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Triangulations for the Cube

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1. INTRODUCTION

In this note we consider the problem of determining a minimal triangulation of I^n, the n-dimensional cube. While the problem seems intrinsically interesting, our purpose in presenting it is motivated by the interest evinced in connection with the simplicial approximation of fixed points of continuous mappings [5, 7]. Several algorithms for locating simplices which approximate fixed points have recently been given [1, 2, 3, 6]. It is expected that by minimizing the number of simplices which fill a cube, the number of pivoting steps in the implementation of a fixed-point algorithm will generally be nearly minimal and that the resulting algorithm will generally perform with optimal efficiency. We consider here only triangulations with vertices of simplices coincident with vertices of the cube.

We indicate techniques yielding triangulations of I^3, I^4, I^5, consisting of 5, 16, 68 simplices of the respective dimensions. We show that 5 is the minimum number of simplices for a triangulation of I^3 and that 16 is the minimum number for I^4 subject to an additional hypothesis. We also give motivation for the conjecture that I^n has a triangulation having $(n! + 2^{n-1})/2$ simplices of dimension n.

2. NOTATION

To facilitate our treatment we will hereafter use the following notation. Each vertex of I^n will be associated with the number for which it is the binary representation. Thus in I^2,

$$(0, 0) \leftrightarrow 0,$$
$$(0, 1) \leftrightarrow 1,$$
$$(1, 0) \leftrightarrow 2,$$
$$(1, 1) \leftrightarrow 3.$$

We associate with each simplex in a triangulation of I^n, the $(n + 1) \times (n)$ matrix whose rows are the coordinates of the vertices. We will call such a matrix the *coordinate matrix* of a simplex. We denote the convex hull of the points $p_1, \ldots, p_n$ by $[p_1, \ldots, p_n]$.

Clearly, there is a minimum triangulation of I^2 containing the triangles $[0, 1, 3]$ and $[0, 2, 3]$ with coordinate matrices (see Fig. 1)

$$\begin{array}{c|cc|} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 3 & 1 & 1 \end{array} \quad \text{and} \quad \begin{array}{c|cc|} 0 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 1 & 1 \end{array}.$$

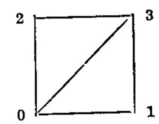

FIG. 1. Minimum triangulation of I^2.

Notice that these two triangles are $\{(x_1, x_2): x_1 \leqslant x_2\}$ and $\{(x_1, x_2): x_2 \leqslant x_1\}$ intersected with I^2. In general we can always construct a triangulation of I^n containing $n!$ simplices by intersecting it with each set of the form $\{(x_1, x_2, \ldots, x_n): x_{\pi(1)} \leqslant \cdots \leqslant x_{\pi(n)}\}$, where π runs over all permutations in the full symmetric group on n elements. We will call this triangulation the *standard triangulation*.

It can be shown that a set of simplices triangulates I^n if and only if

(1) the $(n - 1)$-faces lying on the interior of I^n belong to exactly two simplices, and

(2) the $(n - 1)$-faces lying on the exterior of I^n triangulate each of the $(n - 1)$-dimensional faces of I^n.

With the use of coordinate matrices we can restate properties (1) and (2) as

(i) If a row of a coordinate matrix is deleted and the resulting $n \times n$ matrix has no column of all zeros or all ones, then this $n \times n$ submatrix is shared with exactly one other coordinate matrix.

(ii) For each $i = 1, 2, \ldots, n$ and $e = 0, 1$, the set of all $n \times n$ submatrices obtained from the set of coordinate matrices by deleting a row that yields a submatrix with all e's in the ith column forms a triangulation of the $(n - 1)$-dimensional cube

$$I^n \cap \{(x_1, x_2, \ldots, x_n): x_i = e\}.$$

The standard triangulation of I^3 yields the following six 3-simplices (see Fig. 2).

$$[0, 1, 3, 7], \quad [0, 4, 5, 7],$$
$$[0, 2, 3, 7], \quad [0, 2, 6, 7],$$
$$[0, 1, 5, 7], \quad [0, 4, 6, 7].$$

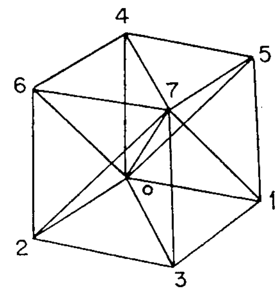

FIG. 2. Standard triangulation of I^3.

We may construct a triangulation containing only five simplices, however, by first "slicing off" four corners and then observing that what is left is a simplex. The following simplices are thus formed (see Fig. 3).

$$[0, 1, 2, 4], \quad [1, 2, 3, 7],$$
$$[2, 4, 6, 7], \quad [1, 2, 4, 7].$$
$$[1, 4, 5, 7],$$

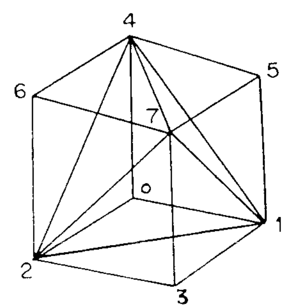

FIG. 3. Minimal triangulation of I^3.

3. DIMENSION 3

We now prove the minimality of five simplices in a triangulation of I^3 and attempt to extend this result to dimension 4.

LEMMA 1. *For $n \geqslant 2$, a simplex in a triangulation of I^n has at most n exterior $(n-1)$-faces.*

LEMMA 2. *For $n \geqslant 3$, if a simplex in a triangulation of I^n has n exterior $(n-1)$-faces, then any simplex sharing its interior $(n-1)$-face has fewer than n exterior $(n-1)$-faces.*

Proof. For a simplex to have n exterior $(n-1)$-faces, each column in the coordinate matrix must have exactly one element different from the rest. The elements that are different in each column all appear in different rows, for if two appeared in the same row, deleting this row would yield an $(n-1)$-face of the simplex lying in two different $(n-1)$-dimensional hyperplanes, which is impossible. Without loss of generality we can assume that matrix is of the form

$$\begin{vmatrix} 0 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{vmatrix}.$$

We obtain the interior $(n-1)$-face by deleting the first row. Let us now build a simplex on this interior $(n-1)$-face by adding a row. For the row added to be different from the row deleted, it must have a 1 in some column. The simplex thus obtained cannot have n exterior $(n-1)$-faces, for at least one column contains two 1's ∎

THEOREM 1. *If P_n denotes the number of simplices in the minimum triangulation of I^n, E_n denotes the total number of exterior $(n-1)$-faces, and F_n denotes the number of interior $(n-1)$-faces, then*

(a) $P_n(n+1) = E_n + 2F_n$,

(b) $E_n \geqslant 2nP_{n-1}$,

(c) $P_n \geqslant 2P_{n-1}$.

Proof. (a) Every simplex has $n+1$ $(n-1)$-faces. Every interior $(n-1)$-face belongs to exactly two simplices, while every exterior $(n-1)$-face belongs to only one; hence, $P_n(n+1) = E_n + 2F_n$.

(b) Since the set of exterior $(n-1)$-faces must triangulate each of the $2n$ $(n-1)$-dimensional faces of I^n and it takes at least $P_{n-1}(n-1)$-simplices to triangulate each of these, $E_n \geqslant 2nP_{n-1}$.

(c) Since there are at least $2nP_{n-1}$ exterior $(n-1)$-faces and by Lemma 1 a given simplex can contain at most n of these, $P_n \geqslant (2nP_{n-1}/n) = 2P_{n-1}$.

THEOREM 2. *The minimum number of simplices in a triangulation of I^3 is five.*

Proof. By Theorem 1(c), $P_3 \geqslant 4$. Since by Theorem 1(b) the number of exterior faces is at least 12, and by Lemma 1, a simplex can contain at most three of these, if a triangulation has exactly four simplices, then each would have exactly three exterior faces. However, by Lemma 2 these four simplices cannot share their interior faces. Hence, there must be at least one other simplex to share these interior faces.

4. DIMENSION 4

Theorem 1(c) says the minimal triangulation of I^4 has at least 10 simplices. This is a rough lower bound, however, that can be improved through the use of the following lemmas and an additional assumption.

LEMMA 3. *Every simplex with n exterior $(n-1)$-faces in a triangulation of I^n contains n edges of I^n.*

This is easily seen by observing the coordinate matrix of a simplex with n exterior $(n-1)$-faces and remembering that two vertices in I^n are connected if and only if they differ in only one coordinate.

LEMMA 4. *In a given triangulation of I^n, no two simplices having n exterior $(n-1)$-faces contain the same edge of I^n.*

Proof by induction on n. This is obvious for $n = 2$. Now assume that lemma true for $n = k - 1$, and without loss of generality assume that two k-simplices each had k exterior $(k-1)$-faces but shared the edge connecting $(0, 0, \ldots, 0)$ to $(1, 0, \ldots, 0)$. Their coordinate matrices would be

$$\left|\begin{array}{ccccc} 0 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{array}\right| \quad \text{and} \quad \left|\begin{array}{ccccc} 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & 0 & 0 & \cdots & 1 \end{array}\right| .$$

Now delete the last row in both matrices. The $(k-1)$-simplices remaining are two simplices in the triangulation of one of the $(k-1)$-dimensional exterior cubes of I^k. Each contains $k-1$ exterior $(k-2)$-faces and they share an edge of the cube, which under the induction hypothesis is the needed contradiction. ∎

LEMMA 5. *In a given triangulation of I^n, at most 2^{n-1} simplices contain n exterior $(n-1)$-faces.*

Proof. This is a direct consequence of Lemmas 3 and 4.

LEMMA 6. *If a triangulation of I^n contains 2^{n-1} simplices with n exterior $(n-1)$-faces, then any other simplex contains at most $n-3$ exterior $(n-1)$-faces.*

Proof. Assume that a triangulation contains 2^{n-1} simplices with n exterior $(n-1)$-faces and that simplex s_1 contains at least $n-2$ exterior $(n-1)$-faces. Without loss of generality we can assume the coordinate matrix of s_1 to be

$$\begin{array}{c|cccccc|} x_0 & 1 & 0 & \cdots & 0 & a_{1,n-1} & a_{1,n} \\ x_1 & 0 & 1 & \cdots & 0 & a_{2,n-1} & a_{2,n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ x_{n-3} & 0 & 0 & \cdots & 1 & a_{n-3,n-1} & a_{n-3,n} \\ x_{n-2} & 0 & 0 & \cdots & 0 & a_{n-2,n-1} & a_{n-2,n} \\ x_{n-1} & 0 & 0 & \cdots & 0 & a_{n-1,n-1} & a_{n-1,n} \\ x_n & 0 & 0 & \cdots & 0 & a_{n,n-1} & a_{n,n} \end{array} .$$

Since vertices x_{n-2}, x_{n-1}, and x_n differ in only two coordinates, two of them differ from the third in only one. Again without loss of generality, assume that s_1 takes on the following form:

$$\begin{array}{c|cccccc|} x_0 & 1 & 0 & \cdots & 0 & a_{1,n-1} & a_{1,n} \\ x_1 & 0 & 1 & \cdots & 0 & a_{2,n-1} & a_{2,n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ x_{n-3} & 0 & 0 & \cdots & 1 & a_{n-2,n-1} & a_{n-2,n} \\ x_{n-2} & 0 & 0 & \cdots & 0 & 1 & 0 \\ x_{n-1} & 0 & 0 & \cdots & 0 & 0 & 1 \\ x_n & 0 & 0 & \cdots & 0 & 0 & 0 \end{array} .$$

Now since our triangulation contains a maximal set of simplices with n exterior $(n-1)$-faces, the simplex s_2 exists with the incidence matrix

$$\begin{array}{c|cccccc|} y_0 & 1 & 0 & \cdots & 0 & 0 & 0 \\ y_1 & 0 & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ y_{n-3} & 0 & 0 & \cdots & 1 & 0 & 0 \\ x_{n-2} & 0 & 0 & \cdots & 0 & 1 & 0 \\ x_{n-1} & 0 & 0 & \cdots & 0 & 0 & 1 \\ x_n & 0 & 0 & \cdots & 0 & 0 & 0 \end{array} .$$

Since $s_1 \neq s_2$, there exists an integer k such that $1 \leqslant k \leqslant n-3$ and $x_k \neq y_k$.

In s_1 delete all rows except for x_k, x_{n-2}, x_{n-1}, and x_n and in s_2 delete the corresponding rows. What remains is two distinct 3-simplices in the triangulation of one of the 3-dimensional exterior cubes of I^n that share the exterior 2-face with coordinate matrix

$$\begin{matrix} x_{n-2} \\ x_{n-1} \\ x_n \end{matrix} \begin{vmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{vmatrix}.$$

This contradicts the definition of a triangulation, so our proof is complete. ∎

We now use the above lemmas to obtain a lower bound for F_n, the number of interior $(n-1)$-faces in a minimum triangulation of I^n. Recall that any triangulation of I^n has at least $2nP_{n-1}$ exterior $(n-1)$-faces. By Lemma 1, at most n of these are contained in any simplex and by Lemma 5, at most 2^{n-1} simplices can contain this maximum number. So as soon as $2nP_{n-1} > n2^{n-1}$, which it is for $n = 4$, then there exist some simplices with fewer than n exterior $(n-1)$-faces.

Now let us assume we get the smallest number of interior $(n-1)$-faces by first constructing the maximum number (2^{n-1}) of simplices containing the maximum number (n) of exterior $(n-1)$-faces. By Lemma 6 we get

$$F_n \geqslant \frac{1}{2}\left[\frac{(2nP_{n-1} - n2^{n-1})\,4}{n-3} + 2^{n-1}\right].$$

Note. The $\frac{1}{2}$ comes from the fact that we may have counted each interior $(n-1)$-face twice, and the 4 comes from the fact that an n-simplex with $n-3$ exterior $(n-1)$-faces contains four interior $(n-1)$-faces.

Now from Theorem 1a, Theorem 1b, and the above lower bound on F_n, we have

$$P_n \geqslant \frac{2nP_{n-1} + [4(2nP_{n-1} - n2^{n-1})/(n-3)] + 2^{n-1}}{n+1}.$$

Hence, for $n = 4$, $P_4 \geqslant 16$.

Notice that the additional assumption which was made is equivalent to saying that the most efficient way to start a triangulation of I^n is to "slice off" 2^{n-1} corners as we did in I^3. So, in order to construct a new triangulation of I^4, we first "slice off" the sequences of vertices (0, 3, 5, 6, 9, 10, 12, 15) and form the eight simplices

[0, 1, 2, 4, 8], [4, 8, 12, 13, 14], [2, 8, 10, 11, 14], [2, 4, 6, 7, 14],
[1, 8, 9, 11, 13], [1, 4, 5, 7, 13], [1, 2, 3, 7, 11], [7, 11, 13, 14, 15].

We then triangulate what is left by passing three cutting planes through it. These eight simplices are thus constructed:

[1, 2, 4, 8, 14], [1, 4, 8, 13, 14], [1, 2, 8, 11, 14], [1, 2, 4, 7, 14],
[1, 8, 11, 13, 14], [1, 4, 7, 13, 14], [1, 2, 7, 11, 14], [1, 7, 11, 13, 14].

5. Conjecture

The author has constructed a triangulation of I^5 in a fashion completely analogous to the constructions in I^3 and I^4 [4]. This triangulation contains 68 simplices. Now what is the connection between the numbers 2, 5, 16, and 68 for dimensions 2, 3, 4, and 5? The answer is found when we compute the volumes of the simplices. Each of the 2^{n-1} corners that we first sliced off has volume $1/(n!)$, while the remaining simplices have volume $2/(n!)$. Hence, there is a total of $(n! + 2^{n-1})/2$ simplices in each triangulation. The conjecture is that a triangulation containing this many simplices can be constructed for any n and that this is the minimal triangulation.

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