JOURNAL OF COMBINATORIAL THEORY (B) 14, 216–240 (1973)

Matroid Basis Graphs. I*

STEPHEN B. MAURER

Department of Combinatories and Optimization, University of Waterloo, Waterloo, Ontario, Canada N2L 3G1

Communicated by A. W. Tucker

Received June 16, 1972

A matroid may be defined as a collection of sets, called bases, which satisfy a certain exchange axiom. The basis graph of a matroid has a vertex for each basis and an edge for each pair of bases that differ by the exchange of a single pair of elements. Two characterizations of basis graphs are obtained. The first involves certain local subgraphs and how they lie when the given graph is leveled with respect to distance from a particular vertex. The second involves the existence of a special mapping from the given graph to some "full" basis graph. It is also shown that in a natural sense all basis graphs are homotopically trivial.

INTRODUCTION

There are several approaches to the study of matroids. The approach emphasizing bases has the advantage that any one basis of a given matroid can be transformed into any other, without ever ceasing to be a basis, by exchanging elements one pair at a time. Thus it seems appropriate to regard each basis as a vertex and each pair of bases differing by a single exchange as adjacent. The graph obtained in this fashion is called the basis graph of the matroid.

It is well known that for any connected graph the spanning trees, viewed as sets of edges, form the bases of a matroid. The basis graphs of such matroids, called tree graphs, have been studied for several years. Cummins [3] showed that every tree graph (with two trivial exceptions) is Hamiltonian. Shank [13] simplified the proof. Very recently several researchers investigated basis graphs in general. Bondy [1] showed not only that every basis graph is Hamiltonian, but also that most are pancyclic. Independently Holzmann and Harary [7] showed that for every edge in a basis graph there is a Hamiltonian cycle containing it and one excluding it.

The main goal of this paper is to characterize basis graphs. Section 1 contains preliminary definitions and lemmas. Section 2 contains the statement and proof of our first characterization, which we call the Main Theorem. In Section 3 we prove some partial strengthenings of the Main Theorem and make some conjectures. Section 4 contains the second characterization, which involves mappings. Finally, in Section 5 we study a notion of homotopy which arises naturally from the methods of the previous sections.

One may ask to what extent basis graphs faithfully represent their matroids. We have answered this in [10]. It has also been answered independently by Holzmann, Norton, and Tobey [8] and by Cunningham [4]. In a sequel [11] to this paper we will investigate the relationships between matroids and their basis graphs further.

* This paper stems from Chapter I of the author's Ph.D. dissertation, [illegible] Princeton under Professor A. W. Tucker, whose suggestions and encouragement are gratefully acknowledged. Support was provided in part by the Office of Naval Research

1. PRELIMINARIES

DEFINITION 1.1. *A matroid* $\mathscr{M}$ on a finite set of *elements* E is a collection $\mathscr{B}$ of subsets of E, called *bases*, which satisfy the following *exchange axiom*:

For all $B, B' \in \mathscr{B}$ and each $e' \in B' - B$, there exists some $e \in B - B'$ such that $B - e + e' \in \mathscr{B}$.

We write $\mathscr{M} = (E, \mathscr{B})$ or simply $\mathscr{M}(E, \mathscr{B})$. We say that $B'' = B - e + e'$ is obtained from B by a *pivot step*; e' is pivoted in, e is pivoted out. We also express this diagrammatically by

$$B \xrightarrow{(e,e')} B''. \qquad (1)$$

Our definition is equivalent to the original basis definition given by Whitney [16]. In particular, one can easily show that all bases of a matroid have the same cardinality, called the *rank*.

$G(\mathscr{V}, \mathscr{E})$ will be a finite graph with vertices $\mathscr{V}$ and edges $\mathscr{E}$. We denote edges in the form vv'. Neither loops nor multiple edges are allowed. Paths are written $v_1v_2 \cdots v_n$. We do allow repetition of both vertices and edges in a path. $\delta(v, v')$ is the distance between v and v'. Given $G(\mathscr{V}, \mathscr{E})$, $\langle \mathscr{V}' \rangle$ is the induced subgraph on $\mathscr{V}' \subset \mathscr{V}$. Let $|B|$ be the cardinality of set B. G is *properly labeled* if each $v \in \mathscr{V}$ is labeled with a distinct finite set B so that

$$vv' \in \mathscr{E} \quad \text{iff} \quad |B - B'| = |B' - B| = 1.$$

14 3-3

In practice it will not be necessary to consider both $|B - B'|$ and $|B' - B|$, for it will be known that $|B| = |B'|$. Finally, we say that $\mathscr{V}'$ is properly labeled if $\langle \mathscr{V}' \rangle$ is. Clearly we may apply the notation (1) to any properly labeled graph.

DEFINITION 1.2. *The basis graph* of the matroid $\mathscr{M}(E, \mathscr{B})$ is the properly labeled graph with labeled vertices $\mathscr{B}$. It is denoted $BG(\mathscr{M})$ or $BG(E, \mathscr{B})$. A graph is *a basis graph* if it can be labeled to become the basis graph of some matroid. Clearly two bases of $\mathscr{M}$ are adjacent in $BG(\mathscr{M})$ iff they differ by a pivot step. It follows that every basis graph is connected.

DEFINITION 1.3. In a given graph suppose $\delta(v, v') = 2$ and $\mathscr{V}'$ consists of v, v' and all vertices adjacent to both. Then $\langle \mathscr{V}' \rangle$ is called the *common neighbor subgraph* $CN(v, v')$ or simply a CN. The vertices adjacent to both v and v' are their *common neighbors* or the *intermediate vertices.*

In Figure 1 we display three graphs that will be of constant use. It should be clear which name applies to which.

LEMMA 1.4. *In a basis graph each* CN *is a square, pyramid, or octahedron.*

Proof. Suppose $\delta(B, B') = 2$. We may write B' as $B-(b_1+b_2)+(c_1+c_2)$. There are only four possible common neighbors: $B - b_1 + c_1$, $B - b_1 + c_2$, $B - b_2 + c_1$, and $B - b_2 + c_2$. By the exchange axiom there must exist bases $B - b + c_1$ and $B - b' + c_2$, where b and b' are each either b_1 or b_2. If no other common neighbors exist, $b \neq b'$ and we have a square: otherwise the exchange axiom is violated with the roles of B and B' reversed. If there are three common neighbors altogether, no matter which three they are we get a pyramid. If all four exist, we get an octahedron. ▮

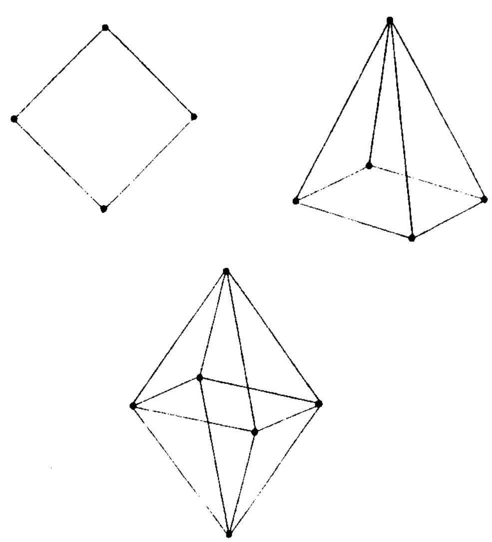

FIG. 1. A *square*, a *pyramid*, and an *octahedron*.

DEFINITION 1.5. A *leveling* of $G(\mathscr{V}, \mathscr{E})$ from v_0 is a partition of $\mathscr{V}$ into sets $\mathscr{V}_k$, $k = 0, 1, 2,\ldots$, such that

$$\mathscr{V}_k = \{v \in \mathscr{V} \mid \delta(v_0, v) = k\}.$$

If G is leveled, and properly labeled up through level k, we adopt the following conventions. For $0 \leqslant j \leqslant k$ we write $\mathscr{B}_j$ instead of $\mathscr{V}_j$. B_0 will be (the label of) the single vertex in $\mathscr{B}_0$. C will be the set of elements in at least some label but not in B_0. We may pick any one labeled vertex and write it as $A \cup D$, where $A \subset B_0$, $D \subset C$. Then using the letters a, b, c, d (with various subscripts and superscripts) to name elements of A, $B_0 - A$, $C - D$, and D respectively, we may write any other labeled vertex in the form

$$A \cup D - (a_1 + a_2 + \cdots + d_1 + \cdots) + (b_1 + b_2 + \cdots + c_1 + \cdots).$$

We will abbreviate this as

$$(a_1 a_2 \cdots d_1 \cdots / b_1 b_2 \cdots c_1 \cdots).$$

When a labeled graph is unleveled, or its leveling is temporarily irrelevant, the letters a, d will not be used, and the letters b, c may take on other meanings to be explained when needed.

With a few explicit exceptions, all figures will be subgraphs of leveled graphs. Vertices will be grouped into distinct horizontal layers. The lower the grouping on the page, the higher is the level number.

LEMMA 1.6 (The Positioning Condition). *Let* $BG(E, \mathscr{B})$ *be leveled from* B_0. *Then every octahedral* CN *lies*

(1) *entirely in some* $\mathscr{B}_k$,

(2) *across two levels as in Figure* 2, *or*

(3) *across three levels as in Figure* 1.

Moreover, every other CN *lies as an induced subgraph of an octahedron positioned as above.*

Proof. Let $\delta(B, B') = 2$. Suppose $B \in \mathscr{B}_{k-1}$ and $B' \in \mathscr{B}_{k+1}$. Then by the definition of leveling all common neighbors must be in $\mathscr{B}_k$. Now suppose $B \in \mathscr{B}_k$, $B' \in \mathscr{B}_{k+1}$. Writing $B = A \cup D$ we have $B' = (a_1a_2/bc)$ or (ad/c_1c_2). In either case an inspection of the labels of the four possible common neighbors shows that we get either Figure 2 or an induced

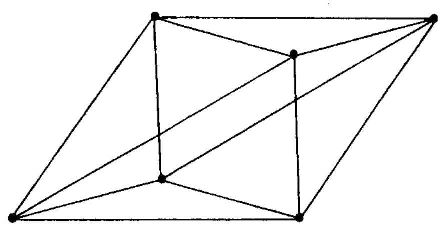

FIG. 2. An octahedron lying across two levels.

square or pyramid. Finally suppose $B, B' \in \mathscr{B}_k$. If $B = A \cup D$ then $B' = (a_1a_2/b_1b_2)$, (d_1d_2/c_1c_2) or (ad/bc). In the first two cases the CN lies entirely in $\mathscr{B}_k$. In the last case we get the octahedron of Figure 1 or an induced subgraph. ∎

DEFINITION 1.7. Let $G(\mathscr{V}, \mathscr{E})$ be leveled from v. Then $\langle \mathscr{V}_1 \rangle$ is called the *neighborhood subgraph* $N(v)$.

Recall that the *line graph* $L(G)$ has a vertex for each edge of G and an edge for each pair of edges in G which share an end-point.

LEMMA 1.8. *Suppose B_0 is a vertex in some* BG$(E, \mathscr{B})$. *Then $N(B_0)$ is the line graph of a bipartite graph.*

Proof. Define $G'(E, \mathscr{E}')$ by $bc \in \mathscr{E}'$ iff $B_0 - b + c \in \mathscr{B}$, where $b \in B_0$, $c \in E - B_0$. Clearly G' is bipartite with partition B_0, $E - B_0$. Moreover $bc \leftrightarrow B_0 - b + c$ is a bijection between the vertex sets of $L(G')$ and $N(B_0)$. We have

$$\begin{aligned} &bc \text{ is adjacent to } b'c' \Leftrightarrow \\ &b = b' \text{ or } c = c' \text{ but not both} \Leftrightarrow \qquad (2) \\ &|(B_0 - b + c) - (B_0 - b' + c')| = 1. \end{aligned}$$

Thus $L(G')$ and $N(B_0)$ may be identified. ∎

2. THE MAIN THEOREM

THEOREM 2.1 (The Main Theorem). *$G(\mathscr{V}, \mathscr{E})$ is a basis graph if and only if:*

(1) *it is connected;*

(2) *each common neighbor subgraph is a square, a pyramid, or an octahedron,*

(3) *in every leveling each common neighbor subgraph meets the Positioning Condition; and*

(4) *for some v_0 the neighborhood subgraph $N(v_0)$ is the line graph of a bipartite graph.*

Necessity has already been shown. The proof of sufficiency is the rest of this section.

THEOREM 2.2. *Suppose G is connected and properly labeled with a collection $\mathscr{B}$ of subsets of some set E. Then $(E, \mathscr{B})$ is a matroid, and hence G is a basis graph if and only if every* CN *is a square, pyramid, or octahedron.*

Proof. First we show that, if $\delta(B, B') = 2$ and

$$B \xrightarrow{(b_1, c_1)} B_1 \xrightarrow{(b_2, c_2)} B',$$

then there are vertices B_2 and B_3, not necessarily distinct, such that:

$$\begin{array}{ccc} B & \xrightarrow{(b_2, x)} & B_2 \\ {\scriptstyle (z, c_2)}\downarrow & & \downarrow{\scriptstyle (b_1, y)} \\ B_3 & \xrightarrow{(w, c_1)} & B' \end{array}$$

Here $\{x, y\} = \{c_1, c_2\}$ and $\{z, w\} = \{b_1, b_2\}$. For consider CN(B, B'). $B_1 = B - b_1 + c_1$ is an intermediate vertex. If there is only one other, B_4, then it must be $B - b_2 + c_2$ and we may set $B_2 = B_3 = B_4$. If B_4 does not exist, there must exist two intermediate vertices adjacent to B_1, and they must be labeled $B - b_2 + c_1$ and $B - b_1 + c_2$. These satisfy the conditions for B_2 and B_3, respectively.

Now let B, B' be any vertices. We must show that for any $c_0 \in B' - B$ there exists $b_0 \in B - B'$ so that $B - b_0 + c_0$ is in $\mathscr{B}$. There must be a path P from B to B', say

$$B = B_1 \xrightarrow{(b_1, c_1)} B_2 \cdots B_n \xrightarrow{(b_n, c_n)} B_{n+1} = B'.$$

We first consider the case in which P is *non-redundant*, that is,

$$\{b_1, \ldots, b_n\} \cap \{c_1, \ldots, c_n\} = \varnothing .$$

In this case each b_k must be in $B - B'$ and each c_k in $B' - B$. Surely $c_0 = c_k$ for some k, $1 \leqslant k \leqslant n$. If $k = 1$, simply pick $b_0 = b_1$. If $k \neq 1$, apply the first part of the proof to

$$B_{k-1} \xrightarrow{(b_{k-1}, c_{k-1})} B_k \xrightarrow{(b_k, c_0)} B_{k+1}$$

to get

$$B_{k-1} \xrightarrow{(x, c_0)} B_k' \xrightarrow{(y, c_{k-1})} B_{k+1} .$$

Performing such a shift $k - 1$ times altogether, we make $c_1 = c_0$.

Now suppose P is redundant. There is an element e and indices k, p such that either

$$B_k \xrightarrow{(e,c)} B_{k+1} \cdots B_{p-1} \xrightarrow{(b,e)} B_p ,$$

or

$$B_k \xrightarrow{(b,e)} B_{k+1} \cdots B_{p-1} \xrightarrow{(e,c)} B_p ,$$

and no proper subsequence of $B_k \cdots B_p$ is redundant. We handle the first case; the second is similar. Suppose $p = k + 2$. If $b = c$, then $B_p = B_k$ and we may simply delete B_k, B_{k+1} from P. If $b \neq c$, then $B_p = B_k - b + c$ and we may take the shortcut

$$B_k \xrightarrow{(b,c)} B_p .$$

If $p > k + 2$, we may use the shifting technique on $B_{k+1} \cdots B_p$ to obtain

$$B_{k+1} \xrightarrow{(b',e)} B_{k+2} .$$

Then the case $p = k + 2$ applies. In any event this redundancy of e, and likewise every other redundancy, can be eliminated. ∎

To prove the Main Theorem we now need "merely" construct a proper labeling on G. Here, and for the rest of the section, the letters G, $\mathscr{V}$, $\mathscr{E}$ (without superscripts) refer to some graph satisfying (1)–(4). Also, $\mathscr{V}_i$ will refer specifically to the leveling of G from v_0 of (4). The purpose of (4) is to get the labeling started.

LEMMA 2.3. *$\mathscr{V}_0 \cup \mathscr{V}_1$ can be properly labeled.*

Proof. By (4), $N(v_0) = L(G')$ for some bipartite $G'(E, \mathscr{E}')$. Arbitrarily let one set in the partition of E be called B_0, the other C. Rename the vertices in $N(v_0)$ by the rule $bc \to B_0 - b + c$. It is immediate from the definitions that $N(v_0)$ is now properly labeled. Finally, label v_0 with B_0. ∎

Remarks. First, were the roles of B_0 and C reversed, the labeling of $\mathscr{V}_0 \cup \mathscr{V}_1$ would be *complemented*, that is, each B would be replaced by $E - B$. Indeed, complementation gives another proper labeling for any properly labeled $\mathscr{V}'$. This is essentially matroid duality [16].

Second, if G' is disconnected, not only can the order of the partition change, but so can the sets themselves. This important observation is pursued elsewhere [8, 10].

Finally, Lemmas 1.8 and 2.3 are not really new. In essence they were first proved by Kishi and Kajitani [9].

We will now properly label G by induction on the following:

LABELING HYPOTHESIS. *$\bigcup_{j=0}^{k} \mathscr{V}_j$ can be properly labeled. For any proper labeling, if $B \in \mathscr{B}_j$ then $| B - B_0 | = j$.*

Lemma 2.3 proves the hypothesis for $k = 1$.

LEMMA 2.4. *Assume the Labeling Hypothesis for k. Given a particular labeling, suppose $\delta(v, B) = 2$, where $v \in \mathscr{V}_{k+1}$, $B \in \mathscr{B}_{k-1}$. Then there is a unique label for v which extends the given labeling to a proper labeling on CN(v, B).*

Proof. B and v must have two common neighbors B_1, B_2 such that BB_1vB_2 is a square. Let $B = A \cup D$. We must have $B_1 = (a/c)$; any other choice, (d/c) say, makes $| B_1 - B_0 | \neq k$. Likewise $B_2 = (a'/c')$ and $a' \neq a$, $c' \neq c$. Now v is a common neighbor of B_1 and B_2. Of the four labels it might thus have—$A \cup D$, (a/c'), (a'/c), (aa'/cc')—the last is always proper and none of the others ever is. ∎

Let CN(v, B) as above be called an *upward* CN on v. We note that $|(aa'/cc') - B_0| = k + 1$, so the Labeling Hypothesis will be true for $k + 1$ if all the locally proper labelings on upward CNs from $\mathscr{V}_{k+1}$ are globally consistent. We will prove global consistency in the following steps:

STEP I. Each $v \in \mathscr{V}_{k+1}$ is given the same label by every upward CN on v.

STEP II. For all $B' \in \mathscr{B}_k$ and $v \in \mathscr{V}_{k+1}$, if v has label B by Step I, then

$$B'v \in \mathscr{E} \quad \text{iff} \quad | B - B' | = 1.$$

STEP III. If $v, v' \in \mathscr{V}_{k+1}$ and $v \neq v'$, then their labels are different. Thus we may use the names B, B', $\mathscr{B}_{k+1}$ instead.

STEP IV. For all $B, B' \in \mathscr{B}_{k+1}$,

$$BB' \in \mathscr{E} \quad \text{iff} \quad |B - B'| = 1.$$

Before taking these steps we need more preparatory work.

We will often consider configurations in which appear two adjacent common neighbors B_1, B_2 of some upward CN(v, B). We may write $B = A \cup D$ and $B_1 = (a/c)$. Thus B_2 is either (a'/c) or (a/c'). In the former case, CN(v, B) contains either another common neighbor (a/c'') adjacent to B_1 or (a'/c'') adjacent to B_2, so v becomes (aa'/cc'') for some c'' not determined by B, B_1, B_2. In the latter case, v becomes (aa''/cc'), where a'' is not determined.

Actually, in most situations we need not even consider the latter case. Suppose the goal is simply to show that the whole configuration can be labeled consistently or that some two labels therein differ by just so many elements. If the desired conclusion obtains in the case $B_2 = (a'/c)$, then by complementing the entire labeled subgraph we get the same conclusion for the case $B_2 = (a/c')$. Henceforth we produce a label for v, and skip the redundant case, without comment.

We introduce one more convention. In the diagram of a subgraph a dashed line between v, v' will mean that vv' is not even an edge of the supergraph G. Of course, it will not be necessary to use this convention if two vertices are two or more levels apart. When v, v' are less than two levels apart and no line is drawn between them, solid or dashed, no claim is made about the existence of vv'.

LEMMA 2.5. *In Figure* 3, (a) *implies* (b).

Proof. If $uv \notin \mathscr{E}$, then CN(u, v) has at least two vertices, x and w, in the next level down. This contradicts the Positioning Condition, whether or not $wx \in \mathscr{E}$. The same argument, turned upside down, shows that wx is in $\mathscr{E}$. ∎

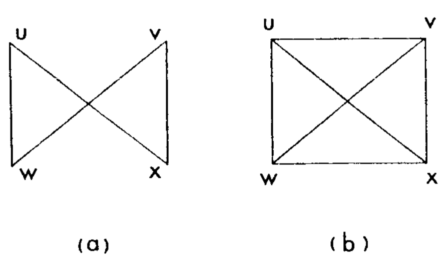

FIG. 3. (a) implies (b) by Lemma 2.5.

The unleveled subgraph in Figure 4 is called a *Propeller* with *shaft uv* and *tips w, x, y*. The edges wx, xy, yw, if they exist in the supergraph, are called *tip edges*.

LEMMA 2.6 (The Propeller Condition). *Any Propeller in G has exactly one or three tip edges.*

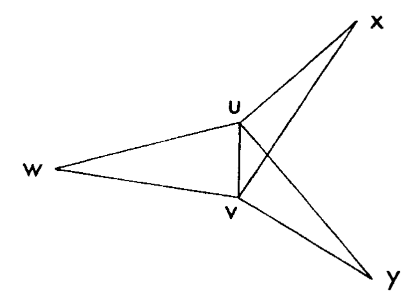

FIG. 4. A Propeller.

Proof. Suppose Figure 4 has two tip edges, say wx and xy. Then CN(w, y) is improper: the common neighbors u, x, v form a triangle. On the other hand, if Figure 4 has no tip edges, relevel G from y. Then $\langle u, v, w, x\rangle$ violates Lemma 2.5. ∎

LEMMA 2.7 (The Book Condition). *Suppose Figure* 5(a) *is an unleveled subgraph of G. Then both vw, xz* $\in \mathscr{E}$ *or neither is. In short, there are no "half open Books."*

Proof. Suppose vw, say, is an edge but xz is not. Leveling G from x we get Figure 5(b). But now square $uwzy$ violates the Positioning Condition. ∎

LEMMA 2.8 (The Siblings Condition). *In any leveling of G, if* $u, v \in \mathscr{V}_{k+1}$ *and* $uv \in \mathscr{E}$, *then there is a* $w \in \mathscr{V}_k$ *such that* $uw, vw \in \mathscr{E}$.

We think of the levels as representing generations. Thus this lemma says that every pair of siblings has a common parent.

Proof. The case $k = 0$ is trivial. Assume the case $k = p - 1$ and let u, v be adjacent in $\mathscr{V}_{p+1}$. Let x be any parent of v. If x is not also a parent of u, consider CN(u, x). If this is a pyramid or octahedron, by the Positioning Condition one of the intermediate vertices is a common parent of u, v. If

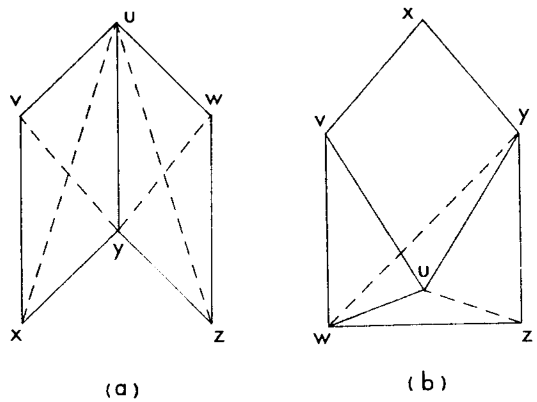

FIG. 5. The Book Condition.

this is a square $uvxy$, then $y \in \mathscr{V}_p$, and by assumption x, y have a common parent z. If CN(u, z) contains a vertex w not adjacent to y, we get Figure 6(a). By the Book Condition w is a common parent of u, v. If no such w exists, we get Figure 6(b). Consider the Propeller with shaft yz and tips w_1, w_2, x. By the Propeller Condition and symmetry we may assume $w_1 x \in \mathscr{E}$. But then CN(u, x) is a pyramid or octahedron after all. ∎

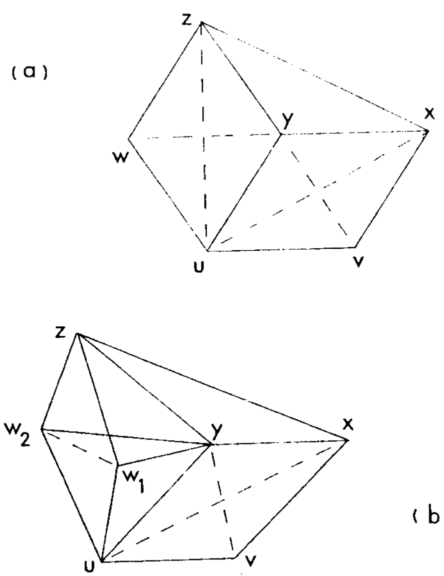

FIG. 6. Diagrams for the proof of Lemma 2.8.

We can now take Steps I–IV. Although Step I is logically first, it is the most intricate and we present it last.

STEP II. By Step I assume that every upward CN on $v \in \mathscr{V}_{k+1}$ labels v with B. Although for all we know so far some other vertex in $\mathscr{V}_{k+1}$ gets the same label, we may without fallacy use v and B interchangeably below.

That $B'v \in \mathscr{E}$ implies $| B - B' | = 1$ is easy: for any parent B'' of B', CN(v, B'') is properly labeled. As for the converse, we first prove

LEMMA 2.9. *Suppose $\bigcup_{j=0}^{k} \mathscr{V}_j$ is properly labeled. If $v \in \mathscr{V}_{k+1}$ has label $A \cup D$, then for any $b \in B_0 - A$ there exists $d \in D$ such that v has a parent (d/b).*

Proof. Let $B_{k+1} = B$. Let $B_{k+1}B_k \cdots B_1B_0$ be a path ascending directly from B to B_0. Somewhere along this path b is pivoted in. If this happens at the first step, $B_k = (d/b)$. If this happens later, we shift b forward by the technique of Theorem 2.2. Although we do not know that all of G is properly labeled, the technique still works, for at least each CN we consider is properly labeled. ∎

Now suppose $B = A \cup D$ is in $\mathscr{V}_{k+1}$, $B' \in \mathscr{B}_k$, and $| B - B' | = 1$. Then B' is some (d/b). Applying the lemma to b, we get the existence of an $X = (d'/b)$. Since $\mathscr{B}_k$ is properly labeled, $XB' \in \mathscr{E}$. Also, by the Siblings Condition X and B' have a common parent B''. If $BB' \notin \mathscr{E}$, CN(B, B'') must contain Y such that $B''XBY$ is a square. Moreover, $YB' \in \mathscr{E}$. (See Figure 7.) But now CN(X, Y) is improper, whether or not B, B'' have a third common neighbor. Thus $BB' \in \mathscr{E}$ after all.

STEP III. Suppose $v, v' \in \mathscr{V}_{k+1}$ have the same label B. Pick any upward CN on v. It must contain a square with v at the bottom. Call the top B''

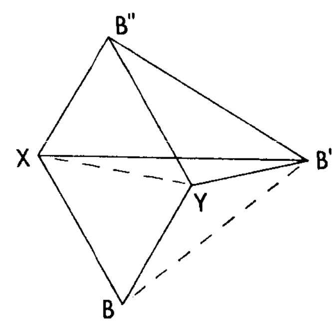

FIG. 7. Step II: CN(X, Y) is improper.

and the intermediate vertices B_1, B_2. By Step II, $v'B_1$, $v'B_2 \in \mathscr{E}$ also. But then $\langle B_1, B_2, v, v'\rangle$ violates Lemma 2.5.

STEP IV. First we show that $BB' \in \mathscr{E}$ implies $|B - B'| = 1$. As siblings, B and B' have a parent Z. Let W be any parent of Z. Suppose CN(W, B) is a pyramid with apex Z; see Figure 8(a). Consider the Propeller

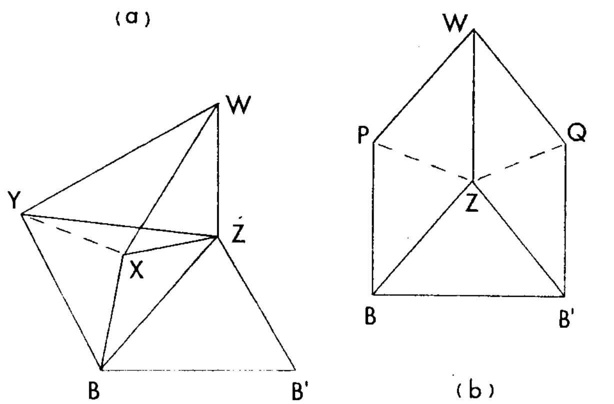

FIG. 8. Diagrams for the proof of Step IV.

with shaft BZ and tips X, Y, B'. By symmetry we may conclude that $XB' \in \mathscr{E}$. Write $W = A \cup D$, $X = (a/c)$, $Z = (a'/c)$. Then $B = (aa'/cc')$. Similarly $B' = (aa'/cc'')$ where $c'' \neq c'$ by Step III. Thus $|B - B'| = 1$.

If CN(W, B') is a pyramid with apex Z, analogous reasoning applies. In the only remaining cases, both CN(W, B) and CN(W, B') include squares containing Z; see Figure 8(b). By the Book Condition $PQ \in \mathscr{E}$. Now choose $W = A \cup D$, $P = (a/c)$, $Q = (a'/c)$. We then deduce, in order, $Z = (a''/c')$, $B = (aa''/cc')$, and $B' = (a'a''/cc')$. Thus $|B - B'| = 1$ as claimed.

Conversely we show that $|B - B'| = 1$ implies $BB' \in \mathscr{E}$. Temporarily let $B = A \cup D$. Then $B' = (a/b)$. By Lemma 2.9 some $Z = (d/h)$ exists, and by Step II Z is a common parent of B, B'. Let W be any parent of Z. Relabeling, we get $W = A \cup D$, $Z = (a/c)$, $B = (aa'/cc')$, $B' = (aa''/cc')$. If $X = (a/c')$ exists, then it too is a common parent of B, B', and by Lemma 2.5 $BB' \in \mathscr{E}$. If X does not exist, CN(W, B) and CN(W, B') must contain squares $WZBP$ and $WZBQ$, respectively (Figure 8(b) with line BB' deleted). We have $P = (a'/c')$ and $Q = (a''/c')$. Since $|P - Q| = 1$ and $P, Q \in \mathscr{B}_k$, $PQ \in \mathscr{E}$. By the Book Condition we still have $BB' \in \mathscr{E}$.

STEP I. Two upward CNs on v are said to *overlap* if they have some intermediate vertex in common.

LEMMA 2.10. *If each pair of overlapping upward* CNs *on v assign it the same label, then all upward* CNs *on v assign it the same label.*

Proof. Let vWB be a path in some fixed upward CN(v, B). Let $vW'B'$ be a path in any upward CN(v, B') not overlapping CN(v, B). If W, W' have a common parent X, then CN(v, X) overlaps both CN(v, B) and CN(v, B'), forcing both to assign v the same label. The only other possibility is that $\delta(W, W') = 2$ and CN(W, W') is a pyramid with apex v. In particular, W, W' have a common neighbor W_1 on their level. But then there exist common parents Y, Z of W, W_1 and W_1, W', respectively. Therefore CN(v, Y) and CN(v, Z) provide an overlapping link between CN(v, B) and CN(v, B'). ∎

We now need to show that two overlapping CNs on $v \in \mathscr{V}_{k+1}$ give v the same label. We will consider several cases.

LEMMA 2.11. *Suppose $\bigcup_{j=0}^{k} \mathscr{V}_j$ is properly labeled, $X, Y \in \mathscr{V}_k$, $v \in \mathscr{V}_{k+1}$ and XvY is a path. Then $|X - Y| \leqslant 2$.*

Proof. It suffices to show that there is a path of length 1 or 2 between X and Y in the properly labeled region. Clearly $\delta(X, Y) \leqslant 2$. If $XY \notin \mathscr{E}$, the Positioning Condition ensures the existence of such a path in CN(X, Y). ∎

Case 1. CN(v, B) and CN(v, B') have at least two intermediate vertices W and W' in common. By Lemma 2.5, WW', $BB' \in \mathscr{E}$. Also, at least one of W, W' is adjacent to another intermediate vertex in CN(v, B). We may assume W is. Likewise at least one of them is adjacent to another intermediate vertex in CN(v, B').

Case 1a. We have Figure 9, where all the vertices on the lower level are understood to be adjacent to v in the next level down. We follow this uncluttering convention until further notice.

Let some parent of B, B' be called $A \cup D$. Then $B = (a/c)$ and $B' = (a'/c)$. Thus $W = (aa'/cc')$, $W' = (aa'/cc'')$. Furthermore $X = (aa''/cc')$ and $Y = (a'\hat{a}/cc'')$. (Except where noted, different symbols in the same argument have always represented necessarily distinct objects. Here and for the rest of the section we allow the possible exceptions $\hat{a} = a''$ and $\hat{c} = c'$.) In fact, $|X - Y| \leqslant 2$ by Lemma 2.11, so $\hat{a} = a''$. Therefore both CN(v, B) and CN(v, B') give v the label $(aa'a''/cc'c'')$.

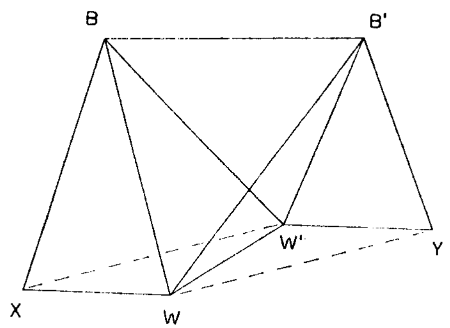

FIG. 9. Case 1a: both W and W' are adjacent to another intermediate vertex.

Case 1b. We have Figure 10. The proof given for Case 2c also covers this case; simply ignore vertex W.

Case 2. CN(v, B) and CN(v, B') have one intermediate vertex B_1 in common.

Case 2a. Both CNs are pyramids and B_1 is the apex of both (Figure 11). $BB' \notin \mathscr{E}$, else there is a Propeller with shaft $B'B_1$, tips W, Z, B, and no tip

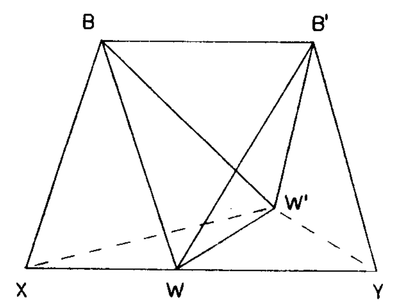

FIG. 10. Case 1b: W is adjacent to both additional vertices.

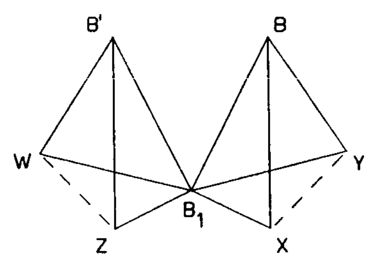

FIG. 11. Case 2a: B_1 is the apex of two pyramid CNs.

edges. Also, either WX or ZX is in $\mathscr{E}$, else there is another improper Propeller with shaft B_1v and tips W, Z, X. We may assume $ZX \in \mathscr{E}$. Likewise, either WY or ZY is an edge. However, given that $ZX \in \mathscr{E}$ we cannot have $ZY \in \mathscr{E}$, for then the Propeller with shaft B_1v and tips Z, X, Y would have exactly two tip edges.

If B, B' have a common parent, call it $A \cup D$. Even if they do not, let $A \cup D$ represent the unique name it would have. Then $B = (a/c)$, $B' = (a'/c')$, $B_1 = (aa'/cc')$, and we may pick $X = (aa'/cc'')$. Considering CN(v, B'), Z is either (aa''/cc') or $(aa'/c'\hat{c})$. Since $XZ \in \mathscr{E}$, we must have the second choice with $\hat{c} = c''$. Now $W = (a'a''/cc')$ and $Y = (a\hat{a}/cc')$, but by analogous reasoning $\hat{a} = a''$. Finally we find that in both CNs v gets labeled $(aa'a''/cc'c'')$.

Case 2b. Only one CN, say CN(v, B'), is a pyramid with B_1 as apex. We must have at least Figure 12. As before $B = (a/c)$, $B' = (a'/c')$, $B_1 = (aa'/cc')$. Then $X = (aa''/cc'')$. We may pick

$$Z = (aa'/c'\hat{c}), \qquad W = (a'\hat{a}/cc').$$

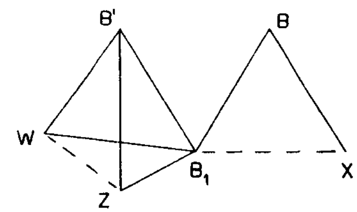

FIG. 12. Case 2b: B_1 is the apex of one pyramid CN.

By Lemma 2.11, $|Z - X|, |W - X| \leqslant 2$, so $\hat{c} = c''$ and $\hat{a} = a''$. Again $v = (aa'a''/cc'c'')$ in both CNs.

Case 2c. B_1 is the apex of neither CN. We have at least Figure 13, where now we put v back in the picture. First suppose $\delta(B, B') = 2$. We may write $B = (a/c)$, $B' = (a'/c')$. Then $B_1 = (aa'/cc')$, $X = (aa''/cc'')$,

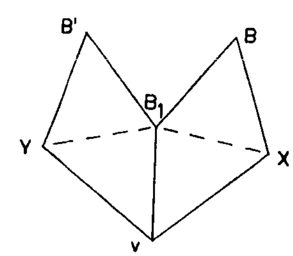

FIG. 13. Case 2c: B_1 is the apex of neither CN.

and $Y = (a'\hat{a}/c'\hat{c})$. Using Lemma 2.11 again, $\hat{a} = a''$, $\hat{c} = c''$, and v is labeled consistently.

On the other hand, suppose $BB' \in \mathscr{E}$. We may write $B = (a/c)$, $B' = (a'/c)$. Once more $B_1 = (aa'/cc')$, $X = (aa''/cc'')$, but now $Y = (a'\hat{a}/c\hat{c})$. By the Book Condition $XY \in \mathscr{E}$, so we still have $\hat{a} = a''$, $\hat{c} = c''$, and $v = (aa'a''/cc'c'')$ in both CNs.

This completes the proof of the Main Theorem.

3. Strengthenings and Conjectures

The Positioning Condition for levelings other than from v_0 has been used solely to show that there are no Propellers without tip edges and no half open Books. Either configuration, were it to exist, would be an induced subgraph. This is immediate for Propellers. For Books, consider Figure 5(a). Whether or not vw and xz are edges, we have $vz, xw \notin \mathscr{E}$; for instance, were $vz \in \mathscr{E}$, then CN(v, y) would be improper. Therefore we have

THEOREM 3.1 (Main Theorem, Second Form). *G is a basis graph if and only if:*

(1) *it is connected;*

(2) *each* CN *is a square, pyramid, or octahedron;*

(3) *no induced subgraph is a Propeller or a half open Book;*

(4) *for some vertex v_0*

  (i) *$N(v_0)$ is the line graph of a bipartite graph; and*

  (ii) *in the leveling from v_0 each* CN *meets the Positioning Condition.*

We now consider redundancies in condition 4(i). Recall that a *clique* is a maximal complete subgraph.

LEMMA 3.2. *Suppose each* CN *of $G(\mathscr{V}, \mathscr{E})$ is a square, pyramid, or octahedron, and no induced subgraph of G is a Propeller. Then, for any $v_0 \in \mathscr{V}$, $N(v_0)$ satisfies the following conditions:*

(1) *no two cliques have an edge in common;*

(2) *each vertex is in at most two cliques.*

Proof. (Part 1). Suppose vv' is in distinct cliques C, C'. There must be vertices w in C and x in C' such that $wx \notin \mathscr{E}$, else neither C nor C' would be maximal. But then CN(w, x) in G is improper, for the common neighbors v, v', v_0 form a triangle.

(Part 2) Suppose v is in three cliques C_1, C_2, C_3. No $v' \neq v$ is in two of these cliques, for then vv' would violate (1). Thus we may pick v_1, v_2, v_3, one from each clique and all distinct from v and each other. No two of these are adjacent. For suppose $v_1v_2 \in \mathscr{E}$. Then there would exist some C_4 containing $\{v, v_1, v_2\}$ and C_1, C_4 would have vv_1 in common. Thus the Propeller with shaft v_0v and tips v_1, v_2, v_3 is left with no tip edges, which is impossible. ∎

COROLLARY 3.3. *Given the hypotheses above, $N(v_0)$ is a line graph.*

Proof. Krausz's Theorem [6, p. 74] says that a graph is a line graph if its edges can be partitioned into complete subgraphs in which no vertex appears more than twice. By the above lemma, the set of all cliques of $N(v_0)$ provides just such a partition. ∎

We call a cycle a *clique cycle* if no three of its vertices are in the same clique.

THEOREM 3.4. *Suppose that each* CN *of $G(\mathscr{V}, \mathscr{E})$ is a square, pyramid or octahedron, that every pyramid* CN *containing v_0 has v_0 as its apex, and that no induced subgraph of G is a Propeller. Then $N(v_0)$ is the line graph of a bipartite graph.*

Proof. By Corollary 3.3, $N(v_0) = L(G')$ for some G'. We prove the theorem by establishing two claims. First, G' may be constructed so that each cycle in it corresponds to a clique cycle in $N(v_0)$, in the sense that the edges of the former become the vertices of the latter. Second, $N(v_0)$ has no clique cycles of odd length. It follows immediately that G' has no odd cycles whatsoever and is thus bipartite.

To show the first claim let H be any graph meeting the condition of Krausz's Theorem. A graph G' such that $L(G') = H$ is constructed by setting down a vertex v' for each complete subgraph K in the partition of H, and an edge incident to v' for each vertex in K. Now suppose specifically that $H = N(v_0)$. Recall that in this case the partition is the set of all cliques in H. Let C' be a cycle in G' and C the corresponding cycle in $N(v_0)$. Clearly no three edges of C' have a common vertex, so no three vertices of C are in the same subgraph in the partition, i.e., no three are together in any clique. This proves the first claim.

As for the second, since no graph has a clique cycle of length 3, it suffices to show that, if $N(v_0)$ has a clique cycle of length $n \geq 5$, then it also has one of length $n - 2$. Figure 14(a) shows a clique cycle in $N(v_0)$ with $n \geq 5$. Consider CN(u, v) in the supergraph G. Because it includes v_0, it is either an octahedron or a pyramid with apex v_0. In either case u, v

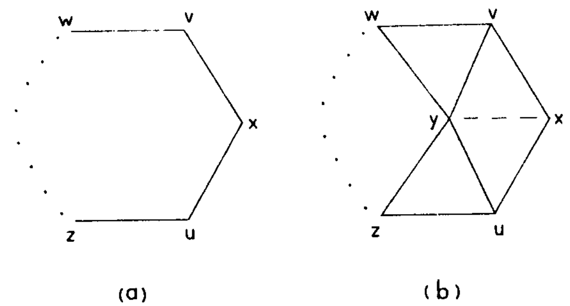

FIG. 14. Clique cycles in $N(v_0)$.

have another common neighbor y in $N(v_0)$ and $xy \notin \mathscr{E}$. Since there are only two cliques containing v, and neither wv or yv is cliqued with x, we get $wy \in \mathscr{E}$. Likewise $zy \in \mathscr{E}$. We have Figure 14(b). Now replace wxu by wyz. The new cycle has length $n - 2$. Moreover, y cannot be in the same clique with two other vertices of the new cycle, because either u or x is in a clique with any set of vertices y is cliqued with. Thus the new cycle is a clique cycle. ∎

We note that the Propeller Condition depends only on (2) and (3) of the Main Theorem (first form). Thus, as a special case of Theorem 3.4 we get

THEOREM 3.5. *If no CN of G is a pyramid, then the condition on $N(v_0)$ in either form of the Main Theorem is redundant.* ∎

CONJECTURE 1. *The condition on $N(v_0)$ is redundant for all basis graphs.*

We believe there is further redundancy in our Main Theorem. For instance, condition (3) of the second form is perhaps unnecessary. We are most interested, though, in eliminating the Positioning Condition, since this seems the strongest and most global of the conditions. In the second form the scope of the Positioning Condition is at least curtailed.

For a time we conjectured that a connected graph G is a basis graph simply iff every CN is a square, pyramid, or octahedron. Unfortunately, this is false. Consider Figure 15. Augment this graph by adding an edge between each two vertices on the bottom level whose names, as sets of digits, are disjoint. Call the result H. It is not hard to see that every CN of H is a square. Moreover, $N(\phi)$ is trivially a line graph of a bipartite graph, and there are certainly no induced Propellers or half open Books in H—there are not even any triangles. Nonetheless, H is not a basis graph, for not every CN satisfies the Positioning Condition. The square with vertices 5, 45, 12, 35 is an example.

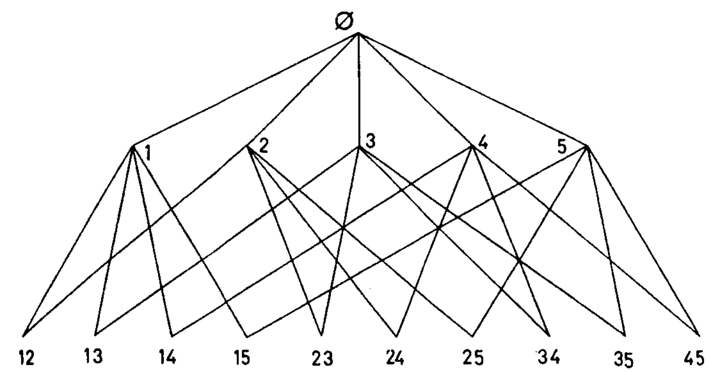

FIG. 15. Part of a graph whose CNs are all squares but which is not a basis graph.

H has been discovered many times, for instance in [5]. We have found a way of viewing it, perhaps new, which generalizes in as much as its usefulness to matroids is concerned. Briefly put, H is the edge graph of the 4-dimensional cube with all the major diagonals added in. One may show that for any $n \geq 4$ the edge graph of the n-cube with the major diagonals added has squares for all its CNs. However, in every leveling each of these has improperly positioned squares lying across $\mathscr{V}_{n-1} \cup \mathscr{V}_n$.

This construction generalizes even further. Take any basis graph in which each vertex has a unique "antipodal" vertex farthest away. Another example (the n-cube is one) is the basis graph of the matroid of all n-subsets of some $2n$-set. If v_a is the antipode of v, one can show that $\delta(v, v_a)$ is constant. If $\delta(v, v_a) \geq 4$, main diagonals can be added without violating anything but the Positioning Condition. In the additional example just given, one gets some octahedra that lie incorrectly. However all these graphs still have some square CNs. Thus the best we can hope for in the way of eliminating the Positioning Condition is

CONJECTURE 2. *Suppose each CN of a connected graph is a pyramid or octahedron. Then the graph is a basis graph.*

4. A MAPPING CHARACTERIZATION

A matroid $\mathscr{M}(E, \mathscr{B})$ of rank r is *full* if $\mathscr{B}$ consists of all r-subsets of E. If $\mathscr{M}$ is full, BG($\mathscr{M}$) is said to be full also. In this section we characterize the class of all basis graphs in terms of mappings into the small subclass of full basis graphs.

Clearly every full basis graph has octahedral CNs only. The converse is also true, as is not hard to show [11]. Thus this mapping characterization will be especially interesting if Conjecture 2, or some modification thereof, should prove true.

In any basis graph, two adjacent vertices B, B' are together in at most two cliques; moreover, if B_1, B_2 are other vertices, one from each clique, then B_1, B_2 are neither equal nor adjacent. All this follows by leveling from B and noting Lemma 3.2. Or, setting $B' = B - b_0 + c_0$, one can show directly that the vertex sets of the two cliques are

$$\mathscr{B}' = \{B\} \cup \{B - b_0 + c \in \mathscr{B} \mid c \in E - B\},$$
$$\mathscr{B}'' = \{B\} \cup \{B - b + c_0 \in \mathscr{B} \mid b \in B\}.$$

(If either $\mathscr{B}'$ or $\mathscr{B}''$ is $\{B, B'\}$, then there is only one clique.)

THEOREM 4.1. *Suppose* $G = \mathrm{BG}(E, \mathscr{B})$ *is full and* $\mathscr{B}' \subset \mathscr{B}$. *Then* $(E, \mathscr{B}')$ *is a matroid if and only if*

(1) $\langle \mathscr{B}' \rangle$ *is connected, and*

(2) *for every adjacent pair* $B_1, B_2 \notin \mathscr{B}'$, *at most one clique containing both intersects* $\mathscr{B}'$.

Proof. Suppose $(E, \mathscr{B}')$ is not a matroid. Since $\langle \mathscr{B}' \rangle$ is properly labeled, by Theorem 2.2 some $\mathrm{CN}(B', B'')$ in $\langle \mathscr{B}' \rangle$ has just one intermediate vertex or else exactly two and they are adjacent. In either case $\mathrm{CN}(B', B'')$ in G contains an adjacent pair $B_1, B_2 \in \mathscr{B} - \mathscr{B}'$. But now $B' \in \mathscr{B}'$ is in one clique containing B_1, B_2, and $B'' \in \mathscr{B}'$ is in the other.

Conversely, suppose $B_1, B_2 \notin \mathscr{B}'$ are adjacent and two cliques containing them intersect $\mathscr{B}'$. Pick $B' \in \mathscr{B}'$ from the first clique, $B'' \in \mathscr{B}'$ from the second. Then $|B' - B''| = 2$, but in $\langle \mathscr{B}' \rangle$ either $\mathrm{CN}(B', B'')$ is improper or $\delta(B', B'') > 2$. In either case $(E, \mathscr{B}')$ is not a matroid. ∎

COROLLARY 4.2. *Let* $\mathrm{BG}(E, \mathscr{B})$ *be full. Suppose* $\mathscr{B}'' \subset \mathscr{B}$ *has the property that* $B', B'' \in \mathscr{B}''$ *implies* $\delta(B', B'') \geq 2$. *Then* $(E, \mathscr{B} - \mathscr{B}'')$ *is a matroid.*

Proof. Condition (2) above is satisfied vacuously. As for (1), we show that, if $B_1, B_2 \in \mathscr{B} - \mathscr{B}''$ and $|B_1 - B_2| = k > 1$, then there exists $B_3 \in \mathscr{B} - \mathscr{B}''$ such that B_3 is adjacent to B_1 and $|B_3 - B_2| = k - 1$.

Pick any $b, b' \in B_1 - B_2$ and $c, c' \in B_2 - B_1$. Let

$$B_4 = B_1 - (b + b') + (c + c').$$

At least two of the intermediate vertices of $\mathrm{CN}(B_1, B_4)$ are in $\mathscr{B} - \mathscr{B}''$, whether or not B_4 is. Any one of these is a B_3. ∎

Recently Piff and Welsh [12] showed that for any $\lambda < 1$, and n large enough, the number of non-isomorphic matroids on n elements is greater than $2^{(2^{n\lambda})}$. This greatly improves all previous lower bounds. Although they state matters differently, their argument involves counting the number of ways a set $\mathscr{B}''$ can be extracted from $\mathscr{B}$ as in the corollary above.

We call an injection $f: \mathscr{V} \to \mathscr{V}'$ a *monomorphism* of $G(\mathscr{V}, \mathscr{E})$ into $G'(\mathscr{V}', \mathscr{E}')$ should $f(u)f(v) \in \mathscr{E}'$ iff $uv \in \mathscr{E}$. Clearly $G \approx \langle f(\mathscr{V}) \rangle$.

THEOREM 4.3 (The Mapping Characterization). *A connected graph* $G(\mathscr{V}, \mathscr{E})$ *is a basis graph if and only if there is a monomorphism* f *of* G *into some full* $\mathrm{BG}(E, \mathscr{B})$ *such that for any adjacent pair* $B_1, B_2 \notin f(\mathscr{V})$, *at most one clique containing both intersects* $f(\mathscr{V})$.

Proof. If $G = \mathrm{BG}(E', \mathscr{B}')$, pick $\mathscr{M}(E, \mathscr{B})$ to be the full matroid with $E = E'$ and $\mathscr{B}' \subset \mathscr{B}$. Then by Theorem 4.1 the identity injection $\mathscr{B}' \to \mathscr{B}$ suffices for f. Conversely, if some f exists, $\langle f(\mathscr{V}) \rangle$ is a basis graph and thus so is G. ∎

C. A. Holzmann informs us that he too has obtained this characterization (unpublished). It is simpler than the Main Theorem but clearly much harder to test.

5. HOMOTOPY

If $\delta(v_{k-1}, v_{k+1}) = 2$, we say that paths

$$P_1 = v_1 \cdots v_{k-1} v_k v_{k+1} \cdots v_n \quad \text{and} \quad P_2 = v_1 \cdots v_{k-1} v_k' v_{k+1} \cdots v_n$$

differ by a *2-switch*. If

$$\delta(v_{k-1}, v_{k+1}) = 1 \quad \text{and} \quad P_3 = v_1 \cdots v_{k-1} v_{k+1} \cdots v_n,$$

we say that P_1 and P_3 differ by a *shortcut*. If

$$v_{k-1} = v_{k+1} \quad \text{and} \quad P_4 = v_1 \cdots v_{k-1} v_{k+2} \cdots v_n,$$

we say P_1 and P_4 differ by a *deletion*. In all three cases we say that two paths differ by an *elementary deformation*. Finally, two paths are *homotopic* if one can be transformed into the other by a finite sequence of elementary deformations.

Of course, the classical notion of path homotopy applies to graphs, but our notion is not the same. Nor is it the same as Tutte's [14].

THEOREM 5.1 (The Homotopy Theorem). *If G is a basis graph, then any two paths with the same end-points are homotopic.*

Proof. Suppose $G = \mathrm{BG}(E, \mathscr{B})$ and let P_1, P_2 go between B and B'. An inspection of the proof of Theorem 2.2 shows that all path changes there are elementary deformations. In particular, by the last part of that proof we may assume that P_1, P_2 are non-redundant. Clearly, any two non-redundant paths with the same end-points have the same length. Thus we may write

$$P_1 : B = B_1 \xrightarrow{(b_1, c_1)} B_2 \cdots B_n \xrightarrow{(b_n, c_n)} B_{n+1} = B',$$

$$P_2 : B = B_1 \xrightarrow{(b_1', c_1')} B_2' \cdots B_n' \xrightarrow{(b_n', c_n')} B_{n+1} = B'.$$

We do induction over n. P_1 and P_2 are equal if $n = 1$ and homotopic by definition if $n = 2$.

We may even assume that $c_1' = c_1$. If not, we may use the shifting technique of Theorem 2.2 to deform P_2 until c_1' does equal c_1. We now have several cases:

Case 1. $b_1' = b_1$. Then $B_2' = B_2$ and any homotopy of $B_2B_3 \cdots B_{n+1}$ with $B_2B_3' \cdots B_{n+1}$ gives a homotopy of P_1 and P_2.

Case 2. $b_2' = b_1$. P_1 begins

$$B \xrightarrow{(b_1, c_1)} B_2 \xrightarrow{(b_2, c_2)} B_3,$$

and P_2 begins

$$B \xrightarrow{(b_1', c_1)} B_2' \xrightarrow{(b_1, c_2')} B_3'.$$

We note that $B_3' = B_2 - b_1' + c_2'$. Therefore we may perform a 2-switch on P_2 so that it begins

$$B \xrightarrow{(b_1, c_1)} B_2 \xrightarrow{(b_1', c_2')} B_3'.$$

We are now back to Case 1.

Case 3. $b_k' = b_1$ for some k, $3 \leqslant k \leqslant n$. Just as the pivoting in of some c can be shifted forward, so can the pivoting out of some b. (See the first paragraph of the proof of Theorem 2.2.) In particular, we may shift b_k' forward to become b_2'. We are now back to Case 2. ∎

Remark. In Case 3 we could shift b_k' forward to become b_1', but we have no guarantee, without the special argument of Case 2, that this can be done without dislodging c_1 in the last step.

Just as in the classical case we can define homotopy groups, and for a connected graph G, $\pi(G)$ is independent of the base vertex chosen. A less obvious similarity is the fact that

$$\pi(G \times G') \approx \pi(G) \times \pi(G').$$

Here, as usual, the direct product $G(\mathscr{V}, \mathscr{E}) \times G'(\mathscr{V}', \mathscr{E}')$ has vertices (v, v') and edges $(v_1, v_1')(v_2, v_2')$, where either $v_1 = v_2$ and $v_1'v_2' \in \mathscr{E}'$, or $v_1' = v_2'$ and $v_1v_2 \in \mathscr{E}$. Thus in some sense our homotopy may be "right" for graph theory.

We note that $\pi(G)$ is not always trivial: consider any cycle with 5 or more edges.

CONJECTURE 3. *G is a basis graph if and only if*

(1) *it is connected,*

(2) *each* CN *is a square, pyramid, or octahedron, and*

(3) *$\pi(G)$ is trivial.*

Without condition (2), a square with one diagonal ($K_4 - x$) would be a counterexample.

Note added in proof. The author has constructed a set of counterexamples to Conjecture 2. This construction, and the constructions at the end of Section 3, all turn out to involve covering spaces.

REFERENCES

1. J. A. BONDY, Pancyclic graphs. II, University of Waterloo (summary in "Proceedings of the Second Louisiana Conference on Combinatorics, Graph Theory and Computing," Baton Rouge, La., 1971).
2. H. H. CRAPO AND G.-C. ROTA, "On the Foundations of Combinatorial Theory: Combinatorial Geometries," M.I.T. Press, Cambridge, Mass., 1970.
3. R. L. CUMMINS, Hamiltonian circuits in tree graphs, *IEEE Trans. Circuit Theory* **13** (1966), 82–90.
4. W. CUNNINGHAM, The basis graph of a matroid (unpublished), University of Waterloo, 1972.
5. A. GEWIRTZ, Graphs with maximal even girth, *Canad. J. Math.* **21** (1969), 915–934.
6. F. HARARY, "Graph Theory," Addison-Wesley, Reading, Mass., 1969.
7. C. A. HOLZMANN AND F. HARARY, On the tree graph of a matroid, *SIAM J. Appl. Math.* **22** (1972), 187–193.
8. C. A. HOLZMANN, P. G. NORTON, AND M. D. TOBEY, A graphical representation of matroids, to appear in *Siam J. Appl. Math.*
9. G. KISHI AND Y. KAJITANI, On the realization of tree graphs, *IEEE Trans. Circuit Theory* **15** (1968), 271–273.

THEOREM 5.1 (The Homotopy Theorem). *If G is a basis graph, then any two paths with the same end-points are homotopic.*

Proof. Suppose $G = \mathrm{BG}(E, \mathscr{B})$ and let P_1, P_2 go between B and B'. An inspection of the proof of Theorem 2.2 shows that all path changes there are elementary deformations. In particular, by the last part of the proof we may assume that P_1, P_2 are non-redundant. Clearly, any two non-redundant paths with the same end-points have the same length. Thus we may write

$$P_1 : B = B_1 \xrightarrow{(b_1, c_1)} B_2 \cdots B_n \xrightarrow{(b_n, c_n)} B_{n+1} = B',$$

$$P_2 : B = B_1 \xrightarrow{(b_1', c_1')} B_2' \cdots B_n' \xrightarrow{(b_n', c_n')} B_{n+1} = B'.$$

We do induction over n. P_1 and P_2 are equal if $n = 1$ and homotopic by definition if $n = 2$.

We may even assume that $c_1' = c_1$. If not, we may use the shifting technique of Theorem 2.2 to deform P_2 until c_1' does equal c_1. We now have several cases:

Case 1. $b_1' = b_1$. Then $B_2' = B_2$ and any homotopy of $B_2 B_3 \cdots B_{n+1}$ with $B_2 B_3' \cdots B_{n+1}$ gives a homotopy of P_1 and P_2.

Case 2. $b_2' = b_1$. P_1 begins

$$B \xrightarrow{(b_1, c_1)} B_2 \xrightarrow{(b_2, c_2)} B_3,$$

and P_2 begins

$$B \xrightarrow{(b_1', c_1)} B_2' \xrightarrow{(b_1, c_2')} B_3'.$$

We note that $B_3' = B_2 - b_1' + c_2'$. Therefore we may perform a 2-switch on P_2 so that it begins

$$B \xrightarrow{(b_1, c_1)} B_2 \xrightarrow{(b_1', c_2')} B_3'.$$

We are now back to Case 1.

Case 3. $b_k' = b_1$ for some k, $3 \leqslant k \leqslant n$. Just as the pivoting in of some c can be shifted forward, so can the pivoting out of some b. (See the first paragraph of the proof of Theorem 2.2.) In particular, we may shift b_k' forward to become b_2'. We are now back to Case 2. ∎

Remark. In Case 3 we could shift b_k' forward to become b_1', but we have no guarantee, without the special argument of Case 2, that this can be done without dislodging c_1 in the last step.

Just as in the classical case we can define homotopy groups, and for a ..nnected graph G, $\pi(G)$ is independent of the base vertex chosen. A less ..vious similarity is the fact that

$$\pi(G \times G') \approx \pi(G) \times \pi(G').$$

..ere, as usual, the direct product $G(\mathscr{V}, \mathscr{E}) \times G'(\mathscr{V}', \mathscr{E}')$ has vertices (v, v') ..d edges $(v_1, v_1')(v_2, v_2')$, where either $v_1 = v_2$ and $v_1'v_2' \in \mathscr{E}'$, or $v_1' = v_2'$ ..nd $v_1v_2 \in \mathscr{E}$. Thus in some sense our homotopy may be "right" for graph ..eory.

We note that $\pi(G)$ is not always trivial: consider any cycle with 5 or more ..ices.

CONJECTURE 3. *G is a basis graph if and only if*

(1) *it is connected,*

(2) *each* CN *is a square, pyramid, or octahedron, and*

(3) *$\pi(G)$ is trivial.*

..thout condition (2), a square with one diagonal ($K_4 - x$) would be a ..nterexample.

Note added in proof. The author has constructed a set of counterexamples to Con..cture 2. This construction, and the constructions at the end of Section 3, all turn .. to involve covering spaces.

REFERENCES

1. J. A. BONDY, Pancyclic graphs. II, University of Waterloo (summary in "Proceedings of the Second Louisiana Conference on Combinatorics, Graph Theory and Computing," Baton Rouge, La., 1971).
2. H. H. CRAPO AND G.-C. ROTA, "On the Foundations of Combinatorial Theory: Combinatorial Geometries," M.I.T. Press, Cambridge, Mass., 1970.
3. R. L. CUMMINS, Hamiltonian circuits in tree graphs, *IEEE Trans. Circuit Theory* **13** (1966), 82–90.
4. W. CUNNINGHAM, The basis graph of a matroid (unpublished), University of Waterloo, 1972.
5. A. GEWIRTZ, Graphs with maximal even girth, *Canad. J. Math.* **21** (1969), 915–934.
6. F. HARARY, "Graph Theory," Addison-Wesley, Reading, Mass., 1969.
7. C. A. HOLZMANN AND F. HARARY, On the tree graph of a matroid, *SIAM J. Appl. Math.* **22** (1972), 187–193.
8. C. A. HOLZMANN, P. G. NORTON, AND M. D. TOBEY, A graphical representation of matroids, to appear in *Siam J. Appl. Math.*
9. G. KISHI AND Y. KAJITANI, On the realization of tree graphs, *IEEE Trans. Circuit Theory* **15** (1968), 271–273.

10. S. B. MAURER, Matroid basis graphs, Ph.D. Dissertation, Princeton University, May, 1972.
11. S. B. MAURER, Matroid basis graphs. II, to appear in *J. Combinatorial Theory Ser. B*.
12. M. J. PIFF AND D. J. A. WELSH, On the number of combinatorial geometries, *Bull. London Math. Soc.* **3** (1971), 55–56.
13. H. SHANK, A note on Hamiltonian circuits in tree graphs, *IEEE Trans. Circuit Theory* **15** (1968), 86.
14. W. T. TUTTE, A homotopy theorem for matroids. I–II, *Trans. Amer. Math. Soc.* **88** (1958), 144–174.
15. W. T. TUTTE, Lectures on matroids, *J. Res. Nat. Bur. Standards, Sect. B* **69** (1965), 1–48.
16. H. WHITNEY, The abstract properties of linear independence, *Amer. J. Math.* **57** (1935), 509–533.