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INTERSECTION HOMOLOGY THEORY

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INTRODUCTION

WE DEVELOP here a generalization to singular spaces of the Poincaré–Lefschetz theory of intersections of homology cycles on manifolds, as announced in [6].

Poincaré, in his 1895 paper which founded modern algebraic topology ([18], p. 218; corrected in [19]), studied the intersection of an i-cycle V and a j-cycle W in a compact oriented n-manifold X, in the case of complementary dimension ($i+j=n$). Lefschetz extended the theory to arbitrary i and j in 1926[10]. Their theory may be summarized in three fundamental propositions:

0. If V and W are in general position, then their intersection can be given canonically the structure of an $i+j-n$ chain, denoted $V \cap W$.

1(a). $\partial(V \cap W) = 0$, i.e. $V \cap W$ is a cycle.

1(b). The homology class of $V \cap W$ depends only on the homology classes of V and W.

Note that by 0 and 1 the operation of intersection defines a product

$$H_i(X) \times H_j(X) \xrightarrow{\cap} H_{i+j-n}(X).$$

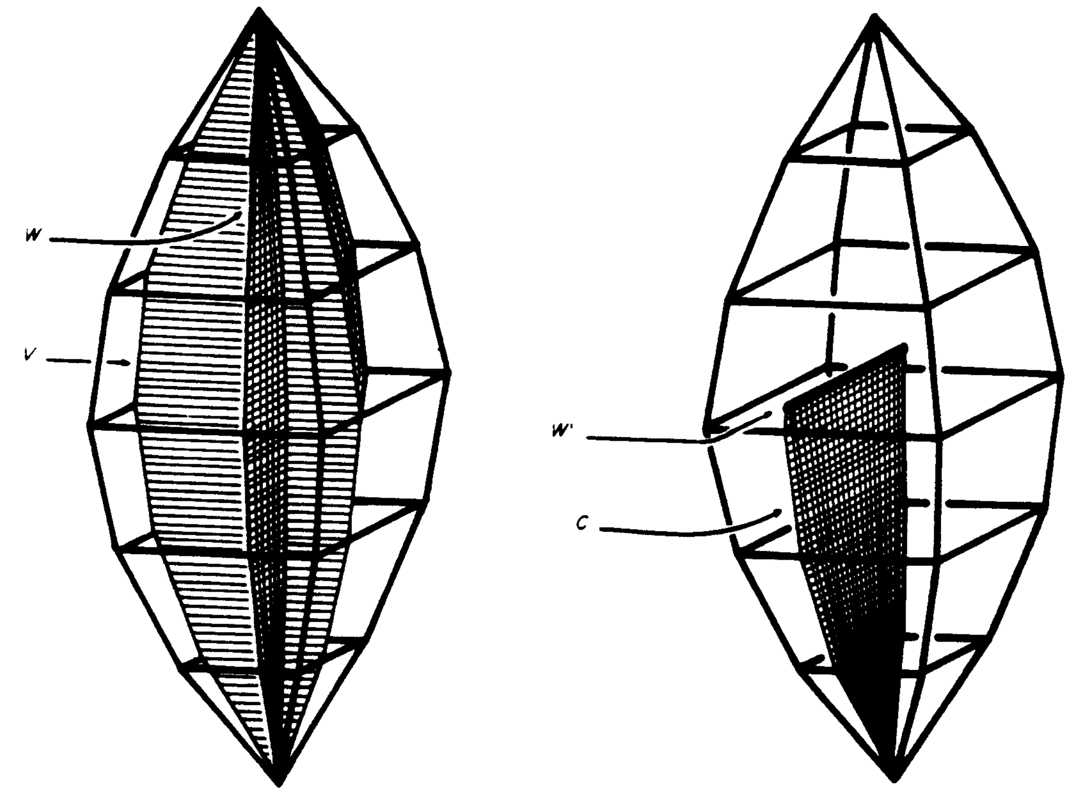
(2) *Poincaré Duality*. If i and j are complementary dimensions ($i+j=n$) then the pairing

$$H_i(X) \times H_j(X) \xrightarrow{\cap} H_0(X) \xrightarrow{\epsilon} Z$$

is nondegenerate (or "perfect") when tensored with the rational numbers. (Here, ϵ is the "augmentation" which counts the points of a zero cycle according to their multiplicities).

We will study intersections of cycles on an n dimensional oriented *pseudomanifold* (or "n-circuit", see §1.1 for the definition). Pseudomanifolds form a wide class of spaces for which proposition 0 of the Poincaré–Lefschetz theory still holds. However, propositions 1 and 2 are false for pseudomanifolds as stated. For example, if X is the suspension of the torus (see Fig. 1) $V \cap W$ is not a cycle: it has boundary at the singularities of X which persist no matter how V and W are moved within their homology class. $V \cap W'$ is a nonvanishing zero-cycle, but since $W' = \partial C$, the homology class of $V \cap W'$ does not depend only on the homology class of W'. Even if we avoid this problem by replacing homology and intersections by cohomology and cup products, Poincaré duality is false. For example, the Betti numbers in complementary dimensions of the suspension of the torus are not equal.

In this paper we study a collection of groups called the *intersection homology groups of* X, denoted $IH_i^{\bar{p}}(X)$. Here, $\bar{p}$ is a multi-index called the perversity. It lies between $\bar{0} = (0, 0, 0, \ldots)$ *and* $\bar{t} = (0, 1, 2, 3, \ldots)$. The intersection homology groups are



(a) (b)

Fig. 1. The suspension of the torus—opposite faces are to be identified. This space with a nonsingular point blown up was Poincaré's original example of how singularities can cause the failure of Poincaré duality ([18], p. 232).

defined as certain cycles modulo certain homologies, the cycles and homologies being restricted as to how they meet the singular set of X.

The fundamental propositions of the Poincaré–Lefschetz intersection theory hold for the intersection homology groups of a pseudomanifold X in the following sense:

(1) (§2.1) Suppose $\bar{p} + \bar{q} \leq \bar{r}$ are perversities and suppose V is a cycle for $IH_i^{\bar{p}}(X)$ and W is a cycle for $IH_j^{\bar{q}}(X)$. If V and W are in general position then $V \cap W$ is a cycle for $IH^{\bar{r}}_{i+j-n}(X)$ and its intersection homology class depends only on the intersection homology classes of V and W.

(2) *Generalized Poincaré duality* (§3.3). If i and j are complementary dimensions ($i + j = n$) and if $\bar{p}$ and $\bar{q}$ are complementary perversities ($\bar{p} + \bar{q} = \bar{t}$) then the pairing

$$IH_i^{\bar{p}}(X) \times IH_j^{\bar{q}}(X) \xrightarrow{\cap} IH_0^{\bar{t}}(X) \xrightarrow{\epsilon} \mathbf{Z}$$

is nondegenerate when tensored with the rationals.

For a compact oriented manifold, all the intersection homology groups coincide with ordinary homology and cohomology (§1.3). For normal spaces (a class of pseudomanifolds including normal complex algebraic varieties, see §4.1) the minimal perversity group $IH_i^{\bar{0}}(X)$ is the cohomology group $H^{n-i}(X)$ and the maximal perversity group $IH_i^{\bar{t}}(X)$ is the homology group $H_i(X)$. The intersection products for these values of the perversity are the usual cup and cap product (§4.3).

In general, the intersection homology groups share many of the properties of ordinary homology, but they are not homotopy invariants. It seems that homotopy invariants are not sufficiently fine for the detailed study of singular spaces.

The definition of the intersection homology groups uses a stratification of X but the groups turn out to be independent of the stratification chosen (§3.2). Let $X = \cup S_k$

where S_{n-k} is the stratum of codimension k (i.e. of dimension $n-k$). Let $\bar{p}=(p_2, p_3, p_4, \ldots)$ be a perversity. Then $IH_i^{\bar{p}}(X)$ is defined to be i-cycles which, for $k \geq 2$, meet each S_{n-k} in a set of dimension at most $i-k+p_k$, modulo boundaries of chains which, for $k \geq 2$, meet each S_{n-k} in a set of dimension at most $i-k+p_k+1$.

A primary motivation of this study was a remarkable problem posed by Sullivan[23]: to find a class of spaces with singularities for which the signature of manifolds extends as a corbordism invariant. The intersection homology groups provide an answer: pseudomanifolds stratifiable with only even codimension strata (for example, complex varieties). If X is such a space of dimension $4k$ then by the generalized Poincaré duality theorem there is a group $IH_{2k}^{\bar{m}}(X)$ which is dually paired to itself. The index of this pairing has the appropriate cobordism invariance (§5.2).

Using this signature, we construct (§5.3) a Hirzebruch L-class for pseudomanifolds with even codimension singularities, following the procedure of Thom[27, 16]. Like the other characteristic classes of singular spaces (the Whitney class[1, 25], the Chern class[12], and the Todd class[2]) this L class lies in ordinary homology.

In this paper we develop intersection homology theory using geometric cycles and their intersections as in Lefschetz. Our proof of generalized Poincaré duality is entirely geometric and somewhat similar to the cell-dual cell proof of Poincaré: it uses certain 'basic sets" (§3) constructed from a triangulation of X. For convenience, we work in the piecewise linear category. The subanalytic category[8] would work as well.

In a later paper we will follow an algebraically more natural approach suggested Deligne and Verdier. We will show that $IH_*^{\bar{p}}(X)$ is the hypercohomology of a complex of sheaves on X which is canonical up to quasi-isomorphism. The generalized Poincaré duality theorem is deduced from Verdier duality. We defer the proof of the topological invariance of the intersection homology groups to this later paper where it is easier.

We are happy to thank the I.H.E.S. for their generous hospitality and support during 1974–75 when the ideas in this paper were being formulated. We have profited from many useful conversations with Dennis Sullivan and Clint McCrory, and we wish to thank Clint McCrory for helping us with several difficult technical aspects of the paper.

§1. THE INTERSECTION HOMOLOGY GROUPS

1.1. Pseudomanifolds and stratifications

In this paper, all spaces and subspaces are piecewise linear, except in §5.3.

Definition. A *pseudomanifold of dimension n* is a compact space X for which there exists a closed subspace Σ with $\dim(\Sigma) \leq n-2$ such that $X-\Sigma$ is an n-dimensional oriented manifold which is dense in X. (Equivalently, X is the closure of the union of the n-simplices in any triangulation of X, and each $n-1$ simplex is a face of exactly two n-simplices). For example, every complex irreducible algebraic variety has a P.L. structure which makes it into a pseudomanifold.

A *stratification* of a pseudomanifold X^n is a filtration by closed subspaces

$$X = X_n \supset X_{n-1} = X_{n-2} = \Sigma \supset X_{n-3} \supset \cdots \supset X_1 \supset X_0$$

such that for each point $p \in X_i - X_{i-1}$ there is a filtered space

$$V = V_n \supset V_{n-1} \supset \cdots \supset V_i = \text{a point}$$

and a mapping $V \times B^i \to X$ which, for each j, takes $V_j \times B^i$ ($PL-$) homeomorphically to a neighborhood of p in X_j. (Here, B^i is the P.L. i-ball and p corresponds to $V_i \times$ (an interior point of B^i)).

Thus, if $X_i - X_{i-1}$ is not empty, it is a manifold of dimension i, and is called the i-dimensional *stratum* of the stratification. Every pseudomanifold admits a stratification. *Henceforth we assume X is an n-dimensional pseudomanifold with a fixed stratification.*

1.2. Piecewise linear chains

If T is a triangulation of X, let $C_*^T(X)$ denote the chain complex of simplicial chains of X with respect to T as in [21]. A P.L. geometric chain is an element of $C_*^T(X)$ for some triangulation T; however we identify two P.L. chains $c \in C_*^T(X)$ and $C' \in C_*^{T'}(X)$ if their canonical images in $C_*^{T''}(X)$ coincide, for some common refinement T'' of T and T'. The group $C_i(X)$ of all P.L. geometric chains is thus the direct limit under refinement of the $C_*^T(X)$ over all triangularions of X (which are compatible with the PL structure).

If $\xi \in C_i^T(X)$ define $|\xi|$ (the *support* of ξ) to be the union of the closures of those i-simplices σ for which the coefficient of σ in ξ is non-zero. If $\bar{\xi} \in C_i^{T'}(X)$ corresponds to ξ under a refinement T' of T, Then $|\xi| = |\bar{\xi}|$. Thus, each $\alpha \in C_*(X)$ has a well defined support, $|\alpha|$.

The following remark is clear from the simplicial construction of homology: If $C \subset X$ is an i-dimensional PL subset and if $D \subset C$ is an $i-1$ dimensional subset, then there is a one-to-one correspondence between chains $\alpha \in C_i(X)$ such that $|\alpha| \subset C$, $|\partial\alpha| \subset D$, and between homology classes $\tilde{\alpha} \in H_i(C, D)$. Furthermore, $\partial\alpha$ corresponds to the class $\partial_*(\tilde{\alpha}) \in H_{i-1}(D)$ under the connecting homomorphism $\partial_*: H_i(C, D) \to H_{i-1}(D)$.

1.3. Definition of $IH_i^{\bar{p}}(X)$

A *perversity* is a sequence of integers $\bar{p} = (p_2, p_3, \ldots, p_n)$ such that $p_2 = 0$ and $p_{k+1} = p_k$ or $p_k + 1$. If i is an integer and $\bar{p}$ is a perversity, a subspace $Y \subset X$ is called *$(\bar{p}, i)$-allowable* if $\dim(Y) \leq i$ and $\dim(Y \cap X_{n-k}) \leq i - k + p_k$ for all $k \geq 2$. Define $IC_i^{\bar{p}}(x)$ to be the subgroup of $C_i(X)$ consisting of those chains ξ such that $|\xi|$ is $(\bar{p}, i)$-allowable and $|\partial\xi|$ is $(\bar{p}, i-1)$-allowable.

For example, in Fig. 1, V, W, and C are in $IC_2^{(0,1)}(X)$ but not in $IC_2^{(0,0)}(X)$. W' is of course in $IC_1^{\bar{p}}(X)$ for any $\bar{p}$.

Definition. The ith *Intersection Homology Group* of perversity $\bar{p}$, denoted $IH_i^{\bar{p}}(X)$, is the ith homology group of the chain complex $IC_*^{\bar{p}}(X)$.

It will be shown later that $IH_i^{\bar{p}}(X)$ is finitely generated and is independent of the stratification of X. ($IH_i^{\bar{p}}(X)$ can be defined for sequences $\bar{p}$ which do not satisfy the condition $p_k \leq p_{k+1} \leq p_k + 1$ but these groups will depend on the stratification of X.)

For preversities $\bar{a}$ and $\bar{b}$, we say $\bar{a} \leq \bar{b}$ if $a_k \leq b_k$ for each k. Thus the smallest perversity is $\bar{0} = (0, 0, \ldots, 0)$ and the largest perversity is $\bar{t} = (0, 1, 2, \ldots, n-2)$.

It is not always possible to add two perversities $\bar{p} + \bar{q}$: the sequence $(\bar{p} + \bar{q})_k = p_k + q_k$ may not be a perversity. However, if $\bar{p} + \bar{q} \leq \bar{t}$ then there is a unique minimal perversity r such that $\bar{p} + \bar{q} \leq \bar{r}$.

Remarks. In certain special cases, some groups $IH_i^{\bar{p}}(X)$ are canonically equal to other groups $IH_i^{\bar{q}}(X)$.

For example, if X has no stratum of codimension k, (i.e. if $X_{n-k} = X_{n-k-1}$) then $IH_i^{\bar{p}}(X)$ does not depend on p_k. (That is, if $p_c = q_c$ for all $c \neq k$ then $IC_*^{\bar{p}}(X) = IC_*^{\bar{q}}(X)$.) In particular, if X is an oriented manifold stratified with only one stratum X then all the $IH_i^{\bar{p}}(X)$ are isomorphoric to $H_i(X)$ (and also to $H^{n-i}(X)$).

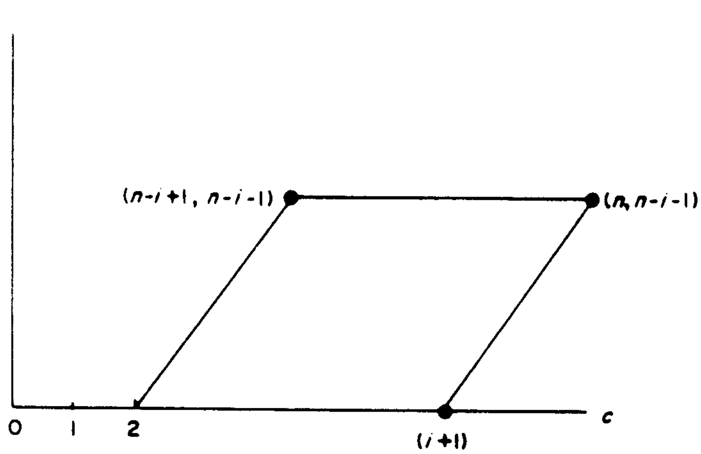
The dimensions $i = 0$ and $i = n$ are also special cases. For any perversity $\bar{p}$ we have

$$IH_0^{\bar{p}}(X) = H_0(X - \Sigma) \cong H^n(X)$$

$$IH_n^{\bar{p}}(X) = H_n(X) \cong H^0(X - \Sigma).$$

Since the minimum perversity is $\bar{0} = (0, 0, \ldots, 0)$ and the maximum perversity is $\bar{t} = (0, 1, 2, \ldots, n-2)$, the graph of p_c as a function of c must lie in a triangular region of c, p-space.

If $1 \leq i \leq n-1$ and if $\bar{p}$ and $\bar{q}$ are perversities such that the portion of the graph of $\bar{p}$ which lies inside the closed region below coincides with the portion of the graph of $\bar{q}$ which lies inside the closed region, then $IH_i^{\bar{p}}(X) = IH_i^{\bar{q}}(X)$.



1.4. The map to homology and the map from cohomology

The fundamental class of X is defined to be the unique class $[X] \in H_n(X)$ which restricts to the local orientation class in $H_n(X, X - p)$ for every "nonsingular point" $p \in X - \Sigma$. The *Poincaré duality map* is defined to be the cap product with this fundamental class,

$$\cap [X]: H^{n-i}(X) \rightarrow H_i(X)$$

although this map is not necessarily an isomorphism since X is singular.

In this section we describe a family of compatible homomorphisms

$$H^{n-i}(X) \xrightarrow{\alpha} IH_i^{\bar{p}}(X) \xrightarrow{\omega} H_i(X)$$

which factor the Poincaré duality map.

The homomorphism $\omega_{\bar{p}}$: $IH_i^{\bar{p}}(X) \rightarrow H_i(X)$ is induced from the inclusions $IC_*^{\bar{p}}(X) \subset C_*(X)$.

The homomorphism $\alpha_{\bar{p}}$: $H^{n-i}(X) \rightarrow IH_i^{\bar{p}}(X)$ has a geometric description in terms of Mock bundles[3]. Each cohomology class $\eta \in H^{n-i}(X)$ is representable by an oriented embedded Mock bundle ξ whose total space $E(\xi)$ is $(\bar{p}, i)$-allowable for any $\bar{p}$ because it is "tranverse to each stratum of X". There is a canonical orientation on $E(\xi)$ which makes it into a geometric cycle whose homology class is $\eta \cap [X]$. With this orientation, $E(\xi)$ is the desired cycle in $C_i^{\bar{p}}(X)$.

We now give an alternate description of $\alpha_{\bar{p}}$: $H^{n-i}(X) \rightarrow IH_i^{\bar{p}}(X)$ using a generalization (to singular spaces) of the *dual cells* of Poincaré which were used in the original geometric proof of the Poincaré duality theorem. Let T be a triangulation of

X such that each X_i is a subcomplex. Let T' be the first barycentric subdivision of T and let $\hat{\sigma}$ denote the barycentre of the simplex $\sigma \in T$. Let T_i be the i-skeleton of T, thought of as a subcomplex of T'. It is spanned by all vertices $\hat{\sigma}$ such that $\dim(\sigma) \le i$. Define the codimension i coskeleton D_i to be the subcomplex of T' spanned by all vertices $\hat{\sigma}$ such that $\dim(\sigma) \ge i$. There are canonical simplex preserving deformation retracts

$$X - |T_i| \to |D_{i+1}|$$
$$X - |D_{i+1}| \to |T_i|$$

since T_i and D_{i+1} are spanned by complementary sets of vertices. (Each simplex in T' is the join of its intersection with $|T_i|$ and of its intersection with $|D_{i+1}|$. Retract along the join lines).

If $C_T^i(X) = \mathrm{Hom}(C_i^T(X), \mathbf{Z})$ is the group of simplicial i-cochains on X, define the homomorphism

$$\beta: C_T^i(X) \to C_{n-i}^{T'}(X)$$

to be the composition

$$\begin{aligned} C_T^i(X) = \bigoplus_{\dim(\sigma)=i} H^i(\sigma, \partial\sigma) = {} & H^i(|T_i|, |T_{i-1}|) \\ & \downarrow \cap [X] \text{ (see appendix)} \\ & H_{n-i}(X - |T_{i-1}|, X - |T_i|) \\ \cong {} & \downarrow \text{ (deformation retract)} \\ & H_{n-i}(|D_i|, |D_{i+1}|) \\ & \downarrow \\ & H_{n-i}(|T'_{n-i}|, |T'_{n-i-1}|) \\ & \| \\ & C_{n-i}^{T'}(X). \end{aligned}$$

Note that for any m-simplex $\eta \in T$ we have $\dim(|D^i| \cap |\eta|) \le m - i$. Thus $\dim(|D^i| \cap X_{n-c}) \le n - i - c$ so $|D^i|$ is $(\bar{0}, n-i)$-allowable. Therefore the image of β lies in $IC_{n-i}^{\bar{p}}(X)$ for every perversity $\bar{p}$.

Furthermore, β is a chain map because the cap product $\cap [X]$ is compatible with boundary and coboundary connecting homorphisms (see Appendix).

Thus β induces a map on (co)-homology

$$\alpha_{\bar{p}}: H^i(X) \to IH_{n-i}^{\bar{p}}(X).$$

It is easy to check that the composition $\omega_{\bar{p}} \circ \alpha_{\bar{p}}$ is the Poincaré duality map.

§2. THE INTERSECTION PRODUCT

In this chapter we construct pairings

$$IH_i^{\bar{p}}(X) \times IH_j^{\bar{q}}(X) \to IH_{i+j-n}^{\bar{r}}(X)$$

whenever $\bar{p} + \bar{q} \le \bar{r}$.

Lefschetz defined the intersection of an i-chain C and a j-chain D in a manifold M whenever $|C| \cap |D|$ contains simplices of dimension at most $i + j - n$. He gave a formula for the multiplicity in $C \cap D$ of an $i + j - n$-simplex $\sigma \subset |C| \cap |D|$ which is local, i.e. depends only on the behavior of C and D near an interior point of σ.

We will define the intersection of chains $C \in IC_i^{\bar{p}}(X)$ and $D \in IC_j^{\bar{q}}(X)$ in a pseudomanifold X, whenever C and D are *dimensionally tranverse.* This is a condition that includes the one of Lefschetz as well as similar conditions for each singular

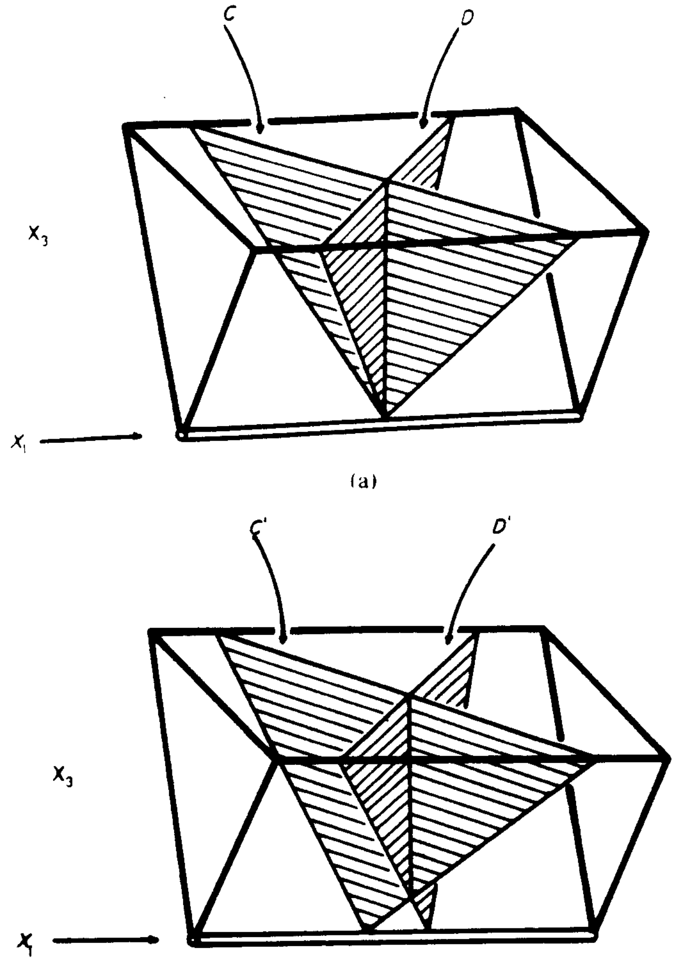


Fig. 2. Sectors of a three dimensional pseudomanifold. C and D are not dimensionally tranverse; C' and D' are dimensionally tranverse.

stratum of X. For dimensionally transverse chains, any simplex in $|C| \cap |D|$which is contained in the singularities of X must have dimension $\leq i + j - n - 2$ (see Fig. 2). Thus the calculation of $C \cap D$ and $\partial(C \cap D)$ can proceed exactly as in Lefschetz.

For the benefit of those unfamiliar with the Lefschetz definition of the multiplicity, we reverse history and define the intersection chain using the cohomology cup product, rather than carry out the Lefschetz procedure.

2.1. Intersection of transverse chains

Definition. Suppose $\bar{p} + \bar{q} = \bar{r}$ and $i + j - n = l$. A chain $C \in IC_i^{\bar{p}}(X)$ and a chain $D \in IC_j^{\bar{q}}(X)$ are said to be ***dimensionally transverse*** (written $C \pitchfork D$) if $|C| \cap |D|$ is $(\bar{r}, l)$-allowable.

If $C \pitchfork D$, $\partial C \pitchfork D$, and $C \pitchfork \partial D$, we define an intersection chain $C \cap D \in IC_l^{\bar{r}}(X)$ as follows:

Let $J = |\partial C| \cup |\partial D| \cup \Sigma$. Let $\bar{C} \in H_i(|C|, |\partial C|)$ and $\bar{D} \in H_j(|D|, |\partial D|)$ be the classes determined by C and D as in §1.2. Define $C \cap D$ to be the chain determined by the image of $(\bar{C}, \bar{D})$ under the following sequence of homomorphisms:

$$
\begin{array}{ccc}
H_i(|C|, |\partial C|) \times H_j(|D|, |\partial D|) & & \\
\downarrow & & \\
H_i(|C|, |C| \cap J) \times H_j(|D|, |D| \cap J) & & \\
\cong \quad \downarrow & & \text{(excision)} \\
H_i(|C| \cup J, J) \times H_j(|D| \cup J, J) & & \\
\cong \quad \uparrow \cap [X] \times \cap [X] & & \text{(see Appendix)}
\end{array}
$$

$$\begin{array}{ccl}
& & H^{n-i}(X-J, X-(|C|\cup J))\times H^{n-j}(X-J, X-(|D|\cup J)) \\
& \downarrow & \qquad\qquad\qquad\qquad \text{cup product} \\
& \downarrow & H^{2n-i-j}(X-J, X-(|C|\cap|D|\cup J)) \\
\cong & & \cap\,[X] \qquad\qquad \text{(see Appendix)} \\
& & H_{i+j-n}((|C|\cap|D|)\cup J, J) \\
\cong & \uparrow & \qquad\qquad\qquad \text{(excision)} \\
\cong & \downarrow & H_{i+j-n}(|C|\cap|D|, |C|\cap|D|\cap J) \\
& & H_{i+j-n}(|C|\cap|D|, (|\partial C|\cap|D|)\cup(|C|\cup|\partial D|))
\end{array}$$

The last is an isomorphism since the third term in the exact sequence is isomorphic (by excision) to $H_{i+j-n-1}(|C|\cap|D|\cap\Sigma, (|\partial C|\cap|D|\cup|C|\cap|\partial D|)\cap\Sigma)$ which is 0 since $\dim(|C|\cap|D|\cap\Sigma)\le i+j-n-2$.

LEMMA (Lefschetz[11]). *Suppose $\bar p+\bar q=\bar r$ and $i+j=l$. Suppose $C\in IC_i^{\bar p}(X)$ $D\in IC_j^{\bar q}(X)$ and $C \pitchfork D$, $C\pitchfork D$, $C\pitchfork \partial D$, (meaning, for example, that $|C|\cap|\partial D|$ is $(\bar r, l-1)$-allowable). Then $\partial(C\cap D)=(\partial C)\cap D+(-1)^{n-i}C\cap(\partial D)$.*

Proof. The proof follows directly from the analogous formula for the relative cup product, which may be found in [28].

2.2. Transversality

The transversality lemma says that given any two P.L. cycles, it is possible to move one of them, while staying within its intersection homology class, so as to make it "transverse" to the other—meaning that the resulting cycles intersect in general position in each stratum of X. We use a relative version of transversality so that the same lemma can be applied to a homology between cycles.

In [14], McCrory proves the following proposition:

LEMMA. *Suppose $\bar p+\bar q=\bar r$, $A\in IC_i^{\bar p}(X)$ and $B\in IC_j^{\bar q}(X)$. Suppose $\partial A\pitchfork B$. Then there is a P.L. isotopy $K\colon X\times[0,1]\to X$ so that for all $t\in[0,1]$,*

(a) $K(x,0)=x$ for all $x\in X$
(b) $K(x,t)=x$ for all $x\in|\partial A|$, for all $t\in[0,1]$
(c) $K(x,t)\in X_i-X_{i-1}$ whenever $x\in X_i-X_{i-1}$
(d) $K(|A|,1)=K_1(|A|)$ is dimensionally tranverse to $|B|$.

K is constructed by induction on the filtered pieces of X, using the technique of Zeeman to move A and B into "general position" within each stratum of X.

COROLLARY. *Suppose $C\in IC_i^{\bar p}(X)$, D is a $(\bar q, j)$-allowable subset of X, and $\bar p+\bar q=\bar r$. Suppose $|\partial C|\pitchfork D$. Then there exists $C'\in IC_i^{\bar p}(X)$ and $E\in IC_{i+1}^{\bar p}(X)$ so that $\partial E=C'-C$, $|C'|\pitchfork D$, and $\partial C'=\partial C$.*

Proof. By the transversality lemma, there is a P.L. isotopy $K\colon X\times[0,1]\to X$ so that $K_0=$ identity, $K_t\Big|\,|\partial C|=$ identity for all $t\in[0,1]$, and $K_1(|C|)$ is dimensionally transverse to D.

K_1 determines a map on chains, K_{1*}: $C_i(X) \to C_i(X)$, which is a limit over triangulations of the induced map on simplicial chains.

Define $C' = K_{1*}(C)$. Clearly $\partial C' = \partial C$ and $|C'| \pitchfork D$. Furthermore E can be taken to be

$$E = (-1)^i K_*(C \times [0, 1])$$

where K_*: $C_{i+1}(X \times [0, 1]) \to C_{i+1}(X)$ is the map on chains induced by K, and where $C \times [0, 1] \in C_{i+1}(X \times [0, 1])$ is the obvious P.L. geometric chain (with product orientation). Then $C' \in IC_i^{\bar{p}}(X)$ and $E \in IC_{i+1}^{\bar{p}}(X)$ since K is stratum-preserving, and

$$\begin{aligned} \partial E &= (-1)^i \partial K_*(C \times [0, 1]) = (-1)^i K_*(\partial(C \times [0, 1])) \\ &= (-1)^i K_*((\partial C) \times [0, 1]) + K_{1*}(C) - K_{0*}(C) \\ &= C' - C \quad \text{since } K_t \Big| |\partial C| \text{ is the identity.} \end{aligned}$$

2.3. Definition of the intersection product

THEOREM 1. *Suppose $\bar{p} + \bar{q} \leq \bar{r}$. Then there is a unique intersection pairing*

$$\cap : IH_i^{\bar{p}}(X) \times IH_j^{\bar{q}}(X) \to IH_{i+j-n}^{\bar{r}}(X)$$

such that $[C \cap D] = [C] \cap [D]$ for every dimensionally transverse pair of cycles $C \in IC_i^{\bar{p}}(X)$ and $D \in IC_j^{\bar{q}}(X)$.

For example, in Fig. 1 $[V]$ *and* $[W]$, which are in $IH_2^{\bar{t}}(X)$ (where $t = (0, 1)$) cannot be intersected since there is no perversity $\bar{r} \geq \bar{t} + \bar{t}$. The class $[W']$ is nonzero in $IH_1^{\bar{0}}(X)$ (where $\bar{0} = (0, 0)$), since C is not in $IC_2^{\bar{0}}(X)$; so $[V] \cap [W']$ is well defined in $IH_0^{\bar{t}}(X)$.

Proof. It is sufficient to define only the pairings

$$IH_i^{\bar{p}}(X) \times IH_j^{\bar{q}}(X) \to IH_{i+j-n}^{\bar{r}}(X)$$

for perversities $\bar{p}$, $\bar{q}$ and $\bar{r}$ such that $\bar{p} + \bar{q} = \bar{r}$. For, if $\bar{p}$ and $\bar{q}$ are general perversities such that $\bar{p} + \bar{q} \leq \bar{r}$ there will exist perversities $\bar{p}' \geq \bar{p}$, $\bar{q}' \geq \bar{q}$ and $\bar{r}' \leq \bar{r}$ so that $\bar{p}' + \bar{q}' = \bar{r}'$. Define a pairing as above to be the composition

$$\begin{array}{ll} IH_i^{\bar{p}}(X) \times IH_j^{\bar{q}}(X) & \\ \quad\downarrow & \\ IH_i^{\bar{p}'}(X) \times IH_j^{\bar{q}'}(X) \to & IH_{i+j-n}^{\bar{r}'}(X) \\ & \quad\downarrow \\ & IH_{i+j-n}^{\bar{r}}(X). \end{array}$$

The result is independent of the choices of $\bar{p}'$, $\bar{q}'$ and $\bar{r}'$. Therefore, assume $\bar{p} + \bar{q} = \bar{r}$.

Fix $\xi \in IH_i^{\bar{p}}(X)$ and $\eta \in IH_j^{\bar{q}}(X)$ and choose representative cycles $C \in IC_i^{\bar{p}}(X)$ and $D \in IC_j^{\bar{q}}(X)$ respectively. By the above corollary, there are chains $C_1 \in IC_i^{\bar{p}}(X)$ and $E \in IC_{i+1}^{\bar{p}}(X)$ so that $\partial C_1 = 0$, $C_1 \pitchfork D$, and $\partial E = C - C_1$. Thus, $\partial(C_1 \cap D) = 0$ by the Lefschetz lemma so we define $\xi \cap \eta = [C_1 \cap D] \in IH_{i+j-n}^{\bar{r}}(X)$.

If C_2 is any other cycle representing ξ which is transverse to D then by the above corollary there is a chain $F \in IC_{i+1}^{\bar{p}}(X)$ which is transverse to D, such that $\partial F = C_1 - C_2$. Thus $\partial(F \cap D) = C_1 \cap D - C_2 \cap D$ so $[C_1 \cap D] = [C_2 \cap D] \in IH_{i+j-n}^{\bar{r}}(X)$.

In general, if $C' \in IC_i^{\bar p}(X)$ and $D' \in IC_j^{\bar q}(X)$ is any pair of transverse cycles representing ξ and η, then there is a cycle $D'' \in IC_j^{\bar q}(X)$ representing η such that $|D''|$ is dimensionally transverse to $|C'| \cup |C_1|$. Then repeated application of the above procedure gives

$$[C' \cap D'] = [C' \cap D''] = [C_1 \cap D''] = [C_1 \cap D].$$

Thus $\xi \cap \eta$ is well defined.

2.4. Elementary properties of the intersection

The following properties of the intersection product follow from similar chain level formulae: If $\xi \in IH_i^{\bar p}(X)$, $\eta \in IH_j^{\bar q}(X)$, and $\omega \in IH_k^{\bar s}(X)$ and if $\bar p + \bar q + \bar s \le \bar t = (0, 1, 2, \ldots, n-2)$, then

$$\xi \cap \eta = (-1)^{(n-i)(n-j)}\eta \cap \xi$$
$$\xi \cap (\eta \cap \omega) = (\xi \cap \eta) \cap \omega.$$

The intersection pairings are compatible with the cup and cap products in the following sense; Let $\alpha_{\bar p}$: $H^*(X) \to IH_*^{\bar p}(X)$ and $\omega_{\bar p}$: $IH_*^{\bar p}(X) \to H_*(X)$ be the homomorphisms of §1.4. Suppose A and B are elements of $H^*(X)$, $C \in IH_*^{\bar p}(X)$ and $\bar p + \bar q = \bar r$. Then

$$\omega_{\bar p}\alpha_{\bar p}(A) = A \cap [X] \qquad \text{(cap product)}$$
$$\alpha_{\bar r}(A \cup B) = \alpha_{\bar p}(A) \cap \alpha_{\bar q}(B)$$
$$A \cap \omega_{\bar p}(C) = \omega_{\bar r}(\alpha_{\bar q}(A) \cap C)$$
$$\langle A, \omega_{\bar p}(C)\rangle = \epsilon\omega_{\bar r}(\alpha_{\bar p}(A) \cap C)$$

where $\langle , \rangle$ is the Kronecker pairing $H^* \times H_* \to Z$ and the $\epsilon: H_*(X) \to Z$ is the augmentation which counts points with multiplicity.

§3. THE BASIC SETS

In this chapter we define, for any triangulation T of X, certain basic subsets

$$\cdots Q_{i-1}^{\bar p} \subset Q_i^{\bar p} \subset Q_{i+1}^{\bar p} \subset \cdots$$

which have the property that

$$I_i^{\bar p}(X) = \text{Image}\,(H_i(Q_i^{\bar p}) \to H_i(Q_{i+1}^{\bar p})).$$

These sets play a key role in the demonstration of the generalized Poincaré duality theorem (§3.3) and in the proof that $IH_i^{\bar p}(X)$ is independent of the stratification of X (§3.2).

3.1. Definition of the basic sets

Definition. Suppose T is any triangulation of X^n. Let T' be the first barycentric subdivision of T. For each perversity $\bar p$ and integer $i \ge 0$ define the function $L_i^{\bar p}$ as follows:

$$L_i^{\bar p}(0) = i, \qquad L_i^{\bar p}(1) = i-1, \qquad L_i^{\bar p}(n+1) = -1,$$

and if $2 \le c \le n$ set

$$L_i^{\bar p}(c) = \begin{cases} -1 & \text{if } i - c + p_c \le -1 \\ n - c & \text{if } i - c + p_c \ge n - c \\ i - c + p_c & \text{otherwise} \end{cases}$$

Define $\Delta L_i^{\bar p}(c) = L_i^{\bar p}(c) - L_i^{\bar p}(c+1)$ (which is either 0 or 1).

Define $Q_i^{\bar p}$ to be the subcomplex of T' spanned by the set of barycentres of simplices

$$\{\hat\sigma | \sigma \in T \text{ and } \Delta L_i^{\bar p}(n - \dim(\sigma)) = 1\}$$

Figure 3 shows the basic sets restricted to any 3-simplex of T in a three dimensional pseudomanifold X. $Q_0^{\bar 0} = Q_0^{\bar t}$ = the barycenter of the simplex; $Q_3^{\bar 0} = Q_3^{\bar t}$ = the whole simplex.

Remarks. (a) $L_i^{\bar p}(c)$ represents the largest possible dimension of intersection of any $(\bar p, i)$-allowable set with X_{n-c} (if $c \ge 2$).

(b) It is the perversity restrictions $p_c \le p_{c+1} \le p_c + 1$ which guarantee $\Delta L_i^{\bar p}(c)$ is either 0 or 1.

(c) $Q_i^{\bar p}$ is a subcomplex of $Q_{i+1}^{\bar p}$ because if $\sigma \in T$ and $\hat\sigma \in Q_{i+1}^{\bar p}$ then $\hat\sigma \in Q_{i+1}^{\bar p}$ as well.

(d) $\dim(Q_i^{\bar p}) = i$ because the intersection of $Q_i^{\bar p}$ with any n-simplex is spanned by $\sum_{c=0}^{n} \Delta L_i^{\bar p}(c) = i + 1$ vertices.

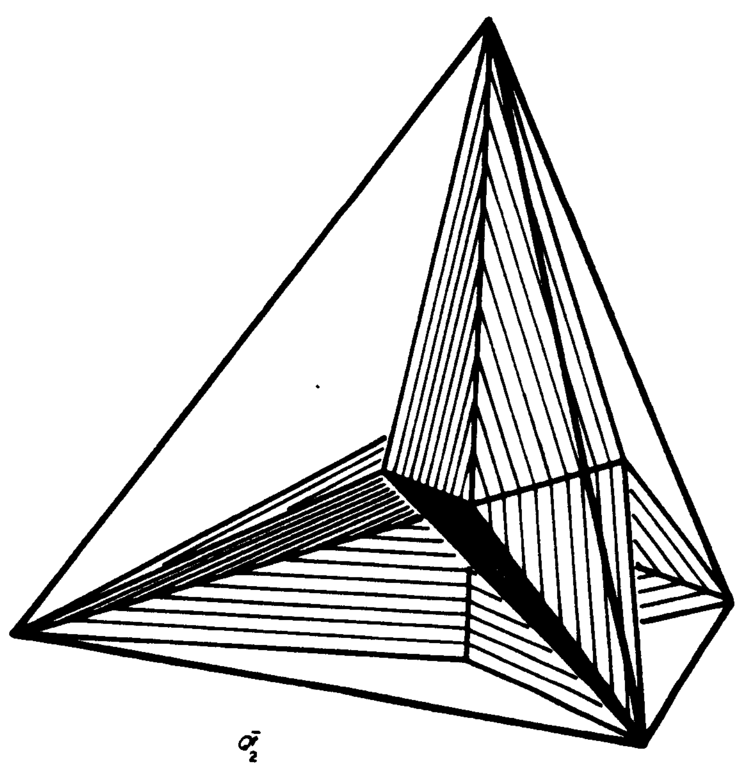
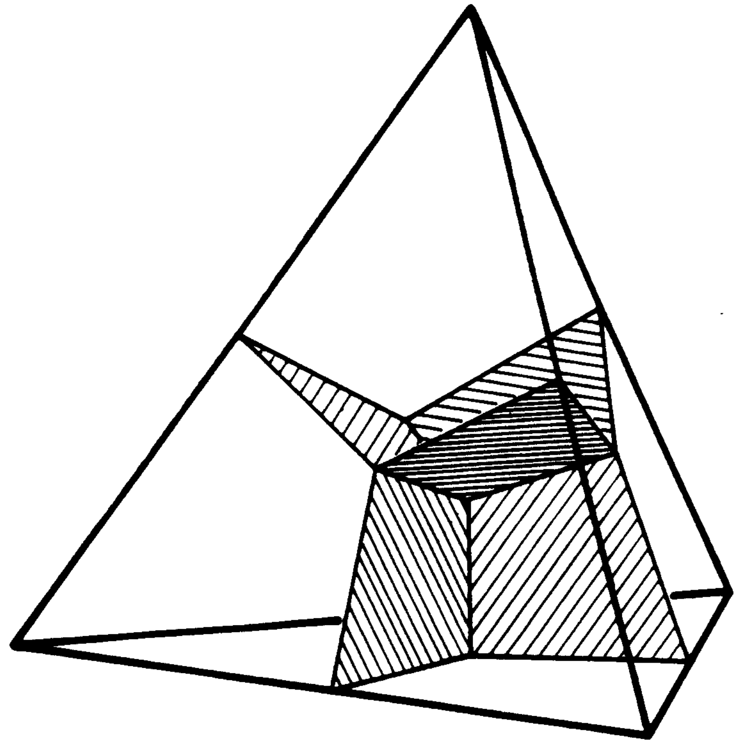
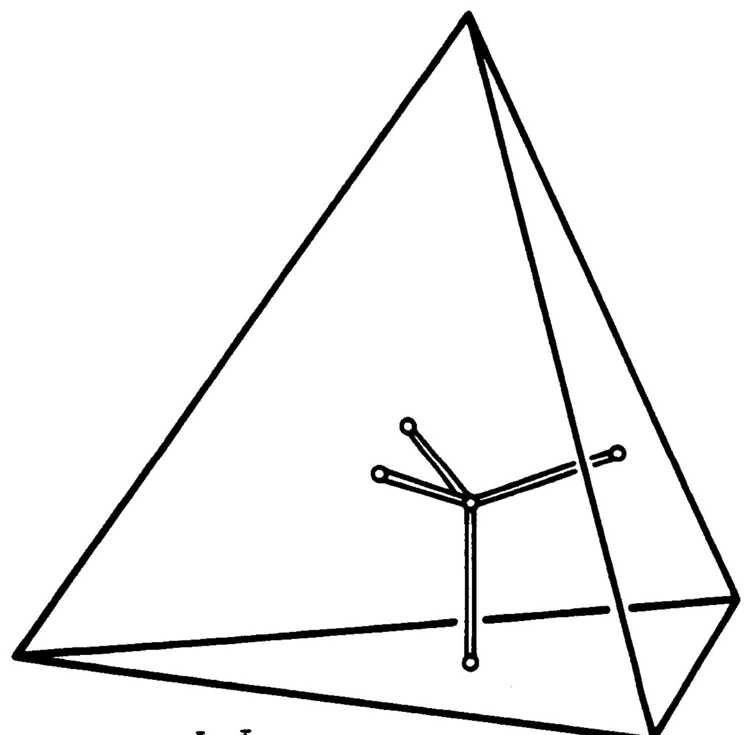


Fig. 3(a). Basic set.



$Q_2^{\bar 0}$

Fig. 3(b). Basic set.



$Q_1^{\bar 0} = Q_1^{\bar t}$

Fig. 3(c). Basic set.

The fundamental technical property of the basic sets is the following lemma:

Lemma. *If* $i \geq 1$ *and if* $\bar p + \bar q = \bar t = (0, 1, 2, \ldots, n-2)$, *then there are canonical simplex-preserving deformation retractions*

$$X - (Q^{\bar q}_{n-i+1} \cap |T_{n-2}|) \to Q^{\bar p}_i$$

$$X - Q^{\bar q}_{n-i+1} \to Q^{\bar p}_i \cap |T_{n-2}|$$

where T_{n-2} is the $n-2$ skeleton of the triangulation T of X.

This may be seen directly from Fig. 3 for a three dimensional pseudomanifold; a more complicated case is illustrated in Fig. 4.

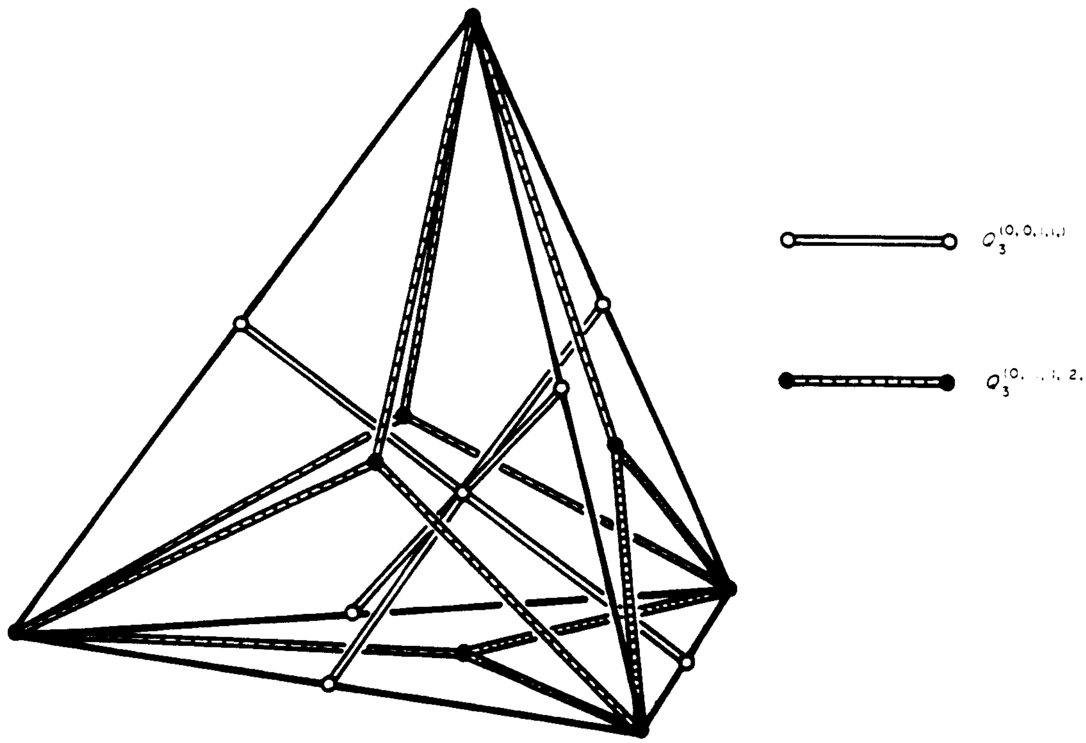


Fig. 4. Basic sets restricted to a simplex of T_{n-2} for $n = 5$.

Proof. Since $p_2 = 0$ for any $\bar{p}$, we have $\Delta L_i^{\bar{p}}(0) = \Delta L_i^{\bar{p}}(1) = 1$ provided $i \geq 1$. Therefore, whenever $\sigma \in T$ and $\dim(\sigma) \geq n-1$, the vertex $\hat{\sigma}$ lies in every $Q_i^{\bar{p}}$.

On the other hand, if $2 \leq c \leq n+1$, then by higher arithmetic, $L_i^{\bar{p}}(c) + L_{n-i+1}^{\bar{q}}(c) = n - c - 1$, so

$$\Delta L_i^{\bar{p}}(c) + \Delta L_{n-i+1}^{\bar{q}}(c) = 1.$$

This means the vertex $\hat{\sigma}$ of any simplex $\sigma \in T_{n-2}$ is in exactly one of $Q_i^{\bar{p}}$ or $Q_{n-i+1}^{\bar{q}}$.

Thus the set of vertices in T' which span $Q_{n-i+1}^{\bar{q}} \cap |T_{n-2}|$ is exactly the complement of the set of vertices which span $Q_i^{\bar{p}}$. Every simplex of T' is therefore the join of its intersection with $Q_i^{\bar{p}}$, and of its intersection with $Q_{n-i+1}^{\bar{q}} \cap |T_{n-2}|$. The first retraction

$$X - (Q_{n-i+1}^{\bar{q}} \cap |T_{n-2}|) \rightarrow Q_i^{\bar{p}}$$

is given in each simplex of T' by retracting along these join lines. A similar remark about complementary sets of vertices gives rise to the other retraction.

3.2 Relation between $Q_i^{\bar{p}}$ and $IH_i^{\bar{p}}$ and independence of the stratification

LEMMA. *Suppose T is a triangulation of X which is subordinate to the stratification, i.e. such that each X_k is a subcomplex of T. Let the $Q_i^{\bar{p}}$ be defined* (as in §3.1) *with respect to this triangulation T. Then $Q_i^{\bar{p}}$ is* $(\bar{p}, i)$*-allowable.*

Proof. For any k-simplex $\sigma \in T$, we have $\dim(Q_i^{\bar{p}} \cap \sigma) = L_i^{\bar{p}}(n-k)$.

Now consider $\dim(Q_i^{\bar{p}} \cap X_{n-c})$. For any simplex $\sigma \subset X_{n-c}$ set $m = n - \dim(\sigma) \geq c$. Then either $Q_i^{\bar{p}} \cap \sigma = \phi$ or

$$\dim(Q_i^{\bar{p}} \cap X_{n-c} \cap \sigma) = L_i^{\bar{p}}(m) \leq i - m + p_m \leq i - c + p_c$$

since $p_{c+1} \le p_c + 1$. Maximizing over all $\sigma \subset X_{n-c}$ gives $\dim (Q_i^{\bar p} \cap X_{n-c}) \le i - c + p_c$.

Definition. Suppose T is a triangulation of X subordinate to the stratification. Let $Q_i^{\bar p}$ be the subcomplex of the first barycentric subdivision T' of T, as defined above. Let $C_i(Q_i^{\bar p})$ denote the group of P.L. geometric chains with support in $Q_i^{\bar p}$. The inclusions

$$\partial^{-1}(C_i(Q_i^{\bar p})) \cap C_{i+1}(Q_{i+1}^{\bar p}) \subset IC_{i+1}^{\bar p}(X)$$

$$(\ker \partial) \cap C_i(Q_i^{\bar p}) \qquad \subset IC_i^{\bar p}(X)$$

define a homomorphism

$$\Psi\colon \text{Image}\,(H_i(Q_i^{\bar p}) \to H_i(Q_{i+1}^{\bar p})) \to IH_i^{\bar p}(X).$$

The main result in this section is

PROPOSITION. *Ψ is an isomorphism.*

Proof. First we show Ψ is surjective. Suppose $\eta \in IH_i^{\bar p}(X)$ is represented by a cycle $Z \in IC_i^{\bar p}(X)$. First we use transversality to make Z miss $Q_{n-i+1}^{\bar q} \cap |T_{n-2}|$ (where $\bar p + \bar q = \bar t = (0, 1, 2, \ldots)$): By the corollary in §2.3 there is a cycle $Z' \in IC_i^{\bar p}(X)$ homologous to Z such that $|Z'|$ is dimensionally transverse to $Q_{n-i+1}^{\bar q} \cap |T_{n-2}|$. This means that $|Z'| \cap Q_{n-i+1}^{\bar q} \cap X_{n-c} = \phi$ if $c \ge 2$ and moreover, that $|Z'| \cap Q_{n-i+1}^{\bar p} \cap |T_{n-2}| \cap (X_n - X_{n-2}) = \phi$ since $\dim (Q_{n-i+1}^{\bar q} \cap |T_{n-2}|) = n - i - 1$. Thus

$$Z' \subset X - Q_{n-i+1}^{\bar q} \cap |T_{n-2}|.$$

Next, we deform Z' back to $Q_i^{\bar p}$. The deformation retraction $r\colon X - Q_{n-i+1}^{\bar q} \cap |T_{n-2}| \to Q_i^{\bar p}$ is not linear. Nevertheless it determines a cycle $r_*(Z') \in C_i(Q_i^{\bar p})$ such that $|r_*(Z')| = r(|Z'|)$ as follows: Choose a triangulation of $|Z'|$ such that each simplex of Z' is contained in a single simplex of T'. Map each vertex v of $|Z'|$ to $r(v) \in Q_i^{\bar p}$ and extend linearly over the simplices of $|Z'|$.

To show $r_*(Z')$ represents the same class in $IH_i^{\bar p}(X)$ as Z', we need to find a chain $\theta \in IC_{i+1}^{\bar p}(X)$ so that $\partial\theta = Z' - r_*(Z')$. The triangulation of $|Z'|$ determines a triangulation of $|Z'| \times [0, 1]$ with exactly two vertices $(v, 0)$ and $(v, 1)$ for each vertex v in $|Z'|$ (see e.g. [7], p. 46). Define $H\colon |Z'| \times [0, 1] \to X$ by $H(v, 1) = r(v)$ and $H(v, 0) = v$ for each vertex v of $|Z'|$ and extend linearly over the simplices of $|Z'| \times [0, 1]$. Then $H_*(Z' \times [0, 1)]$ in a chain in $C_{i+1}(X)$ (if we use the product orientation on $Z' \times [0, 1]$), and $\partial H_*(Z' \times [0, 1]) = r_*(Z') - Z'$. It is easy to check that $H(|Z'| \times [0, 1])$ is $(\bar p, i + 1)$-allowable.

A similar (relative) argument applied to chains in $IC_{i+1}^{\bar p}(X)$ shows Ψ is injective.

The above discussion breaks down for the case $i = 0$ but the proposition still holds (as can be verified by reviewing the remarks of §1.5).

COROLLARY. *$IH_i^{\bar p}(X)$ is finitely generated and is independent of the stratification of X.*

Proof. Given two stratifications of X, there is a triangulation subordinate to both. Define the $Q_i^{\bar p}$ with respect to that triangulation. Then Image $(H_i(Q_i^{\bar p}) \to H_i(Q_{i+1}^{\bar p}))$ is the intersection homology group defined with respect to either stratification.

COROLLARY. *If T is any triangulation of X and if the $Q_i^{\bar{p}}$ are defined with respect to T then*

$$IH_i^{\bar{p}}(X) = \text{Image}\,(H_i(Q_i^{\bar{p}}) \to H_i(Q_{i+1}^{\bar{p}})).$$

Proof. The filtration of X by the skeleta of T,

$$X = |T_n| \supset \Sigma = |T_{n-2}| \supset |T_{n-3}| \supset \cdots \supset |T_0|$$

is a stratification of X, and T is subordinate to it. Thus the preceding proposition applies.

3.3. Poincaré duality

THEOREM. (*Generalized Poincaré Duality*). *Let ϵ: $IH_0^{\bar{t}}(X) \to \mathbf{Z}$ be the "augmentation" which counts points with multiplicities (where $\bar{t} = (0, 1, 2, \ldots, n-2)$). If $i + j = n$ and $\bar{p} + \bar{q} = \bar{t}$ then the augmented intersection pairing*

$$IH_i^{\bar{p}}(X) \times IH_j^{\bar{q}}(X) \to IH_0^{\bar{t}}(X) \to \mathbf{Z}$$

is nondegenerate if these groups are tensored with the rationals $\mathbf{Q}$.

Proof. If $i = 0$, the theorem is clear from the remarks of §1.5. For $i \geq 1$ we use the usual rational duality between homology and cohomology (in the same dimension) of the space $Q_i^{\bar{p}}$. A key step is Lemma 2, below.

Chose a triangulation T of X and define the $Q_i^{\bar{p}}$ with respect to T as in §3.1. Let $S = |T_{n-2}|$ be the $n-2$ skeleton of T. Thus, $\Sigma \subset S$ and $\dim\,(Q_i^{\bar{p}} \cap S) = i - 2$.

LEMMA 1. $IH_i^{\bar{q}}(X) = \text{image}\,(H^j(Q_{j+1}^{\bar{p}}) \to H^j(Q_j^{\bar{p}}))$.

Proof. By §3.2, $IH_i^{\bar{q}}(X) = \text{Image}\,(H_i(Q_i^{\bar{q}}) \to H_i(Q_{i+1}^{\bar{q}}))$ which is isomorphic to

$$\text{Image}\,(H_i(Q_i^{\bar{q}}, Q_i^{\bar{q}} \cap S) \to H_i(Q_{i+1}^{\bar{q}}, Q_{i+1}^{\bar{q}} \cap S))$$

because the necessary terms in the long exact sequences of the pairs vanish for dimensional reasons:

$$\begin{array}{ccccccc} H_i(Q_i^{\bar{q}} \cap S) & \to & H_i(Q_i^{\bar{q}}) & \to & H_i(Q_i^{\bar{q}}, Q_i^{\bar{q}} \cap S) & \to & H_{i-1}(Q_i^{\bar{q}} \cap S) = 0 \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ 0 = H_i(Q_{i+1}^{\bar{q}} \cap S) & \to & H_i(Q_{i+1}^{\bar{q}}) & \to & H_i(Q_{i+1}^{\bar{q}}, Q_{i+1}^{\bar{q}} \cap S) & \to & H_{i-1}(Q_{i+1}^{\bar{q}} \cap S). \end{array}$$

Now apply the deformation retractions of §3.1 and follow that with the relative Duality isomorphism (see Appendix):

$$\begin{array}{ccccc} H_i(Q_i^{\bar{q}}, Q_i^{\bar{q}} \cap S) & \cong & H_i(X - Q_{j+1}^{\bar{p}} \cap S, X - Q_{j+1}^{\bar{p}}) & \cong & H^j(Q_{j+1}^{\bar{p}}, Q_{j+1}^{\bar{p}} \cap S) \\ \downarrow & & \downarrow & & \downarrow \\ H_i(Q_{i+1}^{\bar{q}}, Q_{i+1}^{\bar{q}} \cap S) & \cong & H_i(X - Q_j^{\bar{p}} \cap S, X - Q_j^{\bar{p}}) & \cong & H^j(Q_j^{\bar{p}}, Q_j^{\bar{p}} \cap S) \end{array}$$

Thus

$$\begin{aligned} IH_i^{\bar{q}}(X) &= \text{Image}\,(H^j(Q_{j+1}^{\bar{p}}, Q_{j+1}^{\bar{p}} \cap S) \to H^j(Q_j^{\bar{p}}, Q_j^{\bar{p}} \cap S)) \\ &= \text{Image}\,(H^j(Q_{j+1}^{\bar{p}}) \to H^j(Q_j^{\bar{p}})) \end{aligned}$$

by the same argument with the long exact sequence of a pair.

LEMMA 2. *If $A \subset B$ then the Kronecker pairing $\langle\,,\rangle$: $H^i(A) \times H_i(A) \to Z$*

$$\langle x, y\rangle = \epsilon(x \cap y)$$

induces a perfect pairing between

$$\text{Image}\,(H_i(A) \otimes \mathbf{Q} \to H_i(B) \otimes \mathbf{Q})$$

and

$$\text{Image}\,(H^i(B) \otimes \mathbf{Q} \to H^i(A) \otimes \mathbf{Q}).$$

The proof is an exercise in linear algebra.

The Poincaré duality theorem now follows: by §3.2,

$$IH_i^{\bar p}(X) = \text{Image}\ (H_i(Q_i^{\bar p}) \to H_i(Q_{i+1}^{\bar p})).$$

By the first lemma, $IH_i^{\bar q}(X) = \text{Image}\ (H^i(Q_{i+1}^{\bar p}) \to H^i(Q_i^{\bar p}))$ and by the second lemma, these are perfectly paired over $\mathbf{Q}$. One checks this perfect pairing

$$IH_i^{\bar p}(X) \otimes \mathbf{Q} \times IH_i^{\bar q}(X) \otimes \mathbf{Q} \to \mathbf{Q}$$

agrees (up to sign) with the augmented intersection pairing.

3.4. In this remark we describe a different family of basic sets which is useful for applications. However the result here is not used in this paper.

For any stratification of X and any triangulation T such that the X_i are subcomplexes of T, let $R_i^{\bar p}$ be the subcomplex of T' (the first barycentric subdivision of T) consisting of all simplices which are $(\bar p, i)$-allowable. Then

$$IH_i^{\bar p} \cong \text{Image}\,(H_i(R_i^{\bar p}) \to H_i(R_{i+1}^{\bar p})).$$

§4. NORMAL SPACES

In this section we study the minimum and maximum perversities $\bar 0 = (0, 0, \ldots, 0)$ and $\bar t = (0, 1, 2, \ldots, n-2)$. The main result is that $IH_*^{\bar 0}(X)$ (respectively $IH_*^{\bar t}(X)$) is the cohomology (respectively homology) of the normalization $\tilde X$ of X. It follows that the intersection pairings

$$IH_*^{\bar 0}(X) \times IH_*^{\bar 0}(X) \to IH_*^{\bar 0}(X)$$
$$IH_*^{\bar 0}(X) \times IH_*^{\bar t}(X) \to IH_*^{\bar t}(X)$$

coincide with the cup and cap products on $\tilde X$.

The topological normalization was studied by McCrory[14] following ideas of Sullivan.

4.1. Definition of normal spaces

Recall that X is an oriented n-dimensional pseudomanifold.

Definition. The pseudomanifold X is *normal* if $H_n(X, X - x) = \mathbb{Z}$ for all $x \in X$.

(Using Zariski's Main Theorem, one can see that a normal complex algebraic variety is a normal pseudomanifold.)

If X is normal and T is a triangulation of X, then $L(\sigma)$ (the link of σ) is a normal pseudomanifold for each simplex $\sigma \in T$. If X is normal and connected, then $H_n(X) = \mathbb{Z}$ (otherwise there would exist simplicial n-cycles C and D such that, $[C]+[D] = [X] \in H_n(X)$ and for some n-simplex $\sigma \in T$, the coefficient of σ in C is 1 and the coefficient of σ in D is 0. Then any $x \in |C| \cap |D| \neq \emptyset$ fails to satisfy $H_n(X, X - x) = \mathbb{Z}$).

PROPOSITION. *Suppose T is a triangulation of X^n, with m-skeleton denoted T_m. Then X is normal if and only if the link of each simplex $\sigma \in T_{n-2}$ is connected.*

Proof. It suffices to show $H_n(X, X - \hat{\sigma}) = \mathbb{Z}$ for the barycentre $\hat{\sigma}$ of each i-simplex $\sigma \in T_{n-2}$. Let T' be the barycentric subdivision of T and let $L'(\sigma)$ be the link of σ in T', $\mathrm{St}(\hat{\sigma})$ be the closed star of $\hat{\sigma}$ in T', $L(\hat{\sigma})$ be the link of $\hat{\sigma}$ in T' and let $D(\sigma) = \hat{\sigma} * L'(\sigma)$ be the dual of σ. Then there is an identification of pairs

$$(D(\sigma), L'(\sigma)) * \partial\sigma = (\mathrm{St}(\hat{\sigma}), L(\hat{\sigma})).$$

Thus

$$H_n(X, X - \hat{\sigma}) = H_n(\mathrm{St}(\hat{\sigma}), L(\hat{\sigma})) = \tilde{H}_{n-i-1}(L'(\sigma))$$

(where $\tilde{H}$ denotes reduced homology).

But $L'(\sigma)$ is normal, so when $n - i - 1 > 0$, this group is $\mathbb{Z}$ if and only if $L'(\sigma)$ is connected.

Definition. A *normalization* of an n-dimensional pseudomanifold X is a normal pseudomanifold $\tilde{X}$ together with a finite-to-one projection $\pi\colon \tilde{X} \to X$ such that, for any $p \in X$,

$$\pi_*\colon \bigoplus_{q \in \pi^{-1}(p)} H_n(\tilde{X}, \tilde{X} - q) \to H_n(X, X - p)$$

is an isomorphism.

We define a normalization $\tilde{X}$ given any triangulation T of X, as follows: Let Y be the disjoint union of all the (closed) n-simplices in T and let $f\colon Y \to X$ be the obvious map. Let $\tilde{X}$ be the quotient of Y obtained by identifying the closed $n-1$ simplices τ and τ' whenever $f(\tau) = f(\tau')$. It is easy to see that $\tilde{X}$ is a simplicial complex such that the link of each i-simplex is connected for $i \leq n-2$. The other properties of $\tilde{X}$ are readily verified.

It is easy to see that any normalization of X may be constructed this way, so the normalization is unique. If X is a complex algebraic variety then the algebraic normalization of X is homeomorphic to the topological normalization.

4.2. $IH_i^{\bar{p}}(X)$ does not change under normalization of X

THEOREM. *If X is a pseudomanifold with normalization $\pi\colon \tilde{X} \to X$ then the map $\pi_*\colon C_*(\tilde{X}) \to C_*(X)$ induces isomorphisms $IH_*^{\bar{p}}(\tilde{X}) \cong IH_*^{\bar{p}}(X)$ for any perversity $\bar{p}$.*

(For example, in Fig. 5 the cycle V which disappears in the normalization is not in $IC_1^{\bar{p}}(X)$ for any $\bar{p}$.)

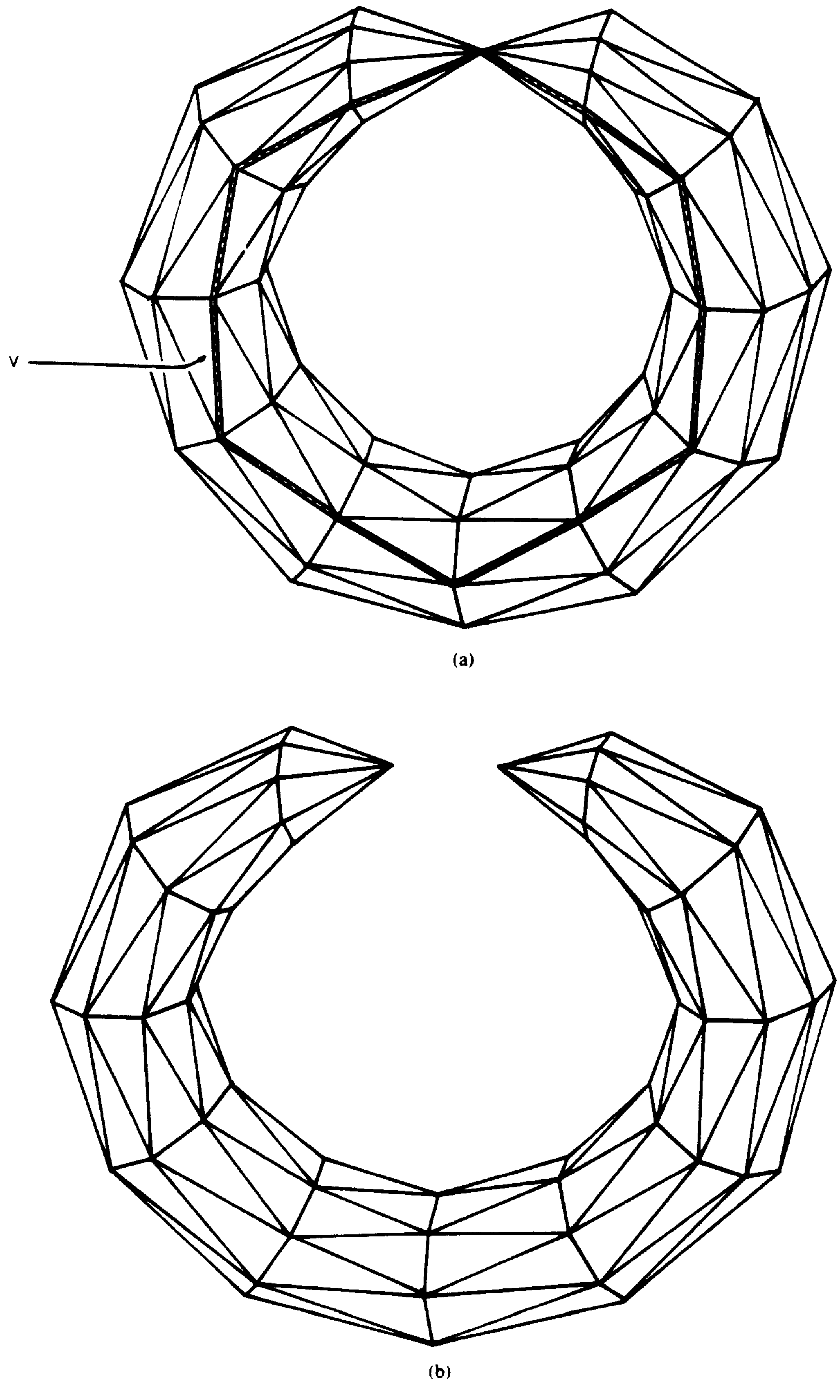


Fig. 5. (a) A two dimensional pseudomanifold X. (b) The normalization of X: a two-sphere.

Proof. Let $\tilde{X}$ be the normalization of X constructed as above with respect to a triangulation T of X. Let $\tilde{T}$ be the corresponding triangulation of $\tilde{X}$. Consider the stratifications of $\tilde{X}$ and X given by the skeleta of these triangulations (as in §3.4). Then π identifies $\tilde{X} - |\tilde{T}_{n-2}|$ with $X - |T_{n-2}|$ by construction.

A P.L. geometric chain $\xi \in C_i(\tilde{X})$ is $(\bar{p}, i)$-allowable in $\tilde{X}$ if and only if $\pi_*(\xi) \in C_i(X)$ is $(\bar{p}, i)$-allowable in X. Furthermore, $|\xi|$ is the closure of $|\xi \cap (\tilde{X} - |\tilde{T}_{n-2}|)$ and the same holds for $|\pi_*(\xi)$. Thus π_* induces an isomorphism $IC_i^{\bar{p}}(\tilde{X}) \cong IC_i^{\bar{p}}(X)$. We conclude $IH_i^{\bar{p}}(\tilde{X}) \cong IH_i^{\bar{p}}(X)$.

4.3. $IH_i^{\bar{p}}(X)$ **and normal spaces**

THEOREM. *If X is normal, then the homomorphism of* §1.4, $\alpha_{\bar{0}}$: $H^{n-i}(X) \to IH_i^{\bar{0}}(X)$ *is an isomorphism.*

Proof. Choose a triangulation T of X. Let T_m be its m-skeleton and T' be the first barycentric subdivision. Recall from §1.4 the homomorphism

$$\begin{array}{rl} C_T^i(X) = \bigoplus_{\dim(\sigma)=i} H^i(\sigma, \partial\sigma) = H^i(|T_i|, |T_{i-1}|) & \\ \big\downarrow \cap [X] & \text{(see Appendix)} \\ H_{n-i}(X - |T_{i-1}|, X - |T_i|) & \\ \cong \big\downarrow & \text{(deformation retract)} \\ H_{n-i}(|D_i|, |D_{i+1}|) & \\ \cong \big\downarrow & \\ \bigoplus_{\dim(\sigma)=i} H_{n-i}(D(\sigma), L(\sigma)). & \end{array}$$

If X is normal, each $H_{n-i}(D(\sigma), L(\sigma)) = Z$ so the above homomorphism is an isomorphism. Thus $H^i(X)$ is the homology of the chain complex

$$\to H_{n-i+1}(|D_{i-1}|, |D_i|) \to H_{n-i}(|D_i|, |D_{i+1}|) \to H_{n-i-1}(|D_{i+1}|, |D_{i+2}|) \to$$

However $|D_i| = Q_{n-i}^{\bar{0}}$. The following diagram with exact rows and columns show that the homology of the complex described above is precisely the image of $H_{n-i}(Q_{n-i}^{\bar{0}})$ in $H_{n-i}(Q_{n-i+1}^{\bar{0}})$.

$$\begin{array}{ccccccc} & & H_{n-i+1}(Q_{n-i+1}^{\bar{0}}, Q_{n-i}^{\bar{0}}) & & & & 0 = H_{n-i-1}(Q_{n-i-2}^{\bar{0}}) \\ & & \downarrow & \searrow & & & \downarrow \\ H_{n-i}(Q_{n-i-1}^{\bar{0}}) & \longrightarrow & H_{n-i}(Q_{n-i}^{\bar{0}}) & \longrightarrow & H_{n-i}(Q_{n-i}^{\bar{0}}, Q_{n-i-1}^{\bar{0}}) & \longrightarrow & H_{n-i-1}(Q_{n-i-1}^{\bar{0}}) \\ \| & & \downarrow & & & \searrow & \downarrow \\ 0 & & H_{n-i}(Q_{n-i+1}^{\bar{0}}) & & & & H_{n-i-1}(Q_{n-i-1}^{\bar{0}}, Q_{n-i-2}^{\bar{0}}) \end{array}$$

We conclude $H^i(X) \cong IH_{n-i}^{\bar{0}}(X)$.

THEOREM. *If X is normal, then* $IH_i^{\bar{t}}(X) \to H_i(X)$ is an isomorphism.

Proof. Let T be a triangulation of X with first barycentric subdivision T'. Let $Q_i^{\bar{p}}$

be the basic sets constructed with respect to T. Consider the filtration

$$Q_i^{\bar t} = P_1 \subset P_2 \subset \cdots \subset P_{n-i+1} = X$$

where P_k is the full subcomplex of T' spanned by all barycentres of simplices $\hat\sigma$ such that $\sigma \in T$ and

$$\dim(\sigma) \le i-2 \quad \text{or} \quad \dim(\sigma) \ge n-k.$$

For any simplex $\sigma \in T_{n-1}$ let $K(\sigma) = \partial\sigma \cap Q_i^{\bar t}$ be the $i-2$ skeleton (in T') of $\partial\sigma$. Let $D(\sigma)$ be the dual of σ (see §4.1) with $L(\sigma) = \partial D(\sigma)$ the link of σ in T'. Thus $D(\sigma) = \hat\sigma * L(\sigma)$.

By excision,

$$H_i(P_k, P_{k-1}) = \bigoplus H_i(D(\sigma) * K(\sigma), L(\sigma) * K(\sigma)),$$

the direct sum is taken over all simplices $\sigma \in K$ with $\dim(\sigma) = n-k$.

But $K(\sigma)$ is a skeleton of a triangulation of the sphere $\partial\sigma$ so it has no homology except in dimensions 0 and $i-2$, i.e.

$$\begin{aligned} H_i(P_k, P_{k-1}) = {} & \bigoplus H_1(D(\sigma), L(\sigma)) \otimes H_{i-2}(K(\sigma)) \\ & \oplus \operatorname{Tor}(H_0(D(\sigma), L(\sigma)), H_{i-2}(K(\sigma))) \\ & \oplus \operatorname{Tor}(H_{i-2}(D(\sigma), L(\sigma)), H_0(K(\sigma))). \end{aligned}$$

(The $\oplus$ taken over all $\sigma \in T$ with $\dim(\sigma) = n-k$).

The Tor terms are 0 and $H_1(D(\sigma), L(\sigma)) = \tilde H_0(L(\sigma))$ is the reduced homology of the link of σ. This is 0 if X is normal. Consequently

$$H_i(Q_i^{\bar t}) \to H_i(X) \quad \text{is surjective.}$$

A similar argument shows $H_i(Q_{i+1}^{\bar t}) \to H_i(X)$ is injective and we conclude

$$IH_i^{\bar t}(X) = \operatorname{Image}(H_i(Q_i^{\bar t}) \to H_i(Q_{i+1}^{\bar t})) \to H_i(X)$$

is an isomorphism.

§5. THE SIGNATURE

In this chapter we extend the signature of oriented manifolds to pseudomanifolds which can be stratified with strata of even codimension. The signature of such a space X of dimension $4k$ is the index of the intersection pairing on a group $IH_{2k}^{\bar m}(X)$ with itself. The signature is a cobordism invariant for cobordisms with even codimension strata and collared boundaries.

Thus, if Ω_*^{ev} denotes the resulting cobordism group, then the signature $\sigma: \Omega_* \to Z$ factors through the natural map $\Omega_* \to \Omega_*^{ev}$. In fact, the signature is the only integral invariant that survives in Ω_*^{ev}, i.e. the image of Ω_{4k} in Ω_{4k}^{ev} is $\mathbf{Z}$. This is because any $4k$ dimensional oriented manifold with signature 0 is cobordant to one which fibers over the circle[9]; the mapping cylinder of this fibration provides the cobordism to zero in Ω_{4k}^{ev}. However Ω_*^{ev} has many new generators and is difficult to compute. Paul Siegel[20] has found a wider class of singular spaces for which $IH_{2k}^{\bar m}(X)$ is still self-dual and yields a cobordism invariant signature.

The resulting cobordism groups are the Witt ring of the rationals. John Morgan[17] has used refinements of the groups $IH^{\bar{m}}_{2k}(X)$ to give a geometric bordism with singularities construction of the generalized homology theory represented by G/PL.

5.1. Definition of the signature

Defination. Suppose A is a pseudomanifold which has a stratification consisting only of strata with even codimension, i.e. $\dim(A) - \dim(\text{stratum})$ is even. If $\dim(A) \neq 4k$ define the signature of A to be 0. If $\dim(A) = 4k$, let $\bar{m} = (0, 0, 1, 1, 2, 2, \ldots, 2k-1)$ and $\bar{n} = (0, 1, 1, 2, 2, \ldots, 2k-1)$. Then $\bar{m} + \bar{n} = \bar{t} = (0, 1, 2, \ldots, 4k-2)$ and $IC_*^{\bar{m}}(A) = IC_*^{\bar{n}}(A)$ since $\bar{m}$ and $\bar{n}$ differ only with respect to strata of odd codimension. Thus by the generalized Poincaré duality theorem, the pairing

$$IH^{\bar{m}}_{2k}(A) \times IH^{\bar{m}}_{2k}(A) = IH^{\bar{m}}_{2k}(A) \times IH^{\bar{n}}_{2k}(A) \to \mathbf{Z}$$

is symmetric and nonsingular over $\mathbf{Q}$. (The determinant is *not* necessarily ± 1 however.) The index of this quadratic form is defined to be the signature of A (denoted $\sigma(A)$). If A is a manifold this agrees with the usual signature.

5.2. Cobordism invariance

Definition. An n-dimensional pseudomanifold with boundary is a compact P.L. pair (X^n, A) with the following properties: (1) A is an $n-1$ dimensional pseudomanifold (with singular set $\Sigma(A)$). (2) There is a closed subspace $\Sigma(X)$ with $\dim(\Sigma(X)) \leq n-2$ so that $X - (\Sigma(X) \cup A)$ is an oriented n-manifold which is dense in X. (3) A is collared in X, i.e. there is a closed neighborhood N of A in X and an orientation preserving PL isomorphism θ: $A \times [0, 1] \to N$ such that

$$\theta(\Sigma(A) \times [0, 1]) = \Sigma(X) \cap N.$$

A stratification of an n-dimensional pseudomanifold with boundary (X, A) is a descending filtration

$$X = X_n \supset X_{n-1} = X_{n-2} = \Sigma(X) \supset X_{n-3} \supset \cdots \supset X_0$$

such that the filtration of A given by $A_{j-1} = X_j \cap A$ stratifies A, the $X_j - A_{j-1}$ stratify $X - A$, and the filtrations respect the collaring of A in X, i.e.

$$\theta(A_{j-1} \times [0, 1]) = X_j \cap N.$$

A stratification of (X, A) is said to consist of only strata with even codimensions if

$$X_j = X_{j-1} \quad \text{whenever } n - j \text{ is odd.}$$

THEOREM. *Suppose (X, A) is a $4k+1$ dimensional pseudomanifold with boundary and suppose (X, A) has a stratification with only strata of even codimension. Then $\sigma(A) = 0$.*

Proof. Let $Y = X \cup c(A)$ be the space obtained from X by adding the cone on A

with cone vertex y_0. Then Y is a pseudomanifold stratified by

$$Y_0 = \{y_0\}$$
$$Y_p = X_p \cup c(A_{p-1}) \qquad \text{for } p \geq 1.$$

We shall now construct an exact sequence

$$\longrightarrow IH^{\bar{m}}_{2k+1}(Y) \rightarrow IH^{\bar{n}}_{2k+1}(Y) \xrightarrow{\beta} IH^{\bar{m}}_{2k}(A) \longrightarrow IH^{\bar{m}}_{2k}(Y) \rightarrow \cdots$$

which is dually paired over $\mathbf{Q}$ with the exact sequence

$$\leftarrow IH^{\bar{n}}_{2k}(Y) \leftarrow IH^{\bar{m}}_{2k}(Y) \leftarrow IH^{\bar{m}}_{2k}(A) \xleftarrow{\beta} IH^{\bar{n}}_{2k+1}(Y) \leftarrow \cdots$$

Then image (β) will be its own annihilator in $IH^{\bar{m}}_{2k}(A)$ so $\sigma(A) = 0$.

The exact sequence is the long exact sequence on homology corresponding to the inclusion of chain complexes $IC_*^{\bar{m}}(Y) \subset IC_*^{\bar{n}}(Y)$. We need to identify the homology of the quotient complex.

Note that $m_{4k+1} = 2k - 1$ and $n_{4k+1} = 2k$, and that Y has no strata of odd codimension except the cone vertex. Therefore

$$IC^{\bar{m}}_{2k+2}(Y) = \{\xi \in IC^{\bar{n}}_{2k+2}(Y) \,\big|\, |\partial\xi| \cap \{y_0\} = \phi\}$$

$$IC^{\bar{m}}_{2k+1}(Y) = \{\xi \in IC^{\bar{n}}_{2k+1}(Y) \,\big|\, |\xi| \cap \{y_0\} = \phi\}$$

$$IC^{\bar{m}}_a(Y) = IC^{\bar{n}}_a(Y) \quad \text{if} \quad a \geq 2k+3 \quad \text{or} \quad a \leq 2k.$$

The quotient chain complex has only two nonzero terms which will be identified as

$$IC^{\bar{n}}_{2k+2}(Y) \big/ IC^{\bar{m}}_{2k+2}(Y) = IB^{\bar{m}}_{2k}(A)$$

$$IC^{\bar{n}}_{2k+1}(Y) \big/ IC^{\bar{m}}_{2k+1}(Y) = IZ^{\bar{m}}_{2k}(A)$$

where

$$IB^{\bar{m}}_{2k}(A) = \partial(IC^{\bar{m}}_{2k+1}(A))$$

$$IZ^{\bar{m}}_{2k}(A) = (\ker \partial) \cap IC^{\bar{m}}_{2k}(A).$$

This identification will complete the construction of the long exact sequences above.

First we define the map

$$IC^{\bar{n}}_{2k+1}(Y) \big/ IC^{\bar{m}}_{2k+1}(Y) \rightarrow IZ^{\bar{m}}_{2k}(A).$$

Let $\xi \in IC^{\bar{q}}_{2k+1}(Y)$. If $|\xi| \cap \{y_0\} = \emptyset$ then $\xi \in IC^{\bar{m}}_{2k+1}(Y)$, so assume $|\xi| \cap \{y_0\} \neq \emptyset$. Choose a triangulation T of Y subordinate to the stratification such that A is a subcomplex, ξ is a sum of simplices, and

$$|\xi| \cap \overline{\mathrm{St}(y_0)} = y_0 * (|\xi| \cap L(y_0))$$

where $\mathrm{St}(y_0)$ is the star of y_0 in T and $L(y_0)$ is the link of y_0 in T.

Pseudo-radial retraction along the cone lines gives a P.L. stratum preserving isomorphism $L(y_0) \cong A$ which takes $|\xi| \cap L(y_0)$ to some P.L. chain (if properly oriented) $f(\xi) \in IC^{\bar{n}}_{2k}(A)$. Since $\xi \in IC^{\bar{q}}_{2k+1}(Y)$, $|\partial\xi| \cap \{y_0\} = \phi$ so $|\partial\xi| \cap L(y_0) = \phi$ so $\partial f(\xi) = 0$. Therefore $f(\xi) \in IZ^{\bar{n}}_{2k}(A)$. Modifying ξ by any P.L. chain in $IC^{\bar{n}}_{2k+1}(Y)$ (which does not intersect y_0) or choosing a finer triangulation does not change the chain $f(\xi)$. The inverse map

$$IZ^{\bar{m}}_{2k}(A) \to IC^{\bar{q}}_{2k+1}(Y)/I^{\bar{m}}_{2k+1}(Y)$$

assigns to any cycle η the cone over that cycle, $y_0 * \eta$, with the product orientation.

A similar procedure *applied to the boundary* of any chain $\xi \in IC^{\bar{q}}_{2k+2}(Y)$ gives the identification

$$IC^{\bar{q}}_{2k+2}(Y)\Big/IC^{\bar{m}}_{2k+2}(Y) \cong IB^{\bar{m}}_{2k}(A).$$

The differential on the quotient complex identifies with the inclusion $IB^{\bar{m}}_{2k}(A) \subset IZ^{\bar{m}}_{2k}(A)$, and all the intersection pairing are compatible.

5.3. The L class

In this section we outline the construction of a rational homology class $L(V) \in H_*(V; \mathbb{Q})$ for any Whitney stratified pseudomanifold V which has only strata with even codimensions (see Thom[26] or Mather[15] for details on Whitney stratifications). For example, any compact complex analytic variety has a canonical Whitney stratification with even codimension singularities, and thus has an L class. If V is a manifold then $L(V)$ is the dual of the Hirzebruch L class, however in general $L(V)$ need not lie in the image of the duality map. Using the ideas of Sullivan[24] one can find, for such a space V, a canonical element in $KO(V) \otimes \mathbb{Z}[1/2]$ whose Pontrjagin character is $L(V)$.

Unfortunately, we have so far defined the signiature only for spaces with a PL structure. (In fact, the signature is a topological invariant since $IH_*^{\bar{m}}(V)$ is a topological invariant—but the proof of topological invariance is not in the spirit of the present paper). We are forced to find a natural (concordance class of) P.L. structures on V.

Definition. For the remainder of this section, we let V denote a ***Whitney stratified pseudomanifold***, i.e. a compact subset of some C^∞ manifold M, together with a filtration by closed subsets

$$V = V_m \supset V_{m-1} = V_{m-2} \supset V_{m-3} \supset \cdots \supset V_0$$

such that each stratum $S_j = V_j - V_{j-1}$ is a (possibly empty) smooth j-dimensional submanifold of M, such that Whitney's conditions A and B hold at each point $x \in S_j \cap \overline{S_k}$ and such that V is the closure of the nonsingular part $V - V_{m-2}$. We say V has only strata of even codimensional if $V_j = V_{j-1}$ whenever $n - j$ is odd.

Definition. Suppose V has only strata of even codimension. Define the *signature* of V as follows:

By Goresky[5], V can be triangulated. It is not hard to see that the filtration of V given by the Whitney stratification defines a PL stratification of V, and hence one with even codimensional strata. Thus $\sigma(V)$ is defined by §5.2. Furthermore, the techniques of [5] may be used to show that any two triangulations of the type defined in [5] are concordant. This triangulation of $V \times [0, 1]$ is therefore a *PL* cobordism with even codimension strata between the two given triangulations of V. We conclude $\sigma(V)$ is the same for either triangulation.

Similarly, the signature is a cobordism invariant for Whitney stratified cobordisms with even codimension singularities.

Let us say that a continuous map f: $V \to S^k$ is *transverse* if

(a) f is the restriction of a C^∞ map $\bar{f}$: $U \to S^k$ for some neighborhood U of V in M.

(b) $\bar{f}$ is transverse regular to the north pole $N \in S^k$.

(c) $\bar{f}^{-1}(N)$ is transverse to each stratum of V.

Suppose V has only strata of even codimension. For each transverse map f: $V \to S^k$ we get a Whitney stratification of $f^{-1}(N) = \bar{f}^{-1}(N) \cap V$ with a stratum of even codimension $\bar{f}^{-1}(N) \cap A$ for each stratum A of V. Thus $\sigma(f^{-1}(N))$ is defined.

LEMMA. *There is a unique map θ: $[V, S^k] \to \mathbf{Z}$ which assigns the number $\sigma(f^{-1}(N))$ to each transverse map f: $V \to S^k$. (Here, $[V, S^k]$ denotes the set of homotopy classes of maps from V to the k-sphere).*

Proof. By standard techniques, every continuous map $V \to S^k$ may be approximated by a transverse map in the same homotopy class. If f and g are homotopic transverse maps then there is a transverse homotopy H: $V \times [0, 1] \to S^k$ between them. Thus, $H^{-1}(N)$ is a cobordism with even codimension singularities between $f^{-1}(N)$ and $g^{-1}(N)$; so $\sigma(f^{-1}(N)) = \sigma(g^{-1}(N))$. See Fig. 6.

Definition. Define the L-class $L_k(V) \in H_k(V; \mathbf{Q})$ to be the homomorphism $\theta \otimes \mathbf{Q}$: $H^k(V; \mathbf{Q}) \to \mathbf{Q}$ where we have identified

$$[V, S^k] \otimes \mathbf{Q} \cong H^k(V; \mathbf{Q}) \quad \text{when } 2k > m + 1.$$

The restriction $2k > m + 1$ can be removed by crossing V with a sphere, as in Milnor[16].

§6. EXAMPLES AND COMMENTS

6.1 Isolated singularities

If an n dimensional pseudomanifold X is nonsingular except at a finite set of points $x_1, x_2, \ldots, x_r$ then X is said to have *isolated singularities*. In this case, X has a stratification with $X_n = X$ and $X_{n-1} = X_{n-2} = \cdots X_0 = \{x_1, x_2, \ldots, x_r\}$, so $IH_*^{p}(X)$ depends only on p_n. If we delete the open stars of $x_1, \ldots, x_r$ with respect to the second barycentric subdivision T'' of some triangulation T of X, then we are left with a manifold $\hat{X}$ with boundary $\partial\hat{X}$. The intersection homology groups of X are given by:

$$\text{If } p_n > n - i - 1 \text{ then } IH_i^{p}(X) = H^{n-i}(\hat{X}, \partial\hat{X}) = H_i(\hat{X})$$
$$\text{If } p_n < n - i - 1 \text{ then } IH_i^{p}(X) = H_i(\hat{X}, \partial\hat{X}) = H^{n-i}(\hat{X}).$$

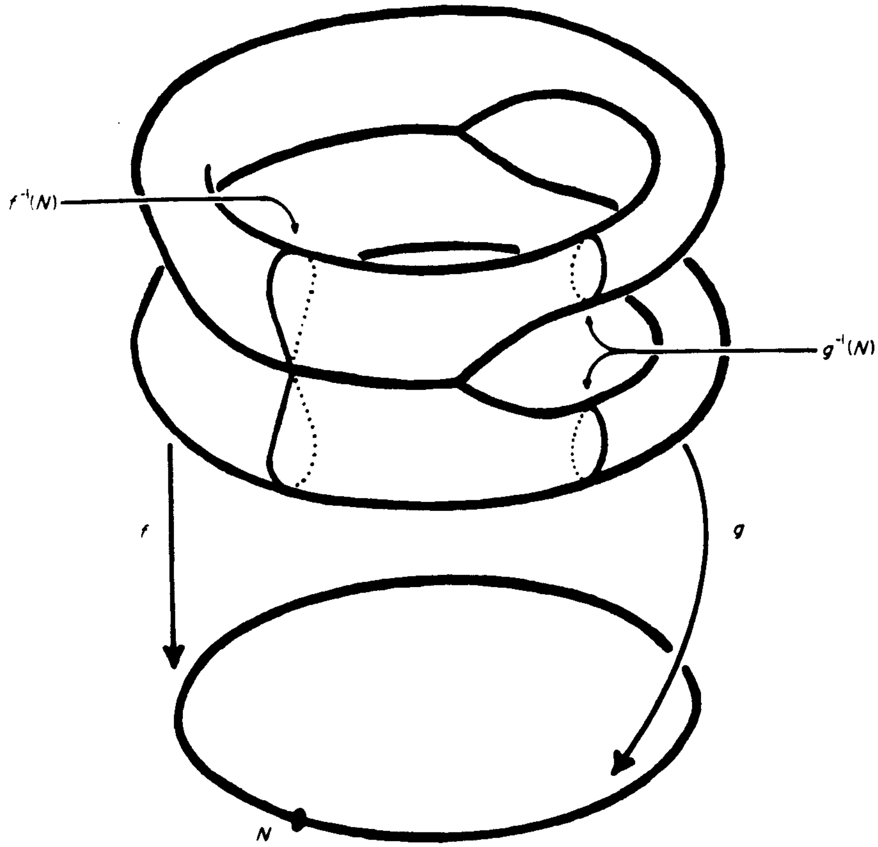


Fig. 6.

If $p_n = n - i - 1$ then

$$IH_i^{\bar{p}}(X) = \text{Image}\,(H^{n-i}(\hat{X}, \partial\hat{X}) \to H^{n-i}(\hat{X}))$$
$$= \text{Image}\,(H_i(\hat{X}) \to H_i(\hat{X}, \partial\hat{X})).$$

If $p_n + q_n \leq n - 2$, the intersection pairing is given by the cup or cap product. In particular the signature of X is the Novikov signature of $\hat{X}$ as a manifold with boundary.

6.2. Nonhomotopy invariance

Let X be the pseudomanifold obtained by gluing the 4-sphere S^4 to complex projective 2-space, CP^2 along a common S^2 which is contained in the S^4 in the standard way and which is contained in the CP^2 as CP^1. Let Y be the one point wedge of two copies of S^4. Then X and Y are homotopy equivalent since S^2 is collapsible in S^4. But by the normalization isomorphism (§4.2) we have

$$IH_2^{\bar{m}}(X) = IH_2^{\bar{m}}(S^4 \cup CP^2) = H_2(S^4) \oplus H_2(CP^2) = Z$$
$$IH_2^{\bar{m}}(Y) = IH_2^{\bar{m}}(S^4 \cup S^4) = H_2(S^4) \oplus H_2(S^4) = 0.$$

Moreover, $\sigma(X) = 1$ and $\sigma(Y) = 0$.

6.3. Torsion questions

For any ring R we can define $IC_i^{\bar{p}}(X; R)$ to be the subgroup of chains $\xi \in C_i(X) \otimes R$ such that $|\xi|$ is $(\bar{p}, i)$-allowable and $|\partial\xi|$ is $(\bar{p}, i-1)$-allowable. This is not the same as $IC_i^{\bar{p}}(X) \otimes R$. It is still true, however, that

$$IH_i^{\bar{p}}(X; R) = \text{Image}\,(H_i(Q_i^{\bar{p}}; R) \to H_i(P_{i+1}^{\bar{p}}; R))$$

and in particular if R is a field then the generalized Poincaré duality theorem holds with coefficients in R.

We now discuss an example illustrating the behavior of torsion in the intersection homology groups. Let M be a smooth connected n-manifold and let X be the Thom space of an oriented n-dimensional vectorbundle on M with nonzero Euler number l. Then $IH_n^{\bar{m}}(X) = Z$, $IH_n^{\bar{m}}(X; Z/(l)) = 0$, and the determinant of the intersection pairing on $IH_n^{\bar{m}}(X; Z)$ is l. Furthermore the duality map $IH_n^{\bar{0}}(X) \to IH_n^{\bar{t}}(X)$ is multiplication by l.

This example shows the universal coefficient theorem fails and the generalized Poincaré duality theorem is not true over Z.

6.4. Homology manifolds

The condition on a normal pseudomanifold X that all the natural maps (for $\bar{p} \leq \bar{q}$), $IH_i^{\bar{p}}(X; R) \to IH_i^{\bar{q}}(X; R)$ be isomorphisms is stronger than the condition that X be an R-Poincaré duality space. If X is an R-homology manifold then all the intersection homology groups are isomorphic. We do not know of a counter-example to the converse of this statement.

6.5. Algebraic intersections

Other geometric questions in intersection theory are considered in the paper, *Defining Algebraic Intersections*, by W. Fulton and R. MacPherson (Tromso conference in algebraic geometry—to appear in Springer Lecture Notes in Mathematics). We do not know if this algebraic intersection theory has a common synthesis with the present paper.

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§7. APPENDIX: RELATIVE CAP PRODUCT AND DUALITY

In this section we deduce a version of the "Duality isomorphism" between homology and cohomology, for relative manifolds. This isomorphism is needed for the construction of the intersection product (§2.1) where it is necessary to think of a geometric chain as both a homology class and a cohomology class in the nonsingular part of X.

We use a relative version of the cap product with the (relative) fundamental class of X as in Whitehead[28]. The isomorphism is actually proven by Dold[4].

7.1. The cap product

Suppose K is a simplicial complex and suppose $M \subset L$ are subcomplex. Whitehead[28] defines a relative cap product

$$H^i(M, L) \otimes H_n(K) \rightarrow H_{n-i}(K-L, K-M).$$

This cap product may be deduced from the usual relative cap product[20, 4] by a sequence of excisions and deformation retractions using regular neighborhoods. (There is a choice of sign, however—which is purely a matter of taste. We prefer to use the sign convention of Steenrod which differs from Whitehead's cap product by $(-1)^{ni+i}$).

7.2. Compatibility with cup and cap products

We include for convenience several relations between cup and cap products, boundary and coboundary, using our sign conventions. These formulae are needed in §2.4. The proofs are standard—(see [27] or [21]).

1. If $L_3 \subset L_2 \subset L_1$ are subcomplexes of K and if $c \in H_n(K)$, the following diagram commutes with no sign correction:

$$\begin{array}{ccc} H^q(L_2, L_3) & \xrightarrow{\delta} & H^{q+1}(L_1, L_2) \\ \downarrow \cap c & & \downarrow \cap c \\ H_{n-q}(K-L_3, K-L_2) & \xrightarrow{\partial} & H_{n-q-1}(K-L_2, K-L_1) \end{array}$$

2. If $a \in H^i(K)$, $b \in H^j(K)$ and $c \in H_n(K)$ then

$$a \cup b = (-1)^{ij} b \cup a$$

$$(a \cup b) \cap c = a \cap (b \cap c).$$

7.3. Constructible sets and duality

Definition. A *constructible subset* of a simplicial complex K is a union of interiors of simplices. A constructible subset of a *PL* space X is one which is constructible with respect to some triangulation of X. In either case, the class of constructible subsets is closed under the operations of union, intersection, and complementation.

Duality theorem. Suppose (X, Σ) is an n-dimensional oriented pseudomanifold. Let $A \subset B$ be constructible subsets of X and assume $(B-A) \subset (X-\Sigma)$, i.e. $A \cap \Sigma = B \cap \Sigma$. Then the cap product with the fundamental class is an isomorphism:

$$\cap [X]: H^i(B, A) \rightarrow H_{n-i}(X-A, X-B).$$

Proof. Choose a triangulation K of X so that Σ is a closed subcomplex and so that A and B are unions of interiors of simplices. Let K' be the barycentric subdivision of K. Define $k(A)$ to be the span in K' of the barycentres of simplices

$$\{\hat{\sigma} | \sigma \in K \text{ and interior } (\sigma) \subset A\}.$$

We have a canonical simplex preserving deformation retraction $A \to k(A)$ defined as follows: If $\tau \in K'$ and interior $(\tau) \subset A$ then τ is the join of its intersection with $k(A)$ and its intersection with $k(X - A)$. Retract along the join lines and restrict to $A \cap \tau$. Similarly, we have a deformation retraction $X - k(A) \to k(X - A)$.

Lemma. *The cap product with the fundamental class gives an isomorphism*

$$\cap [X]: H^i(k(B \cup \Sigma), k(A \cup \Sigma)) \to H_{n-i}(X - k(A \cup \Sigma), X - k(B \cup \Sigma)).$$

Proof. $k(B \cup \Sigma)$ is compact so Dold [4] (VIII, 7.4) applies. (It is necessary to check that the map defined by Dold agrees, up to sign, with the (Whitehead) cap product $\cap [X]$.)

We now complete the proof of the duality theorem:

$H^i(B, A) \cong H^i(B \cup \Sigma, A \cup \Sigma)$ by excision of $\Sigma - (A \cap \Sigma)$
$\cong H^i(k(B \cup \Sigma), k(A \cup \Sigma))$ by deformation retraction
$\cong H_{n-i}(X - k(A \cup \Sigma), X - k(B \cup \Sigma))$ by lemma above
$\cong H_{n-i}(k(X - A \cup \Sigma), k(X - B \cup \Sigma))$ by deformation retraction
$\cong H_{n-i}(X - A \cup \Sigma, X - B \cup \Sigma)$ by deformation retraction
$\cong H_{n-i}(X - A, X - B)$ by excision of $\Sigma - (A \cap \Sigma)$.

The deformation retractions and excisions are all induced by inclusions, so the composition

$$H^i(B, A) \to H_{n-i}(X - B, X - A)$$

coincides with the cap product with the fundamental class of X.