Topological Methods in Combinatorics and Geometry

(Lecture Notes)

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Introduction

In last few decades several authors succeeded in proving theorems from combinatorics and combinatorial geometry by a surprising use of tools from algebraic topology. In these lecture notes we cover a part of such results, mainly those proved using the Borsuk-Ulam theorem and its generalizations. The proofs presented here require a minimum of material from algebraic topology (and this minimum is covered as well); we use only one more advanced fixed point theorem (which we will not prove), and everything is derived from it by relatively elementary means.

These are lecture notes for a course I taught at the Faculty of Mathematics and Physics of the Charles University in Prague in fall 1993. The first version of the notes (in Czech) was written by the participants of this course, I have then modified, combined and moderately expanded these notes and translated them into the present English text (a Czech version also exists).

The understanding of this text should require no specialized knowledge of topology or combinatorics. The reader should suffice with the basics of the topology of metric spaces.

The main sources (which are not explicitly mentioned in the sequel) are the survey papers [4] a [3] (in which also the references to original works and a plenty of further material can be found). Other references are given in the text.

I will be grateful for comments and suggestions from the kind readers.

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1 Simplicial complexes

1.1 Notation, conventions

$\mathbb{R}$ will denote the set of all real numbers, $\mathbb{R}^d$ the Euclidean space of dimension d. For $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$, the Euclidean norm is defined by $\|x\| = \sqrt{x_1^2 + \cdots + x_d^2}$. The distance of points $x, y \in \mathbb{R}^d$ will also be denoted by $\mathrm{dist}(x, y)$.

All the considered topological spaces are subspaces of $\mathbb{R}^d$, thus, in particular, metric spaces. The existence of a homeomorphism (i.e. of a continuous bijection with a continuous inverse mapping) between spaces X, Y will be written $X \cong Y$.

1.2 Geometric simplicial complexes

Definition. A point set $\{v_0, v_1, \ldots, v_k\} \subset \mathbb{R}^d$ is called *affinely independent* if the vectors $v_1 - v_0, v_2 - v_0, \ldots, v_k - v_0$ are linearly independent.

Remark. We present 2 equivalent definitions of the affine independence (the equivalence is easy to see).

- A point set $\{v_0, v_1, \ldots, v_k\}$ is affinely independent iff there are no real numbers $\alpha_0, \alpha_1, \ldots, \alpha_k$ such that $\sum_{i=0}^k \alpha_i v_i = 0$, $\sum_{i=0}^k \alpha_i = 0$ and $(\alpha_0, \ldots, \alpha_k) \neq (0, \ldots, 0)$.

- A point set $\{v_0, v_1, \ldots, v_k\}$ is affinely independent iff the $(d+1)$-dimensional vectors $(1, v_0), (1, v_1), \ldots, (1, v_k)$ are linearly independent.

Definition. The convex hull of a finite affinely independent set A in $\mathbb{R}^d$ is called a *simplex*. The points of A are called the *vertices* of this simplex. The *dimension* of this simplex is equal to $|A| - 1$.

Example. Simplices in $\mathbb{R}^2$ are triangles, segments, points and the empty set.

Definition. The convex hull of a subset of the set of vertices of a simplex is called a *face* of that simplex.

Example. The faces of a triangle are the whole triangle, its edges, its vertices and the empty set.

Definition. The *relative interior* of a simplex σ arises from σ by removing all its faces of dimension smaller than $\dim \sigma$.

Definition. A family of simplices $\Delta = \{\sigma_1, \sigma_2, \ldots, \sigma_m\}$ is called a *simplicial complex* if the following conditions hold:

(1) Each nonempty face of any simplex $\sigma \in \Delta$ is also a simplex of Δ.

(2) $\sigma_1, \sigma_2 \in \Delta \Rightarrow \sigma_1 \cap \sigma_2$ is a face of both σ_1 and σ_2.

The union of all simplices in a simplicial complex Δ is called the *polyhedron* of Δ and is denoted by $\|\Delta\|$. The *dimension of a simplicial complex* $\dim \Delta = \max\{\dim \sigma; \sigma \in \Delta\}$.

Observation. The relative interiors of all simplices of a simplicial complex Δ form a partition of $\|\Delta\|$, i.e. for each point $x \in \|\Delta\|$ there exists exactly one simplex $\sigma \in \Delta$ containing x in its relative interior. This simplex will be denoted by $\mathrm{supp}(x)$ and called the *support* of the point x.

Remark. A simplest example of a simplicial complex is the set of all faces of an n-dimensional simplex. This simplicial complex will be denoted by σ^n. The n-dimensional simplex itself, as a geometric object, can thus be denoted by $\|\sigma^n\|$.

Definition. $\Delta^{\leq k} = \{\sigma \in \Delta; \dim \sigma \leq k\}$ is called the *k-skeleton* of Δ (it is also a simplicial complex). Further we set $\Delta^k = \{\sigma \in \Delta;\ \dim \sigma = k\}$. In particular, Δ^0 is the set of all vertices of a simplicial complex Δ.

Remark. Any homeomorphic image of the "standard" (geometric) simplex will also be called a simplex.

1.3 Abstract simplicial complexes

Definition. An *(abstract) simplicial complex* is a pair (V, Δ), where V is a set, and $\Delta \subseteq 2^V \setminus \{\emptyset\}$ is a hereditary system of nonempty subsets of V, i.e. $\sigma \in \Delta, \emptyset \neq \sigma' \subseteq \sigma \Rightarrow \sigma' \in \Delta$. Further we define the *dimension* $\dim(\Delta) = \max\{|\sigma| - 1; \sigma \in \Delta\}$.

Remarks. In the literature, one often includes also the empty simplex $\emptyset$ into a simplicial complex (abstract or geometric one), here we exclude it (following [3]).

We will assume $V = \bigcup \Delta$; the it suffices to write just Δ instead of (V, Δ).

We will consider only the case when V is finite. From the topological point of view, this is quite a restrictive assumption (since then we cannot express e.g., the space $\mathbb{R}^d$ as the polyhedron of a simplicial complex), but it is sufficient for our combinatorial applications.

Each geometric simplicial complex Δ determines an abstract simplicial complex: The points of the abstract simplicial complex are all vertices of the simplices of Δ, and the sets in the abstract simplicial complex are just the vertex sets of the simplices of Δ. The abstract simplicial complex obtained in this way is also denoted by Δ. The following theorem considers the opposite transition.

Theorem 1.1 *Every finite d-dimensional abstract simplicial complex can be realized as a geometric simplicial complex embedded in* $\mathbb{R}^{2d+1}$.

Remark. Later on we show that $2d + 1$ is the smallest possible dimension. For $d = 1$ simplicial complexes correspond to graphs. The theorem says that every graph can be represented in $\mathbb{R}^3$, with edges being straight segments.

Definition. The curve $\{(t, t^2, \ldots, t^d); t \in \mathbb{R}\}$ is called the *moment curve* in $\mathbb{R}^d$.

Lemma 1.2 *Every* $(d + 1)$*-tuple of distinct points on the moment curve in* $\mathbb{R}^d$ *forms an affinely independent set.*

Proof. Let $(t_0, t_0^2, \ldots, t_0^d), \ldots, (t_d, t_d^2, \ldots, t_d^d)$ be points of the moment curve. They are affinely independent, as the Vandermonde determinant

$$\begin{vmatrix} 1 & t_0 & t_0^2 & \ldots & t_0^d \\ 1 & t_1 & t_1^2 & \ldots & t_1^d \\ & & & \ldots & \\ 1 & t_d & t_d^2 & \ldots & t_d^d \end{vmatrix}$$

is nonzero whenever the numbers $t_0, t_1, \ldots, t_d$ are pairwise distinct. □

Proof of Theorem 1.1. Let $V = \{v_1, v_2, \ldots, v_n\}$. Distinct points $v_1, v_2, \ldots, v_n$ are placed arbitrarily on the moment curve in $\mathbb{R}^{2d+1}$. Let $\sigma_1, \sigma_2 \subseteq V$ be two simplices; we need to show that the intersection of their convex hulls equals to the convex hull of the set $\sigma_1 \cap \sigma_2$. Let x be a point of the above mentioned intersection, i.e.

$$x = \sum_{v_i \in \sigma_1} \alpha_i v_i = \sum_{v_j \in \sigma_2} \beta_j v_j \,, \tag{1}$$

where $\alpha_i, \beta_j \geq 0$ and $\sum_{v_i \in \sigma_1} \alpha_i = 1 = \sum_{v_j \in \sigma_2} \beta_j$. By subtracting we get from (1)

$$\sum_{v_i \in \sigma_1 \setminus \sigma_2} \alpha_i v_i - \sum_{v_j \in \sigma_2 \setminus \sigma_1} \beta_j v_j + \sum_{v_k \in \sigma_1 \cap \sigma_2} (\alpha_k - \beta_k) v_k = 0 \,.$$

The points $\sigma_1 \cup \sigma_2$ are affinely independent by Lemma 1.2, and thus all the coefficient at the left hand side of this equation must be 0, in particular α_k, β_k can only be nonzero for $v_k \in \sigma_1 \cap \sigma_2$. We have thus shown that an arbitrary point of the intersection of the realizations of σ_1 and of σ_2 is a convex combination of points of $\sigma_1 \cap \sigma_2$. □

1.4 Simplicial mappings

Definition. Let Δ_1, Δ_2 be two abstract simplicial complexes. A *simplicial mapping* of Δ_1 into Δ_2 is a mapping f of the set Δ_1^0 into the set Δ_2^0 such that for each $\sigma \in \Delta_1$ one has $f(\sigma) \in \Delta_2$.

A bijective simplicial mapping whose inverse mapping is also simplicial is called an *isomorphism* of abstract simplicial complexes. Isomorphic abstract simplicial complexes are thus "the same" set systems, they only differ by a renaming of vertices.

Observation. Let Δ_1, Δ_2 be *geometric* simplicial complexes, and let f be a simplicial mapping between the corresponding abstract simplicial complexes (thus f is defined at the vertices of the simplices of Δ_1). Then f can be extended, in a unique way, to the domain $||\Delta_1||$ in such a way that it is linear on each simplex. Such an extension is denoted by $||f||$; this is thus a continuous mapping $||\Delta_1|| \to ||\Delta_2||$.

In particular, an isomorphism between the abstract simplicial complexes corresponding to Δ_1, Δ_2 induces a homeomorphism between the polyhedra $||\Delta_1||, ||\Delta_2||$. In this sense, the abstract simplicial complex determines the geometric simplicial complex and its polyhedron uniquely.

Convention. In the sequel, a simplicial complex will formally be understood as an abstract simplicial complex (i.e. it will be a set system as a mathematical object). However, we will also use topological notions for simplicial complexes (such as "a connected simplicial complex"); in such cases we will mean the corresponding polyhedron. A simplicial mapping f between simplicial complexes Δ_1, Δ_2 is written as $f : \Delta_1 \to \Delta_2$ (although, strictly speaking, f is a mapping $\Delta_1^0 \to \Delta_2^0$).

Remark. If we express a topological space as the polyhedron of some simplicial complex (such a simplicial complex is called a *triangulation* of X), we can translate many topological questions to combinatorial ones. These can, however, remain difficult; for instance, to recognize whether the polyhedron of a given simplicial complex is homeomorphic to the 5-dimensional sphere is an algorithmically undecidable problem.

1.5 Correspondence between posets and simplicial complexes

Definition. To each partially ordered set (poset) P corresponds a simplicial complex $\Delta(P)$, whose vertices are the points of P and whose simplices are all nonempty chains (i.e. linearly ordered subsets) in P.

Conversely, to a simplicial complex Δ corresponds a poset $P(\Delta)$, which is the set of all simplices of Δ ordered by inclusion.

Definition. The *first barycentric subdivision* of a simplicial complex Δ is the simplicial complex $\mathrm{Sd}(\Delta) := \Delta(P(\Delta))$.

Geometrically this means that we add the center of gravity of each simplex as a new vertex, and we subdivide the simplex using this vertex and the already constructed subdivision of its faces, see fig. 1.

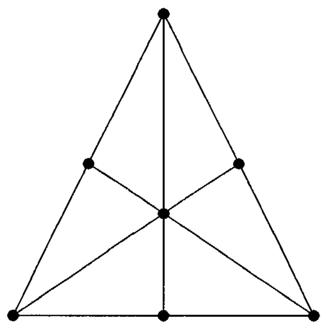

Figure 1: The first barycentric subdivision of a 2-simplex.

Observation. We have $\|\Delta\| = \|\mathrm{Sd}(\Delta)\|$.

Observation. A monotone mapping $f : P_1 \to P_2$ between posets is also a simplicial mapping $f : \Delta(P_1) \to \Delta(P_2)$ between the simplicial complexes.

Corollary. Consider an arbitrary mapping f, which assigns to each simplex $\sigma \in \Delta_1$ a simplex $f(\sigma) \in \Delta_2$, and suppose that if $\sigma' \subseteq \sigma$ then also $f(\sigma') \subseteq f(\sigma)$. Then f can be regarded as a simplicial mapping $f : \mathrm{Sd}(\Delta_1) \to \mathrm{Sd}(\Delta_2)$.

2 The theorem of Borsuk and Ulam

2.1 Several equivalent formulations

Let us denote by $B^n = \{x \in \mathbb{R}^n : \|x\| \leq 1\}$ the n-dimensional *closed ball* and by $S^{n-1} = \{x \in \mathbb{R}^n : \|x\| = 1\}$ the $(n-1)$*-dimensional sphere.*

The Brouwer fixed point theorem claims that every continuous mapping $f : B^n \to B^n$ has a fixed point: $f(x) = x$ for some point $x \in B^n$. The Borsuk-Ulam theorem is a statement of a similar type.

Theorem 2.1 *The following statements are equivalent (and true):*

(1.1) *For every continuous mapping $f : S^n \to \mathbb{R}^n$ there exists a point $x \in S^n$ with $f(x) = f(-x)$.*

(1.2) *For every continuous* antipodal *mapping* $f : S^n \to \mathbb{R}^n$ *(i.e.* $f(x) = -f(-x)\ \forall x \in S^n$*) there exists a point* $x \in S^n$ *satisfying* $f(x) = 0$.

(2) *Whenever a continuous antipodal mapping* $f : S^d \to S^n$ *exists, we have* $d \leq n$.

(3.1) *(Ljusternik-Šnirelmann) For any cover* $A_1, \ldots, A_{n+1}$ *of the sphere* S^n *by* $n+1$ *open sets there is at least one set containing a pair of antipodal points (i.e.* $A_i \cap (-A_i) \neq \emptyset$*).*

(3.2) *Similarly for a covering of the sphere by closed sets.*

Proof of the equivalences (exercise):

(1.1)⇒(1.2) is clear.

(1.2)⇒(1.1) We convert f into an antipodal mapping $g(x) := f(x) - f(-x)$.

(3.2)⇒(3.1) follows from the fact that for any open cover $A_1, \ldots, A_{n+1}$ there exists a closed cover $B_1, \ldots, B_{n+1}$ satisfying $B_i \subset A_i, i = 1, \ldots, n+1$: for each point x of the sphere choose its open neighborhood O_x whose closure is contained in some A_i, and apply the compactness of the sphere.

(3.1)⇒(3.2) follows from the fact that each set of a closed cover $B_1, \ldots, B_{n+1}$ can be wrapped in an open set $A_i^\varepsilon = \{x \in S^n; \operatorname{dist}(x, B_i) < \varepsilon\}$. We let $\varepsilon \to 0$ and we use the compactness of the sphere. We obtain a convergent sequence of points $x_0, x_1, \ldots \in S^n$ with $\operatorname{dist}(x_i, B_j), \operatorname{dist}(-x_i, B_j) \to 0$ for $i \to \infty$ and for some fixed j. The limit point of this sequence provides the required antipodal pair in B_j.

(1.2)⇒(2) If a continuous antipodal mapping $f : S^d \to S^n$, $d > n$, existed, we can assume $d = n+1$. Then we have an antipodal nowhere zero mapping $f : S^{n+1} \to \mathbb{R}^{n+1}$.

(2)⇒(1.2) Should a continuous nowhere zero antipodal mapping $f : S^n \to \mathbb{R}^n$ exist, we consider the mapping $g(x) := \frac{f(x)}{||f(x)||}$. This yields a contradiction with (2).

(1.1)⇒(3.2) For a closed cover $B_1, \ldots, B_{n+1}$ we define a continuous mapping $f : S^n \to \mathbb{R}^n$ by $f(x) := (\operatorname{dist}(x, B_1), \ldots, \operatorname{dist}(x, B_n))$ and we consider a point $x \in S^n$ with $f(x) = f(-x) = y$, which exists by (1.1). If the ith coordinate of the point y is 0, then both x and $-x$ fall into B_i. If all coordinates of y are nonzero, then both x and $-x$ lie in B_{n+1}.

(3.2)⇒ (2) We need an *auxiliary result*: There exists a covering of S^{n-1} by closed sets $B_1, \ldots, B_{n+1}$ such that no B_i contains a pair of antipodal points (to see this, we can use e.g. the projection of the faces of the regular simplex with the center of gravity in the origin). Then if a continuous antipodal mapping $f : S^n \to S^{n-1}$ existed, the sets $f^{-1}(B_1), \ldots, f^{-1}(B_{n+1})$ would contradict (3.2).

□

2.2 A combinatorial proof of the Borsuk-Ulam theorem

Tucker Lemma. For $x \in \mathbb{R}^n$, $||x||_1$ denotes the L_1-norm of x, i.e. $||x||_1 = |x_1| + \cdots + |x_n|$. For the proof, we imagine B^n as the unit ball of the L_1-norm, i.e. $B^n = \{x \in \mathbb{R}^n; ||x||_1 \leq 1\}$. A simplicial complex T is a *special triangulation* of B^n if $||T|| = B^n$, T is a refinement of the triangulation of B^n given by "cutting" by the coordinate hyperplanes (i.e. no simplex of T spans over a boundary of an orthant) and is symmetric around the origin.

Lemma 2.2 *Let the vertices of an arbitrary special triangulation* T *be denoted by labels* $\operatorname{lab}(u) \in \{\pm 1, \pm 2, \ldots, \pm n\}$ *in such a way that the vertices* $u \in \partial B^n$ *(on the boundary) the* *labeling satisfies* $\mathrm{lab}(-u) = -\mathrm{lab}(u)$. *Then there exists a 1-simplex (an edge) in* T *which is* complementary, *i.e. its two vertices are labeled by opposite numbers.*

Proof of the Borsuk-Ulam theorem from the lemma: Let $f : S^n \to \mathbb{R}^n$ be a continuous mapping, let B^n be the unit ball in the "equator" hyperplane of S^n. We define $g : B^n \to \mathbb{R}^n$ by setting $g(x) = f(y) - f(-y)$, where y is the point of the upper hemisphere of S^n whose vertical projection on B^n is x. The mapping g is obviously antipodal on $\partial B^n = S^{n-1}$. For contradiction, let us assume that $g(x) \neq 0$ everywhere; then from the compactness of the ball there exists $\varepsilon > 0$ such that $\|g(x)\|_1 \geq \varepsilon$ for all x. Further, a continuous function on a compact set is uniformly continuous, and thus there exists a number $\delta > 0$ such that if the distance of some two points x, x' does not exceed δ, then $\|g(x) - g(x')\|_1 < \varepsilon/n$.

Let us choose a special triangulation T such that the diameter of each its simplex is at most δ. We define a labeling of the vertices of T as follows: $|\mathrm{lab}(x)| = i$ if $|g_i(x)| = \max\{|g_1(x)|, \ldots, |g_n(x)|\}$, and $\mathrm{sgn}\,\mathrm{lab}(x) = \mathrm{sgn}(g_i(x))$ (if the maximum is attained for more than one index, we take the first such index). From the lemma we know that there exists a complementary edge xx'. Let $\mathrm{lab}(x) = -\mathrm{lab}(x') = i$, then $g(x)_i \geq \varepsilon/n$ and $g(x')_i \leq -\varepsilon/n$, hence $\|g(x) - g(x')\|_1 \geq 2\varepsilon/n$ — a contradiction. Therefore there exists a zero x of the function g, and for the corresponding $y \in S^n$ we have $f(y) = f(-y)$. □

Proof of the Tucker Lemma. [7, Freund-Todd] Let T be a special triangulation of B^n. For a simplex $\sigma \in T$ we set $\mathrm{sgn}\,\sigma = (\mathrm{sgn}\,x_1, \mathrm{sgn}\,x_2, \ldots, \mathrm{sgn}\,x_n)$, where x is an arbitrary point of the relative interior of σ. This definition always makes sense, since a special triangulation refines orthants of $\mathbb{R}^n$ and therefore the signs of the coordinates do not change inside σ. We say that σ is *completely labeled* if the following holds for each $i = 1, 2, \ldots, n$: if $(\mathrm{sgn}\,\sigma)_i = 1$, then some of the vertices of σ is labeled by the number i, and if $(\mathrm{sgn}\,\sigma)_i = -1$, then some vertex of σ is labeled by $-i$.

We define a graph G whose vertices are all completely labeled simplices, and in which vertices $\sigma, \tau \in T$ are connected by an edge if

(a) $\sigma, \tau \in \partial B^n = S^{n-1}$ and $\sigma = -\tau$, or

(b) σ is a k-simplex and τ is its $(k-1)$-face whose vertices are already labeled by all numbers required for a complete labeling of σ .

The simplex $\{0\}$ has degree 1 in G, since it is connected exactly to the edge of the triangulation which is completely labeled by $\mathrm{lab}(0)$. Further we prove that any other vertex σ of the graph G has degree 2 except when σ contains a complementary edge. Since a graph cannot contain only one vertex of an odd degree, this will establish the claim of the lemma.

Let $\mathrm{sgn}\,\sigma$ have k nonzero components, then the dimension of σ can be k or $k-1$. If σ is a $(k-1)$-simplex, it is a face of two completely labeled k-simplices or it is at the boundary of B^n, it is a face of one completely labeled simplex and it has the other neighbor $-\sigma$ according to (a).

If σ is a k-simplex, it has k obligatory labels and one extra label. This extra label either repeats some of the obligatory labels, then σ is adjacent to 2 of its faces, or it is opposite to some of the obligatory labels, then we have a complementary edge, or finally it is yet another number and the neighbors of σ are one its completely labeled face and one adjacent simplex of larger dimension determined by the extra label. For each possibility without a complementary edge we thus have two neighbors. □

2.3 Application: The necklace problem

Two thieves want to divide a necklace composed of d kinds of precious stones on a string (with 2 ends, thus the necklace is not closed) in such a way that each thief gets an equal number of stones of each kind (we assume an even number of stones of each kind). What is the minimum number of cuts needed?

Observation. At least d cuts may be necessary (place the stones of the 1st kind first, then the stones of the 2nd kind, ...).

Theorem 2.3 [Alon] *d cuts always suffice.*

Proof. Let us have k_i stones of the ith kind, $n := \sum_{i=1}^{d} k_i$. We imagine the necklace on the interval $[0, 1]$, the kth stone corresponds to the segment $[(k-1)/n, k/n)$. First we define characteristic functions $f_i(x) : [0, 1] \to \{0, 1\}$: for $x \in [\frac{k-1}{n}, \frac{k}{n})$,

$$f_i(x) = \begin{cases} 1 & \text{if the } k\text{th stone of the necklace is of the } i\text{th kind} \\ 0 & \text{otherwise.} \end{cases}$$

Each function f_i defines a measure μ_i on $[0, 1]$, $\mu_i(A) := \int_A f_i \mathrm{d}x$.

A division of the stones between the thieves will be encoded by a point $x \in S^d$. Let $0 \le z_1 \le z_2 \le \ldots \le z_d \le 1$ be the positions of the d cuts, and further let s_i be $+1$ if the part $[z_{i-1}, z_i]$ goes to the first thief, otherwise let $s_i = -1$, $i = 1, 2, \ldots, d+1$ (where $z_0 := 0, z_{d+1} := 1$). Such a division of the stones is assigned the point $(x_1, .., x_{d+1}) \in S^d \subseteq \mathbb{R}^{d+1}$ such that $x_i^2 = z_i - z_{i-1}$ and $\operatorname{sgn}(x_i) = s_i$.

We define a continuous antipodal function $f : S^d \to \mathbb{R}^d$ by

$$(x_1, .., x_{d+1}) \mapsto \left(\sum_{j=1}^{d+1} \operatorname{sgn}(x_j) \mu_i[z_{j-1}, z_j] \right)_{i=1}^{d},$$

where $z_j = \sum_{k=1}^{j} x_k^2$.

From the Borsuk-Ulam theorem we get the existence of a point $x \in S^d$ with $f(x) = 0$. This x encodes a just division, since $f(x)_i = 0$ means that both thieves get the same number of stones of the ith kind. If x encodes a "nonintegral" division (i.e. some stones would have to be cut) we use a rounding procedure first (we proceed by induction). □

Remark. Surprisingly, the only known proof of the necklace theorem is the above presented topological one.

Remark. For a solution of a similar problem with more than 2 thieves the Borsuk-Ulam theorem needs to be generalized.

2.4 Application: The ham-sandwich theorem

"For every sandwich made of bread, ham and cheese, the bread, the cheese and the ham can all be halved by a single straight cut."

Theorem 2.4 (Ham Sandwich Theorem, a continuous version) *Let $\varphi_1, \ldots, \varphi_d$ be measurable functions*[1] *$\mathbb{R}^d \to [0, \infty)$ with $\int_{R^d} \varphi_i \mathrm{d}x = 1$. Then there exists a hyperplane h such*

[1] We can imagine $\varphi_i(x)$ as the density of some matter of i-th kind, e.g., bread or cheese, at a point x, and the total mass of the matter of the ith kind is 1.

that

$$\int_{h^+} \varphi_i \, \mathrm{d}x = \frac{1}{2}, \quad \text{for } i = 1, 2, .., d$$

(h^+ denotes one of the halfspaces defined by h).

Proof. Let $a = (a_0, a_1, \ldots, a_d)$ be a point of the sphere S^d. If at least one of the components $a_1, a_2, \ldots, a_d$ is nonzero, we assign to the point a the halfspace

$$H(a) = \{(x_1, \ldots, x_d) \in \mathbb{R}^d;\ a_1x_1 + \cdots + a_dx_d \le a_0\}.$$

Obviously antipodal points of S^d correspond to opposite halfspaces. For an a of the form $(a_0, 0, 0, \ldots, 0)$ (where $a_0 = \pm 1$), we have by the same formula

$$\begin{aligned} H((1, 0, \ldots, 0)) &= \mathbb{R}^d \\ H((-1, 0, \ldots, 0)) &= \emptyset. \end{aligned}$$

We define a function $f : S^d \to \mathbb{R}^d$ by

$$f(a) = \left(\int_{H(a)} \varphi_i(x) \mathrm{d}x \right)_{i=1}^{d}.$$

It is easily checked that if we have $f(a) = f(-a)$ for some $a \in S^d$, then the boundary of the halfspace $H(a)$ is the desired hyperplane (clearly it cannot happen that $f((1, 0, \ldots, 0)) = f((-1, 0, \ldots, 0))$, so $H(a)$ is indeed a halfspace). For an application of the Borsuk-Ulam theorem we need to show that f is continuous.

First we note that for every $\varepsilon > 0$ there exists a number $R(\varepsilon)$ such that the integral of each of the functions φ_i over the complement of an $R(\varepsilon)$-ball centered at the origin is $< \varepsilon$. From this it is seen that every component of the function f is continuous at any point $a \neq (\pm 1, 0, \ldots, 0)$. For these exceptional points we note that if the points a approaches e.g., the point $(1, 0, \ldots, 0)$, the halfspace $H(a)$ will contain an arbitrarily large ball centered at the origin, and thus the integral of φ_i over $H(a)$ tends to 1, which is what we need. □

Remark. The proof can also be done using the $(d-1)$-dimensional Borsuk-Ulam theorem.

Theorem 2.5 (Ham Sandwich Theorem, discrete version) *Let $A_1, A_2, .., A_d \subset \mathbb{R}^d$ be finite point sets. Then there exists a hyperplane h halving each A_i ("h halves A_i" means that both the open halfspaces defined by h contain at most $\frac{|A_i|}{2}$ points of A_i).*

Sketch of a proof from the continuous version. For a number $\varepsilon > 0$ we manufacture a density function $\varphi_{i,\varepsilon}$ from A_i, which corresponds to tiny balls around the points $p \in A_i$, each ball has radius ε and mass $1/|A_i|$. For $\varepsilon \searrow 0$ the hyperplanes halving $\varphi_{i,\varepsilon}$ yield a halving hyperplane for the A_i. □

Remark. Recently Živaljević and Vrećica [13] proved (by a more advanced topological means) a nice generalization of the ham-sandwich theorem: for any $k+1$ "reasonable" density distributions $\varphi_1, \ldots, \varphi_{k+1}$ in $\mathbb{R}^d$ there exists a k-flat f such that any hyperplane passing through f has at least $\frac{1}{d-k+1}$ of the ith mass on each side, for all $i = 1, 2, \ldots, k+1$. Ham-sandwich is obtained for $k = d-1$. The case $k = 0$ is another classical result known as the *centerpoint theorem.*

2.5 Consequences of the ham-sandwich theorem

Theorem 2.6 [Akiyama-Alon] *Consider d n-point sets $A_1, \ldots, A_d$ in general position in $\mathbb{R}^d$ (we imagine that the points of each set are colored by one color). Then the points of the union $A_1 \cup \ldots \cup A_d$ can be partitioned into d-tuples in such a way that each d-tuple contains one point of each color, an the convex hulls of these d-tuples are pairwise disjoint.*

Proof sketch. We halve repeatedly using the ham-sandwich theorem, until each piece only contains d points, which are of different colors.

Remark. For $d = 2$ the theorem can be proved in an elementary way (consider the matching with a minimum total edge length). No elementary proof is known in higher dimensions.

Examples of "equipartition theorems".

- Any set in $\mathbb{R}^2$ can be dissected into 4 parts of equal size by 2 lines (this is an application of the ham-sandwich cut theorem).
- Any set in $\mathbb{R}^3$ can be cut into 8 equally large pieces by 3 planes.
- The above cannot in general be done in $\mathbb{R}^5$ (to cut a set into 32 equal parts by 5 hyperplanes).
- It is not known whether a dissection into 16 parts by 4 hyperplanes is possible in $\mathbb{R}^4$.
- A "cobweb partition theorem" — a partition of a set in $\mathbb{R}^2$ into 8 equally large parts by a cobweb [11, Schulman], see fig. 2.
- For every finite set of lines in $\mathbb{R}^3$ there exist 3 perpendicular planes such that the interior of each of the resulting octants is intersected by at most half of the lines [8, Paterson].

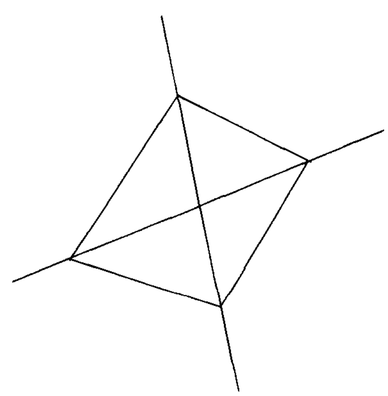

Figure 2: A cobweb equipartition.

3 Further topological notions and results

3.1 Joins

Quotient space. Let T be a topological space and let $\approx$ be an equivalence relation on its elements. We define a topology on the set $T/\approx$ of equivalence classes as follows: A set $U \subseteq T/\approx$ is open iff the union of all the classes of U is open in T. If $(S_i,\, i \in I)$ is some family of subsets of T, we define an equivalence $\approx$ on T corresponding to this family as follows: $x \approx y$ iff $x = y$ or there exists $i \in I$ with $x, y \in S_i$. Then we write $T/(S_i,\, i \in I)$ for $T/\approx$ (meaning: the space $T/(S_i,\, i \in I)$ is obtained from T by shrinking each of the sets S_i into a single point).

Example. Let $U = [0,1] \times [0,1]$ be the unit square. By gluing the two vertical sides together, i.e. by taking $U/(\{(0,y),(1,y)\}_{y\in[0,1]})$ we obtain the surface of a cylinder. The horizontal edges can be further glued either in a "direct" way (that is, a point $(x,0)$ is identified with $(x,1)$ for each $x \in [0,1]$), this produces a torus, or in a "twisted" way (i.e. a point $(x,0)$ is identified with $(1-x,1)$), which leads to the so-called Klein bottle (which cannot be realized in $\mathbb{R}^3$, however).

Example. $B^d/(S^{d-1}) \cong S^d$.

Example. Let X, Y be topological spaces and $x \in X, y \in Y$ be their points. We define the *wedge* $X \vee Y = X \dot{\cup} Y/(\{x,y\})$ (in cases we will consider the choice of the points x, y does not matter).

The join $X * Y$. The *join* of spaces X and Y is formally defined as

$$X * Y = (X \times Y \times [0,1])/(\{(x,y,0), x \in X\}_{y\in Y}, \{(x,y,1), y \in Y\}_{x \in X}).$$

The points of $X * Y$ are thus classes of equivalence on triples of the form (x, y, t), where $x \in X$, $y \in Y$ and $t \in [0,1]$. Such a triple is often written in the form of a "formal convex combination" $tx + (1-t)y$. This reflects the structure of the join somewhat, see an example below, but we must not go too far in the analogy with a usual convex combination; for instance, our new notation is not commutative: while e.g., $\frac{1}{3}x + \frac{2}{3}y$ stands for the triple $(x, y, \frac{1}{3})$, $\frac{2}{3}y + \frac{1}{3}x$ means another triple $(y, x, \frac{2}{3})$. Moreover, we have to remember that for each $y \in Y$ we shrink all points of the form $0x + 1y$ $(x \in X)$ to a single point (depending on y, however), and similarly for each $x \in X$ we identify all points of the form $1x + 0y$ $(y \in Y)$. The introduced notation should help to remember this identification.

Remark. $\dim(X * Y) = \dim(X) + \dim(Y) + 1$.

Example. The join of two segments is a 3-simplex. Geometrically we can imagine the segments being placed on two skew lines in $\mathbb{R}^3$; then their join is the tetrahedron which is the convex hull of these two segments. In general, we have $\|\sigma^n\| * \|\sigma^m\| \cong \|\sigma^{n+m+1}\|$.

Remark. The "join" operation is associative and commutative, more exactly: by rearranging the parentheses or the order of the spaces in a multiple join we obtain homeomorphic spaces.

Special cases of joins.

- Join of X with a one-point space is called the *cone* of X: $\operatorname{cone}(X) := X * \{\cdot\}$. For example, $\operatorname{cone}(S^d) \cong B^{d+1}$.

- Join of X with a 2-point discrete space, i.e. with S^0, is called the *suspension* of X: $\mathrm{susp}(X) := X * S^0$. For example, $\mathrm{susp}(S^d) \cong S^{d+1}$. From this we further get, using the associativity of the join, $S^n * S^m \cong S^{n+m+1}$.

Join of simplicial complexes Δ_1, Δ_2. (We recall that $\emptyset \notin \Delta_1, \Delta_2$.) Let us suppose that $\Delta_1^0 \cap \Delta_2^0 = \emptyset$, then we define the simplicial complex

$$\Delta_1 * \Delta_2 = \Delta_1 \cup \Delta_2 \cup \{\sigma_1 \dot{\cup} \sigma_2, \sigma_1 \in \Delta_1, \sigma_2 \in \Delta_2\}.$$

Fact. We have $\|\Delta_1 * \Delta_2\| = \|\Delta_1\| * \|\Delta_2\|$. (Proof as an exercise.)

If $\Delta_1^0 \cap \Delta_2^0 \neq \emptyset$ and we want to form the join $\Delta_1 * \Delta_2$, we rename the vertices of Δ_2 first. In particular, for joins of a simplicial complex with itself we have:

Definition. Let Δ be a simplicial complex. Its *p-fold join* Δ^{*p} is a simplicial complex whose point set is $\Delta^0 \times \{1, 2 \dots, p\}$ and whose simplices are all nonempty sets of the form

$$\{(v,1);\, v \in \sigma_1\} \cup \{(v,2);\, v \in \sigma_2\} \cup \ldots \cup \{(v,p);\, v \in \sigma_p\}$$

for $\sigma_1, \dots, \sigma_p \in \Delta \cup \{\emptyset\}$ (this agrees with the previous definition of the join; first we make p distinct copies of Δ and then we make the join).

A simplex of the above form can be mnemotechnically written as $\sigma_1 * \sigma_2 * \cdots * \sigma_p$ (but $*$ is not commutative here!).

The points of $\|\Delta^{*p}\|$ can again be written as formal (noncommutative) convex combinations of the form $t_1x_1 + t_2x_2 + \cdots + t_px_p$, where $x_1, \dots, x_p \in X$, $t_1, \dots, t_p \in [0,1]$ and $t_1 + t_2 + \cdots + t_p = 1$ (with an appropriate identification of points with $t_i = 0$).

Join and the cartesian product. The cartesian product $X \times Y$ of topological spaces X and Y can be canonically embedded into the join $X * Y$: a point $(x,y) \in X \times Y$ is mapped to the point $\frac{1}{2}x + \frac{1}{2}y \in X * Y$. Similarly the pth cartesian power X^p can be embedded into X^{*p} by $(x_1, \dots, x_p) \mapsto \frac{1}{p}x_1 + \cdots + \frac{1}{p}x_p$.

Example. As was mentioned above, the join of two segments a, b, i.e. of 1-simplices, is the 3-simplex. The cartesian product $a \times b$ is a square, and the embedding identifies it with the middle slice (parallel to 2 opposite edges) of the 3-simplex, see fig. 3.

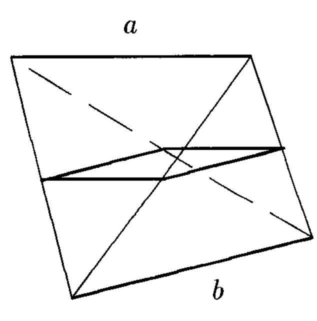

Figure 3: Join of the segments a, b and the embedding $a \times b \to a * b$.

The meaning of the join for posets. If P_1, P_2 are posets, we define $P_1 * P_2$ as "laying P_1 below P_2". We have, using the notation of section 1.5, $\Delta(P_1 * P_2) = \Delta(P_1) * \Delta(P_2)$.

3.2 Homotopy, k-connectedness

In the sequel, X, Y, Z are topological spaces, and all the considered mappings between topological spaces are implicitly continuous.

Definition. Mappings $f, g : X \to Y$ are *homotopic*, (which is written as $f \sim g$), if there exists a mapping $F : X \times [0,1] \to Y$ such that for all $x \in X$

$$F(x,0) = f(x) \quad \text{and} \quad F(x,1) = g(x).$$

Spaces X, Y are *homotopically equivalent*, $X \simeq Y$, if there exist mappings $f : X \to Y$ and $g : Y \to X$ such that $f \circ g \sim id_Y$ and $g \circ f \sim id_X$.

Below we give a more intuitive description of the homotopic equivalence (Theorem 3.1), first we need to introduce one more notion.

Definition. Let $X \subseteq Y$. X is a *deformation retract* of Y if there exists a mapping $F : Y \times [0,1] \to Y$ satisfying

(i) $\forall x \in Y : F(x,0) = x$

(ii) $\forall x \in Y : F(x,1) \in X$

(iii) $\forall x \in X : F(x,t) = x$ for all t.

Meaning: Y can be continuously shrunk onto X with X staying fixed during the shrinking.

Theorem 3.1 *$X \simeq Y$ iff there exists a Z such that both X and Y are homeomorphic to some deformation retracts of Z.* (We omit the proof.)

Definition. A topological space X is *0-connected* if it is nonempty and arcwise connected, i.e. any two points can be connected by an arc (a continuous image of an interval).

X is *k-connected ($k \in \mathbb{N}$)* if it is $(k-1)$-connected and moreover any mapping $f : S^k \to X$ can be extended to a mapping $\bar{f} : B^{k+1} \to X$.

Remark. Intuitively, the k-connectedness of X means that X has no "holes" of dimension $k+1$ and smaller. A hole of dimension k can be surrounded by a $(k-1)$-dimensional sphere within X, which cannot be contracted to a point inside X. This agrees with the following

Fact. S^n is $(n-1)$-connected and not n-connected.

Criteria of k-connectedness. We mention several facts without proofs (references and more criteria can be found in [3]).

Fact. Homotopic equivalence preserves k-connectedness.

The following theorem speaks about "gluing" of simplicial complexes by suitable subcomplexes:

Theorem 3.2 *Let Δ_1, Δ_2 be k-connected (abstract) simplicial complexes (i.e. their polyhedra are k-connected), and let $\Delta_1 \cap \Delta_2$ (which is also a simplicial complex) be $(k-1)$-connected. Then $\Delta_1 \cup \Delta_1$ is k-connected.*

In particular, the wedge of two k-connected polyhedra is k-connected.

For the readers who know the notion of a homological group:

Theorem 3.3 *Let Δ be a 1-connected simplicial complex whose all homological groups up to dimension k vanish. Then Δ is k-connected.*

This criterion has been used in several papers proving combinatorial results. Here we avoid using it.

3.3 Generalizations of the Borsuk–Ulam theorem

Let p be a prime number.

Definition. A $\mathbb{Z}_p$*-space* is a pair (X, ν), where X is a topological space and ν is a *free action of the group* $\mathbb{Z}_p$ *on* X, i.e. $\nu : X \to X$ is a homeomorphism such that for any $x \in X$, the points $x, \nu(x), \nu^2(x), \dots, \nu^{p-1}(x)$ are pairwise distinct and $\nu^p(x) = x$.

A $\mathbb{Z}_p$*-mapping* $f : (X, \nu) \to (Y, \omega)$ is a continuous mapping of X into Y such that $f \circ \nu = \omega \circ f$.

Remark. In the Borsuk-Ulam theorem we had a $\mathbb{Z}_2$-space $(S^n, x \mapsto -x)$. We generalize the statement that there is no antipodal mapping of S^{n+1} into S^n. Later we will have a yet more general theorem, but the proof of the following claim nicely illustrates the meaning of k-connectedness in similar results.

Theorem 3.4 *Let (X, ν) be a k-connected $\mathbb{Z}_2$-space. Then there exists no $\mathbb{Z}_2$-mapping $f : (X, \nu) \to (S^k, x \mapsto -x)$.*

Proof: by contradiction. Suppose that such an f exists. We find $g : S^{k+1} \to X$ such that $f \circ g : S^{k+1} \to S^k$ is antipodal.

First we construct the so-called *equivariant decomposition* of the sphere. Let M_0^+ be a set consisting of one arbitrary point of S^{k+1}, let M_0^- consists of its antipodal point, and set $L_0 = M_0^+ \cup M_0^-$. Having defined L_{i-1}, which is an $(i-1)$-dimensional sphere embedded into S^{k+1} as a "great circle", we consider some placement L_i of the sphere S^i within S^{k+1} as a "great circle" in such a way that L_{i-1} is its "equator". L_{i-1} divides L_i into 2 open hemispheres M_i^+, M_i^-, each homeomorphic to the interior of B^i, and these hemispheres are antipodal, $M_i^+ = -M_i^-$. The sets M_i^+, M_i^- for $i = 0, 1, \dots, k+1$ form a partition of S^{k+1}. (We could also write a definition using coordinates: $M_i^+ = \{(x_0, \dots, x_{k+1}) \in S^{k+1}; x_i > 0, x_j = 0 \text{ for } j = i+1, i+2, \dots, k+1\}$, similarly for M_i^-).

The mapping g is constructed inductively. For $x \in M_0^+$ (which is a single point!) we put

$$g(x) := x_0 \in X \text{ arbitrarily}$$

$$g(-x) := \nu(g(x)).$$

This defines g on L_0. We now want to extend it from L_i to L_{i+1} ($i \le k$); we assume inductively that g is continuous on L_i and $g(-x) = \nu(g(x))$ for every $x \in L_i$. L_i is homeomorphic to S^i and $L_i \cup M_{i+1}^+$ is homeomorphic to B_{i+1}. From the k-connectedness of the space X we get that g can be continuously extended on the domain $L_i \cup M_{i+1}^+$. It remains to put, for $x \in M_{i+1}^-$, $g(x) := \nu(g(-x))$. This defines the mapping g on L_{i+1}.

For $x \in M_{i+1}^+$ we have $g(-x) = \nu(g(x))$ by definition, and for $x \in M_{i+1}^-$ we have $\nu(g(x)) = \nu^2(g(-x)) = g(-x)$ (by the property $\nu^2 = \mathrm{id}$), hence g is a $\mathbb{Z}_2$-mapping on L_{i+1}. It remains to verify the continuity of g which is routine. Let $x, y \in L_{i+1}$ be two close points, $\mathrm{dist}(x, y) < \delta$, we want to show that also $g(x), g(y)$ are close (i.e. for any $\varepsilon > 0$ we can choose δ so small that $\mathrm{dist}(g(x), g(y)) < \varepsilon$). If $x, y \in L_i \cup M_{i+1}^+$ then this follows from the assumed continuity of the extension. For $x, y \in L_i \cup M_{i+1}^-$ we have $-x, -y \in L_i \cup M_{i+1}^+$, so $g(-x), g(-y)$ are close points, and $g(x) = \nu(g(-x))$, $g(y) = \nu(g(-y))$ are also close by the continuity of ν. Finally for $x \in M_{i+1}^+$, $y \in M_{i+1}^-$ we can find a suitable point $z \in L_i$ "in between" (e.g., the intersection of L_i with the great circle arc connecting x and y) and use the triangle inequality.

In this way g is extended on the domain $L_{k+1} = S^{k+1}$.

Let us now consider the mapping $f \circ g$. This is continuous and antipodal: $f(g(-x)) = f(\nu(g(x))) = -f(g(x))$. The existence of such a mapping of S^{k+1} to S^k contradicts the Borsuk-Ulam theorem, however. □

A general theorem of the Borsuk-Ulam type. The following theorem uses the notion of dimension of a topological space. We restrict ourselves to the case when the given space is a polyhedron of a simplicial complex and we let its dimension be the dimension of the underlying simplicial complex. The notion of dimension can however be extended to a much wider class of spaces.

Theorem 3.5 [6, Dold] *Let (X, ν) be a k-connected $\mathbb{Z}_p$-space and let (Y, ω) be a $\mathbb{Z}_p$-space of dimension at most k. Then there is no $\mathbb{Z}_p$-mapping $X \to Y$.* (We omit the proof.)

3.4 Deleted products

The following operation produces a space with a natural $\mathbb{Z}_p$-action from an arbitrary topological space.

Definition. Let X be a topological space. Its *p-fold deleted product* is defined as follows:

$$X^p_\Delta = \underbrace{X \times X \times \cdots \times X}_{p\times} \setminus \{\underbrace{(x, x, \ldots, x)}_{p\times};\ x \in X\}\,.$$

We define a $\mathbb{Z}_p$-action ν on this deleted product as a cyclic shift of the coordinates by one to the left, that is,

$$\nu : (x_1, x_2, \ldots, x_p) \mapsto (x_2, x_3, \ldots, x_p, x_1)\,.$$

Observation. If p is a prime, then ν is a *free* $\mathbb{Z}_p$-action (this need not hold for a composite p!).

Example (important). Let us see how the deleted product $(\mathbb{R}^d)^p_\Delta$ looks like. We interpret the space $\mathbb{R}^{d\times p} = \left(\mathbb{R}^d\right)^p$ as the space of matrices $(x_{ij})_{i=1}^{d}{}_{j=1}^{p}$ with d rows and p columns. The elements of $(\mathbb{R}^d)^p_\Delta$ are matrices of this form except for the matrices with all columns being equal. For instance, for $d = 1$, $p = 3$ we get the 3-dimensional Euclidean space from which the diagonal $\{x_1 = x_2 = x_3\}$ is removed.

Lemma 3.6 *There exists a $\mathbb{Z}_p$-mapping $g : (\mathbb{R}^d)^p_\Delta \to S^{d(p-1)-1}$, where one has a suitable free $\mathbb{Z}_p$-action on the sphere $S^{d(p-1)-1}$.*

Proof. First we consider the orthogonal projection g_1 of the space $R^{d\times p}$ on the $d(p-1)$-dimensional subspace L which is perpendicular to the diagonal. In coordinates, L will be the subspace consisting of all $d \times p$ matrices with zero row sums, and g_1 maps a matrix $x = (x_{ij})$ to the matrix

$$g_1(x) = \left(x_{ij} - \frac{1}{p}\sum_{k=1}^{p} x_{ik}\right)\,.$$

For instance, for $d = 1, p = 3$ is g_1 the orthogonal projection onto the plane $x_1 + x_2 + x_3 = 0$.

Further let g_2 be the mapping contracting $L \setminus \{0\}$ onto the unit sphere in L,

$$g_2 : x \mapsto \frac{x}{\|x\|}\,,$$

and $g = g_2 \circ g_1$. The range of g is thus the $(d(p-1)-1)$-dimensional sphere, which is represented by matrices of L satisfying moreover $\sum_{i,j} x_{ij}^2 = 1$. The $\mathbb{Z}_p$ action on this sphere is the cyclic shift of the columns by 1 column to the left. Clearly g is a $\mathbb{Z}_p$-mapping.

Let us remark that $S^{d(p-1)-1}$ is a deformation retract of $(\mathbb{R}^d)^p_\Delta$ — this can be seen quite similarly. □

As an example of application, we prove a generalization of part (1.1) of the Borsuk-Ulam theorem 2.1 (the above theorem of Dold generalizes part (1.2)).

Theorem 3.7 *Let (X, ω) be a $\mathbb{Z}_p$-space, p a prime, and $f : X \to \mathbb{R}^d$ a continuous mapping. If X is $(d(p-1)-1)$-connected, then there exists $x \in X$ such that $f(x) = f(\omega(x)) = f(\omega^2(x)) = \ldots = f(\omega^{p-1}(x))$.*

Proof. Suppose that there is no point $x \in X$ with $f(x) = f(\omega(x)) = \ldots = f(\omega^{p-1}(x))$. Then we can introduce a mapping $f_p : (X, \omega) \to (\mathbb{R}^d)^p_\Delta$ by

$$x \mapsto (f(x), f(\omega(x)), \ldots, f(\omega^{p-1}(x))) .$$

We consider a $\mathbb{Z}_p$-mapping $g : (\mathbb{R}^d)^p_\Delta \to S^{d(p-1)-1}$ as in Lemma 3.6. Then, however, $g \circ f_p : (X, \omega) \to S^{d(p-1)-1}$ is a $\mathbb{Z}_p$-mapping of a $(d(p-1)-1)$-connected space into a space of dimension $d(p-1)-1$. We have a contradiction with Theorem 3.5. □

4 Non-embeddability theorems and coloring theorems

In this chapter we cover a general technique elaborated by Sarkaria [9], [10]. We explain it step by step on several specific results. For some of the results we also give other, more elementary proofs.

4.1 Topological Radon theorem

Theorem 4.1 (Radon) *Every $(d+2)$-element set in $\mathbb{R}^d$ can be divided into 2 disjoint subsets whose convex hulls intersect.*

(This can be proved in an elementary way, using affine dependence.)

An equivalent formulation of Radon theorem: *Let $f : \|\sigma^{d+1}\| \to \mathbb{R}^d$ be a linear mapping, then there exist two disjoint faces A_1, A_2 of σ^{d+1} such that $f(\|A_1\|) \cap f(\|A_2\|) \neq \emptyset$.*

Proof of the equivalence: Each such f is determined by the mapping of the $d+2$ vertices of the simplex. The image of a face is the convex hull of the images of their vertices.

In this section we prove

Theorem 4.2 (Topological version of Radon theorem [Bajmóczy,Bárány]) *Let $f : \|\sigma^{d+1}\| \to \mathbb{R}^d$ be a* continuous *mapping, then there exist two disjoint faces A_1, A_2 of σ^{d+1} such that $f(\|A_1\|) \cap f(\|A_2\|) \neq \emptyset$.*

First proof. We use the following

Lemma 4.3 *There exists a continuous mapping $g : S^d \to \|\sigma^{d+1}\|$ such that for every $x \in S^d$, $\mathrm{supp}(g(x)) \cap \mathrm{supp}(g(-x)) = \emptyset$ (recall that $\mathrm{supp}(y)$ denotes the simplex containing x in its relative interior).*

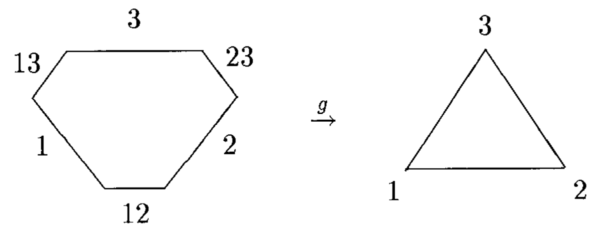

Figure 4: Illustration to Lemma 4.3 for $d = 1$

The mapping g can be defined geometrically, for instance, see fig. 4 for $d = 1$ (we interpret S^1 as the perimeter of a hexagon whose sides are labeled by the faces of σ^2 they are mapped to). We will discuss more general methods how to prove similar statements soon.

Using Lemma 4.3, the topological Radon theorem is proved immediately as follows: $f \circ g : S^d \to \mathbb{R}^d$ is continuous, so by the Borsuk-Ulam theorem we have $f(g(x)) = f(g(-x))$ for some x, and then $A_1 = \operatorname{supp}(g(x))$, $A_2 = \operatorname{supp}(g(-x))$ are the desired faces. □

For the proof of Lemma 4.3 and of more general statements one can use deleted products, this time deleted products of simplicial complexes.

4.2 Deleted product of simplicial complexes

Definition. Let Δ be a simplicial complex. Let the (twofold) *deleted product* of Δ be the topological space denoted by $\|\Delta^2_\Delta\|$ and defined by

$$\|\Delta^2_\Delta\| = \bigcup\{\|\sigma_1\| \times \|\sigma_2\|;\, \sigma_1, \sigma_2 \in \Delta,\, \sigma_1 \cap \sigma_2 = \emptyset\}.$$

This is a subspace of the cartesian product $\|\Delta\| \times \|\Delta\|$ arising by removing cells intersecting the diagonal. It is also a $\mathbb{Z}_2$-space, the action ν is the exchange of the coordinates.

Explanation. We have defined the deleted product of a simplicial complex directly as a topological space, not as a simplicial complex. The deleted product has no natural structure of a simplicial complex (since e.g., the product of 2 segments is a square), but it is a so-called *cell complex*, it is pasted together from cells of the form $\|\sigma_1\| \times \|\sigma_2\|$, which are convex polytopes. In this sense $\|\Delta^2_\Delta\|$ is a "polyhedron of the cell complex" Δ^2_Δ. We will not consider the cell complexes any more here.

Remark. We have $\|\Delta^2_\Delta\| \subsetneq \|\Delta\|^2_\Delta$ (the right hand side is a deleted product of topological spaces, see section 3.4), the inclusion is strict in general.

Example. $(\sigma^1)^2_\Delta$ are 2 isolated points. The deleted product $(\sigma^2)^2_\Delta$ is depicted in fig. 5 on the left (the vertices of σ^2 are labeled $1, 2, 3$, and e.g., the edge labeled 12×3 in the figure arises as the cartesian product of the edge $\{1, 2\}$ and of the vertex $\{3\}$). The deleted product $(\sigma^3)^2_\Delta$ consists of triangles and squares pasted together by their edges into a shape $\cong S^2$ (fig. 5 right shows the graph of the corresponding polytope).

Observation. If (x_1, x_2) is a point of $\|\Delta^2_\Delta\|$, $x_1, x_2 \in \|\Delta\|$, then $\operatorname{supp}(x_1) \cap \operatorname{supp}(x_2) = \emptyset$.

Outline of a proof of Theorem 4.2 using deleted products. As the above example indicates, $\|(\sigma^{d+1})^2_\Delta\| \cong S^d$. We will not prove this here (later on we show how to circumvent

Figure 5: Deleted product: $(\sigma^2)^2_\Delta$ and $(\sigma^3)^2_\Delta$.

this), but suppose for a while that we knew it (it would be sufficient to know that this deleted product is $(d-1)$-connected; this can be proved e.g., using Theorem 3.3).

As we saw in the proof of Theorem 3.4, we can then construct a $\mathbb{Z}_2$-mapping $h : S^d \to \|(\sigma^{d+1})^2_\Delta\|$. The values of the mapping h are pairs of points, $h(x) = (y_1, y_2)$, $y_1, y_2 \in \|\sigma^{n+1}\|$. We claim that if we define a mapping $g : S^d \to \|\sigma^{d+1}\|$ as the first component of the mapping h, we obtain exactly the mapping required in Lemma 4.3. Indeed, if $h(x) = (y_1, y_2)$, then $h(-x) = (y_2, y_1)$ (since h is a $\mathbb{Z}_2$-mapping), and thus $\operatorname{supp}(g(x)) \cap \operatorname{supp}(g(-x)) = \operatorname{supp}(y_1) \cap \operatorname{supp}(y_2) = \emptyset$.

Having the general Theorem 3.5, we can establish the topological Radon theorem more directly as follows. For contradiction, suppose that there exists a mapping $f : \|\sigma^{d+1}\| \to \mathbb{R}^d$, for which the images of any two disjoint faces of σ^{d+1} are disjoint. Then we can define a mapping

$$f_2 : \|(\sigma^{d+1})^2_\Delta\| \to (\mathbb{R}^d)^2_\Delta$$

by

$$f_2 : (x_1, x_2) \mapsto (f(x_1), f(x_2))$$

(for the points x_1, x_2 we have $\operatorname{supp}(x_1) \cap \operatorname{supp}(x_2) = \emptyset$, so $f(x_1) \neq f(x_2)$ by the assumption, and hence the mapping f_2 indeed goes into the deleted product). Apparently this is a $\mathbb{Z}_2$-mapping. The space on the left hand side can be further $\mathbb{Z}_2$-mapped into S^{d-1} (as we know from Lemma 3.6). Thus, if we knew that the deleted product on the left hand side is $(d-1)$-connected, no such $\mathbb{Z}_2$-mapping f_2 may exist, and the resulting contradiction would prove theorem 4.2.

4.3 Deleted joins

In the preceding section we have omitted the proof of the $(n-1)$-connectedness of the deleted product $\|(\sigma^{n+1})^2_\Delta\|$. We now replace the operation of deleted product by the so-called deleted join. This operation might look more complicated on the first sight, but as we will see, the required n-connectedness of the resulting space is proved in a quite simple way.

Definition. Let Δ be a simplicial complex. A (twofold) *deleted join* Δ^{*2}_Δ is a subcomplex of the join Δ^{*2} consisting of all simplices of the form $\sigma_1 * \sigma_2$, $\sigma_1, \sigma_2 \in \Delta \cup \{\emptyset\}$ with $\sigma_1 \cap \sigma_2 = \emptyset$ (thus, unlike the deleted product, the deleted join is a simplicial complex).

Remark. By the notation introduced in section 3.1, we can write the points of $\|\Delta^{*2}_\Delta\|$ in the form $tx_1 + (1-t)x_2$, where $t \in [0,1]$, $x_1, x_2 \in \|\Delta\|$. For the deleted join we moreover have $\operatorname{supp}(x_1) \cap \operatorname{supp}(x_2) = \emptyset$.

Definition. On $\|\Delta_{\Delta}^{*2}\|$ we define a free $\mathbb{Z}_2$-action $\nu : tx_1 + (1-t)x_2 \mapsto (1-t)x_2 + tx_1$.

Example. $(\sigma^1)_{\Delta}^{*2}$ is the perimeter of a square; the maximal (1-dimensional) simplices are $\emptyset * \{1,2\}$, $\{1,2\} * \emptyset$, $\{1\} * \{2\}$ and $\{2\} * \{1\}$, where $1,2$ denotes the vertices of σ^1. The $\mathbb{Z}_2$-action ν is the symmetry around the center of the square.

The following lemma allows us to compute $(\sigma^n)_{\Delta}^{*2}$ easily.

Lemma 4.4 *Let A, B be simplicial complexes. We have*

$$(A * B)_{\Delta}^{*2} = A_{\Delta}^{*2} * B_{\Delta}^{*2}.$$

Proof — clear from the definition.

Corollary 4.5

$$\left\|(\sigma^n)_{\Delta}^{*2}\right\| \cong S^n.$$

Proof. We have $\sigma^n = (\sigma^0)^{*n+1}$. By Lemma 4.4 we obtain

$$((\sigma^0)^{*n+1})_{\Delta}^{*2} = ((\sigma^0)_{\Delta}^{*2})^{*n+1} = (S^0)^{*n+1} \cong S^n.$$

□

Second proof of the topological Radon theorem. For contradiction, we again assume that there exists a continuous mapping $f : \|\sigma^{d+1}\| \to \mathbb{R}^d$ which identifies no points with disjoint supports. We define a new mapping

$$f_2 : \|(\sigma^{d+1})_{\Delta}^{*2}\| \to \left(\mathbb{R}^{d+1}\right)_{\Delta}^{2}$$

by

$$f_2 : tx_1 + (1-t)x_2 \mapsto (v_1, v_2),$$

where $v_1 \in \mathbb{R}^{d+1}$ is the vector $(t, tf(x_1))$ (the first component is the number t, the next d components are the d components of the vector $tf(x_1)$), and similarly $v_2 = ((1-t), (1-t)f(x_2))$. The mapping f_2 goes indeed into the deleted product, since the equality $v_1 = v_2$ would mean $t = \frac{1}{2}$ and thus also $f(x_1) = f(x_2)$. Also, it is not difficult to check that f_2 is continuous, and obviously it is a $\mathbb{Z}_2$-mapping. The space on the left hand side is homeomorphic th S^{d+1} (Corollary 4.5), and the right hand side can be $\mathbb{Z}_2$-mapped into S^d (by Lemma 3.6) — a contradiction to Theorem 3.5. □

4.4 Tverberg theorem; p-fold deleted joins

A generalization of the Radon theorem is

Tverberg theorem. *For any d, p, any $(d+1)(p-1)+1$ points in $\mathbb{R}^d$ can be partitioned into p pairwise disjoint subsets $A_1, \ldots, A_p$ in such a way that* $\mathrm{conv}(A_1) \cap \ldots \cap \mathrm{conv}(A_p) \neq \emptyset$.

The original proof of this result is complicated. Recently Sarkaria found a simple proof using some linear algebra and a lemma due to Bàràny.

Theorem 4.6 (Topological version of Tverberg theorem) *Let p be a prime. Put $N = (d+1)(p-1)$, further let f be continuous,*

$$f : \|\sigma^N\| \to \mathbb{R}^d.$$

Then there exist p pairwise disjoint faces $A_1, \ldots, A_p \in \sigma^N$ such that

$$f(\|A_1\|) \cap f(\|A_2\|) \cap \ldots \cap f(\|A_p\|) \neq \emptyset.$$

Remark. It seems that this theorem might hold for an arbitrary number p (not for primes only), but no proof is known.

The proof of the topological Tverberg theorem can be done very similarly as the proof of the topological Radon theorem. We need, however, p-fold deleted joins. We give a slightly more general definition than we actually need right now.

Definition. Let Δ be a simplicial complex, $p \in \mathbb{N}$, $2 \leq j \leq p$. A *p-fold j-wise deleted join* $\Delta^{*p}_{\Delta(j)}$ is the subcomplex of the p-fold join Δ^{*p} formed by all simplices of the form $\sigma_1 * \cdots * \sigma_p$, where the simplices of $\sigma_1, \ldots, \sigma_p \in \Delta \cup \{\emptyset\}$ are *j-wise disjoint*, that is, each j of these simplices have an empty intersection. In particular, for $j = 2$ we only take p-tuples consisting of pairwise disjoint simplices, and for $j = p$ we exclude p-tuples in which all simplices have a vertex in common. In this section we only need the $j = 2$ case.

Remark. Let us note the difference between the definition of the p-fold 2-wise deleted join and the definition of the p-fold deleted product (section 3.4). In case of the deleted product, we have deleted p-tuples of points with all components being equal, while in case of the p-fold 2-wise deleted join we delete p-tuples of simplices in which at least 2 components intersect. We could thus use the longer name p-fold p-wise deleted product, and in general define a p-fold j-wise deleted product (of topological spaces or of simplicial complexes). In our applications we only need the definitions given above.

Remark. The points of $\|\Delta^{*p}_{\Delta(2)}\|$ are of the form $t_1x_1 + \cdots + t_px_p$, $t_i \in [0, 1]$, $t_1 + \cdots + t_p = 1$, $x_1, \ldots, x_p \in \|\Delta\|$, $\mathrm{supp}(x_1), \ldots, \mathrm{supp}(x_p)$ pairwise disjoint. On $\|\Delta^{*p}_{\Delta(2)}\|$, we define a free $\mathbb{Z}_p$-action $\nu : t_1x_1 + \cdots + t_px_p \mapsto t_2x_2 + \cdots + t_px_p + t_1x_1$.

Analogously to lemma 4.4 we have

Lemma 4.7 *Let A, B be simplicial complexes. Then*

$$(A * B)^{*p}_{\Delta(2)} = A^{*p}_{\Delta(2)} * B^{*p}_{\Delta(2)} .$$

□

Corollary 4.8 *The space $\left\|(\sigma^n)^{*p}_{\Delta(2)}\right\|$ is $(n-1)$-connected.*

Proof. This time we have

$$(\sigma^n)^{*p}_{\Delta(2)} = ((\sigma^0)^{*n+1})^{*p}_{\Delta(2)} = ((\sigma^0)^{*p}_{\Delta(2)})^{*n+1} = (D_p)^{*n+1} ,$$

where D_p denotes the discrete p-point space. One can verify by induction that D_p^{*n+1} is homotopically equivalent to a wedge of finitely many copies of S^n and thus $(n-1)$-connected. □

Proof of the topological Tverberg theorem 4.6. We proceed almost literally as in the proof of the topological Radon theorem (which is the case $p = 2$).

Using f, we define a new mapping

$$f_p : \|(\sigma^N)^{*p}_{\Delta(2)}\| \to \left(\mathbb{R}^{d+1}\right)^p_\Delta ,$$

$$f_p : t_1x_1 + \cdots + t_px_p \mapsto (v_1, \ldots, v_p) ,$$

where $v_i = (t_i, t_if(x_i))$. One again verifies that this mapping is continuous, goes into the deleted product and it is a $\mathbb{Z}_p$-mapping. The space on the left hand side is $(N-1)$-connected according to Corollary 4.8, while the right hand side can be $\mathbb{Z}_p$-mapped into $S^{(d+1)(p-1)-1}$ (Lemma 3.6) — a contradiction to Theorem 3.5. □

4.5 Van Kampen-Flores theorem and its generalizations

The previous result is a theorem about the nonexistence of a certain embedding into $\mathbb{R}^d$ (without p-fold points). Similar methods can be used to show also the following classical result:

Theorem 4.9 [Van Kampen], [Flores] *The simplicial complex $K = (\sigma^{2d})^{\leq d-1}$ (the $(d-1)$-skeleton of the $2d$-dimensional simplex) cannot be realized in $\mathbb{R}^{2d-2}$.*

Remarks. For $d = 2$ we obtain the non-embeddability of the complete graph on 5 vertices into the plane. Generally this theorem shows that the dimension in Theorem 1.1 is best possible.

In fact we prove more: If $f : \|K\| \to \mathbb{R}^{2d-2}$ is continuous, then there exist points $x_1, x_2 \in \|K\|$ with disjoint supports such that $f(x_1) = f(x_2)$.

If we knew that K^{*2}_{Δ} is $(2d-2)$-connected, this claim would immediately follow similarly as in the second proof of the topological Radon theorem in section 4.3. Flores has indeed proved this; he has shown that $\|K^{*2}_{\Delta}\| \simeq S^{2d-1}$.

Here we use a method of Sarkaria [9], which "completes" the simplicial complex K^{*2}_{Δ} to the whole $(\sigma^{2d})^{*2}_{\Delta}$ (for which we know how does it look like).

Auxiliary results. Let Δ be a simplicial complex, $U \subseteq \Delta$ be a subset of simplices (not necessarily a subcomplex). Similarly as we have defined the first barycentric subdivision (see section 1.5), we can put $\mathrm{Sd}(U) = \Delta(P(U))$, this is already a simplicial complex, a subcomplex of $\mathrm{Sd}(\Delta)$. The points of $\mathrm{Sd}(U)$ are thus simplices of U, and the simplices are chains in the ordering of the simplices of U by inclusion.

Lemma 4.10 *Let Δ be a simplicial complex, $\Delta = U_1 \dot\cup U_2$, where U_1, U_2 are sets of simplices (not necessarily subcomplexes). Then one has a (canonical) simplicial embedding*

$$\varphi : \mathrm{Sd}(\Delta) \to \mathrm{Sd}(U_1) * \mathrm{Sd}(U_2)\,.$$

Proof. A vertex of $\mathrm{Sd}(\Delta)$ is some simplex $\sigma \in \Delta$. We define

$$\varphi(\sigma) = \begin{cases} \sigma * \emptyset & \text{for } \sigma \in U_1 \\ \emptyset * \sigma & \text{for } \sigma \in U_2. \end{cases}$$

A simplex $\mathcal{S}$ in $\mathrm{Sd}(\Delta)$ is a chain ordered by inclusion, $\mathcal{S} = (\sigma_1 \subsetneq \sigma_2 \subsetneq \ldots \subsetneq \sigma_k,\ \sigma_i \in \Delta)$. The simplices of $\mathcal{S}$ belonging to U_1 thus form a simplex in $\mathrm{Sd}(U_1)$, and similarly the simplices of $\mathcal{S}$ belonging to U_2 form a simplex in $\mathrm{Sd}(U_2)$. Hence the whole chain $\mathcal{S}$ corresponds to a simplex in the join $\mathrm{Sd}(U_1) * \mathrm{Sd}(U_2)$. □

Definition. Let X, Y, U, V be topological spaces. For mappings $f : U \to X$ a $g : V \to Y$ we define the *join of the mappings f and g*:

$$f * g : U * V \to X * Y,$$

$$f * g : tu + (1-t)v \mapsto tf(u) + (1-t)g(v)\,.$$

The verification of the following lemma is easy and is left to the reader.

Lemma 4.11

(i) *If* (U,ω_U) *and* (V,ω_V) *are* $\mathbb{Z}_p$*-spaces, the mapping* $\omega_U * \omega_V$ *is a free* $\mathbb{Z}_p$*-action on* $U * V$.

(ii) *If* $f : (U,\omega_U) \to (X,\nu_X)$ *and* $g : (V,\omega_V) \to (Y,\nu_Y)$ *are* $\mathbb{Z}_p$*-mappings, then also* $f * g : (U * V, \omega_U * \omega_V) \to (X * Y, \nu_X * \nu_Y)$ *is a* $\mathbb{Z}_p$*-mapping.*

□

Let Δ be a simplicial complex and K its subcomplex. On $\|K_\Delta^{*2}\|$, we have the natural $\mathbb{Z}_2$-action ν. Let us put $U = \Delta_\Delta^{*2} \setminus K_\Delta^{*2}$. We can also define a natural $\mathbb{Z}_2$-action ω on $\mathrm{Sd}(U)$: The vertices of $\mathrm{Sd}(U)$ are simplices of the form $\sigma_1 * \sigma_2$, $\sigma_1, \sigma_2 \in \Delta \cup \{\emptyset\}$; we define $\omega_0(\sigma_1 * \sigma_2) = \sigma_2 * \sigma_1$. This is a simplicial mapping $\mathrm{Sd}(U) \to \mathrm{Sd}(U)$, and it defines a $\mathbb{Z}_2$-action $\omega = \|\omega_0\|$ on $\|Sd(U)\|$, as is easily checked.

By Lemma 4.10 we have the embedding

$$\varphi : \mathrm{Sd}(\Delta) \to \mathrm{Sd}(K_\Delta^{*2}) * \mathrm{Sd}(U)\,.$$

Lemma 4.12 *The subspace* $\|\varphi(\mathrm{Sd}(\Delta))\| \subseteq \|\,\mathrm{Sd}(K_\Delta^{*2}) * \mathrm{Sd}(U)\|$ *is closed under the* $\mathbb{Z}_2$*-action* $\nu * \omega$.

Proof sketch: The mapping $\nu * \omega$ is induced by a simplicial mapping, hence it suffices to verify that each vertex v of the simplicial complex $\varphi(\mathrm{Sd}(\Delta_\Delta^{*2}))$ is mapped to a vertex of the same complex by the mapping $\nu * \omega$. We omit the details. □

Proof of the Van Kampen-Flores theorem 4.9. Set $\Delta = \sigma^{2d}$. We have a mapping $f : \|K\| \to \mathbb{R}^{2d-2}$; we suppose that it identifies no two points with disjoint supports.

First we define, similarly as in the second proof of the topological Radon theorem (section 4.3), a $\mathbb{Z}_2$-mapping

$$f_2 : \|K_\Delta^{*2}\| \to \left(\mathbb{R}^{2d-1}\right)^2_\Delta$$

(by putting $f_2 : tx_1 + (1-t)x_2 \mapsto (v_1, v_2)$, $v_1 = (t, tf(x_1))$, $v_2 = ((1-t), (1-t)f(x_2))$).

As we did before lemma 4.12, we introduce the notation $U = \Delta_\Delta^{*2} \setminus K_\Delta^{*2}$, and by Lemma 4.10 we have the embedding

$$\|\varphi\| : \|\Delta_\Delta^{*2}\| \to \|K_\Delta^{*2}\| * \|\,\mathrm{Sd}(U)\|\,.$$

We define a mapping $g : \mathrm{Sd}(U) \to \{1,2\} \cong S^0$ as follows:

$$g(\sigma_1 * \sigma_2) = \begin{cases} 1 & \text{for } \sigma_1 \in K \\ 2 & \text{for } \sigma_2 \in K\,. \end{cases}$$

This mapping is well-defined for each vertex of $\mathrm{Sd}(U)$: such a vertex is a simplex of U, and has the form $\sigma_1 * \sigma_2$, where σ_1, σ_2 are disjoint faces of the $2d$-dimensional simplex; at least one of σ_1, σ_2 must lie in K, because the dimension of all simplices from $\Delta \setminus K$ is at least d.

Further, it is easy to see that g is a simplicial mapping, and that $\|g\|$ is a $\mathbb{Z}_2$-mapping (the $\mathbb{Z}_2$-action ω on $\mathrm{Sd}(U)$ has been described above Lemma 4.12, the $\mathbb{Z}_2$-action on S^0 is the exchange of the points 1 and 2).

We consider the $\mathbb{Z}_2$-mapping

$$F = f_2 * \|g\| : \|K_\Delta^{*2}\| * \|\,\mathrm{Sd}(U)\| \to \left(\mathbb{R}^{2d-1}\right)^2_\Delta * S^0\,.$$

The right hand side can be $\mathbb{Z}_2$-mapped into $S^{2d-2} * S^0 \cong S^{2d-1}$. The mapping F can be restricted to the domain $\|\varphi(\Delta^{*2}_\Delta)\| \cong S^{2d}$. Since this subspace is closed under the appropriate $\mathbb{Z}_2$-action by Lemma 4.12, we have finally obtained a $\mathbb{Z}_2$-mapping of a $(2d-1)$-connected space into a space of dimension $\leq 2d-1$, and such a mapping cannot exist by Theorem 3.5. □

A generalization of the same proof yields the following:

Theorem 4.13 *Write $\Delta = \sigma^N$ and let K be a subcomplex of Δ. Put $L = \Delta \setminus K$ and let L_0 be the set of all inclusion-minimal simplices in L. Suppose that $\chi : L_0 \to 2^{\{1,2,\ldots,m\}} \setminus \{\emptyset\}$ is a coloring of the simplices of L_0 by nonempty subsets of an m-element set, which satisfies the condition*

$$\sigma_1 \cap \sigma_2 = \emptyset \Rightarrow \chi(\sigma_1) \cap \chi(\sigma_2) = \emptyset . \tag{2}$$

Then for $d \leq N - m - 1$ there is no embedding $\|K\| \to \mathbb{R}^d$.

Applications.

- Van Kampen-Flores theorem was with $m = 1$, $N = 2d$, the condition (2) was void since L had no 2 disjoint simplices.
- As an exercise, let us prove that the complete bipartite graph $K_{3,3}$ is not planar. We put $N = 5, d = 2, m = 2$, K is formed by the vertices and edges of $K_{3,3}$, L is its complement to the 5-simplex, L_0 are the 6 pairs which are not edges of $K_{3,3}$. We color the 3 pairs (non-edges) from one class of $K_{3,3}$ by the color $\{1\}$, and the 3 pairs from the second class by the color $\{2\}$. This works.

Proof of Theorem 4.13. Suppose that an embedding $f : \|K\| \to \mathbb{R}^d$ exists. The mapping

$$f_2 : \|K^{*2}_\Delta\| \to \left(\mathbb{R}^{d+1}\right)^2_\Delta$$

is defined as above. We again write $U = \Delta^{*2}_\Delta \setminus K^{*2}_\Delta$. We consider the values of the mapping χ, i.e. subsets of $\{1, 2, \ldots, m\}$, as faces of σ^{m-1}. First we extend the definition of χ to all simplices of Δ; we put

$$\chi(\sigma) = \bigcup_{\tau \in L_0,\, \tau \subseteq \sigma} \chi(\tau).$$

Note that $\chi(\sigma) = \emptyset$ iff $\sigma \in K$. Further we observe that this extended mapping now satisfies the condition (2) for any 2 simplices $\sigma_1, \sigma_2 \in L$, and moreover it is monotone, i.e. $\sigma \subseteq \sigma'$ implies $\chi(\sigma) \subseteq \chi(\sigma')$ for any $\sigma, \sigma' \in \Delta$. We define a derived mapping

$$\chi_2 : \mathrm{Sd}(U) \to \mathrm{Sd}((\sigma^{m-1})^{*2}_\Delta)$$

by

$$\chi_2(\sigma_1 * \sigma_2) = \chi(\sigma_1) * \chi(\sigma_2) .$$

Since σ_1 or σ_2 belongs to L, $\chi(\sigma_1) * \chi(\sigma_2)$ is indeed a simplex of the deleted join $(\sigma^{m-1})^{*2}_\Delta$. The monotonicity of χ then implies that χ_2 is simplicial (see the remark in section 1.5), and thus it defines a $\mathbb{Z}_2$-mapping between the respective polyhedra. Finally we consider the mapping

$$F = f_2 * \|\chi_2\| : \|K^{*2}_\Delta\| * \|\,\mathrm{Sd}(U)\| \to \left(\mathbb{R}^{d+1}\right)^2_\Delta * \|(\sigma^{m-1})^{*2}_\Delta\| .$$

The domain of F can be restricted to $\|\varphi(\Delta^{*2}_\Delta)\| \cong S^N$ (and F remains a $\mathbb{Z}_2$-mapping). The range of F can further be $\mathbb{Z}_2$-mapped into $S^d * \|(\sigma^{m-1})^{*2}_\Delta\|$. The dimension of this space is $d + (m-1) + 1 = d + m$, therefore Theorem 3.5 requires that $d + m > N - 1$. □

4.6 Kneser conjecture

The explained method can be used not only for proving the impossibility of an embedding from the existence of a certain coloring (as in Theorem 4.13), but also the other way round — for proving the nonexistence of a coloring from a suitable embedding. We illustrate this procedure on the celebrated problem of the chromatic number of the so-called Kneser graphs.

Definition. Let n, k be natural numbers, $n \geq 2k$, $X = \{1, 2, \ldots, n\}$. The *Kneser graph* $G = G(n,k)$ has the set of vertices $V(G) = \binom{X}{k}$, and the set of edges $E(G) = \{(A, B);\ A, B \in \binom{X}{k},\ A \cap B = \emptyset\}$.

Theorem 4.14 (Kneser Conjecture, [Lovász]) *The chromatic number of the graph $G(n,k)$ is equal to $n - 2k + 2$.*

Upper bound for the chromatic number. We color the vertices of the Kneser graph as follows: First we color the points of the set X by assigning color 0 to the first $2k - 1$ points, and by assigning one color of $1, 2, 3, \ldots, n - 2k - 1$ to each of the remaining points. The coloring of a vertex $v \in V(G(n,k))$ is defined by $\chi(v) = \max\{\chi(x),\ x \in v\}$. This is a coloring of the Kneser graph in the graph-theoretic sense, since at least one of two disjoint k-tuples has some element colored by a nonzero color. The maximum of colors cannot be the same for both vertices, since this would mean they have a common point and thus they do not form an edge of the Kneser graph.

Lower bound for the chromatic number. This was first proved by Lovász, all known proofs are topological. The following proof is essentially due to Sarkaria [10].

Let us identify the set X with the set of vertices of an $(n-1)$-dimensional simplex σ^{n-1}. The vertices of $G(n,k)$, i.e. k-tuples, can be regarded as $(k-1)$-dimensional faces. We assume that χ_0 is a coloring of the Kneser graph by m colors, i.e. $\chi_0 : \binom{X}{k} \to \{1, 2, \ldots, m\}$.

We want to apply Theorem 4.13 in a "reverse direction". To this end, we set $N = n - 1$ in that Theorem and we let K be the $(k-2)$-skeleton of σ^N. Theorem 1.1 tells us that K can be realized in $\mathbb{R}^d$ with $d = 2k - 3$. The set L_0 in Theorem 4.13 will be be the set of simplices in σ^N of dimension $k - 1$, which is precisely the vertex set of the Kneser graph. For $\sigma \in L_0$, define $\chi(\sigma) = \{\chi_0(\sigma)\}$ (formally we need coloring by *subsets*). Then the condition (2) says exactly that χ_0 is a proper coloring of the Kneser graph.

Since an embedding $\|K\| \to \mathbb{R}^d$ exists this time, we can conclude from Theorem 4.13 that $d \geq N - m$. Substituing the values $d = 2k - 3$, $N = n - 1$ yields $m \geq n - 2k + 2$. □

The presented method can immediately be generalized to hypergraphs etc. (see the paper [10]). We give one more proof of Kneser's conjecture, which is very simple but for which no similar generalizations are known.

Lower bound for the chromatic number — a second method. [Bárány]

Lemma 4.15 [Gale] *Let $k \geq 1$, $d \geq 0$, then there exists a $(2k+d)$-element set $V \subset S^d$ such that any open hemisphere of S^d contains at least k points of V.*

Assuming this lemma, we proceed by contradiction. Let $\chi : \binom{X}{k} \to \{1, 2, \ldots, d+1\}$ be a coloring of the Kneser graph, where the set X is identical to the set V from the above lemma. Further we define sets $A_1, \ldots, A_{d+1} \subseteq S^d$ as follows: for a point $x \in S^d$ we have $x \in A_i$ iff the open hemisphere centered at the point x contains a k-tuple of points of X colored by the color i. These classes $A_1, \ldots, A_{d+1}$ form an open cover of S^d, and by the Borsuk-Ulam theorem there exist i and $x \in S^d$ such that $x, -x \in A_i$. In this way, we get two disjoint k-tuples colored by the color i, which contradicts the coloring of the Kneser graph. □

Proof of Lemma 4.15. Let us denote

$$H(a) = \{x \in S^d; x \cdot a > 0\},$$

$$S(a) = \{x \in S^d; x \cdot a = 0\},$$

$$e_i = (\underbrace{0, \ldots, 0}_{(i-1)\times}, 1, 0, \ldots, 0), e = e_1 + \ldots + e_{d+1}.$$

Definition. A system of sets $\{U_\alpha\}_{\alpha \in A}$ is a *k-fold cover of* S^d if each point of S^d lies in at least k sets from $\{U_\alpha\}_{\alpha \in A}$.

Lemma 4.16 (An equivalent formulation of Lemma 4.15) *There exists a k-fold cover of S^d by $(2k + d)$ open hemispheres.*

Proof. By induction on k. For $k = 1$ we take the cover $H(e_1), \ldots, H(e_{d+1}), H(-e)$. For the inductive step from k to $k + 1$ we use the following

Claim. Let $H(a_1), \ldots, H(a_n)$ be a k-fold cover of S^d. Then there exist points a_{n+1}, a_{n+2} such that $H(a_1), \ldots, H(a_{n+2})$ form a $(k + 1)$-fold cover of S^d.

Proof of the claim. By induction on d. For $d = 0$ we set $a_{n+1} = 1$, $a_{n+2} = -1$. Suppose that the claim holds for $d - 1$, and let $H(a_1), \ldots, H(a_n)$ be a k-fold cover of S^d. Then $H(a_1), \ldots, H(a_{n-1})$ is a k-fold cover of $S(a_n)$, so by the inductive hypothesis there exist b, b' such that $H(a_1), \ldots, H(a_{n-1}), H(b), H(b')$ form a $(k + 1)$-fold cover of $S(a_n)$.

Let us put $c = a_n - \lambda b$, where λ is positive and sufficiently small so that $S(c)$ is also covered $k+1$ times (recall that we are covering by open sets). We now show that $H(b)$ can be replaced by $H(a_n)$ and we still have a $(k + 1)$-fold cover of $S(c)$; to this end, it suffices to verify that if $x \in S(c) \cap H(b)$ then $x \in H(a_n)$, and this holds since $x \cdot a_n = x \cdot (c + \lambda b) = \lambda x \cdot b > 0$.

Now we set $a_{n+1} = \mu b' + c$, $a_{n+2} = \mu b' - c$, where μ is positive and sufficiently small. If some point $x \in S^d$ is not covered by some $k+1$ of the sets $H(a_1), \ldots, H(a_{n+2})$ then necessarily $x \cdot a_{n+1} \leq 0$, $x \cdot a_{n+2} \leq 0$, or in other words $\mu x \cdot b' \leq x \cdot c \leq -\mu x \cdot b'$, so $x \cdot b' > 0$ and x lies in an arbitrarily thin strip around $S(c)$. The compactness of $S(c)$ implies that μ can be chosen so small that x is covered by some $k + 1$ among the open sets $H(a_1), \ldots, H(a_n), H(b')$ forming a $(k+1)$-fold cover of $S(c)$, but at the same time $x \notin H(b')$, hence x is $(k + 1)$ times covered by $H(a_1), \ldots, H(a_n)$. This finishes the proof of both the claim an Lemma 4.15. □

4.7 Colored Tverberg theorem

The Tverberg theorem implies that if we have sufficiently many points in general position in $\mathbb{R}^d$, we can select r pairwise disjoint $(d + 1)$-tuples of these points such that the simplices determined by them all have a point in common. In this section we discuss a colored version of this statement:

Colored Tverberg theorem. *For any integers $r, d > 1$ there exists an integer t, such that given sets $A_1, \ldots, A_{d+1} \subset \mathbb{R}^d$ in general position (we consider each set as points of one color), consisting of t points each, one can find disjoint $(d + 1)$-point sets $S_1, \ldots, S_r$ such that each S_i contains exactly one point of each A_j, $j = 1, 2, \ldots, d + 1$ (that is, the S_j are multicolored), and the simplices spanned by $S_1, \ldots, S_r$ all have a point in common.*

This theorem was conjectured by Bárány, Füredi and Lovász and proved by Živaljević and Vrećica [12]. Here we present a proof using the above explained approach of Sarkaria.

While the Tverberg theorem can be proved in an elementary way, all known proofs for the colored version are topological. The colored version is essential in proving a nontrivial upper bound in a famous problem in combinatorial geometry, the so-called *k-sets problem*, see Alon *et al.* [5].

We prove the following topological version (we leave the derivation of the colored Tverberg theorem from it to the reader):

Theorem 4.17 *Let d be a positive integer, p a prime. Let $A_1, \ldots, A_{d+1}$ be disjoint sets of cardinality $2p-1$ each. Let K be the simplicial complex with vertex set $A = A_1 \cup \ldots \cup A_{d+1}$, whose simplices are all subsets of A intersecting each A_i in at most 1 point. For any continuous mapping $f : \|K\| \to \mathbb{R}^d$ there exist p vertex-disjoint simplices of K whose f-images have a common intersection.*

Proof. The structure is totally similar to the proofs we've already had, the main twist is to define an appropriate coloring. Let Δ denote the simplex with vertex set A (K is its subcomplex). Suppose for contradiction the existence of a mapping $f : \|K\| \to \mathbb{R}^d$ for which the claim doesn't hold. We first define a $\mathbb{Z}_p$-mapping

$$f_p : \|K^{*p}_{\Delta(2)}\| \to (\mathbb{R}^{d+1})^p_\Delta .$$

We recall that the points of $\|K^{*p}_{\Delta(2)}\|$ can be written as formal convex combinations $t_1x_1 + \cdots + x_pt_p$, where $x_1, \ldots, x_p$ are points of $\|K\|$ with pairwise disjoint supports, and t_i are nonnegative reals with $t_1 + \cdots + t_p = 1$. We define

$$f_p : t_1x_1 + \cdots + x_pt_p \mapsto (v_1, v_2, \ldots, v_p)$$

with $v_i \in \mathbb{R}^{d+1}$ being the concatenation of the 1-element vector (t_i) and the d-element vector $t_if(x_i)$. As in the proofs before, this map is continuous and commutes with the free $\mathbb{Z}_p$-actions.

For every simplex $\sigma \in \Delta$, define a coloring of σ by a subset of $\{1, 2, \ldots, d+1\}$ (which we think of as a face of σ^d):

$$\chi(\sigma) = \{i \in \{1, 2, \ldots, d+1\};\ |\sigma \cap A_i| \geq 2\} .$$

This mapping χ is monotone, and $\chi(\sigma)$ is empty iff $\sigma \in K$.

Put $U = \Delta^{*p}_{\Delta(2)} \setminus K^{*p}_{\Delta(2)}$. To each simplex of U, which is of the form $\sigma = \sigma_1 * \ldots * \sigma_p$, assign a simplex $\chi_p(\sigma) = \chi(\sigma_1) * \ldots * \chi(\sigma_p) \in (\sigma^d)^{*p}$. This simplex is nonempty (since there is at least one component $\sigma_i \notin K$). Moreover, the intersection $\chi(\sigma_1) \cap \chi(\sigma_2) \cap \ldots \cap \chi(\sigma_p) = \emptyset$, since p disjoint simplices cannot each contain 2 points of the same A_i, as A_i has only $2p-1$ points. Therefore, the simplex $\chi_p(\sigma)$ in fact belongs to the p-fold p-wise deleted join $(\sigma^d)^{*p}_{\Delta(p)}$ (see section 4.4).

As usual, χ_p induces a simplicial $\mathbb{Z}_p$-map

$$\chi_p : \mathrm{Sd}(U) \to \mathrm{Sd}((\sigma^d)^{*p}_{\Delta(p)})$$

and we can form a $\mathbb{Z}_p$-map

$$F = f_p * \|\chi_p\| : \|K^{*p}_{\Delta(2)}\| * \|\,\mathrm{Sd}(U)\| \to (\mathbb{R}^{d+1})^p_\Delta * \|(\sigma^d)^{*p}_{\Delta(p)}\| .$$

We have the canonical embedding

$$\varphi : \mathrm{Sd}(\Delta^{*p}_{\Delta(2)}) \to \mathrm{Sd}(K^{*p}_{\Delta(2)}) * \mathrm{Sd}(U)\,,$$

and the domain of F can be restricted to $\|\Delta^{*p}_{\Delta(2)}\|$ with F remaining a $\mathbb{Z}_p$-map (a p-fold analogue of Lemma 4.12 is straightforward). The domain of this restricted $\mathbb{Z}_p$-map is $(\dim \Delta - 1)$-connected. The deleted product $(\mathbb{R}^{d+1})^p_\Delta$ can be $\mathbb{Z}_p$-mapped to $S^{(d+1)(p-1)-1}$, and it is easy to check that the p-fold p-wise deleted join $(\sigma^d)^{*p}_{\Delta(p)}$ has dimension $(d+1)(p-1)-1$. By Theorem 3.5 we get

$$\dim \Delta - 1 \le (d+1)(p-1) - 1 + (d+1)(p-1) - 1 + 1\,,$$

or $(d+1)(2p-1) - 2 \le (d+1)(2p-2) - 1$, which means $d \le 0$ — a contradiction. □

One could formulate a general theorem similar to Theorem 4.13 but concerning mappings of a simplicial complex into $\mathbb{R}^d$ with no p-fold points, whose proof would use p-fold j-wise deleted joins. We trust that the reader can do it by himself by now, and we encourage him to seek more applications or interesting generalizations.

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