Enumeration of Regular Triangulations

Tomonari Masada Hiroshi Imai
Department of Information Science, University of Tokyo

Keiko Imai
Department of Information and System Engineering, Chuo University

Abstract

Regular triangulations form a meaningful wide subclass of triangulations of points in general dimensions. They can be defined as a natural extension of the Delaunay triangulation and also of lexicographic triangulations, a subclass of triangulations well-known in the theory of oriented matroids. Moreover, regular triangulations have interesting algebraic aspects in connection with a famous paradigm of computer algebra, Gröbner bases.

This paper proposes an output-size sensitive and work-space efficient algorithm to enumerate all regular triangulations by reverse search. The algorithm makes full use of the existing results on the secondary polytope [BFS 90, GKZ 94] whose vertices correspond to regular triangulations. These known results are summarized with only using the so-called volume vector, and the algorithm is described in a simple way. Some regular triangulations may not use a point inside the convex hull, which may not be preferable for three-dimensional applications in computer graphics and finite element method. Triangulations using all the points are called spanning, and an algorithm is given to enumerate all spanning regular triangulations. The diameter of the secondary polytope is investigated. Preliminary computational results are also shown. From the viewpoint of computational geometry, these generalizes the results for planar triangulations to higher-dimensional cases by restricting triangulations to be regular.

1 Introduction

Regular Triangulations: Triangulations have been one of main topics in computational geometry and other fields in recent years. Especially, some types of triangulations are found to bridge geometric issues and algebraic ones. Regular triangulations are of such a type [BFS 90, GKZ 94]. For example, this subclass of triangulations has a close connection with a well-known paradigm of computer algebra, Gröbner bases, and also with theory of discriminants, hypergeometric functions, etc. (see [BFS 90, DST 95, GKZ 94, Lee 91, St 91, St 95]).

From the viewpoint of computational geometry, regular triangulations provide a good framework where many known results for triangulations of a planar point set can be generalized to higher dimensional case. For instance, in the planar case, any pair of triangulation can be transformed to each other by a sequence of so-called Delaunay flips, but, even in the three dimensional case, Delaunay triangulation cannot necessarily be obtained from a non-regular triangulation by Delaunay flips [Joe 89]. However, restricting ourselves to the class of regular triangulations in any dimensions, such a result is already shown (see [BFS 90, GKZ 94]). Also, there are several works in computational geometry on regular triangulations such as [ES 92, Fac 95].

Enumeration of all regular triangulations is interesting from the viewpoint of computer-aided mathematical research. As mentioned above, regular triangulations have connection with many mathematical concepts, and by enumerating them mathematical problems can be investigated through computational experiments (e.g., see [DeL 94, DST 95, St 95]). Also, for the three-dimensional case, through the enumeration algorithm, exhaustive and local search can be performed for triangulations of three-dimensional objects in computer graphics, finite element method, etc.

Our Contributions: By extending the original work by Masada [Mas 94, Mas 95], this paper proposes an output-size sensitive and work-space efficient algorithm for enumerating regular triangulations of n points in the $(d-1)$-dimensional space. Its time complexity is $O(ds\mathrm{LP}(r, ds)T)$, where where s is the upper bound of the number of simplices contained in one regular triangulation, i.e., $O(n^{\lfloor d/2 \rfloor})$, LP$(r, ds)$ denotes the time required for solving a linear programming problem with ds strict inequality constraints in $r = n - d$ variables, and T is the number of regular triangulations, which is bounded by $O(n^{(d+1)(r-1)})$. Its work-space complexity is $O(ds)$, which is best possible to retain one triangulation. Our time complexity is proportional to the output size T, and working space is quite small.

Next, we consider regular triangulations using all points. Some regular triangulations may not use a point inside the convex hull, which may not be preferable for three-dimensional applications mentioned above. Triangulations using all the points are called spanning, and an algorithm with similar complexities is given to enumerate all spanning regular triangulations. Also, the diameter of the secondary polytope whose vertices correspond to regular triangulations is shown to be $O(n^{d+1})$. Finally, preliminary computational results are also shown.

Comparisons with Existing Methods: There have been proposed three algorithms for enumerating regular triangulations [BFS 90, DeL 94, Mas 94, Mas 95].

1. The algorithm by Billera et al. [BFS 90] reduces the problem to constructing the arrangement of $O(n^{d+1})$ homogeneous hyperplanes in the r-dimensional space in $O(n^{(d+1)(r-1)})$ time and space. This algorithm is worst-case optimal for the so-called Lawrence polytopes which form a very restricted class. However, the reduction has redundant part for other cases, and the number of regular triangulations may be much smaller than the complexity of the arrangement. Thus, even if a good algorithm for arrangements is available, an output-size sensitive and work-space efficient algorithm is hard to obtain along this line.
2. An output-size sensitive algorithm is given by De Loera [DeL 94]. It is based on the breadth-first search enumeration, and is implemented using `Maple` and `MACAULAY`. Since it is based on the breadth-first search, its work-space complexity is $\Omega(T)$, which becomes prohibitively large even for small-size problems.
3. An output-size sensitive and work-space efficient algorithm is originally developed by Masada [Mas 94, Mas 95]. It is based on the reverse search technique developed in [AF 92], and is implemented in C. This paper presents a more refined version of this algorithm, together with some other interesting results.

The other results in this paper are new as far as the authors know.

Structure of This Paper: To simplify the discussion, in the body of this paper, we assume that given points are in general position. We discuss modifications necessary to degenerate cases after presenting the algorithm with this assumption. Our algorithm fully utilizes the secondary polytope in [BFS 90, GKZ 94]. Regular triangulations have several equivalent definitions by duality such as Gale transforms, etc. This surely adds richness to regular triangulations, and describing all of them would be best to understand them deeply. However, we cannot describe all the definitions due to the space limitations, and only explain the secondary polytope with using the so-called volume vector, which would be best to understand this structure intuitively. In section 2, these existing results are summarized. The arguments used in these descriptions are fully used in the later sections in many ways. In section 3, our main algorithm is given. The last subsection here describes how to handle degenerate cases. In section 4, spanning regular triangulations are considered, and the diameter of the secondary polytope is investigated. Some computational results are give at the end. In the appendix, properties of regular triangulations in connection with other geometric and mathematical concepts are summarized.

2 Regular Triangulations and Secondary Polytope

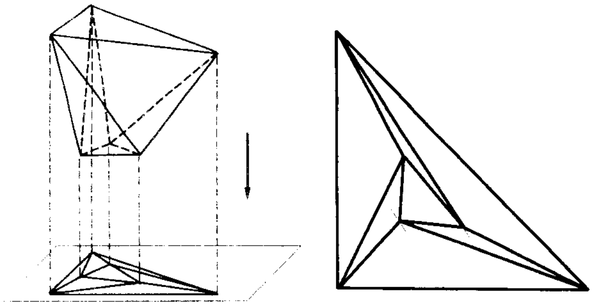
Suppose that n points $V = \{v_1, \ldots, v_n\}$ in general position are given in an affine space $\boldsymbol{R}^{d-1}$. The *convex hull* of a point set U is referred by conv U. First of all, let us review the definition of triangulations.

Definition 1 (Triangulations) *A* triangulation Δ of *V is a collection of sets of d points from V satisfying the following conditions:*

- conv $V = \cup_{\sigma \in \Delta}$conv V_σ, *where* $\sigma = (\sigma_1, \ldots, \sigma_d)$ *and* $V_\sigma = \{v_{\sigma_1}, \ldots, v_{\sigma_d}\}$,
- *for all* $\sigma, \tau \in \Delta$, *either* conv V_σ, conv V_τ *have no intersection or intersect in their common face.*

For this point set, regular triangulations are obtained in the following way.

Definition 2 (Regular triangulations) *For the set V of points, we obtain a point set $W = \{(v_1, w_1), \ldots, (v_n, w_n)\} \subset \boldsymbol{R}^d$ by assigning weights $w_1, \ldots, w_n$ to $v_1, \ldots, v_n \in V$, respectively. Suppose the weights are assigned so that every lower facet (i.e., a facet whose outward normal vector has a negative d-th entry) be a simplex. Then, after projecting the lower facets onto* conv V *in $\boldsymbol{R}^{d-1}$, we obtain a triangulation of V. Triangulations constructed in this manner are called* regular triangulations *of V induced by an assignment of weights $w = (w_1, \ldots, w_n)$. See Fig.1.*



(a) (b)

Figure 1: (a) A regular triangulation obtained by the projection, and (b) a nonregular triangulation

The members of W are called *lifted points*. Definition 2 states that, when an n-dimensional vector w is given, a regular triangulation of V can be constructed if the weights are assigned so that the lifted points be sufficiently generic. Notice that Definition 2 admits regular triangulations which do not use some of given points, while vertices of conv V are necessarily used. The regular triangulations using all points are treated in section 4. The next lemma is an implication of the well-known upper bound theorem of convex polytopes.

Lemma 1 *The number of (any dimensional) simplices in a regular triangulation of V is bounded from above by $O(n^{\lfloor d/2 \rfloor})$.*

For the rest, s denotes the maximum number of simplices in regular triangulations of V, and hence $s = O(n^{\lfloor d/2 \rfloor})$.

With each triangulation, not necessarily regular, a vector is associated as follows.

Definition 3 (Volume vector) *For a triangulation Δ, the volume vector φ of Δ is defined as an n-dimensional vector by:*

$$\varphi_i = \sum_{\sigma \in \Delta,\ v_i \in \sigma} \mathrm{vol}(\sigma), \qquad \varphi = (\varphi_1, \ldots, \varphi_n)$$

where $\mathrm{vol}(\sigma)$ *is the volume of the simplex* $\mathrm{conv}(\{v_{\sigma_1}, \ldots, v_{\sigma_d}\})$.

Notice that i-th entry of the volume vector equals the sum of the volume of all simplices having v_i as its vertex. We adopt *lexicographic ordering* for comparing volume vectors, i.e., for $\varphi, \varphi' \in \boldsymbol{R}^n$, $\varphi > \varphi'$ if and only if there is $i \in \{1, \ldots, n\}$ such that $\varphi_j = \varphi'_j$ for $1 \leq j < i$ and $\varphi_i > \varphi'_i$.

The *secondary polytope* is defined with the volume vectors of all triangulations.

Definition 4 *By constructing a convex hull with the volume vectors of all triangulations of V, we obtain a convex polytope $\Sigma(V)$, called the secondary polytope of V.*

Theorem 1 ([BFS 90, GKZ 94]) *Vertices of the secondary polytope $\Sigma(V)$ correspond to regular triangulations one-to-one.*

Proof: We just give a proof outline. To prove the theorem, it is sufficient to show the existence of a supporting hyperplane for the secondary polytope supported only at the volume vector of a regular triangulation. We claim that a hyperplane whose normal vector is the weight vector realizing the regular triangulation is the one.

To see this, we consider a piecewise linear function $g_\Delta(x)$ on $D = \mathrm{conv}\ V$ in $\boldsymbol{R}^{d-1}$ for a triangulation Δ and a weight vector w such that $g_\Delta(v_i) = w_i$ and, on each simplex of the triangulation Δ, g_Δ is linear. Let Δ_w be the regular triangulation for the weight vector w. Then, it is easy to see that

$$\int_D g_{\Delta_w}(x)\mathrm{d}x < \int_D g_\Delta(x)\mathrm{d}x$$

holds for any triangulation Δ except Δ_w. Noting that this integral is for a piecewise linear function, the following holds

$$\langle w, \varphi_{\Delta_w} \rangle = d \int_D g_{\Delta_w}(x)\mathrm{d}x < d \int_D g_\Delta(x)\mathrm{d}x = \langle w, \varphi_\Delta \rangle,$$

where $\langle w, \varphi \rangle$ is the inner product of w and φ, and hence the claim follows. □

Note that the volume vectors of non-regular triangulations fall into the interior of $\Sigma(V)$ or the relative interior of some face of $\Sigma(V)$.

Next, we consider edges of the secondary polytope. Two vertices are connected by an edge on the polytope if and only if there is a supporting hyperplane whose support are exactly the edge. As in the proof of Theorem 1, suppose that the normal vector of such a supporting hyperplane is made to be a weight vector. For this weight vector and two regular triangulations corresponding to the two vertices, the lower boundary of the convex hull of lifted points become flat for the region consisting of the symmetric difference of families of simplices of these two triangulations. Therefore, affine dependence among lifted points has connection with edges of the secondary polytope, and a careful analysis about the minimality of this supporting hyperplane for this edge shows that the corresponding affinely dependent lifted points should be minimally dependent.

Minimal affinely dependent subsets are known as a circuit in linear algebra and oriented matroid theory,

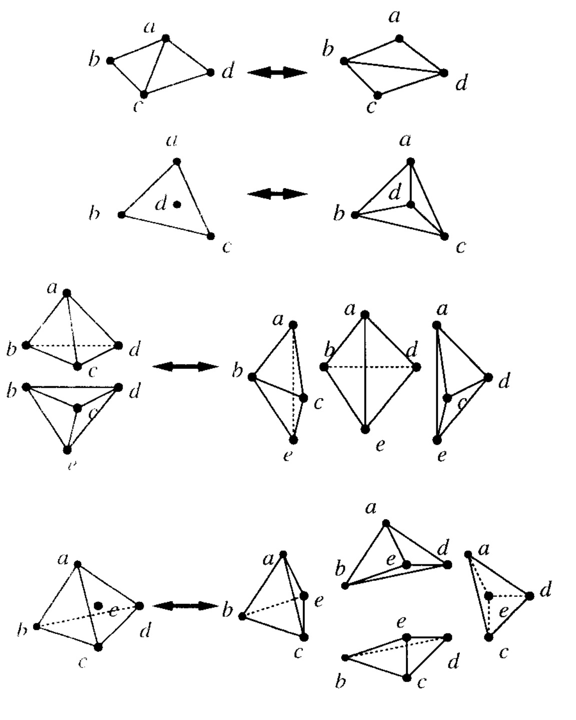


Figure 2: Generalized flips (two- and three-dimensional cases)

that is, a subset Z of V is a circuit if it is affinely dependent and any proper subset is independent (i.e., the proper subset is the set of vertices of a simplex of some dimension). It is known (e.g., see [GKZ 94]) that, for any circuit Z, conv Z has precisely two triangulations $\Delta_+(Z)$ and $\Delta_-(Z)$ with vertices in Z as follows. For circuit Z, a vector $u = (u_v)$ such that $\sum_{v \in Z} u_v \cdot v = 0$ and $\sum u_v = 0$ is determined uniquely up to a constant multiplication. Define Z_+ and Z_- as a partition of Z so that the sign of the corresponding u_v is the same. Z_+ and Z_- are well defined up to interchanging them. Then, the triangulation $\Delta_+(Z)$ consists of all of the faces of the simplices conv $(Z - \{v\})$ $(v \in Z_+)$. Similar for $\Delta_-(Z)$. Every simplex with vertices in Z having maximum dimension is included in exactly one triangulation of Z.

Theorem 2 ([BFS 90, GKZ 94]) *For a point set V in general position, two distinct vertices in the secondary polytope are connected by an edge if and only if, for the corresponding two distinct regular triangulations Δ_1 and Δ_2, there exists a circuit Z satisfying the following conditions:*

(i) *There are no vertices of V inside conv Z except for the elements of Z itself.*

(ii) *conv Z is a union of the faces of the simplices of Δ_1 (and Δ_2) and Δ_1 and Δ_2 coincide outside conv Z.*

Here, conv Z can be triangulated in two ways, which correspond to Δ_1 and Δ_2. These operations are depicted for the two- and three-dimensional cases in Fig.2. These are called generalized flips, and are an extension of the original Delaunay flip in the two-dimensional case. Since the graph formed by vertices and edges of a polytope is connected, by a sequence of generalized flips, any two regular triangulations can be transformed to each other. It should be noted that a new triangulation Δ_2 obtained from a regular triangulation Δ_1 and a circuit satisfying the conditions is not necessarily regular, and hence the regularity of Δ_2 should be checked separately.

Examples of the secondary polytope: A simple example of the secondary polytope of five points in the plane is given. Consider a set of five points in Fig.3(a) where $p_1 = (0,0)$, $p_2 = (1,0)$, $p_3 = (2,1)$, $p_4 = (1,2)$, $p_5 = (0,1)$. There are five triangulations shown in Fig.3(b) for this point set. Regarding the volume of a triangle with the base and weight of length 1 to be 1, then in a triangulation 1, the volumes of $\Delta p_1 p_2 p_3$ and $\Delta p_1 p_4 p_5$ are 1, and that of $\Delta p_1 p_3 p_4$ is 3. p_1 is a vertex of these three triangles, and hence vol$(p_1) = 5$. The volume vector for each triangulation are then described as follows.

$$1 : (5,1,4,4,1),\ 2 : (3,1,5,2,4),\ 3 : (3,4,2,5,1),$$
$$4 : (1,5,2,4,3),\ 5 : (1,3,4,2,5)$$

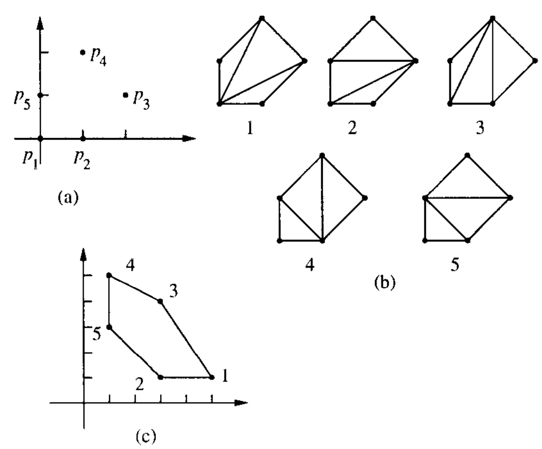


Figure 3: Triangulations of a convex 5-gon and its secondary polytope

The convex hull of these points are two-dimensional, and taking an appropriate coordinate system, it can be depicted as in Fig.3(c). Observe that edges of the secondary polytope correspond to the conventional flip in this case. This polytope is known as an associahedron.

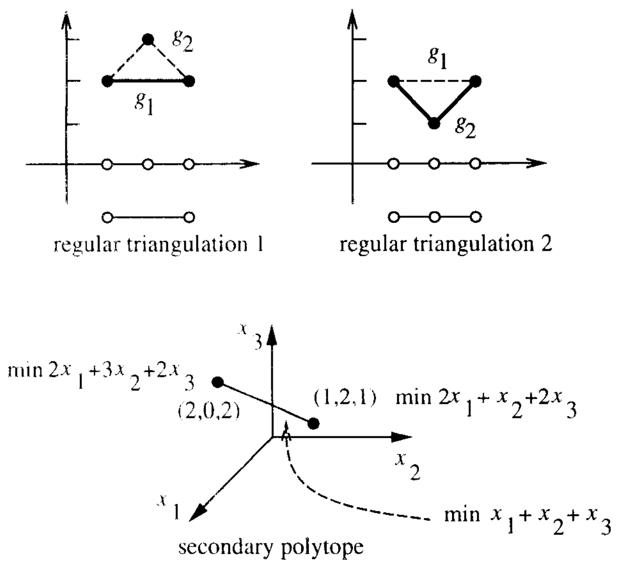


Figure 4: Regular triangulations of 3 points on a line

Fig.4 gives a simpler example for 3 points on a line and its two regular triangulations. This example would give intuitive explanation why edges of the secondary polytope correspond to flips.

Typical secondary polytopes such as permutohedron and associahedron are described in [GKZ 94].

3 Our Algorithm

In this section, we present an algorithm for the enumeration of regular triangulations using reverse search technique developed in [AF 92]. We first describe the data structure for representing a regular triangulation for efficient manipulation. Next, we show that it can be checked by linear programming whether a given triangulation is regular. Then, how to obtain an initial regular triangulation is touched upon. Finally, our reverse search algorithm is presented, together with its complexity analysis.

3.1 Data structure for a triangulation

First of all, we represent each simplex in the triangulation as a set of d points in V. For the set V of points in general position, a graph formed by simplices and facets of the triangulation suffices to represent the incidence relation of simplices of the triangulation. Each facet is a simplex of $d-1$ points, and we represent it by two points which are the complement of $d+1$ points of adjacent two simplices to the $d-1$ points. This data structure for representing the incidence relation requires $O(ds)$ space, where s is the maximum number of simplices of a regular triangulation of V. Note that the number of facets is $O(ds)$.

Besides this graph, we maintain all circuits satisfying the condition (i), (ii) of Theorem 2 for the triangulation. Each circuit is conceptually represented by a $(d+1)$-tuple of points in the circuit sorted in the increasing order of indices of points. Then, all the circuits are maintained by a list in the lexicographic order of the $(d+1)$-tuples. conv Z consists of at most d simplices, and in practice we represent the $(d+1)$-tuple of points implicitly by recording such simplices. The number of circuits is bounded by $O(ds)$ and, if each circuit is represented by $\Theta(d)$ elements, it may take $O(d^2s)$ space. However, a slight careful analysis shows that this implicit representation reduces the total space for this list to $O(ds)$.

For each regular triangulation, we also maintain its volume vector.

By updating triangulations by a generalized flip, we have to maintain these data structures. For example, the volume vector can be updated in $O(d^4)$ time by simply computing necessary changes by the flip. Also, to maintain the list of candidate circuits in the sorted order, at most d^2 circuits are deleted and inserted to the list. Two circuits can be compared with respect to the lexicographic ordering in $O(d)$ time by the implicit representation above. The condition of Theorem 2 for each circuit can be checked in $O(d^2)$ time by simply checking neighbors. Hence, the list for a flip can be updated in $O(d^2s)$ time.

When a new triangulation is computed, we have to check its regularity by solving the linear programming problem in $O(\mathrm{LP}(r,ds))$ time, as described in Lemma 2 below. In the sequel, we assume that the time complexity to update the data structure by a flip is dominated by $O(\mathrm{LP}(r,ds))$, since d^4, $d^2s = O(\mathrm{LP}(r,ds))$ in general.

3.2 Checking the regularity of a triangulation

In the existing literature, the regularity check is done in the dual space. We here give a simple primal approach. For each facet, not on the boundary of conv V, of a given triangulation of the set V of points, there are two simplices sharing the facet. Suppose two simplices have points $\{v_{\sigma_0},\ldots,v_{\sigma_{d-1}}\}$ and $\{v_{\sigma_1},\ldots,v_{\sigma_d}\}$. Then, for variables w_i for v_i as its weight, consider the following determinant for each facet where $(v_{\sigma_i,1},\cdots,v_{\sigma_i,d-1})$ are the coordinate values of v_{σ_i}:

$$D = \begin{vmatrix} 1 & v_{\sigma_0,1} & \cdots & v_{\sigma_0,d-1} & w_{\sigma_0} \\ 1 & v_{\sigma_1,1} & \cdots & v_{\sigma_1,d-1} & w_{\sigma_1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & v_{\sigma_{d-1},1} & \cdots & v_{\sigma_{d-1},d-1} & w_{\sigma_{d-1}} \\ 1 & v_{\sigma_d,1} & \cdots & v_{\sigma_d,d-1} & w_{\sigma_d} \end{vmatrix}.$$

Note that for two simplices having points $\{v_{\sigma_0},\ldots,v_{\sigma_{d-1}}\}$ and $\{v_{\sigma_1},\ldots,v_{\sigma_d}\}$ in a triangulation, the sign of the coefficient of w_{σ_0} is the same as w_{σ_d}'s in the determinant D.

Lemma 2 *A given triangulation is regular if and only if there is a solution w satisfying the following strict linear inequalities defined for each facet:*

$$\operatorname{sign}\left(\begin{vmatrix} 1 & v_{\sigma_0,1} & \cdots & v_{\sigma_0,d-1} \\ 1 & v_{\sigma_1,1} & \cdots & v_{\sigma_1,d-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & v_{\sigma_{d-1},1} & \cdots & v_{\sigma_{d-1},d-1} \end{vmatrix}\right) \cdot D > 0.$$

Proof: The "only if" part is seen just by setting w to the weight vector realizing the regular triangulation. The "if" part can then be shown by standard convex analysis. □

Thus, in a primal way, the regularity can be checked by linear programming. It is easy to see that for a fixed simplex we can set $w_{\sigma_i} = 0$ for each point of the simplex without changing the existence of the solution. Hence, this system of strict linear inequalities is to check the existence of a solution to $r = n - d$ variables and at most ds constraints. In general, a linear programming problem is formulated with non-strict inequalities, and a method of solving this type of problem is developed. It is easy to see that our problem with strict inequalities can also be solved by the general method.

Denote by $\mathrm{LP}(r, ds)$ the time required to solve this linear programming problem.

Theorem 3 *Whether a given triangulation is regular can be judged in* $\mathrm{LP}(r, ds)$ *time.*

3.3 Constructing an initial regular triangulation

Our algorithm requires a regular triangulation to start. This can be an arbitrary regular one. For conceptual simplicity and some technical merits, we may consider two candidates for the initial one. One is a regular triangulation whose volume vector is lexicographically maximum among all volume vectors. The other is the Delaunay triangulation. In the latter case, we can use an algorithm for convex hulls in [AF 92, Cha 91, Sei 86]. [ES 92] devises an algorithm which directly constructs a regular triangulation from an assignment of weight, while its time complexity is analyzed by means of randomized analysis since the algorithm uses flipping operation as a primitive. If one regular triangulation is necessary, this may also be used.

The lexicographically maximum one can be computed with starting any regular triangulation and transforming it by flips towards lexicographic maximization along a path on the secondary polytope. When the input points in V are in general position, the optimal regular triangulation can be obtained simply considering a triangulation formed by points on the convex hull boundary and v_1 such that all simplices have v_1 as a vertex. Such a triangulation is uniquely determined.

In any case, the time necessary for obtaining the initial regular triangulation is negligible in comparison with the time necessary for the rest of the enumeration algorithm.

3.4 Reverse search

Suppose we have a lexicographically maximum regular triangulation. The reverse search technique considers a rooted tree R of the graph of vertices and edges of the secondary polytope, with the root corresponding to this triangulation. The rooted tree can be defined as follows.

Definition 5 (Reverse search tree) *The reverse search tree R of the secondary polytope with respect to the lexicographic maximization of volume vectors is a subgraph of the graph, formed by vertices and edges of the polytope, such that from each vertex except the lexicographically maximum one a directed edge emanates to a vertex adjacent to it whose volume vector is lexicographically maximum among those of adjacent vertices.*

Lemma 3 *R is a directed tree with a unique root with lexicographically maximum volume vector.*

The proof is immediate from linear programming theory (the same fact is used in Rote [Rot 92]).

The reverse search technique traverses this tree R in a depth-first manner. To perform it, we need to arrange children in some order for each vertex in the tree. Each child of a vertex is obtained by flipping with respect to some circuit. For the regular triangulation corresponding to the parent vertex, as described in section 3.1, we maintain all the circuits in the sorted order, and this ordering is adopted in the search. Of course, in performing the depth-first search, we go down to the youngest unvisited child from a vertex.

Now that we have a rooted tree with ordered brother relations, the depth-first search can be performed efficiently with the data structure in section 3.1. Then we obtain the following.

Lemma 4 (1) *For a vertex in the reverse search tree, its parent can be computed in* $O(ds\mathrm{LP}(r, ds))$ *time and* $O(ds)$ *space.*

(2) *During the whole traverse by the depth-first search, the time to spend for listing children of a vertex in the order is* $O(ds\mathrm{LP}(r, ds))$ *time* $O(ds)$ *space.*

Proof: (1) To find the parent, we enumerate all adjacent triangulations, with checking their regularity, and find the lexicographically maximum one. Since there are at most ds adjacent triangulations and each of them can be computed separately, the complexities follow.

(2) Here, a key point is that, when search is returned to a vertex by backtrack in the depth-first search, it returns from the last visited child among its children. From this, we obtain a circuit used to go down to that child. Starting from the circuit we traverse the sorted list of circuits to find a next child. Thus, each circuit is checked only once for flips. The number of children is $O(ds)$, and from the discussion in section 3.1, we see the time complexity is $O(ds\mathrm{LP}(r, ds))$. □

Summarizing the above discussion we now have the following

Theorem 4 *Regular triangulations of n points in $\boldsymbol{R}^{d-1}$ in general position can be enumerated in $O(ds\mathrm{LP}(r, ds)$ $T)$ time and $O(ds)$ working space.*

3.5 How to cope with degenerate cases

In degenerate cases, even in the two-dimensional case, we need another type of flips as shown in Fig.5. With degeneracies, there are lower-dimensional circuits to consider. Concerning the characterization given in Theorem 2, the following condition should be added (see [GKZ 94]):

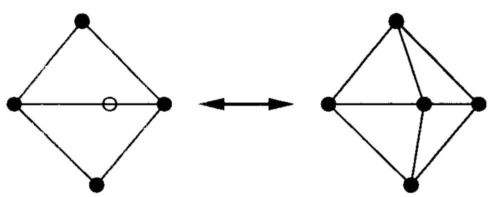


Figure 5: Degenerate flip

(iii) Let conv I and conv I' be two maximal $(\dim(Z)$-dimensional) simplices of one of the two possible triangulations of conv Z. Then for every subset $F \subseteq V - Z$ the simplex conv $I \cup F$ appears in the triangulation Δ if and only if conv $I' \cup F$ appears.

When the number of the elements in Z is $d+1$, (iii) follows from the condition (ii) in Theorem 2. Fig.5 illustrates an example of a triangulation supported on a circuit of smaller cardinality. Thus, in degenerate configuration, more complicated check becomes necessary as in (iii), but still the flippability is characterized well. We have to modify the data structure for representing a regular triangulation so that it represents the whole face lattice. Subsequently, the parameter s should be regarded as the size of this lattice. With these modifications, we can modify the algorithm for nondegenerate case to that for degenerate case without sacrificing any major points.

4 Spanning Regular Triangulations

We call a regular triangulation using all points *spanning*. The first question concerning spanning regular triangulations is whether their corresponding vertices are connected by edges in the secondary polytope. To investigate this, consider the weight vector w_{D} with $w_i = \|v_i\| = \sum_{j=1}^{d-1}(v_{i,j})^2$. The corresponding regular triangulation is the Delaunay triangulation. Consider transforming a spanning regular triangulation into the Delaunay one. Then, the following holds.

Lemma 5 *From a spanning regular triangulation, we can generate a sequence of regular triangulations to the Delaunay triangulation by generalized flips such that*

(1) *all the regular triangulations appearing in this process are spanning, and the inner product of w_{D} and the volume vector of a regular triangulation is strictly decreasing, and furthermore*

(2) *a circuit used in a generalized flip in the sequence is never used again in this process.*

Proof: As in the proof outline of Theorem 1, we consider a piecewise linear function g_Δ for the weight vector w_{D} for each triangulation Δ in the sequence. Since for the w_{D} all the lifted points are on the boundary of lower hull of them, a generalized flip which makes a point unused in any simplex necessarily increases the inner product of w_{D} and the volume vector. By considering a linear programming problem of minimizing a linear function with w_{D} as its cost vector, for a vertex corresponding to a non-Delaunay regular triangulation there exists a adjacent vertex connected by an edge whose inner product with w_{D} strictly decreases. Hence, performing the corresponding generalized flip, a new triangulation with smaller inner product value is obtained and this flip does not destroy the spanning property. Thus, (1) is shown.

For the sequence of triangulations $\Delta_0, \ldots, \Delta_k$ where Δ_k is the Delaunay triangulation, we see

$$g_{\Delta_i}(x) \geq g_{\Delta_j}(x) \quad (i < j;\ x \in \text{conv } V).$$

This is because for lifted points corresponding to the circuit Z their convex hull is a full-dimensional simplex in the lifted space and have the upper and lower boundaries. Each of upper and lower boundaries corresponds to a triangulation of Z in the original space. Since any circuit has two triangulations, these two are such ones, and hence strict above-below relation holds. If a circuit Z is used twice for generalized flips for i and j with $i < j$, $g_{\Delta_i}(x) = g_{\Delta_j}(x)$ for x in the interior of conv Z, while by the argument above $g_{\Delta_i}(x) > g_{\Delta_{i+1}}(x) \geq g_{\Delta_j}(x)$, a contradiction. □

Theorem 5 *All the spanning regular triangulations can be enumerated in $O(ds\mathrm{LP}(r, ds)T')$ time and $O(ds)$ working space, where T' is the number of spanning regular triangulations.*

Proof: As in the case with enumerating all regular triangulations, we consider a rooted tree with the root corresponding to the Delaunay triangulation and from each vertex corresponding to a non-Delaunay spanning regular triangulation choose an edge towards a lexicographically maximum vertex among adjacent vertices whose inner produce with w_{D} is smaller than that at this vertex. By Lemma 5, this forms a rooted tree. Then, by applying the reverse search technique as in the previous section, with noticing that in this case we neglect a circuit formed by a pair of a simplex and a point inside it, we obtain the result. □

The arguments in Lemma 5 can be further utilized as follows.

Theorem 6 *The diameter of the secondary polytope is $O(n^{d+1})$.*

Proof: Since the number of circuits is bounded by $O(n^{d+1})$, and the piecewise linear function monotonically changes downwards also for non-spanning regular triangulations. □

In [BFS 90], they constructed the arrangement of $O(n^{d+1})$ hyperplanes which is a refinement of a fan dual to the secondary polytope, called secondary fan, whose cells correspond to the vertices of the secondary polytope. From these facts, Theorem 6 can also be obtained because any two cells in the arrangement are connected by a sequence of at most $O(n^{d+1})$ adjacent cells. However, it should be noted that the sequence from any regular triangulation to the Delaunay triangulation can be found by the arguments in Lemma 5 and Theorem 6.

5 Preliminary Computational Results

We here describe computational results for randomly generated points. Concerning the results for regularly structured point sets which are interesting from mathematical viewpoints, see Masada [Mas 94, Mas 95]. These are still preliminary results and we just show them here.

Our algorithm is implemented in C language. The experiments are done on Sun SPARCstation 10 with 64MB memory. Exact arithmetics are realized by GNU MP library for arbitrary precision integer and rational number arithmetic. Linear programming problems are solved by a simplex method with Bland's rule. The space complexity is a little more than $O(ds)$ for speeding up the computation in this implementation. Our implementation also works for degenerate inputs.

We here show the number of simplices of regular triangulations when the points are randomly generated in the $(d-1)$-cube with the edges of length 1000. Every coordinate is an integer less than or equal to 500 and more than -500.

- $r = 3$; this is, so to speak, the first non-trivial case, since in the case of $r = 2$ all triangulations are regular.
  - $n = 5, d = 2$: Each of 20 configurations has 8 regular triangulations.
  - $n = 6, d = 3$: 2 of 20 configurations have 16 regular triangulations, 6 of them have 15 ones, and 12 of them have 14 ones.
  - $n = 7, d = 4$: 2 of 20 configurations have 27 regular triangulations, and 18 of them have 25 ones.
  - $n = 8, d = 5$: 3 of 20 configurations have 40 regular triangulations, 3 of them have 41 ones, 7 of them have 42 ones, 3 of them 43 ones, and the other four have 44 ones.
- $r = 4$;
  - $n = 6, d = 2$: Each of 20 configurations has 16 regular triangulations.
  - $n = 7, d = 3$: The number of regular triangulations is quite various. 4 of 20 configurations have 42 ones, 9 of them have 46 ones, 2 of them have 50 ones, one of them has 51 ones, and 2 have 55 one, and two other configurations have 56 regular triangulations, respectively.
  - $n = 8, d = 4$: In this case the number regular triangulations varies from 128 to 168 with some small peak around 133.

Acknowledgments

We would like to thank Nobuki Takayama of Kobe University for introducing us these problems and giving us lectures on many mathematical aspects of them. We are also indebted to reviewers whose comments helped us to improve the original manuscript. The work of the second author was supported in part by the Grant-in-Aid of the Ministry of Education, Japan.

References

[AF 92] Avis, D., and K. Fukuda, A Pivoting Algorithm for Convex Hulls and Vertex Enumeration of Arrangements and Polyhedra, *Discrete & Computational Geometry*, Vol.8 (1992), pp.295–313.

[BFS 90] Billera, L. J., P. Filliman, and B. Sturmfels, Constructions and Complexity of Secondary Polytopes, *Advances in Mathematics*, Vol.83 (1990), pp.155–179.

[Cha 91] Chazelle, B., An Optimal Convex Hull Algorithm and New Results on Cuttings, *Proc. 32nd Annual IEEE Symposium on Foundations of Computer Science*, 1991, pp.29–38.

[DeL 94] De Loera, J., Computing Regular Triangulations of Point Configurations, Preprint, May 1994.

[DST 95] De Loera, J.. B. Sturmfels and R. R. Thomas, Gröbner Bases and Triangulations of the Second Hypersimplex, *Combinatorica*, Vol.15, No.3 (1995), pp.409-424.

[ES 92] Edelsbrunner, H., and N. R. Shah, Incremental Topological Flipping Works for Regular Triangulations, *Proc. 8th Annual ACM Symposium on Computational Geometry*, 1992, pp.43–52.

[Fac 95] Facello, M.A., Implementation of a Randomized Algorithm for Delaunay and Regular Triangulations in Three Dimensions, *Computer-Aided Geometric Design*, Vol.12, No.4 (1995), pp.349–370.

[GKZ 94] Gelfand, I. M., M. M. Kapranov and A. V. Zelevinski, *Discriminants, Resultants and Multidimensional Determinants*, Birkhäuser, Boston, 1994.

[Joe 89] Joe, B., Three-dimensional Triangulations from Local Transformations, *SIAM Journal on Scientific and Statistical Computing*, Vol.10, No.4 (1989), pp.718–741.

[Lee 91] Lee, C., Regular Triangulations of Polytopes, in *Applied Geometry and Discrete Mathematics*, DIMACS Series in Discrete Mathematics and Theoretical Computer Science, Vol.4, 1991, pp.443–456.

[Mas 94] Masada, T., An Output Size Sensitive Algorithm for the Enumeration of Regular Triangulations, Technical Report 94-1, Department of Information Science, University of Tokyo, November 1994.

[Mas 95] Masada, T., *An Algorithm for the Enumeration of Regular Triangulations.* Master's Thesis, Department of Information Science, University of Tokyo, March 1995.

[Rot 92] Rote, G., Degenerate Convex Hulls in High Dimensions Without Extra Storage, *Proc. 8th Annual ACM Symposium on Computational Geometry*, 1992, pp.26–32.

[Sei 86] Seidel, R., Constructing Higher-Dimensional Convex Hulls at Logarithmic Cost per Face, *Proc. 18th Annual ACM Symposium on the Theory of Computing*, 1986, pp.404–413.

[St 91] Sturmfels, B., Gröbner Bases of Toric Varieties, *Tohoku Math. Journal*, Vol.43, No.2 (1991), pp.249–261.

[St 95] Sturmfels, B., Gröbner Bases and Convex Polytopes, Expanded Lecture Notes from the Holiday symposium at New Mexico State University, April 1995

Appendix

A Properties of regular triangulations

A.1 Delaunay triangulation

[ES 92] defines regular triangulations as a natural extension of the Delaunay triangulation. Given a point set $V \subset \boldsymbol{R}^{d-1}$, the Delaunay triangulation can be defined as a subdivision of conv$V$ (conv$V$ means the convex hull of V) dual to the Voronoi diagram of V. If we replace some points by the balls with possibly negative radii, another diagram is obtained by taking into consideration these radii in determining the distances. More precisely, the distance of a point $z = (z_1, \dots, z_{d-1}) \in \boldsymbol{R}^{d-1}$ from $v_i = (v_{i,1}, \dots, v_{i,d-1}) \in V$ is determined by the formula $\sqrt{(z_1 - v_{i,1})^2 + \dots + (z_{d-1} - v_{i,d-1})^2 - r_i}$, where r_i denotes the radius of the ball replacing v_i. This diagram is called a *power diagram.* Then regular triangulations can be defined as subdivisions of conv$V$ dual to power diagrams. Since the power diagram with balls of radius 0 turns out to be a Voronoi diagram, the Delaunay triangulation is obviously regular.

Lemma 6 *Let V be a set of n points in $\boldsymbol{R}^{d-1}$. The Delaunay triangulation of V is regular.*

A.2 Algebraic aspects of regular triangulations

[St 91] reveals the algebraic aspects of regular triangulations. An ideal $\mathcal{I}_V$ in the n-variate polynomial ring $\boldsymbol{C}[x_1, \dots, x_n]$, where each x_i corresponds to $v_i \in V$, called a *toric ideal*, can be obtained from affine dependencies among the points in V. That is, for nonnegative integers $\alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_n$, the binomial $x_1^{\alpha_1} \dots x_n^{\alpha_n} - x_1^{\beta_1} \dots x_n^{\beta_n}$ is contained in $\mathcal{I}_V$ if and only if $\alpha_1 v_1 + \dots + \alpha_n v_n = \beta_1 v_1 + \dots + \beta_n v_n$ such that $\alpha_1 + \dots + \alpha_n = \beta_1 + \dots + \beta_n$ gives an affine dependency among $v_1, \dots, v_n$. Another ideal I_Δ in $\boldsymbol{C}[x_1, \dots, x_n]$, called a *Stanley-Reisner ideal*, can be obtained from a triangulation Δ. This ideal is generated by all monomials of the form $x_{\tau_1} \dots x_{\tau_l}$ where the simplex spanned by $v_{\tau_1}, \dots, v_{\tau_l}$ is not a face of Δ. Suppose Δ is a regular triangulation determined by giving a weight w_i to each $v_i \in V$. If we regard the assignment $w = (w_1, \dots, w_n)$ of weights as a weight vector determining a term order among monomials in $\boldsymbol{C}[x_1, \dots, x_n]$ and calculate the Gröbner basis of $\mathcal{I}_V$ with respect to this term order, then the radical of the initial ideal of this Gröbner basis equals I_Δ. The concept of Gröbner basis is required here only as a tool for computing the initial ideal. Hence, the result can be stated without this concept.

Lemma 7 *Let $w \in \boldsymbol{R}^n$ be a weight vector which defines a term order for the toric ideal $\mathcal{I}_V$. Then the regular triangulation of V induced by the assignment of weights w has a Stanley-Reisner ideal I_Δ which equals the radical of the initial ideal of $\mathcal{I}_V$. (We assumed that w gives a sufficiently generic set of lifted points.)*

It is also remarked that the enumeration of regular triangulations of the products of simplices is an important problem. The product of the $(r-1)$-dimensional simplex and the $(s-1)$-dimensional one is a point configuration

$$V = \{e_i \oplus e'_j \in : 1 \le i \le r, 1 \le j \le s\},$$

where $e_1, \ldots, e_r$ denote the standard basis of $\boldsymbol{R}^{r-1}$ and $e'_1, \ldots, e'_s$ that of $\boldsymbol{R}^{s-1}$.

A.3 Lexicographic triangulations

[Lee 91] gives intensive considerations about regular triangulations. It is shown that lexicographic triangulations can be understood as a special case of regular triangulations, that is, they are obtained by assigning weights to given points in a special manner.

Lemma 8 *Suppose that an assignment of weights $w = (w_1, \ldots, w_n) \in \boldsymbol{R}^n$ is determined as follows: $w_i = \epsilon_i |w_i|$ $(1 \le i \le n)$, where $\epsilon_i \in \{1, -1\}$ and $|w_{\pi_1}| \gg \ldots \gg |w_{\pi_n}|$ for some permutation π of $\{1, \ldots, n\}$. Then the regular triangulation induced by w is lexicographic.*

This observation proceeds from the definition of lexicographic triangulations by *pulling* or *placing* vertices in some order. Details are referred to the original paper. It is a prevalent fact that lexicographic triangulations form one of the subclasses of triangulations admitting a definition in terms of oriented matroids. This subclass also has its own algebraic aspects as is shown in [St 91]. For other applications, see [DeL 94, DST 95, GKZ 94, St 95].