Examples of $\mathbb{Z}$-Acyclic and Contractible Vertex-Homogeneous Simplicial Complexes

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Abstract

It was shown in [11] that there are no (non-trivial) 2- and 3-dimensional $\mathbb{Z}$-acyclic vertex-homogeneous simplicial complexes. In this paper we construct a 5-dimensional example and further examples in higher dimensions, one of which is Oliver's example of dimension 11, the only previously known example of a non-contractible $\mathbb{Z}$-acyclic vertex-homogeneous simplicial complex. We also present infinite series of contractible vertex-homogeneous simplicial complexes by starting with one of the $\mathbb{Z}$-acyclic examples.

1 Introduction

Interst in vertex-homogeneous simplicial complexes with certain topological properties can arise from different perspectives. For example such complexes appear naturally when one studies certain fixed point theorems in algebraic topology but they also show up in connection with the famous Evasiveness Conjecture in complexity theory. This astonishing conjunction was established by Kahn, Saks, and Sturtevant in [10] where they made use of a fixed point theorem by Oliver [12] to settle the Evasiveness Conjecture in the prime power case.

In some sense at the core of the connection is the observation that if the vertex-transitive action of a (finite) group G on a (finite) simplicial complex K (with m vertices) has a fixed point, then K is a simplex. To see this geometrically, we may regard K as a subcomplex of the $(m-1)$-dimensional simplex Δ_{m-1} with vertices $e_1, \dots, e_m$. Any point x of K then has a unique representation $x = \sum_{i=1}^{m} \lambda_i e_i$, with $\lambda_i \geq 0$ and $\sum_{i=1}^{m} \lambda_i = 1$. The group G then acts by permuting the coordinates, $gx = \sum_{i=1}^{m} \lambda_i e_{g(i)}$, $g \in G$. If G is transitive, then for every i, j there is some $g \in G$ such that $e_j = e_{g(i)}$. If, in addition, the action of G has a fixed point y, then $gy = y$ for every group element g, and therefore $\lambda_1 = \ldots = \lambda_m = \frac{1}{m}$. But $y = \frac{1}{m} \sum_{i=1}^{m} e_i$ is a point of K if and only if K is a simplex.

For certain group action the existance of such fixed points can be guaranteed by fixed point theorems. It was shown by Smith [15] that if a p-group P, i. e., a group with prime power order $|P| = p^t$, acts on a $\mathbb{Z}_p$-acyclic complex, then the fixed point

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set for this action is $\mathbb{Z}_p$-acyclic as well. In particular, the fixed point set is not empty — hence, there are no vertex-transitive group actions of a p-group on a $\mathbb{Z}_p$-acyclic simplicial complex (that is not a simplex).

The theorem by Smith has been generalized by Oliver.

Theorem 1 (Oliver [12]) *Let G be a finite group with subsequent normal subgroups $P \lhd Q \lhd G$ such that*

(i) *P is a p-group,*

(ii) *G/Q is a q-group, and*

(iii) *Q/P is cyclic.*

If G acts on a $\mathbb{Z}_p$-acyclic complex K, then the Euler characteristic $\chi(K^G)$ of the fixed point set K^G is equivalent to $1 \pmod q$.

In order to give a brief account on the Evasiveness Conjecture, let $\mathcal{P}$ be any *graph property*, that is, a property of graphs which is invariant under graph-isomorphisms, on a fixed set of nodes V of size $n := |V|$, and let E denote the set of all edges on V, with $m := |E| = \binom{n}{2}$. We identify $\mathcal{P}$ with the set system

$$\mathcal{F}_\mathcal{P} := \{A \subseteq E : \text{Graph } (V, A) \text{ has property } \mathcal{P}\} \subseteq 2^E,$$

and for an unknown graph $\mathcal{G} = (V, A)$ on V we consider the *decision problem* whether $\mathcal{G}$ has the property $\mathcal{P}$ or not. In order to find out if the edge set A of $\mathcal{G}$ belongs to $\mathcal{F}_\mathcal{P}$, we ask questions of the type "Is $e \in A$?", and an oracle answers (correctly) YES or NO.

The number of elements of E that we will have to test in the worst case, if we proceed according to some optimal strategy, is called the *argument complexity* $c(\mathcal{F}_\mathcal{P})$ of $\mathcal{P}$. Then $0 \leq c(\mathcal{F}_\mathcal{P}) \leq m$, and $\mathcal{P}$ is *trivial* if $c(\mathcal{F}_\mathcal{P}) = 0$ and *non-trivial* if $c(\mathcal{F}_\mathcal{P}) > 0$. $\mathcal{P}$ is called *evasive* if $c(\mathcal{F}_\mathcal{P}) = m$ and *non-evasive* otherwise. For general set systems $\mathcal{F} \subseteq 2^E$, these terms are defined analogously. A graph property is *monotone* if it is preserved under deletion of edges.

In the early seventies Richard Karp proposed the following remarkable conjecture.

Evasiveness Conjecture for Graph Properties: *Every non-trivial monotone graph property $\mathcal{P}$ is evasive.*

Extensive work has been done on determining the argument complexity of particular graph properties (see e.g. [1], [2], [5, Ch. VIII], [17]).

Kahn, Saks, and Sturtevant's approach to the Evasiveness Conjecture was by reformulating Karp's Conjecture in the language of simplicial complexes: If $\mathcal{P}$ is a monotone graph property, then the corresponding set system $\mathcal{F}_\mathcal{P}$ is a (finite abstract) simplicial complex with vertex set E. Let us denote $\mathcal{F}_\mathcal{P}$ the *graph complex* associated with $\mathcal{P}$. Invariance under permutation of the nodes of V (what one naturally requires for $\mathcal{P}$ to be a graph property) gives rise to an induced action of the symmetric group S_n on the edge set E, and thus on the simplicial complex $\mathcal{F}_\mathcal{P}$. Clearly, the action of S_n is transitive on E.

Theorem 2 (Kahn, Saks, and Sturtevant [10]) *Let $\mathcal{F}_{\mathcal{P}_n}$ be the graph complex associated with some (non-trivial) graph property $\mathcal{P}_n$ on $n = p^t$ nodes, with p prime. Then $\mathcal{F}_{\mathcal{P}_n}$ is not $\mathbb{Z}_p$-acyclic.*

Proof: Let $G = \mathit{Aff}(GF(p^t)) < S_n$ be the group of affine transformations of $GF(p^t)$ and let $Q := G$ and $P := \{x \mapsto x+b : b \in GF(p^t)\}$. The group G is 2-transitive on $\{1 \dots n\}$ and therefore transitive on the edge set E. Hence, G is a vertex-transitive subgroup of the symmetric group S_n with induced action on all graph complexes $\mathcal{F}_{\mathcal{P}_n}$. But then either $\mathcal{F}_{\mathcal{P}_n}$ is a simplex, and thus $\mathcal{P}_n$ is trivial, or $\mathcal{F}_{\mathcal{P}_n}$ is not $\mathbb{Z}_p$-acyclic by Theorem 1 and the particular choice of Q and P. □

If a graph complex is not $\mathbb{Z}_p$-acyclic, then it cannot be non-evasive.

Corollary 3 (Kahn, Saks, and Sturtevant [10]) *The Evasiveness Conjecture for graph properties holds for every prime power number of nodes.*

By allowing the symmetry group to be any finite group G, one obtains the following more general situation.

Evasiveness Conjecture for Simplicial Complexes [10]: *If $\mathcal{F}$ is a non-evasive vertex-homogeneous simplicial complex on the vertex set $E = \{1, \dots, m\}$ with vertex-transitive action by some group G, then it is the standard $(m-1)$-simplex Δ_{m-1}.*

To be "non-evasive" is a strong *topological* requirement. The following sequence of implications holds for finite simplicial complexes (cf. [3], [10], and [16]):

$$\text{non-evasive} \Rightarrow \text{collapsible} \Rightarrow \text{contractible} \Rightarrow \mathbb{Z}\text{-acyclic} \Rightarrow \mathbb{Q}\text{-acyclic} \Rightarrow \tilde{\chi} = 0$$

and leads to further generalizations of the above conjecture if we replace "non-evasive" with the respective weaker requirements ($\tilde{\chi}$ denotes the reduced Euler characteristic of a simplicial complex).

These generalized conjectures all hold once again for prime power numbers of vertices by a theorem of Rivest and Vuillemin [13] (see also [11]). Yet, for non-prime power numbers there are counterexamples known to all of the generalized conjectures with the execption of the Evasiveness-Conjecture for Simplicial Complexes which still remains open.

There is an abundance of (non-trivial) $\mathbb{Q}$-acyclic vertex-homogeneous simplicial complexes and even more with $\tilde{\chi} = 0$. The smallest $\mathbb{Q}$-acyclic example is the 6-vertex triangulation of the real projective plane (see below and c. f. [11]).

In [11] it was shown that there are no 2- and 3-dimensional $\mathbb{Z}$-acyclic vertex-homogeneous simplicial complexes. other than a simplex. In Section 4 we will present a 5-dimensional example and further examples in higher dimensions, one of which is Oliver's example of dimension 11, the only previously known example of a non-contractible $\mathbb{Z}$-acyclic vertex-homogeneous simplicial complex. In Section 2 we will discuss some topological tools that will be used for our later constructions, and in Section 3 we will give infinite series of contractible vertex-homogeneous simplicial complexes. It remains to mention that Oliver described techniques to construct collapsible vertex-homogeneous simplicial complex (c. f. [10]).

2 Topological Tools

The main tools for our constructions will be the well known nerve operation and duality observations.

2.1 The Nerve Operation

Let K be a (finite abstract) simplicial complex, $\mathcal{F} = (F_j)_{j\in J}$ its collection of maximal faces (facets), and J the corresponding index set. We call the covering of K by its facets $\mathcal{F}$ the *standard covering* of K.

Theorem 4 (Nerve Theorem, Borsuk, cf. [3]) *Let $\mathcal{N}(K)$ be the* ***nerve complex*** *of K (with respect to the standard covering $\mathcal{F} = (F_j)_{j\in J}$), that is, $\mathcal{N}(K)$ is the simplicial complex on the vertex set J such that $\Delta \subseteq J$ is a simplex of $\mathcal{N}(K)$ if and only if $\bigcap_{j\in\Delta} F_j \neq \emptyset$. Then K and $N := \mathcal{N}(K)$ are homotopy equivalent.*

The nerve complex of a simplex of any dimension is a point, and Grünbaum describes in [8] the class of all simplicial complexes having the same nerve. Moreover, by a theorem of Mani (cf. [8]), there is for every simplicial complex N some simplicial complex K such that $N = \mathcal{N}(K)$.

Definition 5 *Let E be the vertex set of K, and for every vertex e of K let $e^1, \dots, e^n$ be n distinct copies. The* ***n-th multiple nK*** *of K is the simplicial complex on the vertex set $nE = \bigcup_{r=1}^{n} E^r$, where E^r denotes the r-th copy of E, that has as its maximal faces the sets $nF = \{e_1^1, e_1^2, \dots, e_1^n, \dots, e_k^1, e_k^2, \dots, e_k^n\}$ for the facets $F = \{e_1, \dots, e_k\}$ of K.*

By construction, $\bigcap_{j\in\Delta} F_j \neq \emptyset$ if and only if $\bigcap_{j\in\Delta} nF_j \neq \emptyset$. Hence, $\mathcal{N}(nK) = \mathcal{N}(K)$, and nK is homotopy equivalent to K.

Although the above examples demonstrate that there are always non-isomorphic simplicial complexes which have the same nerve complex, the nerve operation is injective on a large class of simplicial complexes. Grünbaum [8] calls a simplicial complex *taut* if every vertex is the intersection of the facets containing it.

Lemma 6 (Duality, [8]) *If K is taut, then $\mathcal{N}(K)$ is taut and $K = \mathcal{N}(\mathcal{N}(K))$.*

Proof: Let K be taut with standard covering $\mathcal{F} = (F_j)_{j\in J}$. The nerve $N = \mathcal{N}(K)$ has one vertex j for every facet F_j of K, $j \in J$. If $\Delta_e \subseteq J$ is the collection of all j's so that the corresponding facets F_j contain the vertex $e \in E$, then, by the tautness of K, $\bigcap_{j\in\Delta_e} F_j = \{e\}$ and Δ_e is a facet of N. Hence, to any vertex e of K there uniquely corresponds a facet $\Delta_e = \{j : e \in F_j\}$ of N.

On the other hand, let $j \in J$. Suppose there exists some $j' \in J$, $j' \neq j$, such that $j' \in \bigcap_{e\in E, j\in\Delta_e} \Delta_e$. But then $F_j \subset F_{j'}$, which is a contradiction to the maximality of F_j. Thus, the nerve N is taut, and $\mathcal{N}(\mathcal{N}(K)) = K$. □

The nerve complex $\mathcal{N}(K)$ of a simplicial complex K need not to be taut. For example, the nerve complex of a path of length m is a path of length $(m-1)$. A point is taut, and m-gons C_m are taut for $m \geq 3$.

2.2 The Join and the Dual Join Product

Recall that the join $K * K'$ of two simplicial complexes K and K' (with disjoint vertex sets) is defined as $K * K' := \{\Delta \cup \Delta' : \Delta \in K, \Delta' \in K'\}$.

Lemma 7 *If K and K' are taut simplicial complexes, different from a point, then their join product $K * K'$ is taut as well.*

Proof: The vertex set of $K * K'$ is $E \cup E'$. Let e be a vertex of $K * K'$ with $e \in E$. If $\mathcal{V} = (F_j \cup F'_{j'})_{(j \in J, j' \in J')}$ denotes the collection of maximal faces of $K * K'$ for the facets $\mathcal{F} = (F_j)_{j \in J}$ of K and the facets $\mathcal{F}' = (F'_{j'})_{j' \in J'}$ of K', then

$$\bigcap_{\substack{j \in J, j' \in J', \\ e \in F_j \cup F'_{j'}}} F_j \cup F'_{j'} = \bigcap_{\substack{j \in J, \\ e \in F_j}} F_j = \{e\},$$

and the same is true for $e \in E'$. Therefore, $K * K'$ is taut. □

Definition 8 *Let K and K' be (finite abstract) simplicial complexes. Then the **dual join product** of K and K' is the product*

$$K \bowtie K' := \mathcal{N}(\mathcal{N}(K) * \mathcal{N}(K')). \qquad (1)$$

The dual join product of two (non-trivial) taut complexes K and K' is taut by Lemma 6 and Lemma 7. In particular, the following equality holds for (non-trivial) taut complexes,

$$\mathcal{N}(K \bowtie K') = \mathcal{N}(K) * \mathcal{N}(K'). \qquad (2)$$

3 Vertex-Homogeneous Simplicial Complexes

3.1 The Nerve of a Vertex-Homogeneous Simplicial Complex

Let G be a permutation group of the vertex set E, and let $K \subseteq 2^E$ be a simplicial complex which is invariant under the given G-action.

Lemma 9 *The action of G on K induces an action of G on the nerve $\mathcal{N}(K)$ of K.*

Proof: Let $\mathcal{F} = (F_j)_{j \in J}$ be the standard covering of K. The action of G on the set J of facets of K gives rise to an action of G on the nerve $\mathcal{N}(K)$, since $\bigcap_{j \in \Delta} gF_j \neq \emptyset$ if and only if $\bigcap_{j \in \Delta} F_j \neq \emptyset$ for any $\Delta \in \mathcal{N}(K)$. □

Let, from now on, K be a vertex-homogeneous simplicial complex. Then the induced action of G on the nerve N of K is, in general, not vertex-homogeneous anymore. More precisely, the action of G on N is transitive on the set of vertices of N if and only if K has exactly one orbit of maximal faces.

Lemma 10 (Characterization of vertex-homogeneous simplicial complexes)

(i) *If K is vertex-homogeneous, then its nerve $N = \mathcal{N}(K)$ is facet-homogeneous.*

(ii) *If N is facet-homogeneous, then its nerve $K = \mathcal{N}(N)$ is vertex-homogeneous.*

Proof: (i) Let K be a vertex-homogeneous simplicial complex on m vertices.

We first show that $N = \mathcal{N}(K)$ is pure. Let $\mathrm{Max}(K)$ denote the collection of orbits of maximal faces of K. By transitivity, every vertex $e \in E$ is contained the same number of times, $r_{\mathcal{O}}$, in the k-element sets of any particular orbit $\mathcal{O}$ of facets of K, i.e., $k \cdot |\mathcal{O}| = r_{\mathcal{O}} \cdot m$. Altogether, every vertex e is contained in precisely $r = \sum_{\mathcal{O} \in \mathrm{Max}(K)} r_{\mathcal{O}}$ distinct facets of K, i.e., $\dim(\Delta_e) = r - 1$ for every e. In particular, $\dim(\mathcal{N}(K)) = r - 1$.

Let Δ and Δ' be two different facets of N. Then Δ and Δ' correspond to two distinct sets of r maximal faces of K respectively. Let e be some element in the intersection of the r maximal faces of K corresponding to Δ (there can be more than one such element!), and let e' be an element of E representing Δ'. Since the action of G is transitive on E, there exists a group element $g \in G$ such that $e' = g * e$. But then g maps the facets corresponding to e to the facets corresponding to e', and hence the action of G on N is transitive on the facets of N.

(ii) Trivial, since any facet-homogeneous simplicial complex has only one orbit of maximal faces. □

Example 1: The n-gon C_n, $n \geq 3$, with rotations by elements of $\mathbb{Z}_n$ is a taut vertex-homogeneous and facet-homogeneous simplicial complex with $C_n = \mathcal{N}(C_n)$.

If we replace every edge of the circle C_n by an m-simplex, $m \geq 2$, then the resulting (n,m)-necklace C_n^m is a facet-homogeneous pure simplicial complex with $C_n = \mathcal{N}(C_n^m)$. Thus, (n,m)-necklaces form an infinite class of facet-homogeneous simplicial complexes, which all have the same nerve.

Example 2: For $n \geq 3$, the 2-fold multiple $2\,C_n$ is a chain of tetrahedra:

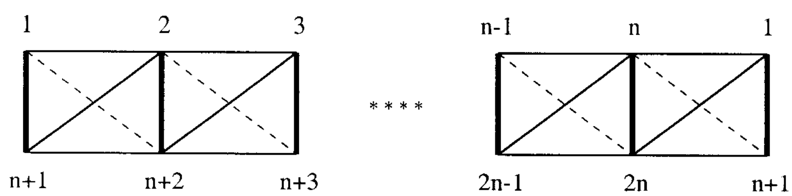

Figure 1: The 2-fold multiple $2\,C_n$.

With respect to the action of $\mathbb{Z}_n \times \mathbb{Z}_2$, which rotates the tetrahedra and flips the upper and lower vertices, the 2-fold multiple $2\,C_n$ is vertex-homogeneous and facet-homogeneous, with $C_n = \mathcal{N}(2\,C_n)$.

Example 3: The 6-vertex triangulation $\mathbb{RP}^2_6$ (see Figure 2) of the real projective plane is vertex-homogeneous and taut.

The symmetry group of $\mathbb{RP}^2_6$ is $A_5(6) = \langle (1,2,3,4,6), (1,4)(5,6) \rangle$, with $A_5(6)$ acting transitively on the set of maximal faces:

1	$\{1,2,4\}$,	2	$\{1,2,5\}$,	3	$\{1,3,4\}$,	4	$\{1,3,6\}$,	5	$\{1,5,6\}$,
6	$\{2,3,5\}$,	7	$\{2,3,6\}$,	8	$\{2,4,6\}$,	9	$\{3,4,5\}$,	10	$\{4,5,6\}$.

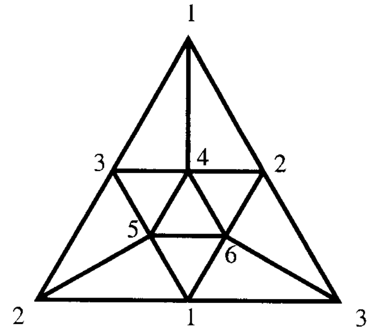

Figure 2: The 6-vertex triangulation of the real projective plane.

The nerve of $\mathbb{RP}^2_6$ is a taut 4-dimensional vertex-homogeneous and facet-homogeneous simplicial complex on 10 vertices:

1	$\{1,2,3,4,5\}$,	2	$\{1,2,6,7,8\}$,	3	$\{3,4,6,7,9\}$,
4	$\{1,3,8,9,10\}$,	5	$\{2,5,6,9,10\}$,	6	$\{4,5,7,8,10\}$.

If we compute the nerve of this complex, then we get $\mathbb{RP}^2_6$ again.

REMARK: We see by this example that the nerve of a vertex-homogeneous complex can have more vertices and can be of higher dimension than the original complex, it also can have less vertices and can be of lower dimension.

EXAMPLE 4: $\mathbb{Q}$-ACYCLIC VERTEX-HOMOGENEOUS SIMPLICIAL COMPLEXES

It turned out in [11] that the 6-vertex triangulation of the real projective plane is the smallest (non-trivial) example of a $\mathbb{Q}$-acyclic vertex-homogeneous simplicial complex. We will make use of Lemma 10 to derive further examples of vertex-homogeneous as well as facet-homogeneous $\mathbb{Q}$-acyclic simplicial complexes, which are homotopy equivalent to $\mathbb{RP}^2$.

Besides that it is vertex-homogeneous, the 6-vertex triangulation of the projective plane **Aa** (cf. Figure 3) is facet-homogeneous as was mentioned above. Hence, the nerve complex **aA** of **Aa** (in Figure 3, vertex-homogeneous and facet-homogeneous simplicial complexes are labeled by capital letters and small letters respectively) is a facet-homogeneous and vertex-homogeneous simplicial complex on 10 vertices with the shaded pentagons depicting the six 4-dimensional facets glued together along the sides of the pentagons.

The nerve complex **B** (**C**) of the facet-homogeneous subdivision **b** (of the facet-homogeneous barycentric subdivision **c**) of **Aa** is vertex-homogeneous on 30 (60) vertices.

If we replace in **b** for any bold edge both neighboring triangles by a tetrahedron, then the resulting simplicial complex **d** is still facet-homogeneous. Its nerve complex **D** is on 15 vertices with two orbits of maximal faces, one consisting of six 4-simplices

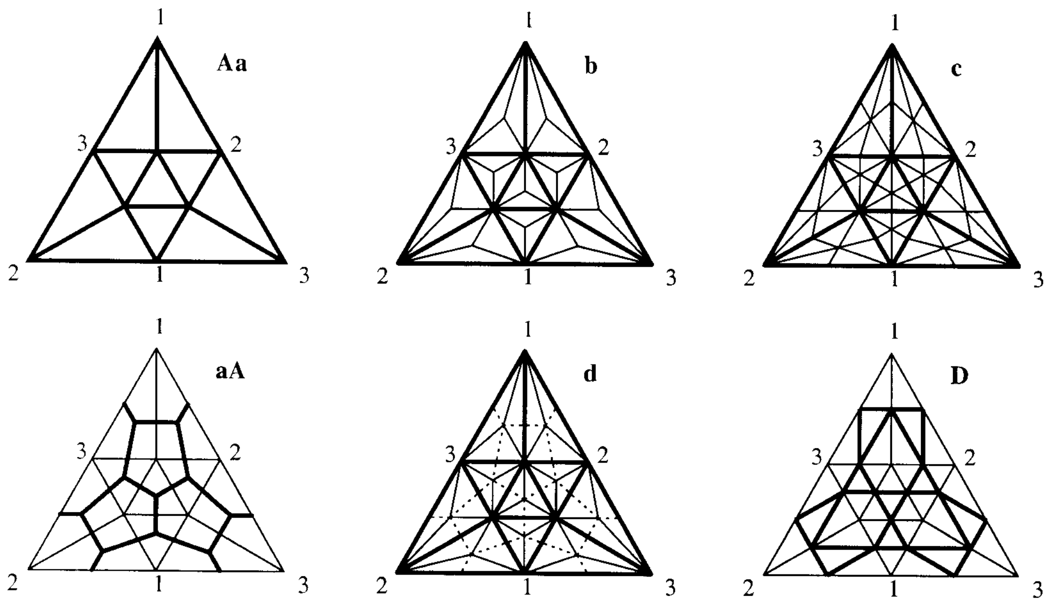

Figure 3: Vertex-/facet-homogeneous simplicial complexes homotopy equivalent to $\mathbb{RP}^2$.

(shaded pentagons) and the other of 10 triangles. The same construction can be carried through for **c** leading to complexes **e** and **E** (the latter on 30 vertices).

A further example **F** of a vertex-homogeneous simplicial complex on 15 vertices, homotopy equivalent to the projective plane, can be obtained from **D** by gluing in 60 tetrahedra in the following way. For any white triangle of **D** we add six tetrahedra with vertex-sets the triangle and in addition one vertex of a neighboring white triangle respectively. The resulting space **F** is still vertex-homogeneous, and it can be worked out easily that it collapses to **D** and thus is homotopy equivalent to **D**.

3.2 Constructions with Vertex-Homogeneous Simplicial Complexes

After we now have seen some simple examples of vertex-homogeneous simplicial complexes, we will next discuss three constructions that allow us to derive further vertex-homogeneous simplicial complexes if we start with a given one.

Proposition 11 *Let (K, G) denote a pair of a simplicial complex K with vertex set E of cardinality m and a group $G < S_m$ that acts vertex-transitively on K. If F is a finite set with $n = |F|$ elements and $H < S_n$ is a transitive permutation group of degree n, which acts on F, then the simplicial complexes*

(i) $(K^{*n}, G \times H)$ (Oliver, cf. [10])

(ii) $(K \bowtie F, G \times H)$

(iii) $(nK, G \times H)$

are vertex-homogeneous for the obvious actions of $G \times H$.

Proof: (i) Let the direct product $G \times H$ act on the n-fold join product K^{*n}, with G acting transitively on every copy of K and with H permuting the n copies of K. The vertex set of K^{*n} is the union $E_{\cup} = \bigcup_{r=1}^{n} E^r$ of n copies of E, and the action of $G \times H$ is clearly transitive on $E_{\cup}$.

(ii) Since (K, G) and (F, H) are vertex-homogeneous complexes, their nerve complexes $(\mathcal{N}(K), G)$ and $(\mathcal{N}(F), H)$ are facet-homogeneous, with $\mathcal{N}(F) = F$. The join product $\mathcal{N}(K) * F$ is facet-homogeneous for the diagonal action of $G \times H$, and thus $(K \bowtie F, G \times H)$ is vertex-homogeneous.

(iii) As in (i), $G \times H$ acts transitively on $E_{\cup} = \bigcup_{r=1}^{n} E^r$. □

Corollary 12 *If K is a $\mathbb{Z}$-acyclic vertex-homogeneous simplicial complex, then the n-fold multiples nK form an infinite series of $\mathbb{Z}$-acyclic vertex-homogeneous simplicial complexes. Moreover, the series K^{*n} and $K \bowtie F$ provide examples of contractible vertex-homogeneous simplicial complexes.*

Proof: It remains to show that K^{*n} and $K \bowtie F$ are contractible for $n \geq 2$. If K is $\mathbb{Z}$-acyclic, then it is connected. Now, the join product of a k-connected complex with an l-connected complex is $(k+l+2)$-connected. In particular, K^{*n} and $K \bowtie F$ are at least $(0-1+2)$-connected, that is, simply connected. But since a simplicial complex is contractible if and only if it is simply connected and $\mathbb{Z}$-acyclic (cf. [3]), the result follows. □

4 The Identified Dodecahedron and Seven Related $\mathbb{Z}$-Acyclic Vertex-Homogeneous Simplicial Complexes

Let us consider the boundary complex of the dodecahedron with 12 pentagonal facets, 30 edges, and 20 vertices. If we identify opposite pentagons by a coherent twist of $\pi/5$ radians, then the resulting cell complex Q is $\mathbb{Z}$-acyclic; see Figure 4 and c. f. [4], [7], and [6, p. 57]. The symmetry group of the *identified dodecahedron* Q is the alternating group A_5.

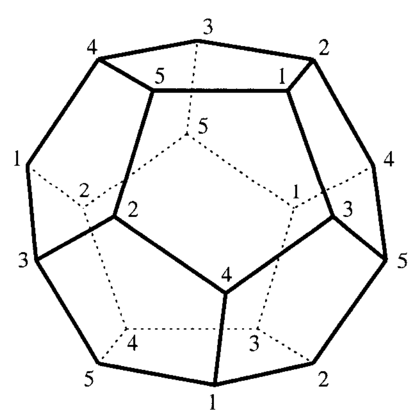

Figure 4: The $\mathbb{Z}$-acyclic identified dodecahedron.

Lemma 13 *There are precisely two A_5-invariant facet-transitive triangulations of the $\mathbb{Z}$-acyclic complex Q.*

Proof: Every edge of the cell complex Q is the intersection of *three* pentagonal cells, which implies that the 1-skeleton of Q is necessarily included in the 1-skeleton of any A_5-invariant triangulation of Q. This is also the case for the five vertices of Q. The action of A_5 on Q is transitive on the pentagons, and any of the pentagons has the dihedral group $D_5 < A_5$ as its isotropy group. It is therefore sufficient to determine D_5-invariant facet-transitive triangulations of a pentagon. There are exactly two such triangulations, one with 5 and the other with 10 triangles. We denote the corresponding triangulations of Q with 30 and 60 triangles by N_I and N_O respectively (see Figures 6 and 5). □

Corollary 14 *The nerve complexes $K_1 := \mathcal{N}(N_I)$ and $K_O := \mathcal{N}(N_O)$ are examples of $\mathbb{Z}$-acyclic but not contractible vertex-homogeneous simplicial complexes on* 30 *and* 60 *vertices respectively.*

Oliver's Example K_O

The complex K_O was first found by Bob Oliver. His construction is algebraic and was mentioned in [10] and in a paper by Segev [14]. In fact, Segev presented an explicit proof that the nerve complex $N_O = \mathcal{N}(K_O)$, and hence K_O, is $\mathbb{Z}$-acyclic. Moreover, it was conjectured in [14] that $\mathcal{N}(K_O)$ is homeomorphic to Q. We will show that this is indeed the case.

Let A_5 be the alternating group of even permutations of the set $\{1, 2, 3, 4, 5\}$. Define the subgroups

$$\begin{array}{rcl|l} U &:=& \{ g \in A_5 : g \cdot 2 = 2 \} & \cong A_4, \\ V &:=& N_{A_5}(\langle (1,2,3,4,5) \rangle) & \cong D_5, \\ W &:=& N_{A_5}(\langle (1,3,5) \rangle) & \cong D_3, \end{array}$$

where $N_{A_5}(H)$ denotes the normalizer in A_5 of a subgroup H of A_5. The stabilizer U of the point 2 is isomorphic to the alternating group A_4 and has 12 elements. The subgroups V and W are isomorphic to the dihedral groups D_5 and D_3 with 10 and 6 elements respectively.

Oliver takes as vertex set E for the simplicial complex K_O the 60 elements of A_5 and lets A_5 act transitively on E by left multiplication. He defines K_O to be the simplicial complex that has the left cosets of U, V, and W as its (orbits of) maximal faces:

$$K_O := \bigcup_{g \in A_5} 2^{g \cdot U} \cup \bigcup_{g \in A_5} 2^{g \cdot V} \cup \bigcup_{g \in A_5} 2^{g \cdot W}.$$

Theorem 15 (Oliver, cf. [10]) *The 11-dimensional simplicial complex K_O is vertex-homogeneous and $\mathbb{Z}$-acyclic.*

Proof: By construction, K_O is 11-dimensional and vertex-homogeneous. To see that K_O is $\mathbb{Z}$-acyclic, we compute the nerve $\mathcal{N}(K_O)$ of K_O. As maximal faces of the nerve we get 60 triangles. By a suitable labeling of the vertices, $\mathcal{N}(K_O)$ turns out to be the triangulation N_O of the $\mathbb{Z}$-acyclic identified dodecahedron Q (see Figure 5). Thus K_O is $\mathbb{Z}$-acyclic by the Nerve Theorem 4. □

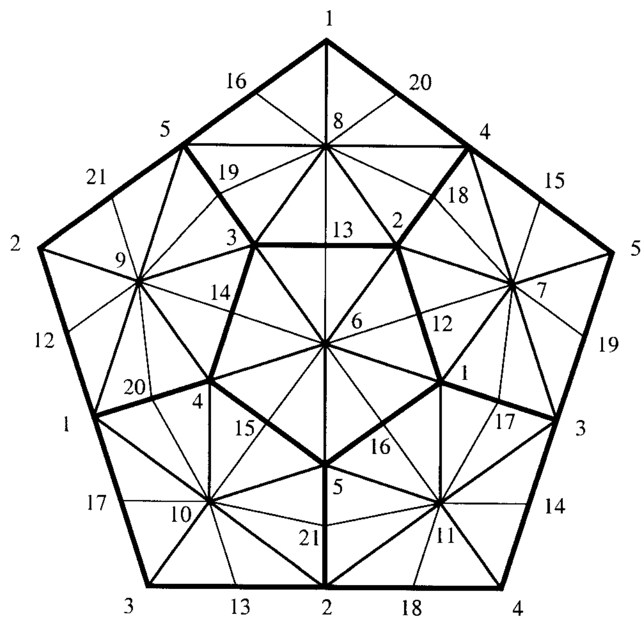

Figure 5: Triangulation N_O of the identified dodecahedron Q with 60 triangles.

New $\mathbb{Z}$-Acyclic Vertex-Homogeneous Simplicial Complexes

In [11], we enumerated all vertex-homogeneous simplicial complexes with reduced Euler characteristic $\tilde{\chi} = 0$ corresponding to a given group action on few vertices. For the A_5-action on 60 vertices it is hopeless to generate all vertex-homogeneous simplicial complexes with $\tilde{\chi} = 0$ and then compute their homology in order to find $\mathbb{Z}$-acyclic examples. But if we restrict our computer search to complexes that have only few orbits of maximal faces with orbit size less than 30, then, in particular, we obtain the above example K_O. Recall that it follows from [11] that an A_5-orbit of k-sets on 60 vertices can have size less than 30 if and only if $\gcd(k, 60) > 2$. We formed combinations of at most six orbits with at most two orbits of maximal faces of the same dimension. For every simplicial complex K corresponding to one of these collections of orbits of facets, we computed the reduced Euler characteristic $\tilde{\chi}(\mathcal{N}(K))$ of the nerve complex of K. Whenever $\tilde{\chi}$ was zero, we computed the homology of $\mathcal{N}(K)$ with the program HOMOLOGY by Heckenbach [9]. Including K_O, we found five $\mathbb{Z}$-acyclic A_5-invariant complexes on 60 vertices that we denote by K_O, K_2, K_4, K_5, and K_6. The examples K_2 and K_4 are not taut, and it turns out that $K_1 := \mathcal{N}(\mathcal{N}(K_2))$ and $K_3 := \mathcal{N}(\mathcal{N}(K_4))$ are taut A_5-invariant $\mathbb{Z}$-acyclic simplicial complexes on 30 vertices. We believe that if we extended our search, then further complexes would appear.

Theorem 16 *There are at least seven non-contractible $\mathbb{Z}$-acyclic simplicial complexes with a vertex-transitive A_5-action that are homotopy equivalent to the identified dodecahedron Q.*

Table 1 gives an overview of the examples. All seven complexes can be characterized algebraicly, and this we will do for K_1 to K_6 in the following. Moreover, we give geometric descriptions of the corresponding facet-homogeneous nerve complexes N_I to N_{IV}.

Complex	# vertices	dim
$K_O = \mathcal{N}(N_O)$	60	11
$K_1 = \mathcal{N}(N_I)$	30	11
$K_2 = 2\,K_1$	60	23
$K_3 = \mathcal{N}(N_{II})$	30	5
$K_4 = 2\,K_3$	60	11
$K_5 = \mathcal{N}(N_{III})$	60	11
$K_6 = \mathcal{N}(N_{IV})$	60	11

Table 1: $\mathbb{Z}$-acyclic vertex-homogeneous simplicial complexes with A_5-action.

REMARK: Although the examples K_O and K_1 to K_6 are not contractible, by Proposition 11 there exist infinite series of contractible vertex-homogeneous simplicial complexes associated with K_O and K_1 to K_6.

THE $\mathbb{Z}$-ACYCLIC COMPLEXES K_1 AND K_2

Consider the subgroups of A_5,

$$\begin{array}{lclcl} U & := & \{\, g \in A_5 : g \cdot 2 = 2 \,\} & \cong & A_4, \\ V & := & N_{A_5}(\,\langle\,(1,2,3,4,5)\,\rangle\,) & \cong & D_5. \end{array}$$

Then the 24-element set

$$A \;:=\; U \cup U \cdot (2,5,3)$$

determines an A_5-orbit of size 5. Define

$$K_2 := \bigcup_{g \in A_5} 2^{g \cdot A} \cup \bigcup_{g \in A_5} 2^{g \cdot V},$$

and

$$K_1 := \mathcal{N}(\mathcal{N}(K_2)).$$

Theorem 17 *The examples K_1 and K_2 are $\mathbb{Z}$-acyclic vertex-homogeneous simplicial complexes on 30 and 60 vertices respectively, with $K_2 = 2K_1$.*

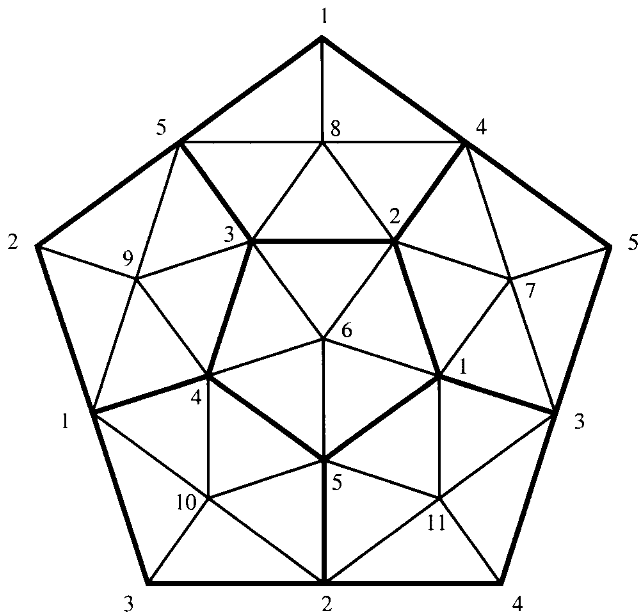

Figure 6: Triangulation N_I of the identified dodecahedron Q with 30 triangles.

Proof: The nerve complex $N_I = \mathcal{N}(K_1) = \mathcal{N}(K_2)$ of K_1 and K_2 is the facet-homogeneous triangulation of the identified dodecahedron Q with 30 triangles (see Figure 6). □

THE $\mathbb{Z}$-ACYCLIC COMPLEXES K_3 AND K_4

Take the subgroups of A_5,

$$\begin{array}{lcll} U & := & \{\, g \in A_5 : g \cdot 2 = 2 \,\} & \cong & A_4, \\ V & := & N_{A_5}(\langle (1,2,3,4,5) \rangle) & \cong & D_5, \\ W & := & N_{A_5}(\langle (1,3,5) \rangle) & \cong & D_3, \end{array}$$

and consider the 12-element set

$$B \;:=\; W \cup W \cdot (3,4,5).$$

Define

$$K_4 := \bigcup_{g\in A_5} 2^{g\cdot U} \cup \bigcup_{g\in A_5} 2^{g\cdot B} \cup \bigcup_{g\in A_5} 2^{g\cdot V},$$

and set

$$K_3 := \mathcal{N}(\mathcal{N}(K_4)).$$

The nerve $N_{II} = \mathcal{N}(K_3) = \mathcal{N}(K_4)$ of K_3 and K_4 is a 3-dimensional facet-homogeneous simplicial complex with 30 tetrahedra (see Figure 7). For every pentagon of N_O, 5 tetrahedra are glued in as indicated by the dashed lines. Since N_{II} collapses to N_O, the complex N_{II} is homotopy equivalent to Q.

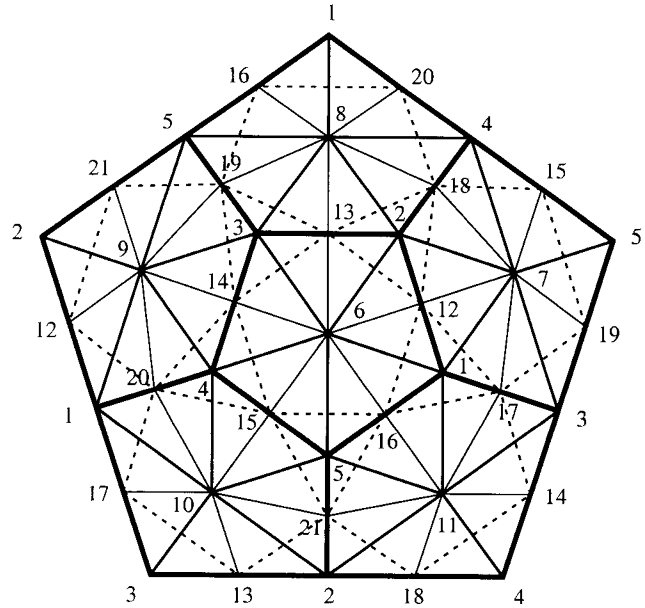

Figure 7: Triangulation N_{II} with 30 tetrahedra replacing the 60 triangles.

Theorem 18 *The example K_3 provides a 5-dimensional non-contractible $\mathbb{Z}$-acyclic vertex-homogeneous simplicial complex on 30 vertices.*

We saw in [11] that there are no (non-trivial) 2- and 3-dimensional $\mathbb{Z}$-acyclic vertex-homogeneous simplicial complexes, and that if there were a 4-dimensional example, then it would have 15, 20, 30, or 60 vertices. Our attempts failed to find a 4-dimensional example.

Conjecture 19 *The complex K_3 with f-vector $f = (1, 30, 195, 340, 255, 96, 15)$ is the smallest example of a non-contractible $\mathbb{Z}$-acyclic vertex-homogeneous simplicial complex, with respect to dimension, the number of vertices, and the total number of faces. The join $K_3 * K_3$ of dimension 11 with 60 vertices is, apart from a simplex, the smallest contractible vertex-homogeneous simplicial complexes.*

The $\mathbb{Z}$-Acyclic Complexes K_5 and K_6

Let U, V, W, and R be subgroups of A_5 with

$$\begin{array}{rcll} U & := & \{ g \in A_5 : g \cdot 2 = 2 \} & \cong \; A_4, \\ V & := & N_{A_5}(\langle (1,2,3,4,5) \rangle) & \cong \; D_5, \\ W & := & N_{A_5}(\langle (1,3,5) \rangle) & \cong \; D_3, \\ R & := & \langle (1,2)(3,5), (1,3)(2,5) \rangle & \cong \; \mathbb{Z}_2 \times \mathbb{Z}_2, \end{array}$$

and consider the 8-element set

$$C \; := \; R \cup R \cdot (2,3,4).$$

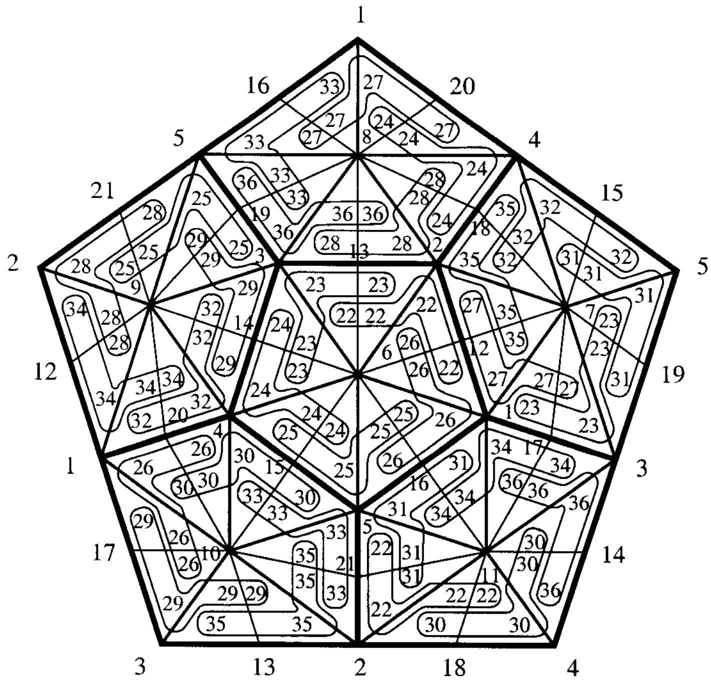

Figure 8: Triangulation N_{III} with 60 4-simplices.

Define

$$K_5 := \bigcup_{g\in A_5} 2^{g\cdot U} \cup \bigcup_{g\in A_5} 2^{g\cdot V} \cup \bigcup_{g\in A_5} 2^{g\cdot C} \cup \bigcup_{g\in A_5} 2^{g\cdot W}.$$

The nerve $N_{III} = \mathcal{N}(K_5)$, composed of 60 4-simplices, is a facet-homogeneous complex homotopy equivalent to Q. Figure 8 gives an illustration of N_{III}. To every of the 60 triangles of the triangulation N_O of Q there uniquely corresponds a 4-simplex that has as vertices the three vertices of the triangle and in addition the two vertices that are placed within the triangle. It can easily be verified that N_{III} collapses to Q.

Let once more U, V, W, and S be subgroups of A_5 with

$$\begin{array}{lcll} U & := & \{\, g \in A_5 : g\cdot 2 = 2\,\} & \cong\ A_4, \\ V & := & N_{A_5}(\,\langle\,(1,2,3,4,5)\,\rangle\,) & \cong\ D_5, \\ W & := & N_{A_5}(\,\langle\,(1,3,5)\,\rangle\,) & \cong\ D_3, \\ S & := & \langle\,(1,3)(4,5),(1,4)(3,5)\,\rangle & \cong\ \mathbb{Z}_2\times\mathbb{Z}_2, \end{array}$$

and consider the 8-element set

$$D \ := \ S \cup S\cdot(2,3,5).$$

Define

$$K_6 := \bigcup_{g\in A_5} 2^{g\cdot U} \cup \bigcup_{g\in A_5} 2^{g\cdot V} \cup \bigcup_{g\in A_5} 2^{g\cdot D} \cup \bigcup_{g\in A_5} 2^{g\cdot W}.$$

The nerve complex $N_{IV} = \mathcal{N}(K_6)$ is again 4-dimensional but combinatorially distinct from N_{III}, and provides another example of a facet-homogeneous simplicial complex homotopy equivalent to Q. The 60 4-simplices of N_{IV} are drawn in Figure 9.

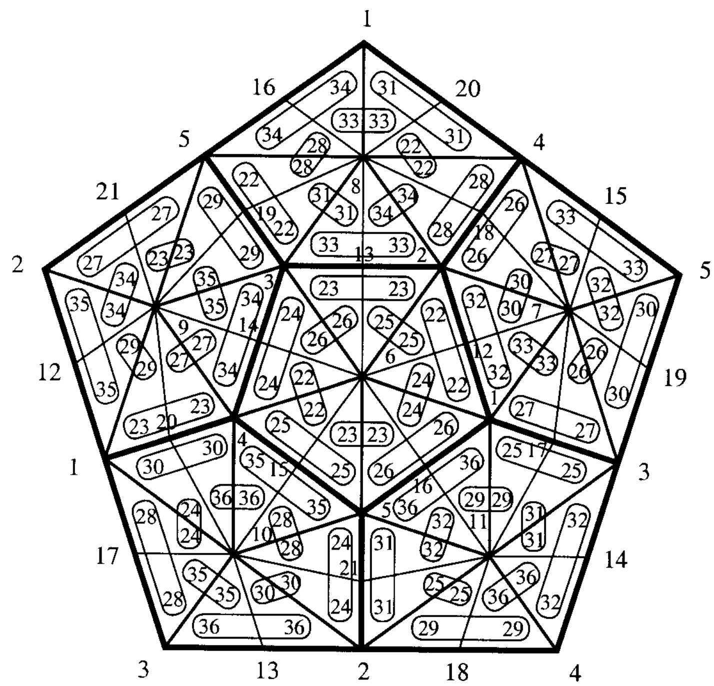

Figure 9: Triangulation N_{IV} with 60 4-simplices.

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