An Output Size Sensitive Algorithm for the Enumeration of Regular Triangulations

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Abstract

This paper proposes an efficient algorithm for the enumeration of every regular triangulation obtained from a given point set $V = \{v_1, ..., v_n\} \subset \boldsymbol{R}^{d-1}$. The regular triangulations form a subclass of triangulations, and can be defined as a natural extension of Delaunay triangulations, which are investigated frequently in computational geometry. Also, this subclass has an interesting algebraic aspects in connection with the well-known paradigm of computational algebra, Gröbner bases. It must be mentioned that the enumeration of regular triangulations of specific configurations of points is an important topic in mathematics [St 91].

Our algorithm achieves a reduced space complexity $O(ds)$, where s is the maximum number of simplices contained in one regular triangulation, i.e., $O(n^{\lfloor d/2 \rfloor})$. The space complexity of the algorithm reported previously [BFS 90] was $\Theta(n^{(r-1)^2})$, although their algorithm completes the enumeration in $\Theta(n^{(r-1)^2})$ time, which is worst-case optimal [1]. Therefore, the space complexity of our algorithm can be taken as a drastic improvement. The time complexity of our algorithm is $O(r^2 s^2 l(s,r) T)$, where $l(s,r)$ denotes the time required for solving a linear programming problem consisting of s constraints with r variables, and T is the number of regular triangulations obtained from the given point set V, which is bounded from above by $O(n^{(r-1)^2})$. This complexity is obviously sensitive to the output size T.

1 Introduction

Triangulations have been one of main topics in computational geometry and other fields in recent years. Especially, some types of triangulations are found to bridge geometric issues and algebraic ones. Regular triangulations are of such a type. It is shown in [St 91] that this subclass of triangulation has a close connection with the well-known paradigm of computational algebra, Gröbner bases. Moreover, [ES 92] defines regular triangulations as a natural extension of Delaunay triangulations, whose remarkable features are often discussed in computational geometry. We proposes an efficient algorithm for enumerating all regular triangulations obtained from a given point set $V = \{v_1, ..., v_n\}$. The chief contribution of this paper is that our algorithm drastically improves the results of [BFS 90] in its space complexity. Throughout the paper, let the dimension of the space where every point is given equal to $d-1$. Then, the number of points spanning a simplex equals d. Regular triangulations form an important subclass of all triangulations with the points in V. In this section, we provide the reason why we concentrate on this subclass by making a list of its remarkable properties.

1.1 Properties of Regular Triangulations

Delaunay triangulations: [ES 92] defines the regular triangulations as a natural extension of the

[1] r equals $n-d$ throughout the paper.

Delaunay triangulations, which are investigated frequently in computational geometry. Given a point set $V \subset \boldsymbol{R}^{d-1}$, the Delaunay triangulation can be defined as a subdivision of conv$V$ (note: conv$V$ means the convex hull spanned by points in V) dual to the Voronoi diagram constructed from V. If we replace some points by balls with possibly negative radiuses, another diagram is constructed by taking into consideration these radiuses in determining the distance between each pair of points in case any of the two are replaced by balls. This diagram is called *power diagram.* Regular triangulations can be defined as subdivisions of conv$V$ dual to power diagrams. Since the power diagram with balls of radius 0 turns out to be a Voronoi diagram, the Delaunay triangulation is obviously a regular triangulation.

Algebraic aspects of regular triangulations: [St 91] reveals the algebraic aspects of regular triangulations. An ideal $\mathcal{I}_V$ in $\boldsymbol{C}[x_1, ..., x_n]$, where x_i corresponds to $v_i \in V$, called *affine toric ideal* can be constructed from the affine dependencies among the given points in V. Another ideal $\mathcal{I}_\Delta$ in $\boldsymbol{C}[x_1, ..., x_n]$, called *Stanley-Reisner ideal*, can be constructed from a triangulation Δ. Suppose Δ is a regular triangulation determined by giving the height w_i to each $v_i \in V$. If we regard the assignment $w := (w_1, ..., w_n)$ of heights as a weight vector determining a term order among monomials in $\boldsymbol{C}[x_1, ..., x_n]$ and calculate the Gröbner basis of $\mathcal{I}_V$ with respect to this term order, then the radical of the initial ideal of this Gröbner basis equals $\mathcal{I}_\Delta$. He also remarks the enumeration of all regular triangulations obtained from the product of simplices is an important problem. Details should be referred to his paper.

Lexicographic triangulations: [Lee 91] gives many intensive considerations about regular triangulations. He shows the lexicographic triangulations can be understood as a subclass of regular triangulations. Lexicographic triangulations are one of subclasses of triangulations admitting a definition in terms of oriented matroid. Later in this paper, we will see regular triangulations are defined by giving heights to all points in V and lifting them up by the assigned heights. Lexicographic triangulations are obtained by assigning heights in a special manner. This subclass also has its own algebraic aspect as shown in [St 91].

1.2 Previous Results

An algorithm for enumerating regular triangulations has already been reported in [BFS 90]. Their algorithm explicitly constructs the face lattice of the *secondary polytope*, the vertices of which are in one-to-one correspondence with regular triangulations of V. A slight consideration reveals that the enumeration not only of regular triangulations, but also of regular *subdivisions*, where each d-dimensional face is not necessarily a simplex, is possible on this face lattice with small amount of computation per subdivision. This is because each face of the secondary polytope corresponds to a regular subdivision. Their algorithm attains the worst-case optimal time complexity $\Theta(d^{(r-1)^2})$, since their analysis shows the number of regular triangulations of a set of n points in $\boldsymbol{R}^{d-1}$ is bounded from above by $O(d^{(r-1)^2})$ when $r \ll n$ is assumed.

However, this algorithm keeps the entire face lattice of the secondary polytope on the storage, and requires an enormous amount of working area. Their estimation of space complexity is regarded as $\Theta(n^{(r-1)^2})$, when $r \ll n$ is not assumed, since this value equals the complexity of an arrangement of hyperplanes containing the origin, each of which is spanned by $r-1$ linearly independent vectors chosen from a vector configuration $\bar{V} = \{\bar{v}_1, ..., \bar{v}_n\} \subset \boldsymbol{R}^r$, called *Gale transform*, computed from the coordinates of points in V. (The definition of Gale transforms will be given in §3.) Here arises a problem of reducing the space complexity of their algorithm.

1.3 Our results

The improvement of space complexity is achieved with the *reverse search* technique [AF 92] [Fuk 93], which is explained in §2. Our algorithm does not construct the secondary polytope explicitly. Hence, finding the incident vertex (i.e., finding the regular triangulation to be enumerated next) is never a trivial task. The reverse search technique enables us to compute the incident vertex based only on the local information, which is, in our case, the regular triangulation corresponding to the current vertex. Our algorithm runs in $O(r^2 s^2 l(s,r)T)$ time and in $O(ds)$ space, when the number of regular triangulations of a given point set $V \subset \boldsymbol{R}^{d-1}$ is T and the maximum number of simplices in one regular triangulation is s. Since each simplex in $\boldsymbol{R}^{d-1}$ is spanned by d vertices, this space complexity is minimum for storing one regular triangulation. The space complexity equals the area for storing only one regular triangulation, and drastically improves that of the previous result. The time complexity is obviously sensitive to the output size T.

1.4 Key Words and Notations

We will provide two definitions of *regular triangulations* in §3. *Gale transforms*, playing an important role in one of the two definitions and also in our enumeration algorithm, is defined in §3. In the same section, it is shown that a polyhedral complex in $\boldsymbol{R}^r$, called *secondary fan*, can be defined based on a Gale transform of V. §4 presents a result that the face lattice of the secondary fan is anti-iscmorphic to that of the secondary polytope mentioned above. Therefore, the discussions about the secondary fan can be rephrased in terms of secondary polytope. It will be shown that the cells, i.e., r-dimensional faces, of the secondary fan correspond to regular triangulations in one-to-one fashion in §3. Our algorithm taking advantage of this property of the secondary fan is proposed in §4. We concentrate on the enumeration of regular triangulations, and do not treat regular subdivisions. Hence, this algorithm enumerates the r-dimensional faces of the secondary fan.

For convenience sake, we introduce some notations. Let $\Lambda(n,k) = \{(i_1,...,i_k) \mid 1 \leq i_1 < \cdots < i_k \leq n\}$, i.e., the collection of every ordered subset of $\{1,...,n\}$ that is of cardinality k. The symbol * represents the operation of taking the complement with respect to $\{1,...,n\}$, e.g., for $\tau \in \Lambda(n,k)$, τ^* is equal to $\{1,...,n\}\backslash\tau$, and is an element of $\Lambda(n, n-k)$.

2 Reverse search

The enumeration algorithm proposed in this paper uses the technique called *reverse search* [AF 92] [Fuk 93]. This technique realizes an exhaustive search of all objects to be enumerated with small amount of storage when successfully applied. In our case, the space complexity is only for storing one regular triangulation, i.e., $O(ds)$, where s denotes the maximum number of simplices contained in one regular triangulation. The reverse search technique consists of two concepts: *adjacency* and *local search*.

Adjacency defined for pairs of objects to be enumerated gives a graph connecting all objects. This graph never be actually constructed, and adjacent objects are computed one by one in each case of necessity. Next, local search determines the object to be visited next everywhere on the graph, and provides a spanning tree of this graph. The root of this tree represents the globally optimal object. (If there are more than one global optima, local search provides a spanning *forest* of this graph. However, this case is ignored here, for our application gives only one global optimum.) This spanning tree is also not constructed explicitly, and the optimum among the adjacent objects is computed every time it is required. Reverse search technique regards this spanning tree as a search tree, and traverse it in the depth-first manner. One of the features making reverse search efficient and useful is that local searches can be executed based only on the local information available at each node of the tree. We show the framework of this technique below. *Adj*(Δ, j) and *Loc*(Δ) mean functions for giving Δ's adjacent object specified by an index j, and for computing the optimum among Δ's adjacent objects, respectively.

By calling *Loc* repeatedly, reach the globally optimal object Δ_0;
    /* Reach the root of the search tree */
$\Delta := \Delta_0$; $j := 0$;
repeat
    while $j < \delta$ **do**

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        j := j + 1;
        Δ' := Adj(Δ, j);
            /* Detect the adjacency */
        if Δ' ≠ NULL then
            if Loc(Δ') = Δ then Δ := Δ'; j := 0;
                /* Descend the search tree */
            endif
    endwhile
    if Δ ≠ Δ_0 then
        Δ' := Δ; Δ := Loc(Δ); j := 0;
            /* Ascend the search tree */
        repeat j := j + 1; until Adj(Δ', j) = Δ
            /* Recover the original index of Δ at Δ' */
    endif
until Δ = Δ_0 and j = δ
```

Here is a typical application: enumeration of basic feasible solutions of a linear programming problem [AF 92]. The search tree consists of roots, i.e., the optimal solutions, nodes representing all basic feasible solutions, and directed edges established from each node to the node whose corresponding basic feasible solution is obtained by one pivoting subject to some prescribed rule, e.g. Bland's rule. Another noteworthy application is found in [Rot 92].

3 Definition of Regular Triangulations and Secondary Fan

A vector configuration in $\boldsymbol{R}^r$ called *Gale transform* will be used to define regular triangulations. Here we review its definition.

Definition 1 *Let V a set $\{v_1, ..., v_n\}$ of n points in $\boldsymbol{R}^{d-1}$. A Gale transform of V is a set $\bar{V} = \{\bar{v}_1, ..., \bar{v}_n\}$ of indexed r-dimensional vectors satisfying the following condition: let A a (d, n)-matrix with columns $(v_i, 1)$, and $\bar{A}$ an (r, n)-matrix with columns $(\bar{v}_i)$. Then, $A\bar{A}^T = 0$ holds.*

Note that this definition suggests how to compute a Gale transform from the given coordinates of points. Many important observations concerning Gale transforms are referred to [Gr 67]. A notation is introduced below for convenience sake.

Definition 2 *Let $\bar{V}$ a Gale transform of the point set $V \subset \boldsymbol{R}^{d-1}$. For a subset $\bar{U}_\tau = \{\bar{v}_{\tau_1}, ..., \bar{v}_{\tau_k}\}$ of $\bar{V}$, the set of vectors*

$$\{\bar{y} \in \boldsymbol{R}^r \mid \bar{y} = \beta_1 \bar{v}_{\tau_1} + \cdots + \beta_k \bar{v}_{\tau_k}, \beta_i \geq 0 \text{ for all } i\}$$

is called a positive hull *spanned by $\bar{U}_\tau$, and denoted by* pos(τ) *with the corresponding index set $\tau = \{\tau_1, ..., \tau_k\} \in \Lambda(n, k)$.*

For example, if the subset $\{\bar{v}_{\mu_1}, ..., \bar{v}_{\mu_r}\}$ of $\bar{V}$ forms a basis of $\boldsymbol{R}^r$, pos(μ) is a simplicial cone with the apex at the origin. The subsets of $\bar{V}$ forming bases in $\boldsymbol{R}^r$ induce a decomposition of $\boldsymbol{R}^r$ called *secondary fan* by simplicial cones they span. We will see all cells (i.e., r-dimensional faces) of the secondary fan are in one-to-one correspondence with all regular triangulations obtained from the given point set V. Before giving this noteworthy fact, we provide two definitions of regular triangulations.

Let conv$U$ mean the convex hull with a point set U. A *triangulation* of conv$V$ with a given point set $V \subset \boldsymbol{R}^{d-1}$ is a collection $\{S_1, ..., S_m\}$ of subsets of V such that (1) every member is of cardinality d, (2) $\bigcup_{i=1}^m \text{conv}S_i = \text{conv}V$, and (3) for $1 \leq i \leq j \leq m$, conv$S_i \cap$ convS_j is empty or some common proper face of convS_i and convS_j. By identifying $v_i \in V$ with its own index, we can regard a triangulation as a subset Δ of $\Lambda(n, d)$. This section provides two equivalent definitions of regular triangulations.

Definition 3 *(Definition by assigning heights). Given a point set* $V = \{v_1, ..., v_n\}$ *in* $\boldsymbol{R}^{d-1}$*. Assign the height* $w_i \in \boldsymbol{R}$ *to each point* v_i *and let* V' *the set of points in* $\boldsymbol{R}^d$ *having coordinates* (v_i, w_i)*. Suppose the heights are assigned so that* convV' *be a simplicial polytope. Then, by projecting the lower facets of* convV' *(i.e.,* $d-1$*-faces having the normal vectors with negative d-th coordinates) down on a hyperplane orthogonal to* x_d*-axis, a triangulation with the points in* V *is obtained. We call this a* regular triangulation *of* V *induced by the assignment* $w = (w_1, ..., w_n)$ *of heights.*

Since the upper bound of the number of faces of a convex polytope in $\boldsymbol{R}^d$ with n vertices is $O(n^{\lfloor d/2 \rfloor})$, the next proposition immediately follows from this definition.

Proposition 1 *The number of simplices contained in one regular triangulation of* $V \subset \boldsymbol{R}^{d-1}$ *is* $O(n^{\lfloor d/2 \rfloor})$.

Let us use the parameter s in place of $O(n^{\lfloor d/2 \rfloor})$. The following is the second definition of regular triangulations.

Definition 4 *(Definition using Gale transforms). Given a point set* $V = \{v_1, ..., v_n\} \subset \boldsymbol{R}^{d-1}$*. Let* $\bar{V} = \{\bar{v}_1, ..., \bar{v}_n\} \subset \boldsymbol{R}^r$ *a Gale transform of* V*. When an* r*-dimensional vector* $\bar{z}$ *is given, let* $\mathcal{P}(\bar{z})$ *denote a collection of sets of* r *linearly independent vectors in* $\bar{V}$ *such that all positive hulls spanned by them contain* $\bar{z}$ *in its interior. Let* $\Delta = \{\mu^* \in \Lambda(n, d) \mid$ *for all* $\mu \in \Lambda(n, r)$ *such that* $\{\bar{v}_{\mu_1}, ..., \bar{v}_{\mu_r}\} \in \mathcal{P}(\bar{z})\}$*. Then,* Δ *comes out a triangulation by identifying* v_i *with its own index. We call* Δ *a* regular triangulation *of* V *induced by* $\bar{z}$*.*

The proofs of the equivalence of these two definitions are found in [BFS 90] [Lee 91]. The propositions concluded from Definition 4 is presented below without proofs. (See [BFS 90].)

Proposition 2 *Two* r*-dimensional vectors belong to the same cell of the secondary fan if and only if they induce the identical regular triangulation.*

Proposition 3 *Each cell* $c \subset \boldsymbol{R}^r$ *of the secondary fan corresponds to the regular triangulation* $\Delta = \{\mu^* \in \Lambda(n, d) \mid c \subset \mathrm{pos}(\mu)\}$*, and this correspondence is one-to-one.*

4 Algorithm

In this section, we present an algorithm for the enumeration of regular triangulations using reverse search technique.

4.1 Detecting Adjacency

Since Proposition 3 says the cells of the secondary fan are in one-to-one correspondence with regular triangulations, we can employ the incidence relations between two cells as the adjacency of the reverse search. A function *Adj* detecting this adjacency is proposed here. The following is the key observation whose elementary proof is presented in §5.

Theorem 1 *Let* $\bar{V}$ *a Gale transform of given point set* $V \subset \boldsymbol{R}^{d-1}$*. Two cells* c *and* c' *of the secondary fan share a common* $(r-1)$*-face, which is the subset of a positive hull* $\mathrm{pos}(\nu)$ *spanned by* $r-1$ *linearly independent vectors* $\{\bar{v}_{\nu_1}, ..., \bar{v}_{\nu_{r-1}}\} \subset \bar{V}$*, if and only if regular triangulations* Δ *and* Δ'*, corresponding to* c *and* c' *respectively, are different from each other by one flipping operation with respect to the point set* $\{v_{\nu_1^*}, ..., v_{\nu_{d+1}^*}\} \subset V$ *defined in [ES 92].*

See the corollary below.

Corollary 1 *A positive hull* $\mathrm{pos}(\nu)$ *spanned by* $r-1$ *linearly independent vectors* $\{\bar{v}_1, ..., \bar{v}_{r-1}\} \subset \bar{V}$ *bounds a cell* c *if and only if the triangulation, given by the flipping operation executed on the regular triangulation corresponding to* c *with respect to the point set* $\{v_{\nu_1^*}, ..., v_{\nu_{d+1}^*}\} \subset V$*, is regular.*

This corollary says that, if the triangulation obtained by one flipping with respect to $\{v_{\rho_1}, ..., v_{\rho_{d+1}}\} \subset V$ is not regular, one can conclude that $\text{pos}(\rho^*)$ does not bound the cell c. Hence, a function *Adj* detecting adjacency can be written down as follows.

Algorithm 1 *(A function Adj).*
Input: *A regular triangulation* Δ, *and a positive integer* j.
Output: *If a flipping executed on* Δ *with respect to* $\{v_{\rho_1}, ..., v_{\rho_{d+1}}\} \subset V$, *which is specified by* j, *gives a regular triangulation* Δ', *output it. Otherwise, return NULL value.*

1. *List up all sets of* $d+1$ *points in* V, *with respect to which flippings can be executed on* Δ. *Let* Γ *be this collection of sets.*
2. *Let* $U_\rho = \{v_{\rho_1}, ..., v_{\rho_{d+1}}\} \subset V$ *be* j*-th element of* Γ.
3. *Execute a flipping with respect to* U_ρ, *and let* Δ' *a triangulation given by this flipping.*
4. *If* Δ' *is regular, output* Δ'. *Otherwise, output NULL.*

In terms of the secondary fan, Step 1 collects the $(r-1)$-faces of the secondary fan which may bound the cell c. As for the total number of such $(r-1)$-faces, the next claim is concluded from Definition 3.

Claim 1 *The number of the sets of* $d+1$ *points in* V, *with respect to which flippings can be executed, is bounded from above by* s, *i.e., the number of* $(d-2)$*-faces of a convex polytope with* n *vertices in* $\boldsymbol{R}^d$.

In Algorithm 1, Step 4 is the most time-consuming part, because a linear programming problem consisting of s constraints with r variables must be solved to check the regularity of the triangulation computed in Step 3. (See Definition 3.) We give three comments as regards Step 4.

- If we omit Step 1, an LP-problem consisting of rs constraints with r variables must be solved.
- Since all constraints in the LP-problem to solve in Step 4 are a central hyperplane spanned by $r-1$ vectors in $\bar{V}$, the normal vectors representing these constraints are not explicitly given, and must be computed every time they are needed. The computation of a normal vector can be done in $O(r^3)$ time. If the only operation executed on a constraint is to determine whether a given point satisfies the constraint (note: this requires $\Theta(r)$ time), the whole computation time is factored by $O(r^2)$, i.e., $O(r^3)/\Theta(r)$. Notice that the exhaustive computation of all normals before Step 4 increases space complexity to $O((d+r)s)$.
- The time complexity of solving an LP-problem consisting of s constraints with r variables is linear in s. However, because there are many theoretical and practical algorithms attaining this bound and they show various dependencies on the parameter r, we leave the time complexity denoted by a function $l(s,r)$.

Verify that, throughout Algorithm 1, only $O(ds)$ storage area is necessary. The following lemma summarizes the discussions in this subsection.

Lemma 1 *Algorithm 1 runs in* $O(r^2 l(s,r))$ *time and in* $O(ds)$ *space.*

4.2 Local search

Local search is an operation for selecting among the adjacent cells the optimal one under the criterion exposed below. This paper's criterion for the optimality is the *volume vector.*

Definition 5 *When a regular triangulation* Δ *is given, the* volume vector *of* Δ *is an n-dimensional vector defined as follows:*

$$\sum_{i=1}^{n}(\sum\{vol(\sigma) \mid \sigma \in \Delta \text{and } i \in \sigma\}) \cdot e_i,$$

where $vol(\sigma)$ *means the volume of the simplex* $\text{conv}(\{v_{\sigma_1}, ..., v_{\sigma_d}\})$*, and* e_i *is an n-dimensional vector with i-th coordinate equal to* 1 *and all other coordinates are* 0.

We decide that the cell whose corresponding regular triangulation has the maximal volume vector under lexicographic ordering is optimal among the adjacent cells. The theorem shown below directly guarantees the correctness of our enumeration algorithm.

Theorem 2 *[BFS 90] When a point set* $V \subset \boldsymbol{R}^{d-1}$ *is given, a convex polytope, called* secondary polytope, *is obtained by constructing a convex hull in* $\boldsymbol{R}^n$ *with volume vectors of all regular triangulations of* V. *Then, it holds that all volume vectors are extreme vertices of the secondary polytope, and that the face lattice of the secondary polytope is anti-isomorphic to that of the secondary fan.*

This theorem implies that the enumeration of cells of the secondary fan by our reverse search is equivalent to the enumeration of vertices of the secondary polytope by a reverse search using lexicographic ordering of coordinates in its local search. (The enumeration of vertices of convex polytopes under lexicographic ordering is found in [Rot 92].) Therefore, our algorithm can complete the exhaustive enumeration of all regular triangulations correctly. A function *Loc* executing local search is described below.

Algorithm 2 *(A function Loc).*
Input: *A regular triangulation* Δ.
Output: *The regular triangulation* Δ_{max} *maximizing the volume vector among the regular triangulations adjacent to* Δ.

1. *Let* $i := 0$ *and* $\Delta_{max} := \Delta$.
2. *Call Adj and obtain the regular triangulation* Δ' *which is specified by the index* i.
3. *If the volume vector of* Δ' *is greater than that of* Δ_{max} *under lexicographic ordering, let* $\Delta_{max} := \Delta'$.
4. *Increment* i *by one and go to Step 2.*

As Claim 1 shows $i \leq s$, let the loop terminate as soon as $i > s$ holds. Moreover, the space complexity never exceeds $O(ds)$. Thus, we are led to conclude the lemma below.

Lemma 2 *Algorithm 2 runs in* $O(r^2 s l(s,r))$ *time and in* $O(ds)$ *space.*

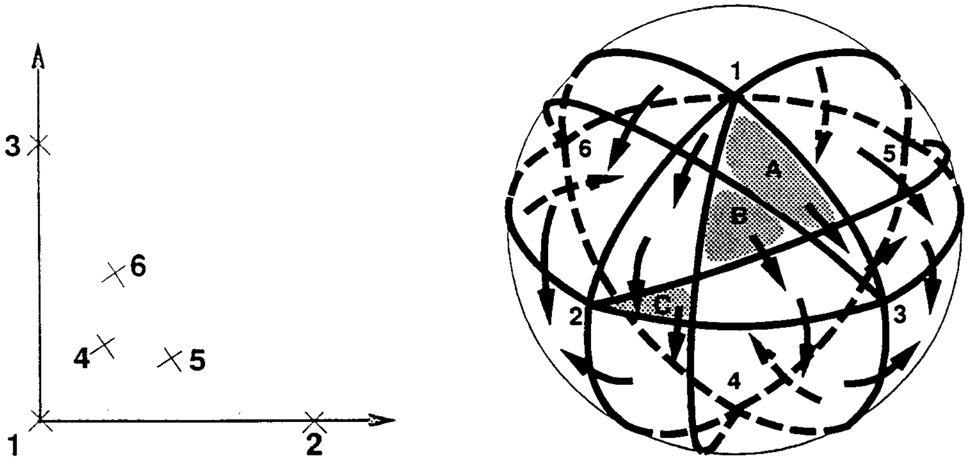
4.3 Total complexity of the enumeration algorithm

The following lemma reveals the complexity of algorithms using reverse search technique.

Lemma 3 *[Fuk 93] If* $t(Adj)$ *time is necessary for detecting adjacency and* $t(Loc)$ *time for local search, the algorithm employing reverse search with these functions, Adj and Loc, completes the enumeration in* $O(\delta(t(Adj)+t(Loc))T)$ *time, where* δ *means the maximum degree of the graph induced by the adjacency between pairs of objects to be enumerated, and* T *denotes the total number of objects.*

Since $\delta = O(s)$ in our case, the total complexity of our enumeration algorithm is concluded from Lemmas 1 and 2.

Theorem 3 *All regular triangulations of a set* V *of* n *points in* $\boldsymbol{R}^{d-1}$ *can be enumerated in* $O(r^2 s^2 l(s,r)T)$ *time and in* $O(ds)$ *space, where* s *equals the upper bound of the number of faces of a convex polytope with* n *vertices in* $\boldsymbol{R}^d$, $l(s,r)$ *denotes the time necessary for solving an LP-problem consisting of* s *constraints with* r *variables, and* T *means the total number of regular triangulations of* V.



(a) (b)

Figure 1: Configuration of points (a) and secondary fan induced by positive hulls (b).

4.4 Example

An example is presented here. Figure 1(a) depicts a configuration of points in V. In this example, $n = 6$ and $d = 3$. The innermost triangle with vertices v_4, v_5, v_6 is rotated clockwise a little relative to the outermost one. Each point has the following coordinate: $v_1 = (0,0)$, $v_2 = (73,0)$, $v_3 = (0,73)$, $v_4 = (20,21)$, $v_5 = (26,20)$, $v_6 = (21,27)$. A Gale transform $\bar{V}$ of this set of points may be computed as follows: $\bar{v}_1 = (-32,-27,-25)$, $\bar{v}_2 = (-20,-26,-21)$, $\bar{v}_3 = (-21,-20,-27)$, $\bar{v}_4 = (73,0,0)$, $\bar{v}_5 = (0,73,0)$, $\bar{v}_6 = (0,0,73)$, while this is not a unique solution. Figure 1(b) depicts the secondary fan induced by $\binom{6}{3} = 20$ positive hulls, each of which 3 vectors in $\bar{V}$ span. The outermost circle represents a ball in $\boldsymbol{R}^3$ (note: $r = n - d = 3$) centered at the origin. The indexed points are the intersection of the corresponding vectors and the boundary of this ball. Every positive hull spanned by two vectors intersects with the sphere in the depicted segment, which is actually an arc as a portion of a circle. Broken segments lie on the invisible hemisphere. In the sequel, we identify the faces of the secondary fan with their intersections with the sphere. Each cell bounded by some facets can be considered as the intersection of several positive hulls.

For example, the cell A is represented as follows: pos$(\{1,2,3\})\cap$ pos$(\{1,2,5\})\cap$ pos$(\{1,3,4\})\cap$ pos$(\{1,3,6\})\cap$ pos$(\{1,4,5\})\cap$ pos$(\{2,5,6\})\cap$ pos$(\{3,5,6\})$. Figure 1(b) shows that an $(r-1)$-face lying on pos$(\{3,6\})$ bounds both cells A and B. Hence, two regular triangulations corresponding to them are adjacent. It is observed in Figure 2, where the regular triangulations corresponding to cells A, B are depicted, that a flipping happens in conv$\{v_1, v_2, v_4, v_5\}$, where the set $\{1,2,4,5\}$ is the complement of $\{3,6\}$.

Now, see the regular triangulation C in Figure 2. The cell C corresponding to it is contained in each positive hull whose interior is a subset of one open halfspace bounded by a hyperplane spanned by $\bar{v}_3$ and $\bar{v}_6$, and has pos$(\{3,6\})$ in its boundary, i.e., pos$(\{2,\mathbf{3},\mathbf{6}\})$ and pos$(\{\mathbf{3},4,\mathbf{6}\})$. For $\{3,6\}^*$ equals $\{1,2,4,5\}$, let us obtain a triangulation Δ' by executing one flipping with respect to the point set $\{v_1, v_2, v_4, v_5\}$ as follows:

$$\Delta = \{\underline{\{1,2,5\}}, \{1,3,6\}, \underline{\{1,4,5\}}, \{1,4,6\}, \{2,3,5\}, \{3,5,6\}, \{4,5,6\}\}$$
$$\Downarrow$$
$$\Delta' = \{\underline{\{1,2,4\}}, \{1,3,6\}, \underline{\{2,4,5\}}, \{1,4,6\}, \{2,3,5\}, \{3,5,6\}, \{4,5,6\}\}$$

Underlines indicate the simplices exchanged by flipping. However, since the intersection of the positive hulls corresponding to the members of Δ' comes out empty, pos(ν) =pos$(\{3,6\})$ fails to bound the

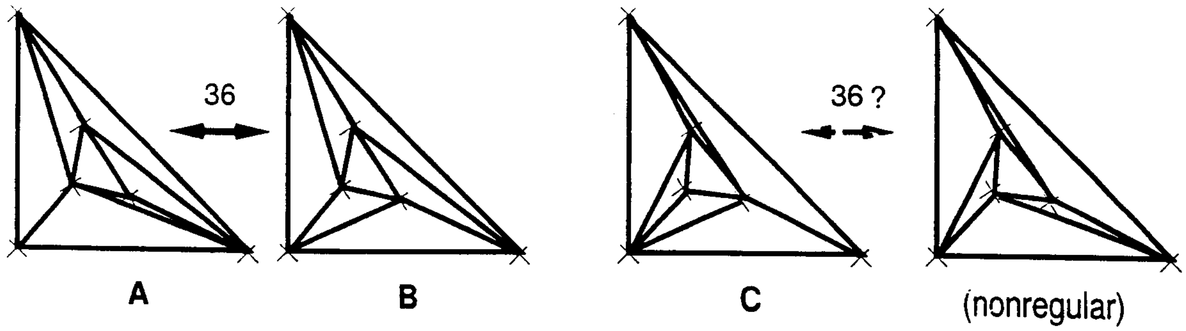


Figure 2: Regular triangulations of the point configuration in Figure 1(a).

cell C. Therefore, the triangulation Δ' is not regular, which is a rightmost triangulation depicted in Figure 2.

Each arrow in Figure 1(b) points at the cell which corresponds to the regular triangulation having the maximal volume vector among the adjacent. These arrows can be considered as directed edges in the search tree of enumeration traversed by our algorithm.

5 Proof of the theorem concerning flipping operation

We give an elementary proof of Theorem 1 here. First of all, let us introduce several notations.

Let $\bar{V} \subset \boldsymbol{R}^r$ be a Gale transform of a given point set V. Fix an index set $\nu = \{\nu_1, ..., \nu_{r-1}\} \in \Lambda(n, r-1)$, and let $\bar{U}_\nu = \{\bar{v}_{\nu_1}, ..., \bar{v}_{\nu_{r-1}}\} \subset \bar{V}$ and $U_{\nu^*} = \{v_{\nu_1^*}, ..., v_{\nu_{d+1}^*}\} \subset V$. Suppose ν is chosen so that the vectors in $\bar{U}_\nu$ be linearly independent. The hyperplane spanned by these vectors determines two subsets of $\bar{V}$: $\bar{U}_\nu^+ = \{\bar{v} \in \bar{V} \mid \det(\bar{v}_{\nu_1}, ..., \bar{v}_{\nu_{r-1}}, \bar{v}) > 0\}$ and $\bar{U}_\nu^- = \{\bar{v} \in \bar{V} \mid \det(\bar{v}_{\nu_1}, ..., \bar{v}_{\nu_{r-1}}, \bar{v}) < 0\}$. Let I_ν^+ and I_ν^- be the corresponding index sets, i.e., $I_\nu^+ = \{k \mid \bar{v}_k \in \bar{U}_\nu^+\}$, $I_\nu^- = \{k \mid \bar{v}_k \in \bar{U}_\nu^-\}$. Now, we present lemmas used in the proof of Theorem 1.

Lemma 4 *Choose $\nu \in \Lambda(n, r-1)$ so that the vectors in $\bar{U}_\nu$ be linearly independent. Then,* $\mathrm{conv}U_{\nu^*} = \bigcup_{k \in I_\nu^+} \mathrm{conv}(U_{\nu^*} \backslash \{v_k\})$ *and* $\mathrm{conv}U_{\nu^*} = \bigcup_{k \in I_\nu^-} \mathrm{conv}(U_{\nu^*} \backslash \{v_k\})$ *hold.*

Proof. A Gale transform of the set $U_{\nu^*} \subset V$ of $d+1$ points is a vector configuration in $\boldsymbol{R}$. (Note that $(d+1) - d = 1$.) Two regular triangulations are read off from there according to Definition 4. It can be shown that they equal two sets of simplices described in the above lemma, namely, $\{\mathrm{conv}(U_{\nu^*} \backslash \{v_k\}) \mid$ for all $k \in I_\nu^+\}$ and $\{\mathrm{conv}(U_{\nu^*} \backslash \{v_k\}) \mid$ for all $k \in I_\nu^-\}$. Therefore, we obtain the conclusion. □

A set of $d+1$ points in $\boldsymbol{R}^{d-1}$ have two regular triangulations from Definition 4. Moreover, [Lee 91] shows the following result using Gale transforms.

Lemma 5 *For a set of $d+1$ points in $\boldsymbol{R}^{d-1}$, there exist only two triangulations.*

In other words, all triangulations are regular for a set of $d+1$ points in $\boldsymbol{R}^{d-1}$. Notice the fact below.

Fact 1 *The flipping operation discussed in [ES 92] is an exchange of these two triangulations of a given set of $d+1$ points in $\boldsymbol{R}^{d-1}$.* ← WRONG.

Now, it is possible to prove Theorem 1. Readers are encouraged to translate the discussions below in terms of cells of the secondary fan and positive hulls into those in terms of regular triangulations and simplices.

Proof.

- Suppose two cells c and c' share a common $(r-1)$-face and this face lies on $\text{pos}(\nu)$, which is spanned by $r-1$ linearly independent vectors $\bar{U}_\nu = \{\bar{v}_{\nu_1}, ..., \bar{v}_{\nu_{r-1}}\} \subset \bar{V}$. Let $\Pi(c) = \{\mu \in \Lambda(n,r) \mid c \subset \text{pos}(\mu)\}$. $\Pi(c)$ represents the set of all positive hulls containing c by corresponding index sets. Let $\Pi_\nu^+ = \{\nu \cup \{k\} \mid \text{ for all } \bar{v}_k \in \bar{U}_\nu^+\}$ and $\Pi_\nu^- = \{\nu \cup \{k\} \mid \text{ for all } \bar{v}_k \in \bar{U}_\nu^-\}$. Each of Π_ν^+ and Π_ν^- represents the set of all positive hulls lying in each of two halfspaces bounded by the hyperplane vectors $\{\bar{v}_{\nu_1}, ..., \bar{v}_{\nu_{r-1}}\}$ span. Then, it can be shown that either $\Pi_\nu^+ \subset \Pi(c)$ and $\Pi_\nu^- \subset \Pi(c')$, or $\Pi_\nu^+ \subset \Pi(c')$ and $\Pi_\nu^- \subset \Pi(c)$ hold. Without loss of generality, assume the former holds. It can be shown that $\Pi(c)\backslash\Pi_\nu^+ = \Pi(c')\backslash\Pi_\nu^-$. Hence, from Lemma 4 and Fact 1, Δ and Δ' are different from each other only by one flipping with respect to $U_{\nu^*} \subset V$.

- Conversely, suppose two regular triangulations Δ and Δ' are different from each other by one flipping operation with respect to $U_{\nu^*} \subset V$. Let c and c' be corresponding cells in the secondary fan respectively. Lemma 4 says either $\Pi_\nu^+ \subset \Pi(c)$ and $\Pi_\nu^- \subset \Pi(c')$, or $\Pi_\nu^+ \subset \Pi(c')$ and $\Pi_\nu^- \subset \Pi(c)$ hold. Without loss of generality, assume the former holds. This assumption implies $\Pi(c)\backslash\Pi_\nu^+ = \Pi(c')\backslash\Pi_\nu^-$. Notice this equality might not lead to the conclusion that both c and c' are bounded by $\text{pos}(\nu)$. However, if $\bigcap_{\mu\in(\Pi(c)\backslash\Pi_\nu^+)} \text{pos}(\mu)$ (note that $\Pi(c)\backslash\Pi_\nu^+ = \Pi(c')\backslash\Pi_\nu^-$) has any intersection with the complement of $(\bigcap_{\mu\in\Pi_\nu^+} \text{pos}(\mu)) \cup (\bigcap_{\mu\in\Pi_\nu^-} \text{pos}(\mu))$, Proposition 3 implies there are more than two triangulations for $d+1$ point set in $\boldsymbol{R}^{d-1}$, and this contradicts to Lemma 5. Hence, $\bigcap_{\mu\in(\Pi(c)\backslash\Pi_\nu^+)} \text{pos}(\mu)$ must be contained in $(\bigcap_{\mu\in\Pi_\nu^+} \text{pos}(\mu)) \cup (\bigcap_{\mu\in\Pi_\nu^-} \text{pos}(\mu))$. This implies $(\bigcap_{\mu\in(\Pi(c)\backslash\Pi_\nu^+)} \text{pos}(\mu)) \cap \text{pos}(\nu)$ is not empty. Thus, both c and c' are bounded by $\text{pos}(\nu)$.

□

References

[AF 92] Avis, D., and K. Fukuda, A Pivoting Algorithm for Convex Hulls and Vertex Enumeration of Arrangements and Polyhedra, *Discrete & Computational Geometry*, Vol.8, 1992, pp.295-313.

[BFS 90] Billera, L. J., P. Filliman, and B. Sturmfels, Constructions and Complexity of Secondary Polytopes, *Advances in Mathematics*, Vol.83, 1990, pp.155-179.

[ES 92] Edelsbrunner, H., and N. R. Shah, Incremental Topological Flipping Works for Regular Triangulations, *Proc. 8th Annual Symposium on Computational Geometry*, 1992, pp.43-52.

[Fuk 93] Fukuda, K., Reverse Search and Its Applications, in *Discrete Structures and Algorithms II*, Kindai-kagaku-sha, Tokyo, 1993 (in Japanese).

[Gr 67] Grünbaum, B., *Convex Polytopes*, Interscience, London, 1967.

[Lee 91] Lee, C., Regular Triangulations of Polytopes, in *Applied Geometry and Discrete Mathematics*, DIMACS Series in Discrete Mathematics and Theoretical Computer Science, Vol.4, 1991, pp.443-456.

[Rot 92] Rote, G., Degenerate Convex Hulls in High Dimensions Without Extra Storage, *Proc. 8th Annual Symposium on Computational Geometry*, 1992, pp.26-32.

[St 91] Sturmfels, B., Gröbner bases of toric varieties, *Tohoku Math. Journal*, Vol.43, No.2, 1991, pp.249-261.