→ join to Keep, ~~v.s. Proposition 3~~.3.

Combinatorial Differential Manifolds

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1 Introduction.

The object of this note is to introduce a new type of geometric structure called a ***combinatorial differential manifold.*** We define tangent bundles and characteristic classes of combinatorial differential manifolds. We show that a differentiable manifold which is appropriately triangulated in the usual sense gives rise to a combinatorial differential manifold.

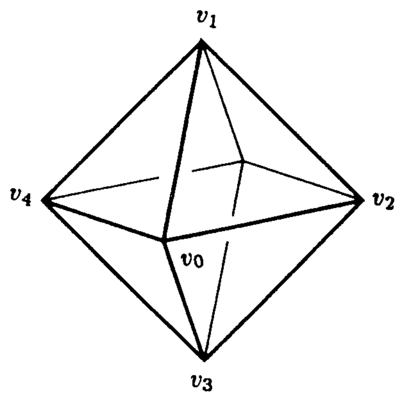
There are more questions than theory about combinatorial differential manifolds at present. However, the theory has already proved its utility in one way: it is the essential basis for the combinatorial formula for the Pontrjagin classes of [GM2].

The reader may wish to start with definition of combinatorial differential manifolds, which is in §2.1, and which may be read after looking at the definition of an oriented matroid in the appendix. The rest of this introduction will be a comparison of combinatorial differential manifolds with usual combinatorial manifolds.

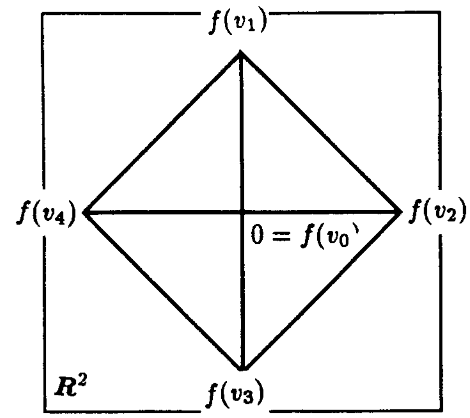
1.1 Combinatorial Manifolds.

DEFINITION. Let X be a simplicial complex, and let Δ be a simplex of X. A ***flattening*** of X at Δ is a simplex-wise linear homeomorphism $f : \text{Star}\,\Delta \longrightarrow U \subset V$ of the star of Δ onto a neighborhood U of the origin in an n dimensional real vector space V, such that the image of the interior of Δ contains the origin of V.

For example, the star of a vertex of the surface of an octahedron may be flattened as follows:



An octahedron

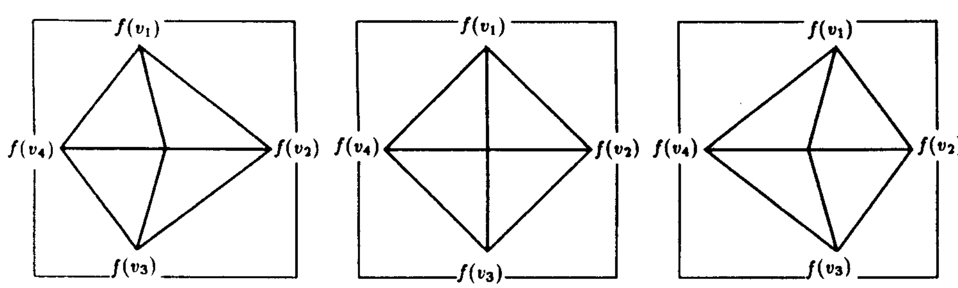


A flattening

An n *dimensional combinatorial manifold* is a simplicial complex X such for every simplex $\Delta \subset X$, there exists an n-flattening of X at Δ. Only the existence of the flattening is required: the flattening itself is not retained as part of the structure (as, for example, coordinate charts are retained as part of the structure of a differentiable manifold). The flattenings at adjacent vertices are not required to be related to each other.

1.2 Oriented Matroids: Combinatorial Remnants of Flattenings.

In a combinatorial differential n-manifold X, only a certain "combinatorial remnant" of a flattening of X at Δ is required. As an example, the "combinatorial remnant" of following three flattenings of the star of a vertex of an octahedron are considered to be different:



Combinatorially distinct flattenings

The difference between them is that in the second picture the line from $f(v_1)$ to $f(v_0)$ to $f(v_3)$ is straight; in the first it bends toward $f(v_4)$; and in the third it bends toward $f(v_2)$. This difference is encoded in a standard combinatorial object called an *oriented matroid*, which is a combinatorial abstraction of a set of vectors in a real vector space. In a combinatorial differential manifold, not only do we require the existence of oriented matroids encoding a "combinatorial remnant" of a flattening of the star of each simplex, but these oriented matroids are kept as part of the structure. They are also assumed to vary in a controlled way as you move around the polyhedron, and the variation is also kept as part of the structure.

1.3 Why study Combinatorial Differential Manifolds?

The real hope, of course, is that the category of combinatorial differential manifolds will have a rich theory. However, there are some *a priori* reasons having to do with the importance of oriented matroids:

1. The linear apparatus of calculus, which is inherent to smooth manifolds, is very important in topology. Micro-bundles or block bundles, which are the tangent objects of combinatorial manifolds, have no linear structure on their fiber. *Matroid bundles* (see §3), which are the tangent objects of combinatorial differential manifolds, have oriented matroids as their fibers. Oriented matroids are the standard combinatorial abstraction of (a set of vectors in) a real vector space.

2. Combinatorial manifolds are not really combinatorial objects: The existence of flattenings is really a question in semi-algebraic geometry rather than combinatorics. Combinatorial differential manifolds are true combinatorial objects since oriented matroids, which are combinatorial, replace the flattenings.

3. Oriented matroid theory is already a well-established branch of combinatorics with some highly non-trivial theorems. There is the potential to establish serious interaction between combinatorics and topology.

2 Combinatorial Differential Manifolds.

2.1 The Definition.

Notations. We assume that the reader has some familiarity with the notions of oriented matroid theory, in the notation given in the appendix. (In particular $M \rightsquigarrow M'$ means that M' is a specialization of M, §5.4.) If Δ is a simplex of a simplicial complex X, then $\mathrm{Star}\,\Delta$ is the star of Δ, i.e. the union of the closed simplices of X that contain Δ as a face; $\partial\mathrm{Star}\,\Delta$ is the boundary of the star of Δ, i.e. the union of the simplices of $\mathrm{Star}\,\Delta$ which do not intersect the interior of Δ.

DEFINITION. An ***n-dimensional combinatorial differential manifold*** is the triple of data $(X, \hat{X}, M)$ as follows:

1. X is a simplicial complex. We denote simplices of X by Δ, Δ', etc.
2. $\hat{X}$ is cell complex which is a refinement of X. We denote the cells of $\hat{X}$ by σ, σ', etc. For any simplex σ of $\hat{X}$, we denote by $\Delta(\sigma)$ the smallest simplex of X containing σ.
3. M is a rule which, to every simplex σ of $\hat{\mathrm{X}}$, assigns a rank n oriented matroid $M(\sigma)$ whose set of elements is S_Δ, the set of vertices of $\mathrm{Star}\,\Delta(\sigma)$.

These data are subject to the following axioms:

A1. The rank in $M(\sigma)$ of the set of vertices of $\Delta(\sigma)$ is the dimension of $\Delta(\sigma)$.

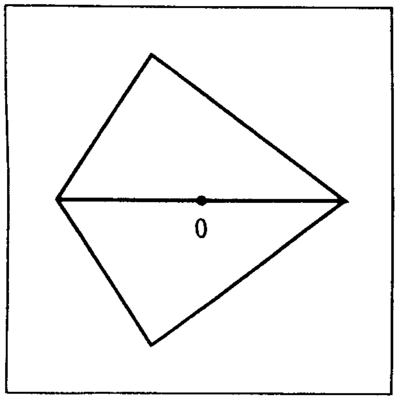
A2. For each simplex Δ' in $\partial\mathrm{Star}\,\Delta(\sigma)$, the set Z of vertices of Δ' is linearly independent in $M(\sigma)$, and no other non-zero element of $M(\sigma)$ is in the convex hull of Z.

A3. Whenever σ' is in the boundary of σ, we have $M(\sigma) \rightsquigarrow M_{\Delta(\sigma)}(\sigma')$, where $M_{\Delta(\sigma)}(\sigma')$ is the submatroid of $M(\sigma')$ whose set of elements is $S_{\Delta(\sigma)}$.

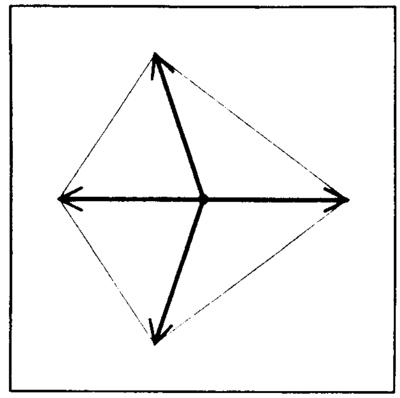
2.2 Interpretation in terms of flattenings.

Recall the notion of a configuration of vectors (§5.2): a map of a finite set S into a real vector space V whose image spans V.

DEFINITION. Suppose that $\Delta \subset X$, is a simplex and $f : \mathrm{Star}\,\Delta \longrightarrow U \subset V$ is a flattening of X at Δ, where V is an n dimensional vector space. Then the vector configuration ***associated to*** f is the configuration $\rho : S_\Delta \longrightarrow V$ where S_Δ is the set of vertices of $\mathrm{Star}\,\Delta$, and ρ is the restriction of f to S_Δ. The following picture shows a flattening and the associated vector configuration:



A flattening



The associated vector configuration

DEFINITION. The rank n oriented matroid $M(f)$ *associated to the flattening* f is the matroid represented by the vector configuration associated to f.

The oriented matroid $M(f)$ is the "combinatorial remnant" of the flattening f referred to in the introduction.

Proposition 2.1 *For any flattening f, the oriented matroid $M(f)$ satisfies axioms A1 and A2 of the definition of a combinatorial differential manifold.*

This proposition shows why axioms A1 and A2 are natural. Axiom A3 is a kind of continuity requirement on the "combinatorial remnants" of flattenings. Consider the case that σ and σ' are both in the same simplex Δ of X, so that $S_{\Delta(\sigma)} = S_{\Delta(\sigma')}$. Suppose that we are given a map $m_\Delta : \Delta \longrightarrow F_\Delta$ from Δ into the space F_Δ of flattenings $f : \mathrm{Star}\,\Delta \longrightarrow U \subset V$ of X at Δ. (By §5.5.2, the set of all flattenings of X at Δ is a manifold.)

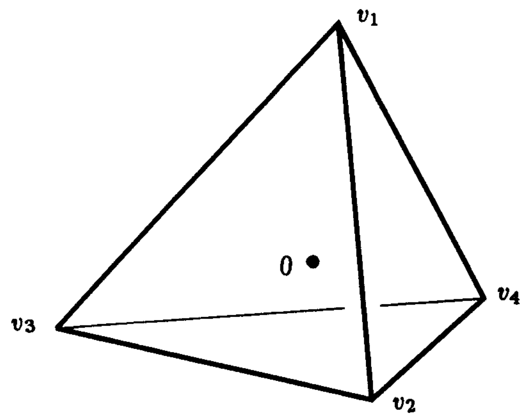
Proposition 2.2 *Suppose that the map m_Δ is continuous. Suppose further that m_Δ is constant on each cell of $\hat{X}$ in Δ. Then the assignation $M(\sigma) = M(m_\Delta(\sigma))$ satisfies axiom A3 for σ and σ' in Δ.*

This follows from proposition 5.2

2.3 An example.

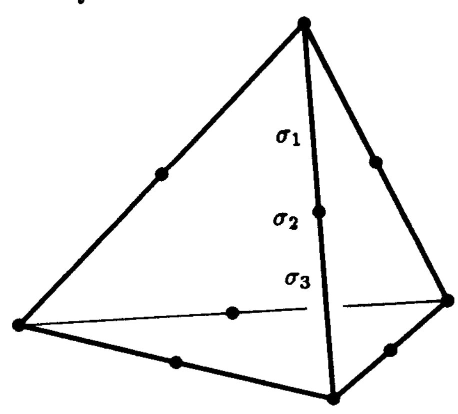
Suppose that X is simplex-wise linearly embedded as the surface of a convex body in $\boldsymbol{R}^{n+1}$ that contains the origin 0 in its interior. Then X has the structure of a combinatorial differential manifold defined as follows: To each point $p \in \Delta$ of X, we obtain a flattening of X at Δ by projecting $\operatorname{Star}\Delta$ into the quotient space $\boldsymbol{R}^{n+1}/(p)$, where (p) is the 1-dimensional subspace of $\boldsymbol{R}^{n+1}$ spanned by p. We obtain an oriented matroid M_p on S_Δ from the flattening as above. The cell complex $\hat{X}$ is defined by the condition that two points p and p' in the interior of Δ lie in the same cell of $\hat{X}$ if $M_p = M_{p'}$. (It is an amusing exercise to verify that $\hat{X}$ so defined is indeed a cell complex.) Finally, $M(\sigma) = M(p)$ for $p \in \sigma$.

For example, suppose that X is the surface of the regular tetrahedron, embedded in $\boldsymbol{R}^3$ with 0 as its center of gravity.



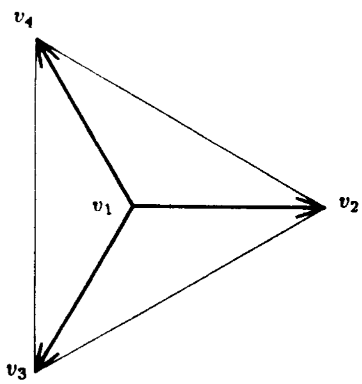
The surface of a regular tetrahedron

Then $\hat{X}$ is the subdivision of X obtained by dividing each edge of X into two 1-cells separated by a vertex.

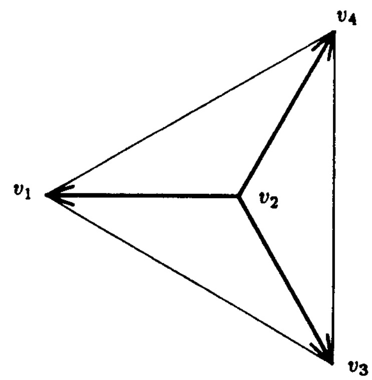


The subdivision $\hat{X}$

The oriented matroids $M(v_1)$ and $M(v_2)$ the are represented by the following vector configurations. (Each vector is labeled by the vertices to which it corresponds.)

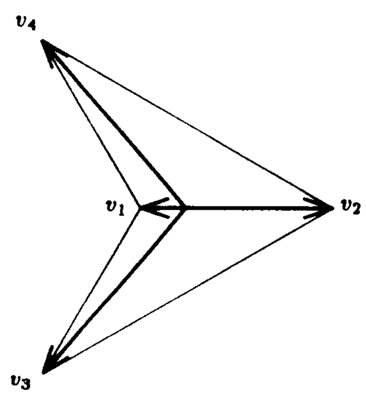


$M(v_1)$

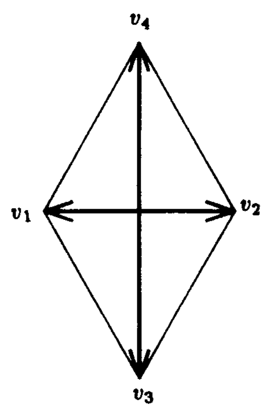


$M(v_2)$

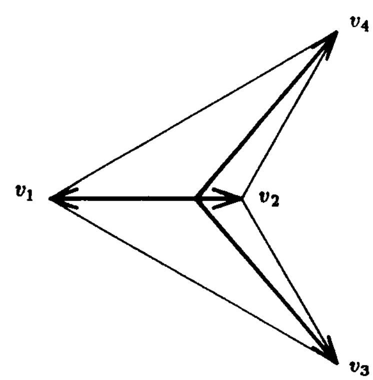
These are, in fact, the only possibilities allowed by the axioms. For example, in $M(v_1)$, the point corresponding to v_1 must be zero axiom A1. The oriented matroids $M(\sigma_1)$, $M(\sigma_2)$, and $M(\sigma_3)$ are represented by the following vector configurations:



$M(\sigma_1)$

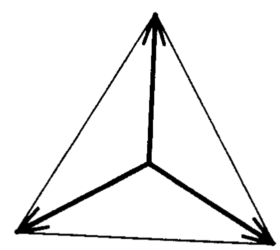


$M(\sigma_2)$



$M(\sigma_3)$

This illustrates why in the definition we are forced to make the cell complex $\hat{X}$ finer than X: we must allow the matroid $M(p)$ to vary as the p moves along the edge in X. Finally, $M(\sigma)$ where σ is the face with vertices v_1, v_2, and v_3 is also determined up to isomorphism by the axioms



$M(\sigma)$

2.4 The connection with differentiable manifolds

Let N be a differentiable manifold. Recall that a ***smooth triangulation*** of N is a simplicial complex X together with a homeomorphism $\eta : X \longrightarrow N$ which is smooth on each closed simplex Δ of X. (The smoothness condition means that $\eta|\Delta$ extends to a smooth map to N of a neighborhood of Δ in an affine space containing Δ as a linearly embedded simplex.)

DEFINITION. Suppose we are given a smooth triangulation $\eta : X \longrightarrow N$. Let $p \in X$ be a point in the interior of a simplex Δ. The flattening $f_p : \mathrm{Star}\,\Delta \longrightarrow U \subset T_{\eta(p)}N$ is said to be ***induced at p by the smooth triangulation*** if for each simplex $\Delta' \in \mathrm{Star}\,\Delta$, the differential at p of $f|\Delta'$ coincides with the differential at p of $\eta|\Delta'$.

For all $p \in X$, there exists a unique flattening induced by p, because $f|\Delta'$ is an affine map. (Purists should note that $T_{\eta(p)}N$ has been identified with $T_0T_{\eta(p)}N$ in the definition.)

Just as in §2.2 above, the flattening f_p of X at Δ has an associated vector configuration, which represents a matroid M_p on the set S_Δ.

DEFINITION. The smooth triangulation $\eta : X \longrightarrow N$ is said to be ***tame*** if there is a cell decomposition $\hat{X}$ which refines X with the property that the matroid M_p is constant on each open cell σ of $\hat{X}$.

For example, if η is piecewise analytic then it is tame. Generic smooth triangulations are tame. These may be both seen by using the analytic constructibility of matroid stratification of the space of equivalence classes of flattenings of §5.5.2.

Proposition 2.3 *Suppose that $\eta : X \longrightarrow N$ is a tame smooth triangulation. Then X has an induced structure of a combinatorial differential manifold (namely $(X, \hat{X}, M)$ where $\hat{X}$ is the cell decomposition showing that η is tame and $M(\sigma) = M_p$ for $p \in \sigma$.*

3 Matroid bundles and the Matroid Grassmannian.

Every good category for geometric topology has an associated bundle theory with a classifying space. Differential manifolds have vector bundles, which are classified by the Grassmannian. Piecewise linear manifolds have micro-bundles (or block bundles) which are classified by BPL.

In this section, we develop the bundle theory associated to combinatorial differential manifolds. The bundles are Matroid bundles, and their classifying space is the Matroid Grassmannian. Both are purely combinatorial constructions.

3.1 Matroid bundles.

DEFINITION. Let X be a simplicial complex. A *rank n matroid bundle* over X is a triple of data $(S, \hat{X}, M)$ as follows:

1. S is a finite set.
2. $\hat{X}$ is cell complex which is a refinement of X.
3. M is a rule which, to every simplex σ of $\hat{X}$, assigns a rank n oriented matroid $M(\sigma)$ whose elements are S.

such that whenever σ' is in the boundary of σ, we have $M(\sigma) \rightsquigarrow M(\sigma')$.

Matroid bundles have some properties that one expects from a topological bundle theory. A matroid bundle over X has an associated sphere bundle, which is a cell complex mapping to X. The fiber over a point in σ is a simplicial complex whose set of vertices is the set of all covectors of $M(\sigma)$. This fiber is homeomorphic to a sphere, by the representation theorem for matroids [BLSWZ]. Likewise, a matroid bundle over X has analogues of associated Grassmann bundles. The constructions of these are sketched in [GM2].

3.2 The tangent bundle.

Let $(X, \hat{X}, M)$ be a combinatorial differential manifold. We want to construct a matroid bundle $TX = (S, \hat{X}', M')$ over X called the *tangent bundle* of X. Tangent bundles are an important source of examples of a matroid bundle.

The set S is the set of vertices of X.

The cell complex $\hat{X}'$ is obtained from $\hat{X}$ by adding cellulated tubular neighborhoods around the simplices Δ of X. More specifically, we proceed as follows: Consider the category C of cell complexes whose objects are the closed simplices Δ of X, cell decomposed as in $\hat{X}$, and whose morphisms are the inclusions. Then the classifying space of C is itself a cell complex K. The cells of K are parameterized by the data

$$\sigma \subset \Delta_0 \subset \Delta_1 \subset \cdots \subset \Delta_k$$

The dimension of this cell is $\dim \sigma + k$. Its boundary is all cells obtained by replacing σ by a cell in the boundary of σ, and by deleting some of the Δ_i. The cell complex K is isomorphic to a cell decomposition of X, which is our $\hat{X}'$.

The matroid $M'(\sigma')$ on S , where σ' is the cell just discussed, is the matroid obtained from $M(\sigma)$ on the set $S_{\Delta(\sigma)} bysettingallelementsof S_{\Delta} - S_{\delta_0}$ equal to zero and adding additional zero elements S-S_{Δ}.

3.3 The Matroid Grassmannian

The Matroid Grassmannian plays the same role for matroid bundles as the ordinary Grassmannian plays for vector bundles.

DEFINITION. Let S be a finite set and let n be a positive integer. The *Matroid Grassmannian* of rank n on the set S, denoted $M^n(S)$, is the order complex of the poset of rank n oriented matroids on the set S, ordered by the relation $\rightsquigarrow$ of specialization.

(Recall that the order complex of a poset $(P, \rightsquigarrow)$ is the simplicial complex whose vertices are elements of p of P, whose edges are pairs (p_0, p_1) such that $p_0 \rightsquigarrow p_1$, whose 2-simplices are triples (p_0, p_1, p_2) such that $p_0 \rightsquigarrow p_1 \rightsquigarrow p_2$, etc.) The Matroid Grassmannian $M^n(S)$ is actually homeomorphic to the ordinary Grassmannian $G^n(\boldsymbol{R}^S)$ if n is 1, 2, $|S| - 2$, or $|S| - 1$, where $|S|$ is the number of elements in S. Otherwise, the topology of the M Grassmannian is a mystery.

Proposition 3.1 *A rank n matroid bundle $(S, \hat{X}, M)$ over X determines a simplicial map $c : \hat{X}' \longrightarrow M^n(S)$, where $\hat{X}'$ is the barycentric subdivision of $\hat{X}$.*

This is clear from the definitions. The map c is called the *classifying map* for the matroid bundle.

3.4 Relations with ordinary Grassmannians.

Let S be a finite set. Denote by $\boldsymbol{R}^S$ the real vector space with a basis indexed by elements of S. Let $G^n(\boldsymbol{R}^S)$ be the Grassmannian of n dimensional quotients of $\boldsymbol{R}^S$.

Proposition 3.2 *There is a canonical (up to homotopy) map $\zeta : G^n(\boldsymbol{R}^S) \longrightarrow M^n(S)$ of the ordinary Grassmannian into the M Grassmannian.*

To prove this proposition, recall from §5.5.1 the decomposition $G^n(\boldsymbol{R}^S) = \bigcup Y_M$ of the Grassmannian into matroid strata, indexed by the set $\mathcal{O}$ of oriented matroids of rank n on the set S. Let T be a triangulation of $G^n(\boldsymbol{R}^S)$ such that for each simplex δ of T, the interior of δ is wholly contained in a single matroid stratum Y_M. Denote the matroid M by $\theta(\delta)$. Such a triangulation exists since the matroid strata are semi-analytically constructible. The proof now hinges on the following statement: There is a partition of unity $\left\{\phi_M : G^n(\boldsymbol{R}^S) \longrightarrow \boldsymbol{R}\right\}$, $1 = \sum \phi_M$ indexed by $\mathcal{O}$, with the property that $\phi_M(g) \neq 0$ only if $M(g) \rightsquigarrow M$. Such a partition of unity may be constructed this way. Take the usual barycentric partition of unity $1 = \sum \psi_\delta$ associated to the triangulation T (with the property that $\psi_\delta(g) \neq 0$ if and only if g is in the interior of Star δ). Then

$$\phi_M = \sum_{\theta(\delta)=M} \psi_\delta$$

This partition of unity has the property that $\phi_M(g) \neq 0$ only if $M(g) \rightsquigarrow M$ because the sets Y_M have the property that Y_M intersects the closure of $Y_{M'}$ if and only if $M' \rightsquigarrow M$.

Now, the map $\zeta : G^n(\boldsymbol{R}^S) \longrightarrow M^n(S)$ is defined by sending $g \in G^n(\boldsymbol{R}^S)$ to the point of $M^n(S)$ whose homogeneous coordinates are $\{\phi_M(g)\}$. The fact that the map is well defined up to homotopy is a consequence of the fact that the partitions of unity with the property that $\phi_M(g) \neq 0$ only if $M(g) \rightsquigarrow M$ form a convex set.

3.5 Characteristic classes and vector bundles.

Since the Grassmannian is the classifying space for vector bundles and the M Grassmannian is the classifying space for matroid bundles, the set of maps ζ of Proposition 3.2 give a map of the whole theory of vector bundles to the theory of matroid bundles. To make this precise, first we need to stabilize.

Any inclusion of S into a larger set S' induces a stabilization map of $M^n(S)$ into $M^n(S')$ (by sending a rank n matroid on S to the rank n matroid on S' determined by making elements of $S' - S$ be zero). Any two maps of S into S'

determine homotopic maps $M^n(S) \longrightarrow M^n(S')$. Therefore, it makes sense to take the limit of the cohomology groups of Matroid Grassmannians

$$\lim_{\leftarrow} H^k(M^n(S))$$

taken as the number of elements of S increases. An element of this ring is called a *characteristic class* for matroid bundles. Such an element gives a cohomology class in X for every matroid bundle B over X. Similarly, there are stabilization maps $G^n(\boldsymbol{R}^S)$ to $G^n(\boldsymbol{R}^{S'})$. The limit $\lim_{\leftarrow} H^k(G^n(\boldsymbol{R}^S))$ is well defined, and elements of this ring are standardly called characteristic classes for vector bundles.

The set of maps ζ of Proposition 3.2 commutes with the stabilization maps, and hence induces a map $\bar{\zeta}$ from the ring of characteristic classes for matroid bundles to the ring of characteristic classes for vector bundles. Eric Babson has announced a proof of the following proposition:

Proposition 3.3 (Babson, [Ba]) *With rational coefficients, map $\bar{\zeta}$ (from the ring of characteristic classes for matroid bundles to the ring of characteristic classes for vector bundles) is a surjection.*

The idea of the proof is the following. With rational coefficients, the ring of characteristic classes of vector bundles is generated by the Pontrjagin classes. In [GM2], a combinatorial formula for the Pontrjagin class of a matroid bundle was given, under the assumption that the associated Grassmannian bundle of 2 planes has a "fixing cycle". Babson has shown that every combinatorial vector bundle has an essentially unique fixing cycle.

Call two matroid bundles on X *equivalent* if their classifying maps m are homotopic (after stabilization). Any vector bundle E on X gives rise to an equivalence class B of matroid bundles on X (by composing the classifying map for E with the map ζ). The Pontrjagin classes of E coincide with the Pontrjagin classes for B.

4 Questions.

1. Does the map $\bar{\zeta}$ from the ring of rational characteristic classes for matroid bundles to the ring of rational characteristic classes for vector bundles have a kernel? In other words, are there stable rational characteristic classes for matroid bundles that are not Pontrjagin classes?

2. What is a good notion of a refinement of a combinatorial differential manifold? Taking the limit, one would like a category that bears the same relationship to combinatorial differential manifolds that the category of piecewise linear manifolds bears to combinatorial manifolds.

3. If X is a combinatorial differential manifold, is its underlying simplicial complex a topological manifold? (See [SZ] for related questions on oriented matroids.)

4. Is the sphere bundle of a combinatorial vector bundle a topological fiber bundle?

5. What does the theory of transversality look like for combinatorial differential manifolds? (Transversality for oriented matroids makes sense.) How about cobordism? Surgery? ...

5 Appendix: Oriented Matroids

For a detailed treatment of the theory and significance of oriented matroids, see [BLSWZ]. We give here a minimal sketch on the material directly relevant to combinatorial differential manifolds.

5.1 The Definition of an Oriented Matroid.

An oriented matroid is defined to be a finite set S together with a finite collection of functions from S to the three element set $\{-,0,+\}$ satisfying certain axioms. Before giving the axioms, we give several definitions about functions $c : S \longrightarrow \{-,0,+\}$, which are all obvious except the last one. We say that $c(v) \geq 0$ if $c(v)$ is 0 or 1. Likewise, $c(v) \leq 0$ if $c(v)$ is 0 or $-$. There is an operator $-$ from $\{-,0,+\}$ to itself defined by $-(-) = +$, $-(+) + 0$, and $-(0) = 0$. For any function $c : S \longrightarrow \{-,0,+\}$, the function $-c$ is defined by $(-c)(v) = -c(v)$. If c and d are two functions from S to $\{-,0,+\}$, then the function $c \circ d$ is defined by $c \circ d\ (v) = c(v)$ if $c(v) \neq 0$ and $c \circ d\ (v) = d(v)$ otherwise.

DEFINITION. An oriented matroid M is a finite set S called the set of *elements* or of M, together with a collection of functions $c : S \longrightarrow \{-,0,+\}$ called the *covectors* of M, subject to the following system of axioms: 1. The constant function with value 0 is a covector.

2. If c is a covector, then $-c$ is a covector.

3. If c and d are covectors, then $c \circ d$ is a covector.

4. If c and d are covectors and $v \in S$ is an element such that $c(v) = +$ and $d(v) = -$, then there exists a covector e with the following properties: i) $e(v) = 0$. ii) If $c(w) = d(w) = 0$, then $e(w) = 0$. iii) If $c(w) \geq 0$ and $d(w) \geq 0$ but $c(w)$ and $d(w)$ are not both zero, then $e(w) = +$. iv) If $c(w) \leq 0$ and $d(w) \leq 0$ but $c(w)$ and $d(w)$ are not both zero, then $e(w) = -$.

5.2 Representations of oriented matroids

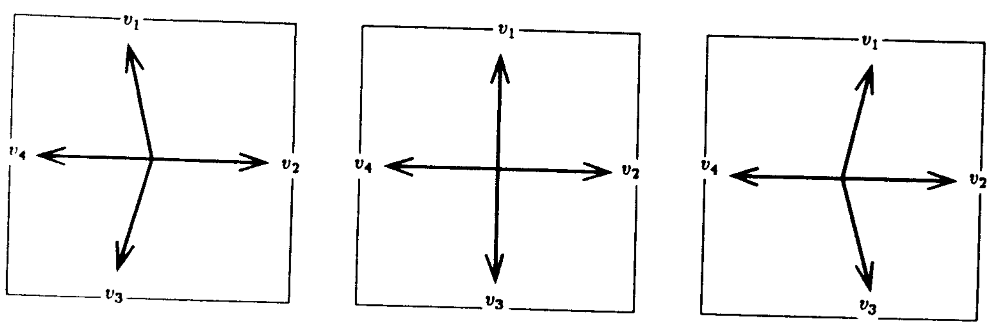
The motivating example of an oriented matroid arises from vector configurations.

DEFINITION. Let S be an n dimensional real vector space and let S be any finite set. A ***vector configuration*** is a map $\rho : S \longrightarrow V$ such that the image of ρ spans V.

DEFINITION. The oriented matroid ***represented by*** ρ is the matroid whose covectors are those functions $c : S \longrightarrow \{-, 0, +\}$ obtained as follows: Take any linear function $f : S \longrightarrow \boldsymbol{R}$. Then $c(v)$ is $-$, 0, or $+$ depending on whether $f(\rho(v))$ is negative, zero, or positive.

If an oriented matroid M arises in this way from a representation of a configuration of vectors, it is called ***representable***. Non representable oriented matroids exist. The problem of finding a purely combinatorial characterization of representable matroids is unsolved.

As an example, consider the three matroids represented as follows, where V is 2- dimensional and S has 4 elements, $v_1, \ldots, v_4$.



These three examples represent different oriented matroids. The second one is distinguished by the fact that any covector which is 0 on v_1 is 0 on v_3. The first (resp. third) are distinguished from each other by the fact that any covector which is $+$ on both v_1 and v_3 is $-$ (resp. $+$) on v_2.

For $v \in S$, if $\rho(v)$ is replaced by $\lambda\rho(v)$ for a real number $\lambda > 0$, then the oriented matroid represented by ρ is also not changed. Therefore, we may equally well imagine the oriented matroid M to be represented by a function $\hat{\rho}$ which takes S to the set of rays in V. (A ***ray*** in V is a set $\{\lambda v | \lambda > 0\}$ for $v \in S$. As a degenerate example, the set consisting only of the zero vector is a ray.)

5.3 Constructions on a single oriented matroid.

An oriented matroid is a combinatorial analogue of (a finite collection of vectors in) a real vector space. Many notions linear algebra have analogous constructions on oriented matroids. In order to define the analogues, it suffices to express the linear algebra constructions in terms of the set of linear functions on the vector space. We will now carry out this procedure for several cases. Let us fix a matroid M whose set of elements is S.

An element $v \in S$ is ***nonzero*** if there is some covector c for which $c(v) \neq 0$. Otherwise, v is said to be ***zero***. There may be many zero elements. A subset $\{v_1, v_2, \ldots, v_j\}$ is said to be ***independent*** if there exists a set of covectors $\{c_1, c_2, \ldots, c_j\}$ such that $c_i(v_k) \neq 0$ if and only if $i = k$. The ***rank*** of a subset W of S is the cardinality of any (and hence every) maximal independent subset of W. The ***rank*** of M is the rank of the set S. Any subset W of S becomes a matroid M' by taking as its covectors all restrictions of covectors of M. We write $M' \subset M$ and say that M' is a ***submatroid*** of M.

For $W \subset S$, the ***linear span*** of W is the set of $v \in S$ such that if $c(w) = 0$ for all $w \in W$, then $c(v) = 0$. A ***flat*** of M is a subset of S that is its own linear span. If $F \subset S$ is a flat, then the ***quotient matroid*** is the matroid $\bar{M}$ whose set of elements is S, and whose covectors are those covectors of M which are zero on F. We have $\text{rank}\bar{M} + \text{rank}F = \text{rank}M$.

For $W \subset S$, the ***convex hull*** $\bar{W}$ of W is the set of $v \in S$ such that if $c(w) \geq 0$ for all $w \in W$, then $c(v) \geq 0$. (If M is a represented matroid, the convex hull of W is all elements of S represented by points in the smallest closed convex union of rays in V containing $\rho(W)$.

5.4 Specialization.

DEFINITION. Suppose that M and M' have the same set of elements S and the same rank. Then we say that M' is a ***specialization*** of M, symbolized $M \rightsquigarrow M'$, if for every covector c' of M' there is a covector c of M such that if $c'(v) \neq 0$ then $c'(v) = c(v)$. (In other words, c' is obtained from c by the process of setting some nonzero values to zero.)

For example, the second oriented matroid whose representation is pictured above is a specialization of both the first one and the third one.

The set of oriented matroids of rank r on a given set S forms a partially ordered set under the relation $\rightsquigarrow$. The relation of specialization has the following interpretation: If the matroids M and M' are both representable, and a representation of M is the limit of a sequence of representations of M', then $M \rightsquigarrow M'$. In the next section, we will develop a topology on the set of representations to make this statement precise.

5.5 The Grassmannian viewed as a space of vector configurations.

5.5.1 Equivalence classes of vector configurations.

Let V and V' be two n dimensional real vector space and let S be a finite set. Two vector configuration $\rho : S \longrightarrow V$ and $\rho' : S \longrightarrow V'$ are said to be *equivalent* if there is a linear isomorphism $\nu : V \longrightarrow V'$ such that $\rho' = \nu \circ \rho$. It is clear that if ρ is equivalent to ρ', then the matroids that they represent are isomorphic: $M(\rho) = M(\rho')$.

Denote by $\boldsymbol{R}^S$ the real vector space with a basis indexed by elements of S. Let $G^n(\boldsymbol{R}^S)$ be the Grassmannian of n dimensional quotients of $\boldsymbol{R}^S$. In other words, a point in $G^n(\boldsymbol{R}^S)$ is a subspace of $\boldsymbol{R}^S$ of dimension $s - n$, where s is the number of elements of S.

Proposition 5.1 *There is a canonical bijection between the set of equivalence classes of n dimensional vector configurations on the set S and the set of points in $G^n(\boldsymbol{R}^S)$.*

In other words, the Grassmannian is the moduli space for vector configurations. This proposition, which appeared in [M] and [GM1], was one of the points of departure for the ideas presented here. The proof is clear once the maps in the two directions are given:

- If ξ is a plane representing a point in the Grassmannian $G^n(\boldsymbol{R}^S)$, let V be the quotient n=space $\boldsymbol{R}^S/\xi$, and let $\rho(s)$ be the image of s in $\boldsymbol{R}^S/\xi$, for $s \in S$. Then, the ξ is mapped to the equivalence class containing ρ.
- If $\rho : S \longrightarrow V$ is a vector configuration, let $\hat{\rho} : \boldsymbol{R}^S \longrightarrow V$ be the linear transformation which sends the basis vector corresponding to s to $\rho(s)$, and let $\xi \subset \boldsymbol{R}^S$ be the kernel of ρ. Then the equivalence class containing ρ is mapped to ξ.

DEFINITION. [GGMS] Let M be a rank n oriented matroid whose set of elements is S. Then the *matroid stratum* Y_M for M is set of points in the Grassmannian $G^n(\boldsymbol{R}^S)$ which correspond to vector configurations representing M.

The matroid strata give a decomposition of $G^n(\boldsymbol{R}^S) = \bigcup Y_M$ into a disjoint collection of semi-analytically constructible subsets. The matroid strata themselves can have almost arbitrary topological complexity [Mn]

Proposition 5.2 *Suppose that the matroids M and M' are both representable, and that the closure of Y_M has nonempty intersection with $Y_{M'}$. Then M' is a specialization of M.*

The converse is false [W].

5.5.2 Equivalence classes of flattenings.

DEFINITION. Let X be a simplicial complex, and let Δ be a simplex of X. Two flattenings of X at Δ, $f : \text{Star}\,\Delta \longrightarrow U \subset V$ and $f' : \text{Star}\,\Delta \longrightarrow U' \subset V'$ are said to be equivalent if there is a linear isomorphism $\nu : V \longrightarrow V'$ such that $f' = \nu \circ f$.

Recall that to a flattening of X at Δ determines a vector configuration on the set S_Δ, by restricting the flattening to the vertices. Equivalent flattenings give rise to equivalent vector configurations.

Proposition 5.3 *The set F_Δ of equivalence classes of flattenings of X at Δ is an open subset of the Grassmannian $G^n(\boldsymbol{R}^S)$ which is a union of Grassmannian strata.*

This makes the set of flattenings equivalence classes of into an analytic space, which is decomposed into semi–analytically constructible subsets by the matroid strata. The space of flattenings was considered in [GGL].

Bibliography

[B] Babson, E., Thesis, MIT, 1992.

[BLSWZ] A. Björner, M. Las Vergnas, B. Sturmfels, N. White, and G. Ziegler, **Oriented matroids**, Encyclopedia of Math. and its Appl., Cambridge Univ. Press (1992).

[GGL] I. Gabrielov, I. Gelfand, and M. Losik, Combinatorial calculation of characteristic classes, Funktsional Anal. i Prilozheniya 9 (1975), no.1, 54-55; no. 2, 12-28; no. 3, 5-26.

[GGMS] Gelfand, I.M., Goresky, M., MacPherson, R., and Serganova, V., Combinatorial geometries, convex polyhedra, and Schubert cells, Advances in Math. 63 (1987) 301-316.

[GM1] Gelfand, I.M. and MacPherson, R., Geometry of the Grassmannian and a generalization of the dilogarithm, Advances in Math., 44 (1982) 279-312.

[GM2] Gelfand, I.M. and MacPherson, R., A combinatorial formula for the Pontrjagin classes, Bull. Amer. Math. Soc. (1992).

[M] R. MacPherson, The combinatorial formula of Gabrielov, Gelfand, and Losik for the first Pontrjagin class, Séminaire Bourbaki No. 497, Springer Lect. Notes Math. 667 (1977).

[Mn] Mnev, M.N., Universality theorems on the classification of configuration varieties and convex polytopes varieties, Springer Lect. Notes in Math. 1346, 527-44. [W] White, N., A non–uniform oriented matroid which violates the isotopy property, Discrete Comput. Geom., 4 (1989) 1-2.

[SZ] Sturmfels, B., and Ziegler, G., Extension spaces of oriented matroids, preprint, 1991.