maximum number of disjoint directed cuts equals the minimum number of edges which cover all directed cuts.*

This result had been conjectured by N. Robertson and D. H. Younger for some years. Also its special case when G is bipartite, which follows from the results in McWhirter and Younger [7], is very interesting.

We remark that the minimum number of edges which cover all directed cuts can be interpreted as the minimum number of edges whose contraction results in a strongly connected graph.

Proof of Theorem 2. We use induction on the number of edges. If this is 0 the assertion can be considered as true. Let $e \in E(G)$. Contract e; this results in a digraph G''_e. It is easily seen that the directed cuts of G''_e are exactly those of G not containing e.

Let k be the maximum number of disjoint directed cuts in G. If there is an edge $e \in E(G)$ such that G''_e contains at most $k - 1$ edge-disjoint directed cuts then, by the induction hypothesis, there exist $k - 1$ edges $e_1, \ldots, e_{k-1}$ covering all directed cuts of G''_e. Then $e, e_1, \ldots, e_{k-1}$ are k edges

[1] This assumption is irrelevant but convenient.

which cover all directed cuts of G. Since, obviously at least k edges are needed, the assertion is proved. Thus we may assume G''_e contains k disjoint directed cuts for each edge e.

If we subdivide all edges of G by a point the arising graph contains $k + 1$ disjoint directed cuts. Hence we can find a subdivision H of G such that H contains at most k disjoint directed cuts but if we subdivide a certain edge f of H by a point then it will contain $k + 1$ disjoint directed cuts. Hence H contains $k + 1$ directed cuts $D_1, \ldots, D_{k+1}$ such that only two of them have a common edge which is f.

Also observe that H''_f arises either from G or from G''_f by subdivision. Hence by the assumption made above, H contains k disjoint directed cuts $C_1, \ldots, C_k$ such that $f \notin C_i$. Thus $D_1, \ldots, D_{k+1}, C_1, \ldots, C_k$ is a collection of directed cuts of G_0 such that any edge belongs to at most two of them. Thus it suffices to show:

LEMMA. *If a digraph G contains at most k disjoint directed cuts, and F is any collection of directed cuts in G such that any edge belongs to at most two of them then $|F| \leqslant 2k$.*

Proof of the lemma. First we replace F by a collection of a simple structure. Let $D_1, D_2 \in F$ be called *laminar* if $S_{D_1} \cap S_{D_2} = \varnothing$ or $S_{D_1} \subseteq S_{D_2}$ or $S_{D_2} \subseteq S_{D_1}$ or $S_{D_1} \cup S_{D_2} = V(G)$. Otherwise, D_1 and D_2 are called *crossing*.

Let D_1, D_2 be a crossing pair. Set

$$D_1' = \Delta_G(S_{D_1} \cup S_{D_2}), \qquad D_2' = \Delta_G(S_{D_1} \cap S_{D_2}),$$

$$F' = F \cup \{D_1', D_2'\} - \{D_1, D_2\}.$$

It is easily checked that D_1', D_2' are directed cuts. Moreover, D_1', D_2' cover any edge the same number of times as D_1, D_2. Hence F' has the same properties as F, and $|F'| = |F|$.

Also,

$$\sum_{D \in F} |S_D|^2 < \sum_{D \in F'} |S_D|^2$$

since

$$|S_{D_1} \cup S_{D_2}|^2 + |S_{D_1} \cap S_{D_2}|^2 > |S_{D_1}|^2 + |S_{D_2}|^2.$$

Hence, if we do the same with F' as we did with F, i.e., we replace two crossing cuts by two new directed cuts and repeat this procedure we cannot go into a cycle, i.e., finally we get a collection F_0 of directed cuts such that any edge belongs to at most two of them, any two are laminar and $|F_0| = |F|$. So it suffices to prove the Lemma in the case when F consists of pairwise laminar cuts.

Let $F = \{D_1, \dots, D_N\}$. We construct a graph G' as follows. $V(G') = \{v_1, \dots, v_N\}$ and we join v_i to v_j iff $D_i \cap D_j \neq 0$. Then G' contains at most k independent points. We show it is bipartite. This will imply it has at most $2k$ points (k in each color class) i.e., $N \leqslant 2k$ as stated.

To show G' is bipartite we consider a circuit $(v_1, \dots, v_m)$ in G' and the corresponding sets $S_{D_1}, \dots, S_{D_m}$. $D_1, \dots, D_m$ must be different. For if $D_\nu = D_\mu$ then each edge of D_ν belongs to both D_ν and D_μ; thus, to no other member of F. Hence v_ν has degree 1 and it cannot occur in any circuit of G'.

Since $D_i \cap D_{i+1} \neq 0$ ($i = 0, \dots, m-1$; $D_0 = D_m$), we have either $S_{D_i} \subset S_{D_{i+1}}$ or $S_{D_i} \supset S_{D_{i+1}}$. We claim the two possibilities occur alternatingly: this will prove m is even. Suppose not, e.g., $S_{D_0} \subset S_{D_1} \subset S_{D_2}$. We say D_i is to the left from D_j if either $S_{D_i} \subset S_{D_j}$ or $V(G) - S_{D_i} \subset S_{D_j}$; D_i is to the right from D_j if $S_{D_i} \subset V(G) - S_{D_j}$ or $V(G) - S_{D_i} \subset V(G) - S_{D_j}$. Since F consists of laminar cuts, each $D_i \neq D_j$ is either to the left or to the right from D_j. Since D_2 is to the right from D_1 but $D_0 = D_m$ is to the left from D_1, there is a j, $1 \leqslant j \leqslant m-1$ such that D_j is to the right from D_1 but D_{j+1} is to the left from D_1. But D_j and D_{j+1} have a common edge e which, therefore, must belong to D_1. Thus e belongs to three cuts, a contradiction.

Remark. The proof uses several ideas which occur in previous papers. Thus crossing and laminar cuts occur in [7]; families F of cuts covering each edge at most twice are considered in [8].

3. A hypergraph H is a finite collection of finite sets. These sets are called *edges*, the elements of edges are called *vertices*. The set of vertices is denoted by $V(H)$. If $E_1, \dots, E_m$ are the edges, $v_1, \dots, v_n$ are the vertices of hypergraph H then we define

$$a_{ij} = \begin{cases} 1, & \text{if } v_i \in E_j; \\ 0, & \text{otherwise.} \end{cases}$$

The matrix $A = (a_{ij})$ is called the *incidence matrix* of H.

A *partial hypergraph* of H is a subcollection of (the collection of edges of) H. The *partial hypergraph induced by* $S \subseteq V(H)$ is the collection of edges contained in S.

If $h > 0$ the *multiplication of a vertex x by h* means that we replace x by h points $x_1, \dots, x_h$ and each edge E containing x by h edges $E - \{x\} \cup \{x_h\}$. The partial hypergraph induced by S can be obtained by multiplying the points of $V(H) - S$ by 0.

Let $\nu(H)$ denote the maximum number of disjoint edges of H and let $\tau(H)$ denote the minimum number of points covering (representing) all edges of H. These numbers can be considered as optima of the linear programs

$$\begin{array}{cc} x \text{ integer} & x \text{ integer} \\ Ax \leqslant 1 & A^T x \geqslant 1 \\ x \geqslant 0 & x \geqslant 0 \\ \hline \max 1.x & \min 1.x \end{array} \tag{2}$$

Let $\nu^*(H)$ and $\tau^*(H)$ denote optima of these programs when dropping the assumption that x is an integer, then

$$\nu(H) \leqslant \nu^*(H) = \tau^*(H) \leqslant \tau(H).$$

Also, let us denote by $\frac{1}{2}\nu_2(H)$ and $\frac{1}{2}\tau_2(H)$ the optima for solutions with coordinates half of an integer, then

$$\nu(H) \leqslant \tfrac{1}{2}\nu_2(H) \leqslant \nu^*(H) = \tau^*(H) \leqslant \tfrac{1}{2}\tau_2(H) \leqslant \tau(H).$$

In [5] a hypergraph was called *normal* if $\nu(H') = \tau(H')$ holds for every partial hypergraph H' of H and *seminormal* if $\nu(H') = \tau(H')$ holds for every induced partial hypergraph. It was proved that

(A) A hypergraph is normal iff $\tau(H') = \tau^*(H')$ holds for every partial hypergraph H';

(B) A hypergraph is seminormal iff $\nu(H') = \nu^*(H')$ holds for every induced partial hypergraph H'.

It is easy to see that (B) is of stronger type than (A); in fact, to show the relation $\nu(H) = \tau(H)$ we have to consider induced partial hypergraphs only. We remark that hypergraphs with totaly unimodular incidence matrix are normal (see [1]).

Now form two hypergraphs as follows. If G is a digraph rooted at a then let B_G consist of the sets of edges of branchings rooted at a. Then Theorem 1 expresses

$$\nu(B_G) = \tau(B_G). \tag{3}$$

If we remove an edge from G this means removal of a point of B_G together with all edges containing it. So, the induced partial hypergraphs of B_G are of form $B_{G'}$, and hence, they also satisfy (3). So B_G is seminormal. It is easy to see that B_G is not always normal.

Define the hypergraph D_G to consist of all directed cuts of the digraph G. Then Theorem 2 says

$$\nu(D_G) = \tau(D_G); \tag{4}$$

and if we contract an edge of G then this will correspond in D_G to the removal of a point together with all edges containing it. Hence [5] holds for the induced partial hypergraphs of D_G as well, i.e., D_G is seminormal.

This raises the question if seminormal hypergraphs have a characterization which would imply Theorems 1 and 2? In other words is there a simple property P of hypergraphs such that the theorem

A hypergraph is seminormal iff each induced partial hypergraph of it has this property P

holds? (B) above is an example but the property $\nu(H) = \nu^*(H)$ is not too easy to verify. Nevertheless, it should be pointed out that our proof of Theorem 2 does something similar.

In fact, the first argument actually proved

THEOREM 3. *If any hypergraph H' arising from H by multiplication of the vertices satisfies $\nu_2(H') = 2\nu(H')$ then $\tau(H) = \nu(H)$.*

It would be possible to give a separate (but related) proof based on Theorem B above. First we show that (1) if F is a collection of pairwise laminar directed cuts then its incidence matrix A is totally unimodular.

A simple proof of this fact was mentioned to me by N. Robertson (private communication). We can find a directed tree T with $V(T) \supseteq V(G)$, $E(T) = \{f_1, \ldots, f_N\}$ so that the cut of G determined by the edge f_i of T in the natural way is exactly D_i and f_i is oriented correspondingly to D_i. Let I be the $N \times N$ identity matrix, then (I, A) is the regular representation of the circuit matroid of $T \cup G$ in the basis T. It is well known that the matrix (I, A) is totally unimodular and hence, so is A.

(2) If F is a collection of laminar directed cuts $k+1$ of which are disjoint and no $s+1$ of which have a common edge then $|F| \leq sk$.

This follows from well knows results on hypergraphs (see [1, Chap. 20]).

(3) If F is any collection of directed cuts of a digraph G with $\nu(D_G) = k$ and no edge of G is contained in more than s members of T then $|F| \leq ks$. In other words, $\nu(D_G) = \nu^*(D_G)$.

This follows from (2) by exactly the same argument as used in the first part of the proof of the Lemma.

(4) $\nu(D_G) = \tau(D_G)$. By Theorem B.

We remark that $\tau(B_G) = \tau^*(B_G)$ is easily verified. One can show even more: the polyhedron

$$A^T \cdot x \geqslant 1$$
$$x \geqslant 0$$

has integral vertices; hence any optimal solution of the program defining $\tau^*(B_G)$ is baricenter of integral solutions. This follows from the results of Fulkerson [3].

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