NEITHER THE GREEDY NOR THE DELAUNAY TRIANGULATION OF A PLANAR POINT SET APPROXIMATES THE OPTIMAL TRIANGULATION

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The classic Greedy Triangulation (GT) of a planar point set S with cardinality n is a triangulation T in which each of the $n(n-1)/2$ undirected edges is considered in sequence from smallest to largest (the case of equal-length edges complicates but does not alter the arguments presented here). Each edge is inserted into the plane provided no edge already in the place intersects it. Assuming no pair of edges has the same length, the GT is clearly unique.

Let EL(T) denote the sum of edge lengths of some triangulation T. Let OT denote the *optimum triangulation,* defined as the triangulation which minimizes EL(T) over all triangulations. The question addressed in this paper is whether the GT is an approximation to the OT; that is, whether there exists some constant c such that R(S) is less than c for all S, where $R(S) \equiv EL(GT(S))/EL(OT(S))$. We show that the answer is no, and exhibit a class of point sets S_0 such that $R(S_0) = \Omega(n^{1/3})$. [1] This result raises the question of whether the dual of the Voronoi diagram [8], known [6] as the Delaunay Triangulation (DT) is approximately optimal. (It is known from [3] that neither the GT nor the DT are optimal.) It turns out to be quite easy to show that the DT is not approximately optimal; we defer to the end of the paper a demonstration that $EL(DT(S_1))/EL(OT(S_1)) = \Omega(n/\log n)$ for a class of point sets S_1. In [3], the distantly related problem of whether a given set of edges contains a triangulation subset is shown to be NP-complete; this result unfortunately sheds no direct light on these matters.

We now provide a construction for which we may readily construct and characterize the GT for a set S_0, and we show that there exists a better triangulation of S_0, $BT(S_0)$, for which

$$EL(GT(S_0))/EL(BT(S_0)) = \theta(n^{1/3}) . \qquad (1)$$

We first define a set S to be *chordal* if it resides entirely on an arc of a circle, such that the arc subtends less than 180°. In fact we shall choose arcs that are almost straight lines (Fig. 1). The real number d is the diameter of the set.

We shall ignore the y position of chordal points in specifying their position, since this is second order for sufficiently thin chords.

Lemma 1. Let S_1 be a chordal point set. Then $EL(T(S_1)) = O(dn)$, where T is any triangulation.

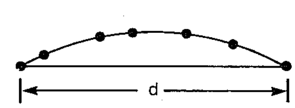

Fig. 1

[1] We use Knuth's O-notation [2]: 'O' means 'of the same or lesser order'; 'θ' means 'exactly of order', and 'Ω' means 'of the same or greater order'.

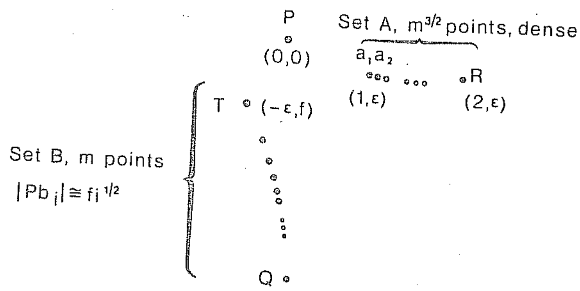

Fig. 2.

Proof. Trivial. There are O(n) edges, all of which must have length equal to or less than d.

The set S_0 is shown in Fig. 2. Its convex hull is TPRQ. It consists of three parts:

(1) an isolated point P at (0, 0);

(2) a chordal set A, which is slightly concave with respect to the convex hull, consisting of about $\lfloor m^{3/2} \rfloor$ points located between $(1, \epsilon)$ and $(2, \epsilon)$. The points in A are denoted $a_1, a_2, ...$, reading left to right. Point P is located above the line defined by a_1a_2. Finally,

(3) there is a chordal subset B of m points, also slightly concave, extending almost vertically as shown in Fig. 2. All $b \in B$ are 'visible' from P in the sense that the edge joining each $b \in B$ to P does not pass through the arc on which the points in B reside. Point b_1 is also called T and point $b_{|B|}$ is also called Q. The vertical position of each point b_i is given by

$$y(b_i) = fi^{1/2} \tag{2}$$

where f is a constant smaller than 1.

Lemma 2. $|b_{i+1}P| < |b_ia_1|$ for all i.[2]

Proof. Follows from the definition of set B and (2);

$$|b_{i+1}P| \approx f(i+1)^{1/2} \quad \text{and} \quad |b_ia_1| \approx f(i+1/f^2)^{1/2} .$$

Now consider any triangulation of S_0. There will be five kinds of edges:

Class 1. Edges joining two points in A.
Class 2. Edges joining two points in B.
Class 3. Edges joining P to some $a \in A$.
Class 4. Edges joining P to some $b \in B$.
Class 5. Edges joining some $a \in A$ to some $b \in B$.

The edges in Class 1 are confined to the small chordal region bounded by A, which is a convex s
A similar statement holds for Class 2.

Now consider an algorithm producing the GT; this a 'greedy algorithm' (GA).

Fact 1. Since for any triple of points a′, a″, b, wit a′, a″ ∈ A and b ∈ B, it is the case that |a′a″| < |a′ and |a′a″| < |a′b|, it follows that a GA will comple triangulate A.

Fact 2. Since for any triple b, b′, b″ ∈ B and point p ∈ {P} ∪ A, edge pb does not cross edge b′b″, it follows that any triangulation must completely triangulate B.

Fact 3. Edges TP, PR, RQ, and QT must be in any triangulation, since they define the convex hull of S

Fact 4. For any $b \in B$, $|ba_1| < |ba_j|$ for all $j > 1$.

Facts 1–4 are almost self-evident and require no proof.

We use Fact 1–3 to expose the role of Classes 1 and 2; predictably a GA will fully triangulate A and

By Lemma 2, $|Pb_2| < |Ta_1|$. Therefore when the GA is considering Pb_2, Ta_1 cannot yet be in the plan Nor, by Fact 4, can Ta_k be in the plane for $k > 1$. Therefore $Pb_2 \in GT(S_0)$. When Pb_3 is considered by the GA, b_2a cannot be in the plane for any $a \in A$ because of Lemma 2 and Fact 4, and b_1a cannot be in the plane because its 'view' is blocked to all $a \in A$ by Pb_2. Applying this reasoning inductively, we find that

Lemma 3. At the time Pb_i is considered by the GA, Pb_{i-1} will be in the plane, $b_{i-1}a_1$ will not be in the plane, and therefore Pb_i will be placed in the plane.

Proof. Foregoing reasoning.

Theorem 1. The GT for S_0 consists of

(1) the GT of A;

(2) the GT of B;

(3) edges Pb_i, for all $1 \leqslant i \leqslant |B|$;

(4) the edges of the convex hull PRQT (edges PR,

[2] If S is a set, |S| denotes the cardinality of the set; if e is an edge, |e| denotes the length of the edge.

RQ, QT and TP);
(5) edges Qa_j, for all $1 \leqslant i \leqslant |A|$;
(6) edge Pa_1.

Proof. By Facts 1–3 and Lemma 3, together with the observation that these leave only the edges Qa_j and Pa_1, none of which cross one another, so that they must all be in the plane.

Lemma 4. $EL(GT(S_0)) = \theta(m^2)$.

Proof. By Lemma 1, $EL(GT(A)) = O(m^{3/2})$ and $EL(GT(B)) = O(m^{3/2})$. The sum of the lengths of edges $Pb_i = \theta(m^{3/2})$, and the length of the convex hull is $\theta(m^{1/2})$. Finally, the sum of the lengths of edges Qa_j is $\theta(m^{1/2})$ times $\theta(m^{3/2}) = \theta(m^2)$.

Lemma 5. There exists a better triangulation BT of S_0 such that

$$EL(BT(S_0)) = \theta(m^{3/2}).$$

Proof. Let the triangulation BT contain
(1) the GT of A;
(2) the GT of B;
(3) Ta_i, for all $1 \leqslant i \leqslant |A|$;
(4) the edges of the convex hull PRQT;
(5) Rb_i, for all $1 \leqslant i \leqslant |B|$;
(6) edge Pa_1.
The new classes are 3 and 5; their edge-length sums may be shown easily to be respectively $\theta(m^{3/2})$ and $\theta(m^{3/2})$. This proves the lemma.

Theorem 2. $R(S_0) = \Omega(n^{1/3})$.

Proof. $EL(GT(S_0))/EL(BT(S_0)) = \theta(m^{1/2})$; since $m = \theta(n^{2/3})$, and BT upper bounds the OT, the theorem follows.

We now show that the DT is not approximately optimal. The construction consists of a class of sets S_1 consisting of $n = 2^k + 1$ points with 2^k forming a regular polygon of diameter d and the remaining point c slightly displaced from the center of the polygon [3]. The Voronoi diagram is shown (solid lines) in

[3] The displacement of the center point is introduced in order to avoid the technical nuisance of three collinear points.

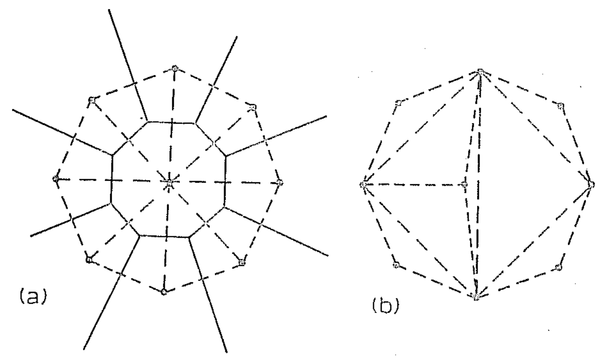

Fig. 3.

Fig. 3(a); it consists of a 2^k-gon about c and 2^k open polygonal regions extending from it. Its dual is the convex hull, together with edges extending from each vertex of the convex hull to c. Hence for each such S_1, $EL(DT(S_1)) = \theta(dn)$.

A better triangulation $BT(S_1)$ is easily constructible (Fig. 3(b)). The convex hull is first constructed, then a second 'shell' consisting of 2^{k-1} edges linking next-nearest neighbors of the convex hull, then a third shell consisting of 2^{k-2} edges linking fourth-nearest neighbors, etc. Finally, three edges connecting c to the triangle in which it resides are constructed. Since each shell contains edges whose length totals $\theta(d)$, and there are $\log_2 n$ shells, we have at once $EL(BT(S_1)) = \theta(d \log n)$. Consequently, we have

Theorem 3. If $R'(S_1) = EL(DT(S_1))/EL(OT(S_1))$, then

$$R'(S_1) = \Omega(n/\log n) \,. \qquad (3)$$

Proof. Follows from the above constructions.

As a concluding remark, we note that no efficient approximate algorithm for the OT now exists, and the existence of one is an open question. However, we note also that if finding an approximation to the OT is comparable to finding an approximation to the Travelling Salesman Problem (TSP) [7], then there is no reason to suppose that the GT would have been optimal, since strictly speaking no greedy algorithm is known for approximating the TSP. On the other hand, there are simple algorithms [5,7] for efficiently finding solutions to the planar TSP good to within a factor of 2. One of the simplest is basically an annexation technique involving 'closest insertion', in which a solution is built up from a subset of the points. The solution is then augmented by annexing the point external to it that lies closest to some point in it. The process terminates when the last point has been annexed. The other known approximation techniques [1,7] are based on the minimum spanning tree.

We conjecture that any efficient (i.e., polynomial time) algorithm for finding an approximation to the OT must be at least as complicated as any of the known approximation algorithms for the TSP.

Reference [4] contains a condensed version of this paper.

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