# Topological Methods in Combinatorics

László Lovász

December 1996

## 1 Preliminaries

### 1.1 Notation

The $r$-dimensional unit ball is denoted by $B^r$. Its boundary, the $(r-1)$-dimensional unit sphere is denoted by $S^{r-1}$. Let $\Sigma^r$ denote the $r$-dimensional standard simplex $\{x \in \mathbb{R}^r : x_i \geq 0, \sum_i x_i \leq 1\}$.

### 1.2 Simplicial complexes

A *simplicial complex* $\mathcal{K}$ is a non-empty finite collection of finite sets such that $X \in \mathcal{K}$, $Y \subseteq X$ implies $Y \in \mathcal{K}$. The union of all members of $\mathcal{K}$ is denoted by $V(\mathcal{K})$. The elements of $V(\mathcal{K})$ are called the *vertices* of $\mathcal{K}$, the elements of $\mathcal{K}$ are called the *simplices* of $\mathcal{K}$.

The *dimension* of a simplex $X \in \mathcal{K}$ is $|S| - 1$. The *dimension* of $\mathcal{K}$ is the maximum dimension of any simplex in $\mathcal{K}$.

Let $n = |V(\mathcal{K})|$. Embed $V(\mathcal{K})$ in $\mathbb{R}^{n-1}$ so that the $n$ points representing the vertices are not on one hyperplane. For each simplex $S$, take the convex hull of the elements in $S$, and take the union of these simplices. This set $G(\mathcal{K}) \subset \mathbb{R}^{n-1}$ is called a *geometric realization* of $\mathcal{K}$. It is clear that all geometric realizations are homeomorphic.

Let $\mathcal{K}_1$ and $\mathcal{K}_2$ be two simplicial complexes and let $f: V(\mathcal{K}_1) \to V(\mathcal{K}_2)$ be a mapping. We say that $f$ is *simplicial* if $f(S) \in \mathcal{K}_2$ for every simplex $S \in \mathcal{K}_1$. Every simplicial mapping can be extended linearly to get a continuous mapping $\hat{f}: G(\mathcal{K}_1) \to G(\mathcal{K}_2)$.

The *baricentric subdivision* $\mathcal{B}(\mathcal{K})$ of a simplicial complex $\mathcal{K}$ is obtained as follows. We create a vertex $v_S$ for every non-empty simplex $S$ (the "baricenter" of $S$), and let $v_{S_1}, \ldots, v_{S_k}$ for a simplex for every $S_1 \subset S_2 \subset \ldots \subset S_k$. It is ckear that $G(\mathcal{B}(\mathcal{K}))$ is homeomorphic with $G(\mathcal{K})$.

Let $(P, \leq)$ be a poset (partially ordered set), and let $\mathcal{C}(P)$ be the set of all chains (totally ordered subsets) of $P$. Clearly $\mathcal{C}(P)$ is a simplicial complex, called the *chain complex* of $P$. If $P$ is the set of non-empty simplices of a simplicial complex $\mathcal{K}$, ordered by inclusion, then the chain complex of $P$ is just the baricentric subdivision of $\mathcal{K}$.

### 1.3 Homology groups

Homology groups, Betti numbers, Euler characteristic.

## 2 Homotopy equivalence

Let $T_1$ and $T_2$ be two topological spaces. Two continuous maps $f_0, f_1 : T_1 \to T_2$ are called *homotopic* (denoted $f_0 \sim f_1$), if they can be deformed into each other. More exactly, there exists a continuous mapping $F : T_1 \times [0,1] \to T_2$ such that $f_0(x) = F(x, 0)$ and $f_1(x) = F(x, 1)$.

We say that $T_1$ and $T_2$ are *homotopically equivalent*, if there exist maps $f : T_1 \to T_2$ and $g : T_2 \to T_1$ such that $f \circ g \sim \mathrm{id}_{T_2}$ and $g \circ f \sim \mathrm{id}_{T_1}$.

A topological space is *contractible* if it is homotopically equivalent to the space with a single point.

We say that a simplicial complex is contractible if its geometric realization is. We define homotopy equivalence of two simplicial complexes and homotopy of two simplicial maps between simplicial complexes in a similar way.

### 2.1 Elementary facts

**Lemma 1** *Let $\mathcal{K}$ be a simplicial complex and $f : V(\mathcal{K}) \to V(\mathcal{K})$, a simplicial map such that for all $S \in \mathcal{K}$, $S \cup f(S) \in \mathcal{K}$. Then $f \sim 0$.*

**Lemma 2** *Let $\mathcal{K}_1$ and $\mathcal{K}_2$ be two simplicial complexes and $f : G(\mathcal{K}_1) \to G(\mathcal{K}_2)$, two continuous maps such that for all $x \in \mathcal{K}_1$, $f(x)$ and $g(x)$ are contained in the same face of $G(\mathcal{K}_2)$. Then $f \sim g$.*

**Lemma 3** *For a simplicial complex $\mathcal{K}$ with $V(\mathcal{K} = V$, the following are equivalent:*

*(i) $\mathcal{K}$ is contractible.*

*(ii) $\mathcal{K}$ is a retract of $2^V$.*

**Lemma 4** *Let $\mathcal{K}$ be a simplicial complex and $U \subseteq V(\mathcal{K})$. If two of $\mathcal{K}$, $\mathcal{K} \cup 2^U$ and $\mathcal{K} \cap 2^U$ are contractible then so is the third.*

**Lemma 5** *Let $\mathcal{K}_1$ and $\mathcal{K}_2$ be simplicial complexes.*

*(a) If $\mathcal{K}_1$, $\mathcal{K}_2$ are contractible and one of $\mathcal{K}_1 \cap \mathcal{K}_2$ and $\mathcal{K}_1 \cup \mathcal{K}_2$ is contractible then so is the other.*

*(c) If $\mathcal{K}_1 \cap \mathcal{K}_2$ and $\mathcal{K}_1 \cup \mathcal{K}_2$ are contractible then so are $\mathcal{K}_1$ and $\mathcal{K}_2$.*

**Lemma 6** *Let $\mathcal{K}_1$ and $\mathcal{K}_2$ be simplicial complexes and $f : \mathcal{K}_1 \to \mathcal{K}2$, a simplicial map. Let $\overline{f}$ be the map from $G(\mathcal{K}_1)$ into $G(\mathcal{K}_2)$ induced by $f$. Assume that for every $x \in G(\mathcal{K}_1)$, the set $\overline{f}^{-1}(x)$ is contractible. Then $K_1$ and $K_2$ are homotopy equivalent.*

### 2.2 The Nerve Theorem

Let $G = (V, E)$ be a bipartite graph with bipartition $V = U \cup W$. The *neighborhood complex on $U$ $\mathcal{N}_U$* of $G$ is the simplicial complex consisting of all subsets $A$ of $U$ such that the elements of $A$ have a common neighbor. $\mathcal{N}_W$ is defined similarly.

**Theorem 1** *The two neighborhood complexes of a bipartite graph are homotopy equivalent.*

Let $\mathcal{H}$ be a set-system. The *nerve* $\mathcal{N}(\mathcal{H})$ of $\mathcal{H}$ is defined as the simplicial complex whose vertices are the sets in $\mathcal{H}$, and $\{X_1, \ldots, X_r\} \in \mathcal{N}(\mathcal{H})$ if and only if $X_1, \ldots, X_r \in \mathcal{H}$ and $X_1 \cap X_2 \cap \ldots \cap X_r$.

**Theorem 2** *Let $\mathcal{K}$ be a simplicial complex and let $A_1, \ldots, A_m$ be its maximal simplices. Then $\mathcal{K}$ is homotopy equivalent to the nerve of $\{A_1, \ldots, A_m\}$.*

**Theorem 3** *Let $\mathcal{K}$ be a simplicial complex and $V_1, \ldots, V_m$, subsets of $V(\mathcal{K})$ such that every simplex in $\mathcal{K}$ is contained in at least one $V_i$. Assume that*

$$\mathcal{K} \cap 2^{V_{i_1} \cap \ldots V_{i_r}}$$

*is contractible for every $1 \leq i_1 < \ldots < i_r \leq m$, $r \geq 1$. Then $\mathcal{K}$ is homotopy equivalent to the nerve of $\{V_1, \ldots, V_m\}$.*

**Theorem 4** *Let $\mathcal{K}$ be a simplicial complex and let $\mathcal{K}_1, \ldots, \mathcal{K}_m$ be subcomplexes such that $\mathcal{K} = \mathcal{K}_1 \cup \ldots \cup \mathcal{K}_m$. Assume that*

$$\mathcal{K}_{i_1} \cup \ldots \cup \mathcal{K}_{i_r}$$

*is contractible for every $1 \le i_1 < \ldots < i_r \le m$, $r \ge 1$. Then $\mathcal{K}$ is homotopy equivalent to the nerve of $\{V(\mathcal{K}_1), \ldots, V(\mathcal{K}_m)\}$.*

A *cross-cut* in a poset $P$ is an antichain that meets every maximum chain.

**Theorem 5 (The Cross Cut Theorem.)** *Let $L$ be a finite lattice, $P = L \setminus \{0, 1\}$, and let $S$ be a cross-cut in $P$. Then the simplicial complex*

$$\mathcal{A}_S = \{X \subseteq S \ : \ \text{either } \vee X \neq 0 \text{ or } \wedge X \neq 1\}$$

*is homotopy equivalent to $\mathcal{C}(P)$.*

**Remark.** Clearly, $\mathcal{A}_S$ consists of those subsets $X$ of $S$ that are bounded from either below or above in $P$. The poset obtained from a 2-element chain by doubling every element shows that the theorem is not true for posets in general.

**Corollary 1** *Let $L$ be a finite lattice, $P = L \setminus \{0, 1\}$, and let $A$ be the set of atoms of $L$. Then the simplicial complex*

$$\mathcal{A}_S = \{X \subseteq A \ : \wedge X \neq 1\}$$

*is homotopy equivalent to $\mathcal{C}(P)$.*

# 3 Topological connectivity

A topological space is *$k$-connected* ($k \ge 0$) if for every $0 \le r \le k$, every continuous $f : S^r \to T$ extends to a continuous map $\bar{f} : B^{r+1} \to T$. Equivalently, $f$ is homotopic to a constant map.

Sometimes it is convenient to define $(-1)$-connected as "non-empty".

**Lemma 7** *For a simplicial complex $\mathcal{K}$ with $V(\mathcal{K} = V$, the following are equivalent:*

*(i) $\mathcal{K}$ is $k$-connected.*

*(ii) The $(k+1)$-dimensional skeleton $(\mathcal{K})_{k+1}$ is $k$-connected.*

*(iii) $(\mathcal{K})_{k+1}$ is a retract of $(2^V)_{k+1}$.*

**Lemma 8** *Let $\mathcal{K}$ is a $(k+1)$-dimensional simplicial complex, $\mathcal{K}'$, a subcomplex of $\mathcal{K}$, and $T$ a $k$-connected space. Then every continuous map from $G(\mathcal{K}')$ into $T$ extends to a continuous map of $G(\mathcal{K})$ into $T$.*

**Lemma 9** *Let $\mathcal{K}$ be a simplicial complex and $U \subseteq V(\mathcal{K})$.*

*(a) If $\mathcal{K}$ is $k$-connected and $\mathcal{K} \cap 2^U$ is $(k-1)$-connected, then $\mathcal{K} \cup 2^U$ is $k$-connected.*

*(b) If $\mathcal{K} \cup 2^U$ and $\mathcal{K} \cap 2^U$ are $k$-connected, then $\mathcal{K}$ is $k$-connected.*

**Lemma 10** *Let $\mathcal{K}_1$ and $\mathcal{K}_2$ be two $k$-connected simplicial complexes and assume that $\mathcal{K}_1 \cap \mathcal{K}_2$ is $(k-1)$-connected. Then $\mathcal{K}_1 \cup \mathcal{K}_2$ is $k$-connected.*

**Theorem 6** *Let $\mathcal{K}$ be a simplicial complex and let $\mathcal{K}_1, \ldots, \mathcal{K}_m$ be subcomplexes such that $\mathcal{K} = \mathcal{K}_1 \cup \ldots \cup \mathcal{K}_m$. Assume that*

$$\mathcal{K}_{i_1} \cup \ldots \cup \mathcal{K}_{i_r}$$

*is $(k-r+1)$-connected for every $1 \leq i_1 < \ldots < i_r \leq m$, $r \geq 1$. Then $\mathcal{K}$ is $k$-connected if and only if the nerve of $\{V(\mathcal{K}_1), \ldots, V(\mathcal{K}_m)\}$ is $k$-connected.*

**Lemma 11** *Let $\mathcal{K}_1$ and $\mathcal{K}_2$ be simplicial complexes and $f:\ \mathcal{K}_1 \to \mathcal{K}2$, a simplicial map. Let $\overline{f}$ be the map from $G(\mathcal{K}_1)$ into $G(\mathcal{K}_2)$ induced by $f$.*

*(a) Assume that $\mathcal{K}_1$ is $k$-connected and for every $x \in G(\mathcal{K}_1)$, the set $\overline{f}^{-1}(x)$ is $(k-1)$-connected. Then $\mathcal{K}_2$ is $k$-connected.*

*(b) Assume that $\mathcal{K}_2$ is $k$-connected and for every $x \in G(\mathcal{K}_1)$, the set $\overline{f}^{-1}(x)$ is $k$-connected. Then $\mathcal{K}_1$ is $k$-connected.*

## 4 Brouwer's fixed point theorem

A simplicial complex $\mathcal{K}$ such that $G(\mathcal{K}) \approx S^r$ is called a *triangulation of the sphere $S^r$*.

**Lemma 12 (Sperner's lemma)** *Let $\mathcal{K}$ be a triangulation of the sphere $S^r$. Let $f:\ V(\mathcal{K}) \to \{0, 1, \ldots, r\}$ be any labelling of the vertices of $S^r$. Then the number of simplices whose vertices are labelled by different numbers is even.*

**Theorem 7 (Brouwer's fixed point theorem)** *Every continuous map $f:\ B^r \to B^r$ has a fixed point.*

**Corollary 2** *Let $\mathcal{K}$ be a contractible simplicial complex. Then every continuous map $f : G(\mathcal{K}) \to G(\mathcal{K})$ has a fixed point.*

**Corollary 3** *$S^{r-1}$ is not a retract of $B^r$.*

**Corollary 4** *Let $f : \Sigma^r \to \Sigma^r$ be a continuous map such that for every face $F$ of $\Sigma^r$, $f(F) \subseteq F$. Then $f$ is surjective.*

**Corollary 5** *Let $T$ be a compact contractible topological space and let $G$ be a finite cyclic group acting on $T$. Then the elements of $G$ have a common fixed point.*

This last corollary does not remain valid for all cyclic groups. But it does remain valid for certain finite groups. One such class of finite groups is the following [38].

**Theorem 8** *Let $T$ be a compact contractible topological space and let $\Gamma$ be a finite group acting on $T$. Assume that $\Gamma$ has a normal $p$-subgroup $\Gamma_1$ such that $\Gamma/\Gamma_1$ is cyclic. Then the elements of $G$ have a common fixed point.*

### 4.1 Partitioning a graph into connected pieces

The following theorem was conjectured by Frank and Maurer.

**Theorem 9** *Let $G$ be a $k$-connected graph, $v_1, \ldots, v_k \in V$, and $n_1, \ldots, n_k$, positive integers with $n_1 + \ldots + n_k = |V(G)|$. Then there exists a partition $V(G) = V_1 \cup \ldots \cup V_k$ such that $v_i \in V_i$ and $|V_i = n_i$ for all $1 \leq i \leq k$.*

### 4.2 Decision trees and evasive properties

([26])

## 5 The Borsuk–Ulam Theorem

Let $T$ be a topological space and let $\iota$ be an antipodality on $T$, i.e., a homeomorphism $\iota : T \to T$ such that $\iota$ is an involution ($\iota \circ \iota = \mathrm{id}_T$) and $\iota$ has no fixed points. Sometimes we denote $\iota(x)$ by $-x$. $T$, together with the antipodality $\iota$, is called an *antipodality space*.

An *antipodal map* between antipodality spaces $T_1$ and $T_2$ is a continuous map $f : T_1$ to $T_2$ that satisfies $f(-x) = -f(x)$.

The following theorems are all essentially equivalent to each other, and are called the *Borsuk–Ulam Theorem.*

**Theorem 10** *There is no antipodal map from $S^r$ into $S^{r-1}$ ($r \geq 1$).*

**Theorem 11** *For every continuous map $f: S^r \to \mathbb{R}^r$ there exists an $x \in S^r$ such that $f(x) = f(-x)$.*

**Theorem 12** *If $S^r$ is covered by $r+1$ open sets, then one of these sets contains an antipodal pair.*

(Analogous statement holds for closed sets.)

The following discrete version of the Borsuk–Ulam theorem is due to Bajmo'czy and Ba'ra'ny. Let $P$ be a full-dimensional convex polytope in $\mathbb{R}^d$. We say that two faces $A$ and $B$ of $P$ are *opposite* if there exists a linear function $\ell: \mathbb{R}^d \to \mathbb{R}$ that is maximized by every point in $A$ and minimized by every point in $B$.

**Theorem 13** *[5] Let $P$ be a full-dimensional convex polytope in $\mathbb{R}^d$ and $f: P \to \mathbb{R}^{d-1}$. Then $P$ has two opposite faces $A$ and $B$ such that $f(A) \cap f(B) \neq \emptyset$.*

The following slight extension of this theorem will be needed later. We say that the map $f: P \to \mathbb{R}^{d-1}$ is *generic* if for every pair of faces $A$ and $B$ with $\dim(A) + \dim(B) < d-1$, we have $f(A) \cap f(B) = \emptyset$, and for every pair of faces $A$ and $B$ with $\dim(A) + \dim(B) = d-1$, the set $f(A) \cap f(B)$ is finite.

**Theorem 14** *[34] Let $P$ be a full-dimensional convex polytope in $\mathbb{R}^d$ and let $f: P \to \mathbb{R}^{d-1}$ be a generic map. Then $P$ has two opposite faces $A$ and $B$ such that $\dim(A) + \dim(B) = d-1$ and $|f(A) \cap f(B)|$ is odd.*

We say that the polytope $P$ in $\mathbb{R}^d$ is *generic*, if for every two opposite faces $A$ and $B$, $\dim(A) + \dim(B) = d-1$.

**Theorem 15** *Let $P$ be a generic full-dimensional convex polytope in $\mathbb{R}^d$ and let $f: P \to \mathbb{R}^{d-1}$ be a generic map. Then*

$$\sum |f(A) \cap f(B)| \equiv 1 \qquad (\mathrm{mod} 2),$$

*where the summation extends over all pairs of opposite faces $A$ and $B$ of $P$.*

### 5.1 Kneser graphs

The following theorem was conjectured by Kneser.

**Theorem 16** *[30] If all $k$-element subsets of a $(2k+r-1)$-element set are divided into $r$ classes, then one of the classes contains two disjoint $k$-sets.*

It is easy to divide all $k$-element subsets of a $(2k+r-2)$-element set into $r$ classes so that none of the classes contains two disjoint $k$-sets.

The *Kneser graph* $K_k^n$ is defined as the graph whose nodes are all $k$-subsets of an $n$-set, and two are adjacent iff they are disjoint $(n \geq k)$.

**Theorem 17** *For $n \geq 2k$, the chromatic number of the Kneser graph $K_k^n$ is $n-2k+2$.*

Let $G=(V,E)$ be any graph. The *neighborhood complex* $\mathcal{N}(G)$ of $G$ is the simplicial complex consisting of all subsets $A$ of $V$ such that the elements of $A$ have a common neighbor.

For a bipartite graph, the neighborhood complex consists of two components.

**Theorem 18** *If the neighborhood complex of a graph $G$ is $r$-connected, then $\chi(G) \geq k+3$.*

**Lemma 13** *If $T$ is a $k$-connected topological space with an antipodality, then there exists an antipodal map of $S^{k+1}$ into $T$.*

**Lemma 14** *Let $P$ be the poset consisting of $t$ consecutive levels of a Boolean algebra on $n$ atoms. Then $\mathcal{C}(P)$ is $(t-2)$-connected.*

**Corollary 6** *The neighborhood complex of $K_k^n$ is $(n-2k-1)$-connected.*

See [33] for another application of theorem 18.

### 5.2 The Ham Sandwich Theorem

**Theorem 19** *Let $A_1,\dots,A_d$ be measurable sets in $\mathbb{R}^d$ with finite measure. Then there exists a halfspace $H$ such that $\mu(H \cap A_i) = \frac{1}{2}\mu(A_i)$ for all $i$.*

See [1] for another combinatorial application of the Borsuk–Ulam Theorem.

## 6 A linked Borsuk theorem and its application to graph embedding

Let $U$ and $W$ be two disjoint embedded copies of $S^r$ in $\mathbb{R}^{2r+1}$. Then we define their (modulo 2) *linking number* $\ell(U,W)$ as follows. Let $f: S^r \to U$ be a homeomorphism. We extend $f$ to a continuous mapping $F: B^{r+1} \to \mathbb{R}^{2r+1}$ such that the number of points $x \in B^r$ for which $F(x) \in W$ is finite; and then let $\ell(U,W)$ be this number. It can be shown that $\ell(U,W)$ is independent of the choice of $f$ and $F$, and $\ell(U,W) = \ell(W,U)$.

**Theorem 20** *Let $P$ be a convex polytope in $\mathbb{R}^{2d+1}$ and let $f$ be an embedding of the $(d-1)$-dimensional skeleton of $P$ in $\mathbb{R}^{2d-1}$. Then there exists a pair of opposite faces $A$ and $B$ with* $\dim(A) = \dim(B) = d$ *such that $f(\partial A)$ and $f(\partial B)$ are linked.*

**Theorem 21** *Let $P$ be a generic convex polytope in $\mathbb{R}^{2d+1}$ and let $f$ be an embedding of the $(d-1)$-dimensional skeleton of $P$ in $\mathbb{R}^{2d-1}$. Then*

$$\sum \ell(\partial A, \partial B) \equiv 1 \qquad (\mathrm{mod} 2),$$

*where the summation extends over all pairs of opposite faces $A$ and $B$ with* $\dim(A) = \dim(B) = d$.

### 6.1 Linkless embeddable graphs

Let $G = (V,E)$ be a graph. An embedding of a graph $G$ in $\mathbb{R}^3$ is called *linkless* if each pair of disjoint circuits in $G$ are unlinked closed curves in $\mathbb{R}^3$. $G$ is *linklessly embeddable* if $G$ has a linkless embedding in $\mathbb{R}^3$.

A Y$\Delta$ transformation of a graph means deleting a vertex of degree 3, and making its three neighbours mutually adjacent. $\Delta$Y is the reverse operation (applied to a triangle). The *Petersen family* consists of all graphs obtainable from $K_6$ by the so-called $\Delta$Y- and Y$\Delta$-operations. One can check that this family consists of seven graphs (and it includes the Petersen graph).

**Theorem 22 (Robertson, Seymour and Thomas)** *A graph $G$ is linklessly embeddable if and only if $G$ does not have any graph in the Petersen family as a minor.*

We call a linear subspace $L$ of $\mathbb{R}^V$ a *connected representation* if for each $x \in L$, $\mathrm{supp}_+(x)$ is nonempty and $\mathrm{supp}_+(x)$ induces a connected subgraph of $G$. (For a vector $x \in \mathbb{R}^V$, $\mathrm{supp}(x)$

is the *support* of $x$, that is, $\text{supp}(x) := \{v \in V | x(v) \neq 0\}$. The *positive support* is $\text{supp}_+(x) := \{v \in V | x(v) > 0\}$ and the *negative support* is $\text{supp}_-(x) := \{v \in V | x(v) < 0\}$.)

We define $\lambda(G)$ as the maximum dimension of any connected representation $L$ of $G$. It is easy to see that $\lambda(G)$ is monotone under taking minors.

**Theorem 23 (Van der Holst, Laurent and Schrijver)** *(a) $\lambda(G) \leq 1$ if and only if $G$ is a forest;*

*(b) $\lambda(G) \leq 2$ if and only if $G$ is series-parallel;*

*(c) $\lambda(G) \leq 3$ if and only if $G$ is a subgraph of a clique sum of planar graphs.*

**Theorem 24** *[34] If $G$ is linklessly embeddable, then $\lambda(G) \leq 4$.*

Among the graphs in the Petersen family, $K_6$ has $\lambda = 5$, but all other graphs have $\lambda = 4$, so the theorem above does not provide a characterization of linklessly embeddable graphs. The following invariant was introduced by Colin de Verdière.

Let $G = (V, E)$ be an undirected graph, $V = \{1, \ldots, n\}$. Then $\mu(G)$ is the largest corank of any symmetric real-valued $n \times n$ matrix $M = (m_{i,j})$ satisfying the following conditions:

(i) $M$ has exactly one negative eigenvalue, of multiplicity 1;

(ii) for all $i \neq j$, $m_{i,j} < 0$ if $i$ and $j$ are adjacent, and $m_{i,j} = 0$ if $i$ and $j$ are nonadjacent,

(iii) there is no nonzero symmetric $n \times n$ matrix $X = (x_{i,j})$ such that $MX = 0$ and such that $x_{i,j} = 0$ whenever $i = j$ or $m_{i,j} \neq 0$.

There is no condition on the diagonal entries $m_{i,i}$. (The *corank* $\text{corank}(M)$ of a matrix $M$ is the dimension of its kernel.) Condition (iii) is called the *Strong Arnold Property*.

**Theorem 25** *[16, 17] (a) $\mu(G) \leq 1$ if and only if $G$ is a path;*

*(b) $\mu(G) \leq 2$ if and only if $G$ is outerplanar;*

*(c) $\lambda(G) \leq 3$ if and only if $G$ is planar.*

The following lemma shows that for a matrix $M$ in the definition of $\mu(G)$, $\mathcal{K}er(M)$ is *almost* a connected representation for $G$; that this is not always true is shown by the Petersen graph.

For any graph $G = (V, E)$ and $U \subseteq V$, let $N(U)$ be the set of vertices in $V \setminus U$ that are adjacent to at least one vertex in $U$. For any $V \times V$ matrix and $I, J \subseteq V$, let $M_{I \times J}$ denote the submatrix induced by the rows in $I$ and columns in $J$, and let $M_I := M_{I\times I}$. For any vector $z \in \mathbb{R}^I$ and $J \subseteq I$, let $z_J$ be the subvector of $z$ induced by the indices in $J$.

**Lemma 15** *[23] Let $G$ be a connected graph, let $M$ be a matrix satisfying (i)–(iii), and let $x$ be a vector in $\mathcal{K}er(M)$ with $G|\mathrm{supp}_+(x)$ disconnected. Then there are no edges connecting $\mathrm{supp}_+(x)$ and $\mathrm{supp}_-(x)$, and each component $K$ of $G|\mathrm{supp}(x)$ satisfies $N(K) = N(\mathrm{supp}(x))$.*

**Theorem 26** *[34] $\mu(G) \leq 4$ if and only if $G$ is linklessly embeddable.*

# 7 Equivariant maps

Let $G$ be a finite group acting on a topological space $T$. We say that the action is *free* if no element of $G$ other than the identity has a fixed point. A *$G$-space* is a topological spaces with a free $G$ action. We are mostly concerned with the case when $G = \mathbb{Z}_k$, the cyclic group with $k$ elements.

**Example 1.** $S_k^0$ consists of $k$ points.

**Example 2.** $S_k^1$ is $S_1$ with rotation by $2\pi/k$.

**Example 3.** Let $T^k = \mathrm{conv}\{e_1, \dots, e_k\}$ and $P_k^{k-1}$ be the boundary of $T^k$, with $\mathbb{Z}_k$ acting through a cyclic permutation of the coordinates. This action is free iff $k$ is a prime.

The *join* $T_1 \vee T_2$ of two topological spaces $T_1$ and $T_2$ is obtained by taking $T_1 \times T_2 \times [0,1]$, and shrinking $T_1 \times \{x_2\} \times \{0\}$ to a single point for all $x_2 \in T_2$, and also shrinking $\{x_1\} \times T_2 \times \{0\}$ to a single point for all $x_1 \in T_1$.

Let $\mathcal{K}_1$ and $\mathcal{K}_2$ be two simplicial complexes; assume that $V(\mathcal{K}_1) \cap V(\mathcal{K}_2) = \emptyset$. Then their *join* $\mathcal{K}_1 \vee \mathcal{K}_2$ is defined as the simplicial complex consisting of all sets $A_1 \cup A_2$, $A_i \in \mathcal{K}_i$. It is strightforward that $G(\mathcal{K}_1 \vee \mathcal{K}_2) \approx G(\mathcal{K}_1) \vee G(\mathcal{K}_2)$. The join of boundary complexes of simplicial convex polytopes is the boundary of a simplicial convex polytope.

Free $Z_k$-actions on two simplicial complexes give free $\mathbb{Z}_k$ action on the join.

**Example 4.** If $n$ is odd, then $S_k^n$ is defined as the join of $(n+1)/2$ copies of $S_k^1$. If $n$ is even, then $S_k^n$ is the join of $n/2$ copies of $S_k^1$ and one copy of $S_0^k$.

**Lemma 16** *For odd $n$, $S_k^n$ is homeomorphic with $S^n$. For even $n$, $S_k^n$ is homeomorphic with the space obtained by gluing together $p$ copies of $B^n$ along their bondaries.*

**Corollary 7** *$S_k^n$ is $n$-dimensional and $(n-1)$-connected.*

**Lemma 17** *If $T$ is any $(n-1)$-connected space with a $\mathbb{Z}_k$ action, then $S_k^n$ has a covariant map into $T$.*

**Lemma 18** *If $\mathcal{K}$ is any $n$-dimensional simplicial complex with a free $\mathbb{Z}_k$ action, then $G(\mathcal{K})$ has a covariant map into $S_k^n$.*

**Lemma 19** *[27] Let $n$ be even. Then every covariant map of $S_k^n$ into itself has index $1 \pmod{p}$.*

**Theorem 27** *[18, 19] $S_k^n$ has no covariant map into $S_k^{n-1}$.*

**Corollary 8** *[45] If $T$ is any $(n-1)$-connected space with a free $\mathbb{Z}_k$ action, and $\mathcal{K}$ is an $(n-1)$-dimensional simplicial complex with a free $\mathbb{Z}_k$ action, then $T$ has no covariant map into $G(K)$.*

**Theorem 28** *[7] Let $p$ be a prime. Then for every continuous map $f: S_p^{d(p-1)} \to \mathbb{R}^d$ there is a point $x \in S_p^{d(p-1)}$ such that $f$ is constant on the orbit of $x$.*

# 8 Kneser hypergraphs

An *$r$-graph* (or $r$-uniform hypergraph) is a pair $(V, \mathcal{H})$ where $V$ is a finite set and $\mathcal{H} \subseteq \binom{V}{r}$. The elements of $V$ are called *nodes*, thye elements of $\mathcal{H}$ are called *edges*.

A subset $A \subseteq V$ in an $r$-graph $(V, \mathcal{H})$ is *independent*, if it does not contain any edge. The *chromatic number* of an $r$-graph $(V, \mathcal{H})$ is the least integer $k$ such that there is a partition $V = A_1 \cup \ldots \cup A_k$ into independent sets.

An *ordered $r$-graph* is a pair $(V, \hat{\mathcal{H}})$ where $V$ is a finite set and $\hat{\mathcal{H}}$ is a set of ordered $r$-subsets of $V$ (the same $r$-set may occur with different orderings). A *complete $r$-partite ordered $r$-graph*, or *$r$-box*, is defined as $(V = V_1 \cup \ldots \cup V_r, \hat{\mathcal{H}} = V_1 \times \ldots \times V_r)$, where $V_1, \ldots, V_r$ are disjoint finite sets. The numbers $|V_1|, \ldots, |V_r|$ are the *class-sizes* of $(V, \hat{\mathcal{H}}$. A *complete $r$-partite $r$-graph* is obtained if we disregard the ordering of the edges.

Given an $r$-graph $(V, \mathcal{H})$, let $(V, \hat{\mathcal{H}})$ be the ordered $r$-graph obtained by including every edge of $\mathcal{H}$ with every ordering. We define the *box complex* $\mathcal{B}$ of $(V, \mathcal{H})$ as the simplicial complex whose vertices are the ordered edges (ordered $r$-sets in $\hat{\mathcal{H}}$, and where a set of ordered edges forms a simplex if and only if it is a subset of an $r$-box contained in $(V, \mathcal{H})$.

The group $\mathbb{Z}_r$ acts on $\hat{\mathcal{H}}$ by cyclically permuting the nodes in every ordered edge. So if $\omega$ is the generator of $\mathbb{Z}_r$ then $\omega(v_1, \ldots, v_r) = (v_2, \ldots, v_r, v_1)$. Clearly, this action is simplicial on $\mathcal{B}$ and so it defines an action on $G(\mathcal{B})$. It is also easy to see that this action is free.

**Theorem 29** *[3] If the box complex of an $r$-graph $(V, \mathcal{H})$ is $t(r-1)-2$-connected, then its chromatic number is at least $t+1$.*

### 8.1 Application to Kneser hypergraphs

**Theorem 30** *[3] If all $k$-element subsets of an $(rk + (r-1)(t-1))$-element set are divided into $t$ classes, then one of the classes contains $r$ disjoint $k$-sets.*

The $k$-subsets of a set $S$ with only $rk + (r-1)(t-1) - 1$ elements can be divided into $t$ classes so that none of the classes contains $r$ disjoint $k$-sets. Let $S = S_1 \cup \ldots S_k$, where $|S_1| = \ldots = |S_{t-1}| = r-1$ and $|S_t| = rk-1$. For $i = 1, \ldots, t-1$, let $H_i$ be the set of $k$-subsets of $S$ intersecting $S_i$ but not $S_1, \ldots, S_{i-1}$. Let $H_t$ be the rest, i.e., the set of all $k$-subsets of $S_t$.

The *Kneser hypergraph* $K(n,k,r) = (V, \mathcal{H})$ is defined as follows. Let $S$ be an $n$-set and $V = \binom{S}{k}$. Let

$$\mathcal{H} = \{\{A_1, \ldots, A_r\} : \ A_i \in V, \ A_i \cap A_j = \emptyset \text{ for all } i \neq j\}.$$

Then theorem 30 (along with the remark following it) can be stated as follows:

**Theorem 31** *Let $n = rk + (r-1)(t-1)$. Then the chromatic number of $K(n,k,r)$ is $t+1$.*

This is first proved for th case when $r = p$ is a prime. Using theorem 29, it suffices to show the following.

**Lemma 20** *Let $p$ be a prime, and $n = pk + (p-1)(t-1)$. Then the box complex of $K(n,k,p)$ is (at least) $t(p-1)-2$-connected.*

To finish the proof, one only needs the following simple lemma.

**Lemma 21** *If theorem 30 is true for two values $r = r'$ and $r = r''$, then it is also true for $r = r'r''$.*

# 9 Colored Tverberg Theorem

## 9.1 The chessboard complex

Let $n, m \geq 1$. The *chessboard complex* $\Delta_{m,n}$ is defined on the fields of an $m \times n$ chessboard, and consists of all sets of fields such that no two are in the same row or column.

**Lemma 22** *[14] The chessboard complex $\Delta_{m,n}$ is (at least) $(\nu - 2)$-connected, where*

$$\nu = \min\left\{m, n, \lfloor\frac{m+n+1}{3}\rfloor\right\}.$$

## 9.2 Colored Tverberg Theorem

**Theorem 32** *[46] Let $p$ be a prime, $d \geq 1$ and let $A_0, \ldots, A_d \subseteq \mathbb{R}^d$, $|A_i| \geq 2p$. Then one can select from each $A_i$ $p$ distinct points $a_1^i, \ldots, a_p^i \in A_i$ such that*

$$\cap_{j=1}^{p} \mathrm{conv}\{a_j^0, a_j^1, \ldots, a_j^d\} \neq \emptyset.$$

## 9.3 An application: bisecting hyperplanes

([28, 6, 46])

# 10 Linear decision trees

([12, 13, 21])

# References


[1] N. Alon, Splitting nechlaces, *Adv. in Math.* **63** (1987), 247–253.

[2] N. Alon, I. Bárány, Z. Füredi, D. Kleitman, *Point selections and weak $\epsilon$-nets for convex hulls*

[3] N. Alon, P. Frankl, L. Lovász, *The chromatic number of Kneser hypergraphs*, Trans. Amer. Math. Soc. **298** (1986), 359–370.

[4] R. Bacher, Y. Colin de Verdière, *Multiplicités des valeurs propres et transformations étoile-triangle des graphes*, preprint, 1994.

[5] E.G. Bajmóczy, I. Bárány, On a common generalization of Borsuk's and Radon's theorem, *Acta Mathematica Academiae Scientiarum Hungaricae* 34 (1979) 347–350.

[6] I. Bárány, Z. Füredi, L. Lovász, On the number of halving planes *Combinatorica* **10**, 175–183.

[7] I. Bárány, S.B. Shlosman and A. Szűcs, On a topological generalization of a theorem of Tverberg, *J. London Math. Soc.* **23**, 158–164.

[8] M. Ben-Or: Lower bounds for algebraic computation trees, *15th ACM STOC* (1983) 80–86.

[9] A. Björner, Topological Methods, in: *Handbook of Combinatorics* (ed. R. Graham, M. Grötschel, L. Lovász), Chapter 34, 1819–1872. North-Holland, Amsterdam (1995).

[10] A. Björner, Some Cohen-Macaulay Complexes arising in Group Theory, in *Commutative Algebra and Combinatorics* (ed. M. Nagata and H. Matsumura), Advanced Studies in Pure Mathematics **11**, North-Holland, Amsterdam, 1987, pp. 13–19.

[11] A. Björner, B. Korte, L. Lovász, *Homotopy properties of greedoids*, Advances in Appl. Math. **6** (1985), 447–494.

[12] A. Björner, L. Lovász, A. Yao, Linear decision trees, hyperplane arrangements, and Möbius functions, in: *Proc. 24th ACM SIGACT Symp. on Theory of Computing*, 170–177.

[13] A. Björner, L. Lovász, Linear decision trees, subspace arrangements, and Möbius functions *Journal of the Amer. Math. Soc.* **7**, 677–706.

[14] The connectivity of chessboard complexes (A. Björner, S. Vrečica, R. Živaljevič) *Journal of the London Math Soc.* **49** (1994) 25–39.

[15] T. Böhme, On spatial representations of graphs, in: *Contemporary Methods in Graph Theory* (R. Bodendieck, ed.), BI-Wiss.-Verl. Mannheim, Wien/Zurich, 1990, pp. 151–167.

[16] Y. Colin de Verdière, Sur un nouvel invariant des graphes et un critère de planarité, *Journal of Combinatorial Theory, Series B* 50 (1990) 11–21.

[17] Y. Colin de Verdière, On a new graph invariant and a criterion for planarity, in: *Graph Structure Theory* (N. Robertson, P. Seymour, eds.), Contemporary Mathematics, American Mathematical Society, Providence, Rhode Island, 1993, pp. 137–147.

[18] A. Dold: *Lectures on Algebraic Topology*, Springer-Verlag, 1972.

[19] A. Dold: Parametrized Borsuk–Ulam theorems, *Comment. Math. Helv.* **63** (1988) 275–285.

[20] J. Folkman, The homology groups of a lattice, *J. Math. Mech.* **15** (1966) 631–636.

[21] M. Goresky and R. MacPherson: *Stratified Morse Theory*, Ergebnisse, Band 14, Springer-Verlag, Berlin (1988)

[22] E. Győri, On division of graphs to connected subgraphs, in: *Combinatorics I* (eds. A. Hajnal and V.T. Sós), Coll. Math. Soc. J. Bolyai **18** (1987) 485–494.

[23] H. van der Holst, A short proof of the planarity characterization of Colin de Verdière, *Journal of Combinatorial Theory, Series B* 65 (1995) 269–272.

[24] H. van der Holst, M. Laurent, A. Schrijver, On a minor-monotone graph invariant, *Journal of Combinatorial Theory, Series B* 65 (1995) 291–304.

[25] H. van der Holst, L. Lovász, A. Schrijver, On the invariance of Colin de Verdière's graph parameter under clique sums, *Linear Algebra and Its Applications* 226 (1995) 509–517.

[26] J. Kahn, M. Saks and D. Sturtevant, A topological approach to evasiveness, it Combinatorica **4** (1984) 297–306.

[27] M.A. Krasnoselsky and P.P. Zabrejko, *Geometricheskie Zadachi Nelinejnogo Analiza*, Nauka, Moscow, 1975.

[28] L. Lovász, On the number of halving lines, *Annales Univ. Eötvös* **14** (1971), 107-108.

[29] L. Lovász, A homology theory for spanning trees of a graph, *Acta Math. Hung.* **30**, 241-251.

[30] L. Lovász, Kneser's conjecture, chromatic number, and homotopy, *J. Comb. Theory A* **25** (1978), 319-324.

[31] L. Lovász, Topological and algebraic methods in graph theory, in: *Graph Theory and Related Topics*, Academic Press, 1979, 1-14.

[32] L. Lovász, Matroids and Sperner's Lemma, *Europ. J. Combin.* **1**, 65-66.

[33] L. Lovász, Self-dual polytopes and the chromatic number of distance graphs on the sphere, *Acta Sci. Math. Szeged* **45**, 317-323.

[34] L. Lovász, A. Schrijver, A Borsuk theorem for antipodal links and a spectral characterization of linklessly embeddable graphs (preprint)

[35] J. Mather: Invariance of the homology of a lattice, *Proc. Amer. Math. Soc.* **17** (1966) 1120–1124.

[36] J. Milnor: On the Betti numbers of real algebraic varieties, *Proc. Amer. Math. Soc.* **15** (1964) 275–280.

[37] J.R. Munkres: *Elements of Algebraic Topology*, Addison-Wesley, Menlo Park (1984).

[38] R. Oliver: Fixed-point sets of group actions on finite acyclic complexes, *Comment. Math. Helvet.* **50** (1975), 155–177.

[39] N. Robertson, P.D. Seymour, R. Thomas, A survey of linkless embeddings, in: *Graph Structure Theory* (N. Robertson, P. Seymour, eds.), Contemporary Mathematics, American Mathematical Society, Providence, Rhode Island, 1993, pp. 125–136.

[40] N. Robertson, P. Seymour, R. Thomas, Sachs' linkless embedding conjecture, *Journal of Combinatorial Theory, Series B* 64 (1995) 185–227.

[41] R.P. Stanley: *Enumerative Combinatorics*, Vol. 1, Wadsworth & Brooks/Cole, Monterey (1986).

[42] M. Steele and A. Yao, Lower bounds for algebraic decision trees, *J. Algorithms* **3** (1982), 1–8.

[43] R. Thom: Sur l'homologie des variétés algébraiques réelles, in: *Differential and Algebraic Topology* (ed. S. S. Cairns), Princeton Univ. Press, Princeton (1965).

[44] K. Wagner, Über eine Eigenschaft der ebene Komplexe, *Mathematische Annalen* 114 (1937) 570–590.

[45] R. Živaljević: User's guide to equivariant methods in combinatorics, *Publ Inst. Math. Belgrade* **59(73)** (1996), 114–130.

[46] R.T. Živaljević, S.T. Vrećica: The colored Tverberg's problem and complexes of injective functions, *J. Combinatorial Theory* Ser. A **61** (1991) 309–318.