Facets With Fixed Defect of the Stable Set Polytope

László Lipták László Lovász
Yale University

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Abstract

The stable set polytope of a graph is the convex hull of the 0-1 vectors corresponding to stable sets of vertices. To any nontrivial facet $\sum_{i=1}^{n} a_i x_i \leq b$ of this polytope we associate an integer δ, called the *defect* of the facet, by $\delta = \sum_{i=1}^{n} a_i - 2b$. We show that for any fixed δ there is a finite collection of graphs (called "basis") such that any graph with a facet of defect δ contains a subgraph which can be obtained from one of the graphs in the basis by a simple subdivision operation.

1 Introduction

In this paper a graph $G = (V, E)$ will be assumed to be simple, finite, and undirected. $V = V(G)$ denotes the set of *vertices* (or *nodes*) and $E = E(G)$ denotes the set of *edges*. The *degree* $d(v)$ of a vertex $v \in V$ is the number of edges incident to v. A subset S of vertices in G is called *stable* or *independent* if no two vertices of S are adjacent. The maximum size of a stable set is the *stability number* of G, and will be denoted by $\alpha(G)$. A graph G is called α-*critical* if it has no isolated nodes and the removal of any edge increases the stability number of G, that is for any edge $e \in E$ we have $\alpha(G - e) > \alpha(G)$.

In Section 2 we review the known properties of α-critical graphs. Gallai introduced the notion of the defect of an α-critical graph, which turned out to be very useful in classifying α-critical graphs. The defect of the α-critical

graph gives an upper bound for the degree of any vertex (Hajnal, Theorem 2), and also an upper bound for the number of vertices of degree at least 3 (Lovász, Theorem 5).

In Section 3 we consider the stable set polytope of a graph, and its facets. A graph together with a facet of its stable set polytope will be called a facet-graph. Chvátal (Theorem 7) showed that every α-critical graph is a facet-graph with all the weights equal to 1. The defect of a facet-graph is a generalization of the defect of an α-critical graph, and turns out to be useful in classifying critical facet-graphs. Similarly as in the case of α-critical graphs, the defect of a critical facet-graph gives an upper bound for the degree of any vertex (Theorem 9), and our main theorem (Theorem 11) gives a finiteness result for critical facet-graphs of fixed defect. The main tool in the proof is a representation of facet-graphs as the homomorphic image of an α-critical graph (Lemma 3).

2 α-critical graphs

Let G be an α-critical graph. The quantity $\delta(G) = |V(G)| - 2\alpha(G)$, called the *defect* of G, was introduced by Gallai and turned out to be very useful in the classification of α-critical graphs.

The first theorem concerning α-critical graphs is due to Erdős and Gallai [4]:

Theorem 1 *If G is α-critical, then $\delta(G) \geq 0$.*

Hajnal [5] proved the following result:

Theorem 2 *If G is α-critical with defect δ, then $d(v) \leq \delta + 1$ for all $v \in V$.*

A more detailed account about these theorems can be found in [9].

Theorem 2 shows that if G is α-critical and connected, then its defect is 0 if and only if G is an edge, and its defect is 1 if and only if G is an odd cycle C_{2n+1} (let C_k denote a cycle of length k). Andrásfai [1] gave a full characterization of α-critical graphs with $\delta = 2$:

Theorem 3 *If G is connected and α-critical, then $\delta = 2$ if and only if G is an odd subdivision of K_4.*

Here *odd subdivision* of a graph means that we replace every edge by a path of odd length (the new vertices have degree 2), and K_n denotes the complete graph on n vertices. This operation can be generalized to the facets of the stable set polytope, which will be discussed in the next section. Using this operation enables us to express the classification of α-critical graphs with defect 1 in a similar form:

Theorem 4 *If G is connected and α-critical, then $\delta = 1$ if and only if G is an odd subdivision of K_3.*

A full classification is known also for $\delta = 3$ (see [10, p. 454], but in general only the following theorem is known:

Theorem 5 *For any fixed δ there exists a finite collection of α-critical graphs such that every α-critical graph with defect δ is an odd subdivision of one of them.*

This theorem was proved for $\delta = 3$ by Surányi [17], and for all δ by Lovász [7]. We give a generalization of this theorem in Theorem 11.

Another interesting property of α-critical graphs is that the number of edges not covered by a maximum stable set is bounded by the defect [8]:

Theorem 6 *A maximum stable set M in an α-critical graph avoids at most $\binom{\delta+1}{2}$ edges.*

We give an easy generalization of this bound to the stable set polytope in Theorem 12.

3 The stable set polytope

First we define the stable set polytope. The motivation is Chvátal's theorem (Theorem 7), which shows that a facet of the stable set polytope can be viewed as a weighted generalization of an α-critical graph.

Given a set of vertices $S \subseteq V$, its *incidence vector* $\chi_S \in \{0,1\}^V$, is defined by

$$\chi_S(u) = \begin{cases} 1 & \text{if } u \in S, \\ 0 & \text{if } u \notin S. \end{cases}$$

The *stable set polytope* of G, denoted by $STAB(G)$, is defined to be the convex hull of the incidence vectors of all stable sets of G:

$$STAB(G) = \text{conv}\{\chi_S : S \text{ is a stable set in } G\}.$$

Since $STAB(G)$ contains the basic unit vectors and the origin ($STAB(G)$ is contained in the unit cube $[0,1]^V$), it is full-dimensional. Hence up to a constant there is a unique system of linear inequalities describing its facets. The facets of $STAB(G)$ can be divided into two classes:

1. **Trivial facets**: $x_v \geq 0$ for all $v \in V$; $x_v \leq 1$, if $v \in V$ is an isolated vertex in G.

2. **Nontrivial facets**: $\sum_{v \in V} a(v)x_v \leq b$, where all coefficients are non-negative, and at least two among the $a(v)$'s are positive. When an inequality of the form $x_v + x_w \leq 1$ defines a facet, it will be called an *edge inequality.*

The coefficients $a(v)$ will be referred to as *weights*. Sometimes we will simply write $a \cdot x$ instead of $\sum_{v \in V} a(v)x_v$. For $S \subseteq V$, define the weight of S by $a(S) := \sum_{v \in S} a(v)$. The graph G with the weighting a will be called a *weighted graph*.

In the classification of the facets the inequalities indicate which halfspace the stable set polytope belongs to. To make these inequalities unique we scale them so that their coefficients become integral and their greatest common divisor is 1. The facet defined by the inequality $a \cdot x \leq b$ will be sometimes called the facet $a \cdot x \leq b$.

The graph G with the weighting a will be called the *weighted graph* (G, a) ($b = \max\{ax : x \in STAB(G)\}$ will be usually omitted from the notation). If $U \subseteq V$ is a stable set in G with $a(U) = b$, then U will be called a *maximum weight independent* (or *stable*) *set* (we will usually restrict ourselves to one facet, so the fact that an independent set may be of maximum weight with respect to a facet, but not with respect to another one will not cause confusion). The edge $e \in E(G)$ in the weighted graph (G, a) will be called *critical* if the maximum weight of a stable set in G is larger then in $G - e$. The weighted graph (G, a) will be called *critical* if every edge $e \in E(G)$ is critical. If (G, a) is not critical, then we can remove edges from G until it becomes critical.

The weighted graph (G, a) with $a > 0$ will be called a *facet-graph* if $\sum_{v \in V} a(v)x_v \leq b$ defines a facet of $STAB(G)$. The condition $a > 0$ can always be achieved by deleting the vertices with zero weight. The *defect* of this facet is defined by $\delta = \sum_{v \in V} a(v) - 2b = a(V) - 2b$. This definition is motivated by Theorem 8, which shows that this number can be viewed as the integrality gap between an integer programming problem and its linear relaxation.

An important class of facet-graphs was found by Chvátal [3]:

Theorem 7 *If G is an α-critical graph, then $\sum_{v \in V} x_v \leq \alpha(G)$ defines a facet of $STAB(G)$.*

In other words, every α-critical graph is a critical facet-graph if we set all weights equal to 1. Notice that in this case the two notions of defect coincide, since $\delta = \sum_{v \in V} 1 - 2\alpha(G) = |V(G)| - 2\alpha(G)$. This shows that the defect of a facet-graph is a generalization of the defect of G.

A relaxation of the stable set polytope is the *fractional stable set polytope*, $FRAC(G)$, which is defined by the trivial facets and the edge inequalities:

$$FRAC(G) = \{x \in [0,1]^{V(G)} : x_u + x_v \leq 1 \text{ for all } uv \in E\}.$$

Nemhauser and Trotter [**?**] proved that $FRAC(G)$ has 1/2-integral vertices (i.e. the vertices of $FRAC(G)$ have components 0, 1, or 1/2). Lovász and Schrijver [11] proved that if the facet $\sum_{v \in V} a(v)x_v \leq b$ is nontrivial, then $\sum_{v \in V} a(v)x_v$ is maximized at $(\frac{1}{2}, \ldots, \frac{1}{2})$. This result can be formulated as follows, which shows that it is a generalization of Theorem 1:

Theorem 8 *If $\sum_{v \in V} a(v)x_v \leq b$ is a nontrivial facet of $STAB(G)$, then the defect of (G, a) is nonnegative and is the maximum of $2(\sum_{v \in V} a(v)x_v - b)$ over the fractional stable set polytope.*

This theorem also shows that the defect of a facet-graph is an integrality gap, and that the defect of a facet-graph is 0 if and only if the graph is an edge.

Sewell [14] proved the following upper bound on the coefficients of the facets:

Theorem 9 *Let $\sum_{v \in V} a(v)x_v \leq b$ define a facet of $STAB(G)$ with defect $\delta > 0$. Then $a(v) \leq \delta$ for all $v \in V$.*

The defect δ of a facet was used in [11] to bound the number of iterations needed to derive the facet from edge constraints by a certain iterative procedure. Structural results about α-critical graphs—such as Hajnal's theorem (Theorem 2)—can be extended to critical facet-graphs with appropriate care. For example, one can prove that every degree is at most $2\delta - 1$ when $\delta > 1$. This will be explored in [6].

Similarly as in the case of α-critical graphs, critical facet-graphs with a fixed defect have a finite classification. The results for small values of δ are rather similar to the results for α-critical graphs, which is shown by the following theorem:

Theorem 10 *Let (G, a) be a critical facet-graph. Its defect is 0 if and only if G is an edge, its defect is 1 if and only if G is an odd cycle, and its defect is 2 if and only if G is an odd subdivision of one of the following five graphs shown in Figure 1 (only weights different from 1 are indicated).*

Here the case $\delta = 0$ is trivial by Theorem 8, and the case $\delta = 1$ follows from the corresponding theorem concerning α-critical graphs, since all coefficients are 1 by Theorem 9. The case $\delta = 2$ was proved by Sewell [14], where an odd subdivision of a facet-graph is defined similarly as an odd subdivision of an α-critical graph as follows:

We first define the *subdivision* of a facet-graph (G, a), introduced first by Wolsey [19]: Let $uv \in E$ be an edge of V, and let $c_{uv} = \max\{ax : x \in STAB(G - uv)\} - b$. The quantity c_{uv} will be called the *surplus* of the edge uv, and it shows by how much the maximum weight of an independent set increases if we delete the edge uv. Clearly the surplus of any edge is at most the minimum of the weights of the two vertices of the edge ($c_{uv} > 0$ if and only if the edge uv is critical). If the surplus of the edge is equal to this minimum, we call the edge *closed*. Introduce now two new vertices on the edge uv with weights c_{uv} (the new vertices are of degree 2), and let (G', a) be the resulting weighted graph. This procedure is demonstrated in Figure 2, where the weights of the vertices are indicated. When this subdivision operation is repeated finitely many times, the final weighted graph will be called an *odd subdivision* of the original facet-graph.

The following lemma shows that the subdivision of a facet-graph is a generalization of the odd subdivision of critical graphs. A proof can be found in [19].

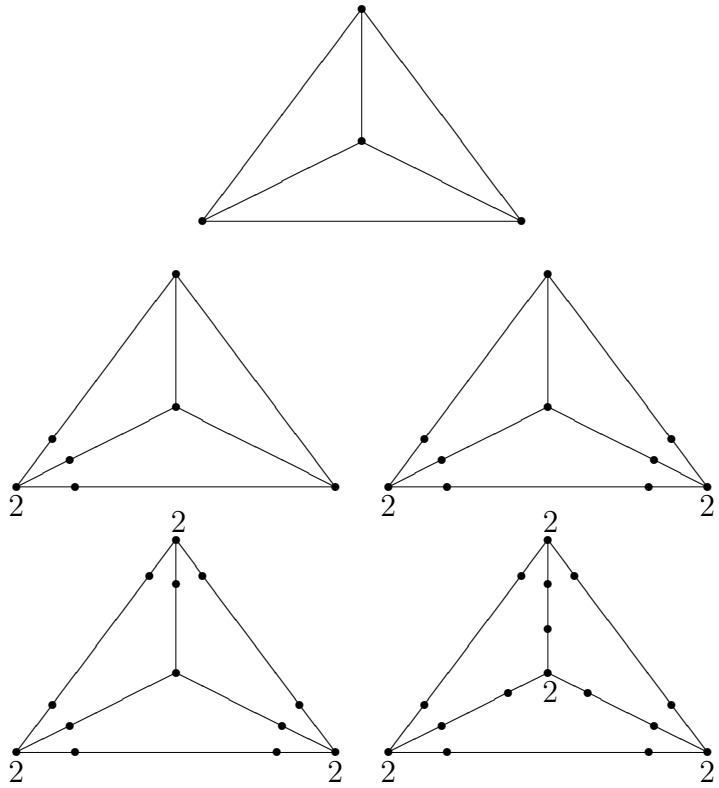

Figure 1: Basic facet-graphs with defect 2.

Lemma 1 *Every odd subdivision of a critical facet-graph with positive defect is again a critical facet-graph with the same defect.*

The converse of this lemma is not always true (a counterexample is the third graph in Figure 1); however, the following partial result holds, proved by Barahona and Mahjoub [2]:

Lemma 2 *Let (G, a) be a facet-graph. If $d(u) = d(v) = d(w) = 2$, and $uv, vw, wz \in E(G)$, then $a(u) = a(v) = a(w)$, and $(G - \{v, w\} + uz, a)$ is also a facet-graph with the same defect. If $a(u) = a(v)$, then the same statement holds even without the assumption that $d(u) = 2$.*

These lemmas show that those paths whose inner vertices are of degree 2 play a special role. Call such a path a *pending* path.

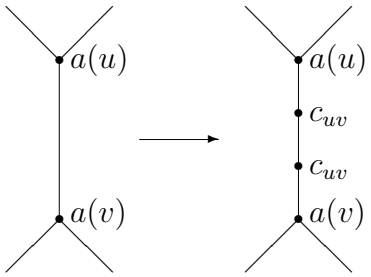

Figure 2: Subdivision of a facet-graph.

Theorem 10 gave a description of all critical facet-graphs for $\delta \leq 2$. For higher values of δ a full classification is not known, and the difficulty of this problem is shown by the fact that even small graphs can have facets with large defect. Examples can be obtained by the following method:

Suppose that (G_1, a_1) and (G_2, a_2) are facet-graphs with facet-defining inequalities

$$\sum_{u \in V(G_i)} a_i(u) x_u \leq b_i, \quad i = 1, 2.$$

Define the graph $G = G_1 \times G_2$ by $V(G) = V(G_1) \cup V(G_2)$ and

$$E(G) = E(G_1) \cup E(G_2) \cup \{uv : u \in V(G_1), v \in V(G_2)\}.$$

It is easy to see that the inequality

$$\sum_{u \in V(G_1)} b_2 \cdot a_1(u) x_u + \sum_{u \in V(G_2)} b_1 \cdot a_2(u) x_u \leq b_1 \cdot b_2$$

defines a facet of $STAB(G)$. If $\gcd(b_1, b_2) = 1$, then this inequality cannot be simplified, hence the defect of this facet becomes

$$\begin{aligned} \delta &= b_2 a_1(V(G_1)) + b_1 a_2(V(G_2)) - 2b_1 b_2 \\ &= b_2(a_1(V(G_1)) - 2b_1) + b_1(a_2(V(G_2)) - 2b_2) + 2b_1 b_2 \\ &= b_2 \delta_1 + b_1 \delta_2 + 2b_1 b_2. \end{aligned}$$

An example is shown on Figure 3, where $G_1 = C_5, G_2 = C_7$. The weights of G become 3 for the vertices of G_1 and 2 for the vertices of G_2, $b = 2 \cdot 3 = 6$,

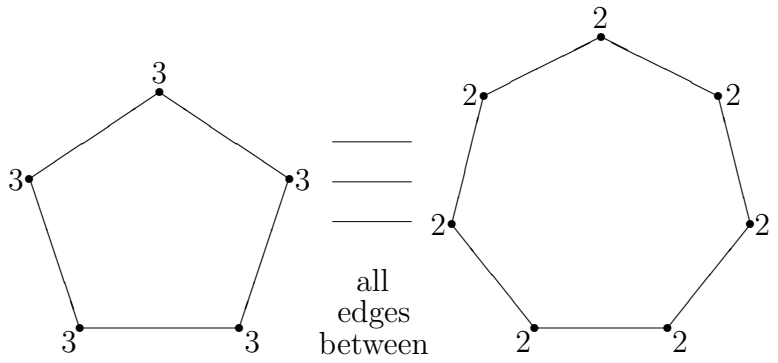

Figure 3: A facet-graph with defect $\delta = 17$

and the defect of this facet is (since the defect of an odd cycle is 1) $\delta = 3 \cdot 1 + 2 \cdot 1 + 2 \cdot 2 \cdot 3 = 17$.

Using this method it is easy to see that the defect of a facet can be exponentially large in the number of vertices. Although the facet-graph (G, a) obtained by this method is usually not critical even if both G_1 and G_2 are, we can always delete some noncritical edges to make the facet-graph critical.

Our main result is the following finiteness theorem, which is a generalization of Theorem 5:

Theorem 11 *For every fixed δ there exists a finite collection of critical facet-graphs with defect δ such that every critical facet-graph with defect δ is an odd subdivision of one of them.*

The main tool in the proof of this theorem is the following lemma:

Lemma 3 *Every critical facet-graph (G, a) can be obtained as a homomorphic image of an α-critical graph with the same defect, i.e. G can be obtained by identifying some nonadjacent vertices of an α-critical graph H, and for any vertex $v \in V(G)$ the weight of v is the number of those vertices in H which were identified to v.*

In addition, we prove the following properties for these identifications:

Lemma 4 *(a) Two vertices of the same pending path of H are never identified.*

(b) If two pairs of vertices are identified from two pending paths of H, then the number of vertices between the identified vertices is the same for these pending paths, and the vertices of the subpaths between these vertices must be pairwise identified as shown in Figure 4.

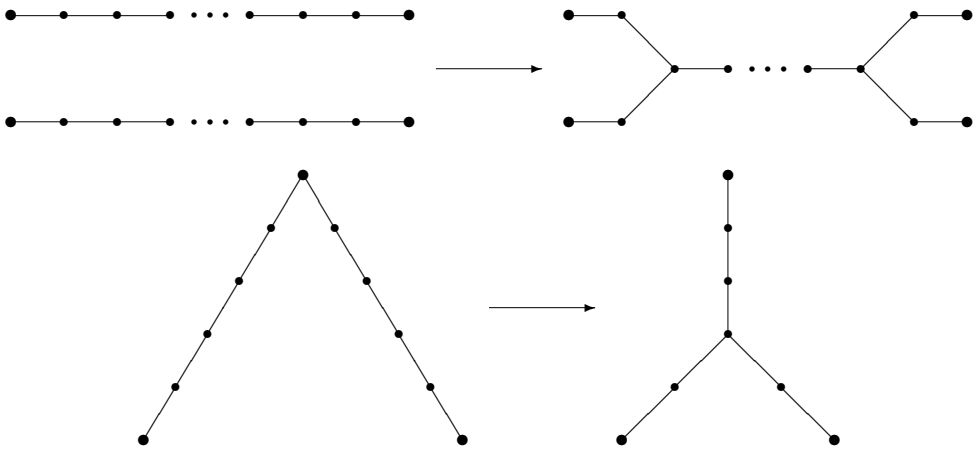

Figure 4: Possible identifications along two pending paths

Before proving these lemmas, we introduce an operation on graphs. Let (G, a) be a facet-graph, and suppose that $vw \in E(G)$ is a critical edge in (G, a) with $a(v) > a(w) \geq 1$. The following operation will be called the *splitting of the node v*:

1. Introduce a new vertex, v', and join it to every neighbor of v except w. Let the resulting graph be G'.

2. Define the new weighting a' by

$$a'(u) = \begin{cases} c_{vw} & \text{if } u = v, \\ a(v) - c_{vw} & \text{if } u = v', \\ a(u) & \text{otherwise.} \end{cases}$$

   This definition makes sense, because $c_{vw} \leq a(w) < a(v)$. The operation is demonstrated on Figure 5.

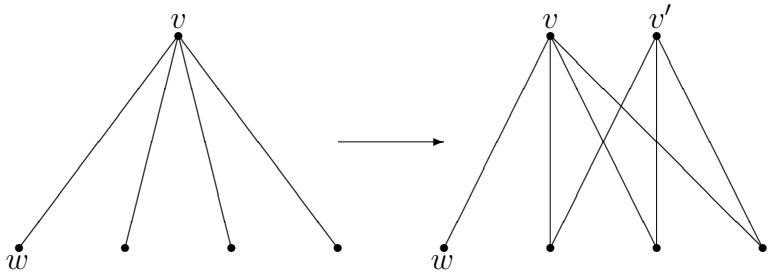

Figure 5: Splitting of a node

The following lemma shows that using this operation we can obtain new facet-graphs:

Lemma 5 *$\sum_{v \in V(G')} a'(v)x_v = a'x \leq b$ defines a facet of $STAB(G')$ with the same defect (i.e. (G', a') is also a facet-graph).*

Proof: We show that the given inequality is valid and gives rise to a facet. First we prove that the given inequality is valid for $STAB(G')$. Take a maximum weight independent set M' in (G', a'). If both v and v' are in M', then $M = M' \setminus \{v'\}$ is an independent set in G, hence $a'(M') = a(M) \leq b$. If neither v nor v' are in M', then $M = M'$ is an independent set in G, hence $a'(M') = a(M) \leq b$. If v is in M', but v' is not, then $M' \cup \{v'\}$ is also independent in G', so $a'(M') \leq a'(M' \cup \{v'\}) \leq b$ follows from the first case. Finally, if v' is in M', but v is not, then $M = (M' \setminus \{v'\}) \cup \{v\}$ is independent in $G - vw$, hence $a(M) \leq c_{vw} + b$, thus $a'(M') = a(M) - a(v) + a'(v') \leq (c_{vw} + b) - a(v) + (a(v) - c_{vw}) = b$.

This proves that the given inequality is valid. To define a facet, there should exist $|V(G')|$ linearly independent maximum weight independent sets in G'. Since $ax \leq b$ defined a facet of $STAB(G)$, there are $|V(G)|$ such sets in G. Now note that each such set can be "lifted" to a similar set in G' by possibly adding v' if v is in the set. Since the number of nodes increased by 1 during the splitting of v, we only have to find one more such set. The criticality of the edge vw and the definition of c_{vw} implies that there exists an independent set M in $G - vw$ with $v, w \in M$ and $a(M) = b + c_{vw}$. But then $M' = (M \setminus \{v\}) \cup \{v'\}$ is independent in G' with $a'(M') = b$. That these maximum weight independent sets remain independent follows from the fact that all lifted maximum weight independent sets contain either both v and

v' or neither of them, whereas this last set contains only v'. Since the defect of the facet clearly has not changed, this completes the proof. □

Remark: This operation does not preserve the criticality of the original facet-graph; moreover, it is easy to see that the new facet-graph is *never* critical (every edge incident to v but not to w is noncritical). However, we can always make it critical by deleting some noncritical edges.

Proof of Lemma 3: Let $G_0 = G, a_0 = a$, and choose any edge $vw \in E(G_0)$ with $a_0(v) > a_0(w)$ (such an edge clearly exists unless all weights are 1). Apply the splitting of a node operation in G_0 using these two vertices. The resulting facet-graph (denote it by (G_0', a_0')) is not necessarily critical, but we can delete noncritical edges from G_0' until all remaining edges become critical. Obviously, the only noncritical edges of (G_0', a_0') are those incident with v or v', and it is easy to see that both copies uv and uv' of an original edge cannot be deleted. Hence the resulting critical facet-graph be (G_1, a_1), with weights $a_1 = a_0'$, is mapped onto (G_0, a_0) when v and v' are identified.

Repeat this process until the defined facet becomes $\mathbf{1} \cdot x \leq b$ (we might stop with G_0 if $a_0(v) = 1$ for all $v \in V(G)$). This way we define a sequence of graphs, $G_0, G_1, \ldots, G_n$, such that each can be obtained from the previous one using one node-splitting operation and deleting some noncritical edges. This means that each graph can be obtained from the *next* one by identifying two vertices (and identifying the resulting multiple edges). Since G_n is α-critical with the same defect as (G_0, a_0), this proves the lemma. □

Proof of Lemma 4: The proof comes from a minor modification of the proof of Lemma 2 in [12].

(a) Suppose that $u_0, u_1, \ldots, u_k$ are consecutive vertices of a pending path in G_n ($k > 1$), and without loss of generality we can assume that u_0 and u_k are the only identified pairs from this path in G_0. Let M be a maximum weight stable set in G_0. This stable set can be "lifted" to a maximum weight stable set in G_n in the following way: Start from M, and whenever we split a vertex v in the current maximum weight stable set during the vertex-splitting in the sequence $G_0, G_1, \ldots, G_n$, add v' to M. Since u_0 and u_k were identified, we will get a maximum weight stable set M' in G_n with either $u_0, u_k \in M'$, or $u_0, u_k \notin M'$. Now consider two cases:

(i) k is odd. If $u_0, u_k \in M'$, then $u_1, u_{k-1} \notin M'$, and since M' is of maxi-

mum weight in G_0, M' must contain exactly half of the vertices $u_2, \ldots, u_{k-2}$. Similarly, if $u_0, u_k \notin M'$, then M' must contain exactly half of the vertices $u_1, \ldots, u_{k-1}$. Hence $\sum_{i=1}^{k} \chi_{M'}(u_i) = (k-1)/2$, and after the identifications this gives a linear relation for all maximum weight independent sets in G_0. Since (G_0, a_0) is a facet-graph, this linear relation must be either trivial or the facet itself. Since the constant on the right hand side cannot disappear, the linear relation must be the facet-defining equality, hence only the images of the vertices $u_0, u_1, \ldots, u_k$ have nonzero weight, and since these vertices are not identified with each other, G_0 must be an odd cycle, which is a contradiction.

(ii) k is even. Then similarly as in part (i), we get that if $u_0, u_k \in M'$, then $u_2, u_4, \ldots, u_{k-2} \in M'$, and if $u_0, u_k \notin M'$, then $u_1, u_3, \ldots, u_{k-1} \in M'$, hence $\chi_M(u_0) + \chi_M(u_1) = 1$ for all maximum weight independent sets M in (G_0, a_0), thus G_0 is an edge, which again is a contradiction.

(b) Suppose that we have two pending paths with vertices $u_0, u_1, \ldots, u_j$ and $v_0, v_1, \ldots, v_k$ $(j, k \geq 1)$ in G_n, while u_0 is identified with v_0, and u_j is identified with v_k in G_0. Let M be a maximum weight stable set in G_0, and lift it as in part (a) to a maximum weight stable set M' in G_n. Consider two cases again.

(i) $j + k$ is odd, thus one of the paths is of odd length and the other one is of even length. By looking at all cases depending on whether u_0 and u_j are in M or not, we can easily check that

$$\sum_{i=0}^{j} \chi_{M'}(u_i) + \sum_{i=1}^{k-1} \chi_{M'}(v_i) = \frac{j+k-1}{2}$$

holds for all maximum weight stable sets in G_0. Since this linear dependence cannot become trivial by other identifications, it must become the facet-defining inequality, so G_0 is again an odd cycle, which is a contradiction.

(ii) $j + k$ is even, hence the paths have the same parity. Then it is easy to check as before that for all lifted maximum weight stable sets in G_n the following linear dependence holds:

$$\sum_{i=1}^{j-1} \chi_{M'}(u_i) - \sum_{i=1}^{k-1} \chi_{M'}(v_i) = \frac{j-k}{2}.$$

Now this dependence cannot be the facet-defining inequality, because then u_0 has zero weight (by part (a) u_0 cannot be identified with any other vertices of

these pending paths beside v_0). Thus it must become trivial by other identifications. Hence $j = k$, and the inner vertices of the paths must be pairwise identified, and since the same must hold for any other pairs of identified vertices, we get that u_i must have been identified with v_i for all $i = 1, \ldots, k$. The two possibilities for these identifications (depending on whether the two pending paths have a common endpoint or not) are shown in Figure 4. □

Remark: An example for this type of identification is shown on Figure 6. The first graph is obtained from one of the five basic α-critical graphs with defect 3 by subdividing two edges (see [10, p. 454] for the complete list). The second graph is obtained then by identifying u_i with v_i for $i = 1, 2$. The new vertices u and v have weight 2, every other weight is 1.

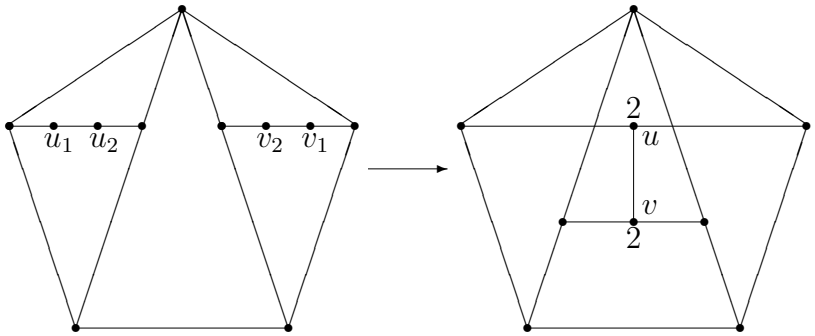

Figure 6: Example for identifications along two pending paths

Proof of Theorem 11: Let (G, a) be a critical facet-graph. The main part of the proof is to show that the number of vertices of degree at least 3 is bounded by a constant $c(\delta)$ depending only on δ. Then it will be easy to argue that the number of vertices of degree 2 is either bounded or we can obtain this graph as an odd subdivision of a smaller graph.

By Lemma 3 we know that the facet-graph (G, a) is a homomorphic image of an α-critical graph G_n with the same defect. Hence Theorem 5 implies that the number of vertices in G_n with degree at least 3 is bounded.

Now to prove that the number of vertices in $G = G_0$ with degree at least 3 is also bounded, it is enough to show that the identifications of vertices in $G_n, G_{n-1}, \ldots, G_0$ cannot introduce too many new vertices of degree at least 3.

The identification of two vertices results in a new vertex of degree at least 3 only if both vertices were of degree 2. From Theorem 5 we know that G_n can be obtained from a graph of bounded number of vertices by replacing edges with paths of odd length. Hence only the interior points of these paths are of degree 2. Choose any pair of pending paths. Part (b) of Lemma 4 shows that at most two new vertices of degree at least 3 are introduced during the identifications along these paths. Since the number of edges in the original graph is also bounded, so is the number of pairs that can be formed from its edges, hence the number of new vertices of degree at least 3 is also bounded.

This shows that the number of vertices of degree at least 3 is also bounded in the original graph, G_0.

Now to finish the proof, note that if a path contains only vertices of degree 2 in G_0, then by Lemma 2 we can apply the "inverse" of the subdivision operation until we get a graph where every pending path has at most two vertices of degree 2. Hence the total number of vertices of degree 2 is also bounded, and (G_0, a_0) can be obtained as an odd subdivision of this facet-graph. This completes the proof. □

In [10, p. 455] the upper bound $2^{8\delta^2}$ was obtained for the cyclomatic number of any α-critical graph with defect δ. This implies that the number of vertices with degree at least 3 in G_n is at most $k = 2^{8\delta^2+1}$, and then during the identifications we obtain at most

$$2 \cdot \binom{\binom{k}{2}}{2} = \binom{k}{2} \cdot \left(\binom{k}{2} - 1 \right) \leq \frac{k^4}{4} - k$$

new vertices of degree at least 3 in G_0, hence it has at most $2^{32\delta^2+2}$ vertices of degree at least 3. Thus (G_0, a) can be obtained as an odd subdivision of a facet-graph which has at most $c(\delta) = 2^{2(32\delta^2+2)}$ vertices. The largest known facet-graph has $O(\delta^2)$ vertices of degree at least 3 (see [18]), and it is an interesting open problem whether the real upper bound is polynomial in δ or not.

As an application of Theorem 11 we can easily generalize Theorem 6 as follows:

Theorem 12 *If M is a maximum weight independent set in the critical facet-graph (G, a) whose defect is δ, then M avoids at most $\binom{\delta+1}{2}$ edges of G.*

Proof: This follows trivially from Lemma 3 and Theorem 6, since during the splitting of vertices the number of avoided edges can only increase. □

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