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Fractional Power Series Expansions and Resultants

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ABSTRACT

Fractional Power Series Expansions and Resultants

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In this work, we consider two problems which, on the surface, seem quite unrelated, but which hopefully form the beginnings of a unified approach to studies of generalized discriminants.

The first section explores the question of constructing fractional power series expansions for systems of algebraic equations. We give primary attention to a generalization of Newton's polygon construction for equations in two variables. The nature of this construction, and the series solutions it produces allows us to make some surprising connections between collections of power series solutions and normal fans of various fiber polytopes.

From these results, we also derive a relationship between these normal fans and ramification loci of projections of the variety in question. The first of these loci corresponds to the vanishing of the classical discriminant of the polynomial with respect to one of its variables. More general loci, however, correspond to a common generalization of classical discriminants and resultants – the mixed discriminant. Moreover, we propose a generalization of the fiber polytope, called the mixed fiber polytope which would extend the above relationships to this more general locus.

The second section makes the first step into investigating generalizations of the resultants studied in [12]. In a direction proposed by Deligne in [7], we attach an algebraic line bundle to the direct image of a product of Chern classes. This

line bundle is given as a generalized resultant. Moreover, we are able to attach a determinantal representation to these line bundles, which greatly facilitates both interpretation and calculation.

Conveniently, these determinantal representations yield generalizations of the cube theorem for the determinant bundle on the Picard group of an algebraic variety. These cube theorems are key tools for deriving the properties of the generalized resultants. Moreover, through these theorems, we are able to derive a few general formulas connecting discriminants and generalized resultants.

It is hoped that the methods in this second section will be extendible, through the use of more general resolutions of loci, to the case of the above mixed discriminant.

Acknowledgements

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Fiber Polytopes and Fractional Power Series

CHAPTER 1

Preliminary Concepts

1. Introduction and history of the problem

Consider k algebraic equations in $k + \ell$ variables

$$F_i(x_1, \ldots, x_\ell, y_1, \ldots, y_k) = 0 \text{ for } i = 1, \ldots, k.$$

We wish to construct k fractional power series expansions, $y_i = \phi_i(x_1, \ldots, x_\ell)$, for $i = 1, \ldots, k$, such that $F_j(x_1, \ldots, x_\ell, \phi_1, \ldots, \phi_k) = 0$ formally for all j. We are particularly interested in constructing complete sets of fractional power series solutions for these equations which converge in some common region of $(\mathbb{C}^*)^\ell$, where $\mathbb{C}^* = \mathbb{C} - \{0\}$. On the common domains of convergence of these series, we require that

$$F_j(x, y) = 0 \quad \text{for all} \quad j \Longleftrightarrow y_i = \phi_i(x) \quad \text{for all} \quad i.$$

This classical setup, which dates back to Newton, turns out to be related to the structure of various polytopes. The series solutions themselves are intimately related with edges on the Newton polytopes of the F_i. Also, in developing such expansions, a relationship to a fiber polytope (as introduced in [3]) emerges.

The classical case of two variables and one equation is well known, and is is due to Newton. We will briefly sketch his original construction later in this section. More recently Bruno and Soleev considered k equations in $k+1$ variables in [6]. In each of these situations, the fiber polytopes involved are simply line segments.

In chapter 2 we treat the case of a single equation in N variables. In this case we construct a single power series in N variables. It may, on the surface, appear that this case could be handled by simply iterating Newton's construction. However, such a method would produce only series with increasing powers in all variables. That is, such series would have support only in the the first "quadrant" of the space of exponents.

We wish to consider more general solutions with exponents in some arbitrary convex cone. For this we give a generalization of Newton's polygon construction. We demonstrate that full collections of series solutions correspond to the vertices of a certain fiber polytope, and that the normal cones of these vertices determine the regions on which they converge.

In chapter 3 we demonstrate the relationship in a more general setting. For computing such series expansions we again give an extension of Newton's construction. Whereas, in the first case, only a special class of fiber polytopes arise, here more general fiber polytopes appear.

In the most general case, we cannot assure that the construction gives a series solution. However, under certain explicit conditions, we can prove that the construction can be carried out and that the series built have common domains of convergence. We will see that, generically, the number of systems of series solutions converging in a given cone is equal to the mixed volumes of the projections of $P(F_1), \ldots, P(F_k)$ to $\mathbb{R}^k$. This agrees with the theorem due to Bernstein [2] on the number of solutions to a system of equations.

In this case, however, we do not have as convenient a correspondence as before. Only certain vertices of the fiber polytope correspond to complete systems of solutions. The results here suggest existence of a "Mixed Fiber Polytope" $\Sigma(P_1, \ldots, P_k)$ of k polytopes. This polytope should be a summand of the fiber polytope of their Minkowski sum, and should be equal to the Fiber polytope $\Sigma(P)$ in the event that the k polytopes are equal. Finding such a polytope would conveniently generalize the relationship in chapter 2 between power series expansions and the normal fans of polytopes. Moreover the results here suggest a possible relationship between this mixed fiber polytope and a generalization of the classical discriminant, called the mixed discriminant.

2. Polytopes, cones, and convex geometry

1.2.1. Polytopes. Consider a real vector space $V = \mathbb{R}^m$. We will always work with convex polytopes and will therefore often leave out the term "convex". For this development, the most convenient definition is

DEFINITION 1.2.2. A polytope $P \subset V$ is the convex hull of a finite set of elements of V, i.e. for some finite set $S = \{v_1, \ldots, v_n\}$ we have

$$P = \text{conv}\{v_1, \ldots, v_n\} = \Big\{\lambda_1 v_1 + \cdots + \lambda_n v_n : v_i \geq 0, \sum v_i = 1\Big\}.$$

A polytope is called rational if all of the v_i have rational coordinates. (Actually, it is only strictly necessary for those v_i which are vertices of P, but this makes no difference for our purposes.)

Under this definition all polytopes are bounded. Often polytopes are defined as intersections of half-spaces. There the possibility exists that the resulting set will not be bounded. In this case, the resulting objects are usually called polygons or polyhedra. The word "polytope" is often reserved, in these cases, for bounded sets.

DEFINITION 1.2.3. Let P be an m-dimensional polytope, and let $\gamma \in (\mathbb{R}^m)^*$ be a linear functional. Then the set

$$f_\gamma = \{p \in P : \langle \gamma, p \rangle \geq \left\langle \gamma, p' \right\rangle \text{ for all } p' \in P\}$$

is called the extreme face of P in the direction γ. Note that f_γ is itself a polytope, being the intersection of P with a hyperplane. As a polytope, it has a well defined real dimension k, and is hence called a k-face of P. A vertex of P is a face of dimension 0, while a facet is an $m-1$-dimensional face.

Through any k-face f there passes a unique k plane. (Take a point p on the relative interior of the face, and consider the k-plane spanned by all vectors from p to points on the face.) We say that this k-plane is determined by f. A hyperplane $H \subset \mathbb{R}^m$ is said to support a polytope P if H contains points of P, and P lies entirely in one of the half-spaces determined by H.

The next definition gives a useful construction for forming a new polytope out of a collection of polytopes in such a way that the structure of the new polytope reflects an amalgamation of the structures of each of the former polytopes.

DEFINITION 1.2.4. Let $P_1, \ldots, P_n$ be n polytopes of dimension m. The Minkowski sum of these polytopes is the set of all vector sums of elements of $P_1, \ldots, P_n$, i.e.

$$P_1 + \cdots + P_n = \{p_1 + \cdots + p_n : p_i \in P_i\}$$

It is easy to show that $P_1 + \cdots + P_n$ is itself a polytope. In fact, it is the convex hull of the sums of the vertices of the P_i. For information see [31].

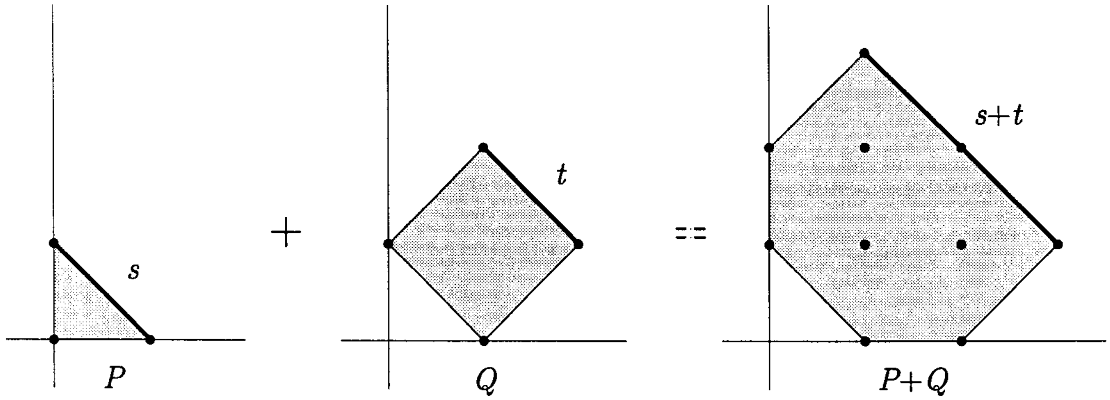

FIGURE (1.1). The Minkowski sum of two polytopes

For an example of the Minkowski sum of two 2-dimensional polytopes, see figure (1.1). On the Minkowski sum of a set of polytopes we can distinguish two classes of faces of dimension greater than 0. Extreme points on $P_1 + \cdots + P_n$ in some direction γ are always sums of points of the P_i that are extreme in the direction γ. Thus, a face f of P is uniquely the sum of faces of $P_1, \ldots, P_n$, though some of these faces may be vertices (and though the points in f are not uniquely a sum of points of the P_i).

DEFINITION 1.2.5. Let $P_1, \ldots, P_n$ be m-dimensional polytopes, and let P be their Minkowski sum. Let f be a face of P. We say that f is *decomposable* if $f = f_1 + \cdots + f_n$ where f_i is a face of P_i and $\dim(f_i) > 0$ for all i, otherwise f is called *indecomposable*.

To analyze number of common solutions to collections of polynomials we will consider the *mixed volume* of the polytopes $P(F_1), \ldots, P(F_n)$.

DEFINITION 1.2.6. Let $P_1, \ldots, P_n$ be m-dimensional polytopes. The mixed volume $\mathrm{Vol}(P_1, \ldots, P_n)$ is the alternating sum

$$\mathrm{Vol}(P_1, \ldots, P_n) = \frac{1}{m!} \sum_{k=1}^{n} (-1)^{n-k} \sum_{1 \le i_1 < \cdots < i_k \le n} \mathrm{Vol}(P_{i_1} + \cdots + P_{i_k}).$$

Here the volume $\mathrm{Vol}(P)$ of a polytope is normalized so that the standard n-simplex has volume 1. It is well known that $\mathrm{Vol}(P, \ldots, P) = \mathrm{Vol}(P)$. Again, for details see [31].

1.2.7. The Newton polytope of a polynomial. The following polytope is one of the main tools used in this work for connecting the study of polynomials and their power series expansions to convex geometry.

DEFINITION 1.2.8. Let F be a polynomial in m variables, i.e.

$$F = \sum_I a_I x^I$$

with $I \in \mathbb{Q}^m$ ranging over some finite subset. The Newton polytope of F is the polytope

$$P(F) = \mathrm{conv}\{I \in \mathbb{Q}^n : a_I \neq 0\}$$

$P(F)$ is a rational polytope because F is a polynomial with rational exponents and so has only finitely many non-zero coefficients.

The set $S_F = \{I \in \mathbb{Q}^m : a_I \neq 0\}$ is called the *support* of F. We will also use the notation $\mathrm{Supp}(F)$ to denote the support.

Consider the connection between the polytopes of a collection of k polynomials in ℓ variables and the number of common roots of these equations. let $F_1, \ldots, F_k$ be k polynomials in ℓ variables. Let $S_i \subset \mathbb{R}^\ell$ be their supports. By considering the coefficients of these polynomials as variables, we can consider F_i as a point in $\mathbb{C}^{S_i}$. Hence we can consider the system $F_1, \ldots, F_k$ as an element in the space $\Pi\mathbb{C}^{S_i}$. By a generic system of equations, we will mean a system $F_1, \ldots, F_k$ which lies on some specific Zariski open subset of $\Pi\mathbb{C}^{S_i}$.

THEOREM 1.2.9. *Consider k generic equations in ℓ unknowns,*

$$F_i(y_1, \ldots, y_\ell) = 0 \text{ for } i = 1, \ldots, k.$$

and let $P_i = P(F_i) = \operatorname{conv}(S_i)$ be the Newton polytope of F_i for all i. Then the number of non-zero solutions of this system is generically equal to the mixed volume $\operatorname{Vol}(P_1, \ldots, P_k)$ *of the polytopes $P_1, \ldots, P_k$.*

For the proof of this theorem see [2].

Once we have the language of normal cones and linear functionals, we will be able to demonstrate the beginnings of the tight relationship which exists between polytopes and series expansions.

1.2.10. Convex polyhedral cones. A convex polyhedral cone in $\mathbb{R}^m$, is a set of the form

$$C = \{r_1 v_1 + \cdots + r_n v_n : r_i \in \mathbb{R}, r_i \geq 0\},$$

where $v_1, \ldots, v_n \in \mathbb{R}^m$ are fixed vectors. A cone is rational if $v_i \in \mathbb{Q}^m$ for every i, and is strongly convex if it contains no non-trivial linear subspaces.

We identify the dual space $(\mathbb{R}^m)^*$ with $\mathbb{R}^m$, by means of the usual pairing $\langle u, x\rangle = \sum u_i x_i$. Let C be a strongly convex rational polyhedral cone in $\mathbb{R}^m$. Define the dual cone, $C^* \subset (\mathbb{R}^m)^*$, to be the set

$$C^* = \{u \in \mathbb{R}^m : \langle u, x\rangle \leq 0 \;\; \forall \; x \in C\}.$$

This is the cone consisting of all linear functionals which have a maximum on V.

We will often work with polytopes in $\mathbb{R}^m = \mathbb{R}^{k+\ell}$ where k is the number of equations and ℓ is the number of independent variables. Therefore, we will assume throughout this work that we have chosen a direct sum decomposition

$$\mathbb{R}^m = \mathbb{R}^\ell \oplus \mathbb{R}^k.$$

The coordinates in $\mathbb{R}^\ell$ and $\mathbb{R}^k$ will be denoted by $\alpha_1, \ldots, \alpha_\ell$ and $\beta_1, \ldots, \beta_k$ respectively.

Let Π be a k-plane in $\mathbb{R}^{k+\ell}$, then Π is called admissible if the projection $\pi : \Pi \to \mathbb{R}^k$ is injective. Thus on such a Π we have a parameterization

$$\alpha_i = \sum_i (\delta_{ij}\beta_j + \epsilon_i) = \epsilon + \sum_i (\delta_{ij}\beta_j).$$

The matrix $||\delta_{ij}||$ is called the matrix of slopes of Π. On a polytope P we will say that a k-face is admissible if the k-plane it determines is admissible.

Let f be an admissible face of P, and let $w \in (\mathbb{R}^\ell)^*$ be a linear functional. Then w determines a unique hyperplane $H_{f,w}$ in $\mathbb{R}^m$ which contains f and on which w is constant on every fiber of the projection to $\mathbb{R}^k$. (The equation $\langle w, x\rangle = z$ determines a hyperplane in $\mathbb{R}^\ell$.) Such a hyperplane is called *w-constant.*

Consider the case $k = 1$. Let w be any linear function on $\mathbb{R}^\ell$. We extend w trivially to a linear function of $\mathbb{R}^{\ell+1}$ by defining for $x \in \mathbb{R}^{\ell+1}$

$$\langle w, x\rangle := \langle w, (x_1, \ldots, x_\ell)\rangle$$

A hyperplane H in $\mathbb{R}^{\ell+1}$ is then w-constant if for each $c \in \mathbb{R}$

$$\langle w, H \cap \{x_{\ell+1} = c\}\rangle := \{\langle w, x\rangle : x \in H \cap \{x_{\ell+1} = c\}\} = \{d_c\}$$

for some $d_c \in \mathbb{R}$. i.e. w is constant on each "vertical" section of H. Since we will be using the $x_1, \ldots, x_\ell$-hyperplane frequently, we will call it the null-hyperplane.

1.2.11. Normal and barrier cones. Let $w \in (\mathbb{R}^\ell)^*$ be a linear functional on $\mathbb{R}^\ell$ such that the coordinates of w are linearly independent over $\mathbb{Q}$. Such a linear functional is called irrational. The equation $\langle w, x\rangle = z$ for any fixed z has at most one solution in $\mathbb{Q}^\ell$. Therefore, w induces a linear ordering on $\mathbb{Q}^\ell$.

Consider the projection

$$\pi : P \longrightarrow Q = \pi(P) \subset \mathbb{R}^k$$

onto the last k coordinates. The fiber of π over any interior point $q \in Q$ is a $d-k$ dimensional polytope. Note that this fiber is given by a system of linear inequalities with rational coefficients, but possibly with irrational right hand sides. Since w is irrational, there exists a unique point $p_q \in \pi^{-1}(q)$ such that

$$\langle w, p_q\rangle > \langle w, p\rangle \text{ for all } p \in \pi^{-1}(q)$$

Therefore, w defines a section of π which is called the maximal section of π with respect to w. This section is denoted by $\mathcal{S}_{w,\pi}(P) = \mathcal{S}_w(P)$.

DEFINITION 1.2.12. Let P be an m-dimensional polytope, and let f be an admissible k-face of P, with $m = k + \ell$. The normal cone of f is the closure of the set of all irrational linear functionals in $(\mathbb{R}^\ell)^*$ such that f is contained in $\mathcal{S}_{w,\pi}(P)$. The normal cone of f is denoted by $N(f)$. The barrier cone, denoted $B(f)$, is defined to be the dual $N^*(f)$ of $N(f)$.

In particular, this means that for a vertex v of P, the normal cone $N(v)$ is the cone in $(\mathbb{R}^m)^*$ consisting of all linear functionals which achieve a maximum on P at v. Notice that, under this definition, the barrier cone of a k-face does not lie in the same space as the polytope. Rather, it lies in $\mathbb{R}^\ell$. Likewise, the normal cone of a k-face lies in $(\mathbb{R}^\ell)^*$ rather than $(\mathbb{R}^m)^*$.

There is another, more geometric definition of the barrier cone of a vertex v.

$$C(v) = \{\lambda(p - v) : \lambda \in \mathbb{R}_+, p \in P(F)\}$$

i.e. the cone spanned by the vectors from v to points in $P(F)$. The dual of $C(v)$ is obviously the normal cone of v.

Let $K(f)$ be the k-plane in $\mathbb{R}^m$ which contains the face f. We can consider the following subset of $\mathbb{R}^m$.

$$N(f) \times K(f) := \left\{z \in \mathbb{R}^m = \mathbb{R}^\ell \oplus \mathbb{R}^k : z \in N(f) + y, \text{ where } y \in K(f)\right\}$$

We will call this set the barrier wedge of f, denoted $W(f)$. Note that the barrier wedge of a k-face is just the cone spanned by all vectors from an interior point of f to points in P.

As an example, consider the case of the barrier wedge of an admissible edge of a polytope are considering admissible edges. Let e be any admissible edge of $P(F)$. The vertices of e with the largest and smallest $x_{\ell+1}$ coordinates will be respectively called the major and minor vertices of e, and will be denoted by $m(e)$ and $M(e)$ respectively. Write $m(e) = (p_1, \ldots, p_{\ell+1})$ and $M(e) = (q_1, \ldots, q_{\ell+1})$. The slope vector $S(e)$ of e with respect to $x_{\ell+1}$ is

$$S(e) = \frac{1}{q_{\ell+1} - p_{\ell+1}}(q_1 - p_1, \ldots, q_\ell - p_\ell)$$

For such an edge the barrier cone of e is the following subset of $\mathbb{R}^\ell$.

Let L be the line in $\mathbb{R}^{\ell+1}$ determined by e, and let y be the point of intersection of L with the null-hyperplane (such a point exists since e was assumed not to be parallel to this plane). Then the barrier wedge of e in $\mathbb{R}^{\ell+1}$ is

$$W(e) = \{\lambda(p - x) + x : \lambda \in \mathbb{R}_+, p \in P(F), x \in L\}$$

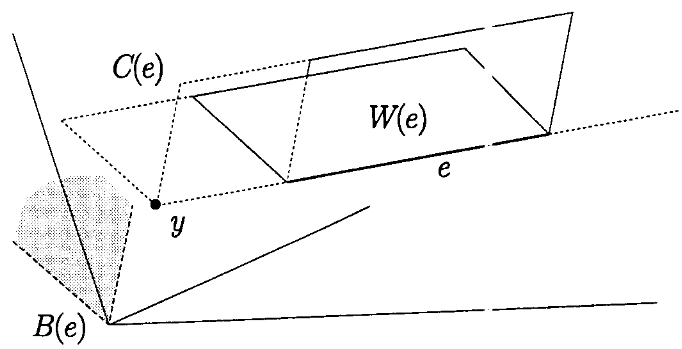

FIGURE (1.2). The barrier wedge and cone associated to an edge

The intersection of this wedge with the null-hyperplane is a convex rational polyhedral cone, $C(e) = B(e) + y$, which has its vertex at y. See figure (1.2). This is a translate of the barrier cone $B(e)$. For convenience, we will often relax our definition and refer to translates of cones as cones themselves.

EXAMPLE 1.2.13. Also, in the case $k = 1$ we have a more familiar interpretation of the maximal section of a linear functional. Let P be a polytope in $\mathbb{R}^{\ell+1}$, and let $\psi : \mathbb{R}^{\ell+1} \longrightarrow \mathbb{R}$ be the projection onto the last coordinate. A monotone edge path on P is a sequence $E = \{e_1, \ldots, e_n\}$ such that for each i, $M(e_i) = m(e_{i+1})$ and e_i does not lie parallel to the $x_1, \ldots, x_\ell$-plane. Therefore the edge path is increasing with respect to ψ. A monotone edge path is called coherent if

$$\bigcap_{i=1}^{n} N(e_i) \neq \{0\}$$

That is, a coherent edge path is a maximal section of the projection of the polytope to the line $\mathbb{R}$.

1.2.14. The normal fan. One important aspect of the normal cones of the vertices of a polytope is that they knit together to form a fan of $(\mathbb{R}^m)^*$, since every linear functional attains a maximum on some face of P, and hence at some vertex of P.

DEFINITION 1.2.15. Let $\{v_1, \ldots, v_n\}$ be the set of vertices of P. Then the collection of pairwise disjoint cones $N(v_1), \ldots, N(v_n)$ forms a fan that covers $(\mathbb{R}^m)^*$. This fan is called the normal fan of P and is denoted Δ_P. These definitions coincide with the standard notions [31].

As an example of how the structure of the Minkowski sum of a collection of polytopes is related to their individual structures, notice that the normal fan of the Minkowski sum of n polytopes $\Delta_{P_1+\cdots+P_n}$ is the smallest common refinement of $\Delta_{P_1}, \ldots, \Delta_{P_n}$. That is, cones in the normal fan of the Minkowski sum are intersections of cones in the normal fans of each of the summands. For a more complete discussion of these ideas see [12], pp. 190-191, and [31].

1.2.16. Fiber Polytopes. In the following two chapters we extensively use the notion of the fiber polytope of a projection of two polytopes $P \xrightarrow{\psi} Q$ as defined in [3], for a projection ψ from X to Y. Let us recall the definitions.

Let $P \subset \mathbb{R}^N$ be a convex polytope. Let $\psi : \mathbb{R}^N \to \mathbb{R}^M$ be a surjective linear map and let $Q = \psi(P)$. The Minkowski integral it the set of vector integrals

$$\int P = \int_Q P = \int_Q \gamma(x)\, dx$$

where γ ranges over all continuous sections of ψ.

The fiber polytope $\Sigma_\psi(P, Q)$ is defined to be the normalized Minkowski integral

$$\Sigma(P, Q) := \frac{1}{\mathrm{Vol}(Q)} \int_Q P.$$

The following are some of the important properties and results concerning fiber polytopes. We leave out most of the proofs here, as they can be found in [3].

Two polytopes are called normally equivalent if they have the same normal fan.

PROPOSITION 1.2.17. *The fiber polytope $\Sigma(P,Q)$ is a nonempty convex polytope in R^{N-M}. Moreover, there exists a finite subset $\{x_1,\ldots,x_n\} \subset Q$ such that the Minkowski sum of the fibers $P_{x_1}+\cdots+P_{x_n}$ is normally equivalent to $\Sigma(P,Q)$.*

Let $\psi : P \longrightarrow Q$ and let $F \subset P$ such that F_x is a face of P_x for every x in Q. Then the projection $F \longrightarrow Q$ is called a face bundle of P. If there exists a linear functional ψ on R^{N-M} such that F_x is extreme in the direction ψ (in the same sense as above), then F is called a coherent face bundle. Notice that a coherent face bundle such that each F_x consists of a single point (i.e. a vertex of P_x) is a maximal section of $P \longrightarrow Q$ in some direction.

PROPOSITION 1.2.18. *The faces of $\Sigma(P,Q)$ are in one to one correspondence with the coherent face bundles of P. In fact, the faces of the Minkowski integral $\int_Q P$ are the integrals of the coherent face bundles of P. In particular, the vertices of $\Sigma(P,Q)$ correspond to the maximal sections of $P \longrightarrow Q$.*

Putting these first two propositions together yields

COROLLARY 1.2.19. *Let v be a vertex of $\Sigma(P,Q)$, and let ϕ be the corresponding maximal section of $P \longrightarrow Q$. The normal cone of v is the intersection of the normal cones of the $\phi(x) \subset P_x$. Equivalently, the normal cone of v is the intersection of the normal cones of the M-faces of P in $\phi(Q)$.*

1.2.20. Rational functions and polytopes. The following example demonstrates, for the special case where $x_{\ell+1}$ is a rational function in $x_1,\ldots,x_\ell$, the relationship between the Newton polytope of a polynomial and the power series expansions of $x_{\ell+1}$. Let

$$x_{\ell+1} = \frac{1}{F(x_1,\ldots,x_\ell)}. \tag{1.1}$$

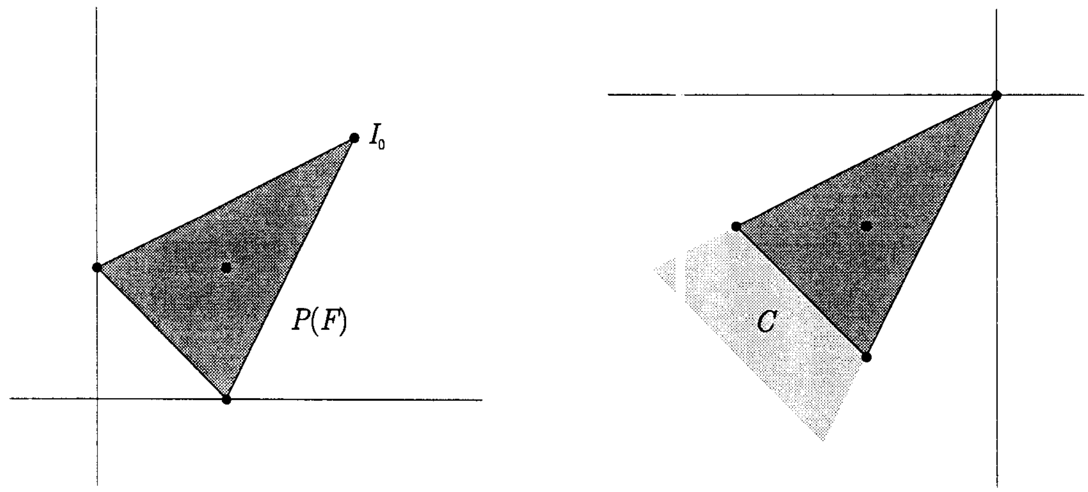

FIGURE (1.3). The Newton polytopes for a rational function

Consider the Newton polytope of $F(x)$, see figure (1.3). We can expand $1/F(x)$ as a power series in different ways using the geometric series expansion. Factor out of $f(x)$ a monomial which corresponds to one of the vertices of $P(F)$.

$$x_{\ell+1} = \frac{1}{a_{I_0} x^{I_0}} \cdot \frac{1}{1 + g(x)}$$

where $g(x) = \sum_{I \neq I_0} a_I x^I / a_{I_0} x^{I_0}$. By the geometric expansion we get

$$x_{\ell+1} = \frac{1}{a_{I_0} x^{I_0}} \sum_{i=0}^{\infty} (-1)^i g^i(x) \tag{1.2}$$

By factoring out one monomial we have shifted the exponents in the denominator so that the new denominator has the polytope shown in figure (1.3), a translate of the original polytope. We are taking the geometric progression of terms that lie in the barrier cone of the chosen vertex, so all monomials appearing in series (1.2) also lie in this cone.

If we had tried to factor out a monomial corresponding to a point of $P(F)$ that wasn't a vertex we would have constructed a series in which finding a coefficient would involve summing an infinite number of terms. For obvious reasons, we avoid such expansions. The construction for rational functions can be summarized by the following theorem.

THEOREM 1.2.21. *The series expansions of* $x_{\ell+1} = 1/f(x_1, \ldots, x_\ell)$ *with monomials given by points in some convex cone correspond to the vertices of the Newton polytope of* $f(x_1, \ldots, x_\ell)$. *If a vertex* v *of* $P(f)$ *corresponds to the series expansion* $\phi(x_1, \ldots, x_\ell)$, *then the monomials in* ϕ *correspond to points lying in the barrier cone of* v, *and converge in some translate of the normal cone of* v.

For more information see [12], chapter 6.

Note that the equation for this rational function can also be written in the following polynomial form

$$F(x) = x_{\ell+1} f(x_1, \ldots, x_\ell) - 1 = 0.$$

The Newton polytope $P(F)$ is a cone over the Newton polytope of f. Therefore, the vertices of $P(f)$ correspond to edges of $P(F)$. Since the series expansions of x_{N+1} correspond to the vertices of $P(f)$, they correspond to the edges of $P(F)$. Moreover, the fiber polytope of the projection of this cone to the line, is just the Newton polytope of f. So, in the setting of fiber polytopes, the series expansions of F correspond to the vertices of the fiber polytope $\Sigma(P(F))$, and converge in some translate of the corresponding normal cone. We will extend this correspondence in the next two chapters.

3. Fractional power series expansions

1.3.1. Newton's original construction. For the convenience of the reader, we recall the well known construction of fractional power series via Newton polygons, and illustrate the construction with an example. Let $f(x, y) = 0$ be a polynomial equation in two variables, with

$$f(x,y) = \sum a_{ij} x^i y^j. \tag{1.1}$$

We wish to write y as a fractional power series $\phi(x)$, so that formally

$$f(x, \phi(x)) = 0.$$

The game is to determine what the lowest order term in the series expansion must be, and then proceed by induction, building the series one term at a time. We write our prospective series expansion as

$$\phi(x) = cx^\beta + \psi(x).$$

Where $\psi(x)$ contains all higher order terms of ϕ.

Substituting this expression into equation 1.1 yields

$$\begin{aligned} f(x, \phi(x)) &= \sum a_{ij} x^i \left(cx^\alpha + \phi(x)\right) \\ &= \sum a_{ij} c^j \left(x^{j\alpha + i}\right) + \text{higher order terms.} \end{aligned}$$

In order for this to be identically 0, the lowest order terms in this expression must cancel. So, there must be at least two terms (i, j) and (i', j') in the above expression such that

$$\beta = j\alpha + i = j'\alpha + i' \leq j''\alpha + i'', \tag{1.2}$$

for some fixed β and for all $(i'', j'') \in \S_f$.

All of the (i'', j'') lie on $P(F)$ by definition, so equation 1.2 defines an line containing two points on the polytope (namely (i, j) and (i', j')) and assures that the rest of the points (i'', j'') lie on or above this line. Hence, equation 1.2 defines an edge e on the bottom of $P(F)$.

Since the terms whose exponents satisfy equation 1.2 must sum to 0,

$$\left(\sum a_{ij}c^j\right) x^\beta = 0,$$

the coefficient in this term must be 0

$$F_e = \sum a_{ij}c^j = 0, \tag{1.3}$$

where (i, j) ranges over all indices whose terms satisfy the equality in 1.2. We therefore find c as a root of equation 1.3. This equation must have a non-zero root, because the expression has at least two distinct terms. Equation 1.3 is called the edge equation of e.

Once we have found the term of least order in the series we make the substitution

$$cx^\beta + y$$

in place of y in the original equation, and proceed as before to find the next term.

We will prove all the remaining details of this construction as a special case in the next chapter, but this is the procedure.

1.3.2. Parameterizations of branches of varieties – an example. The following example illustrates this connection between Newton polytopes and fractional power series, and also indicates the necessity for using fractional powers, instead of just ordinary integer powers in the expansions of branches of the varieties in question.

Consider the folium of Descartes.

$$x^3 + y^3 - 3xy = 0.$$

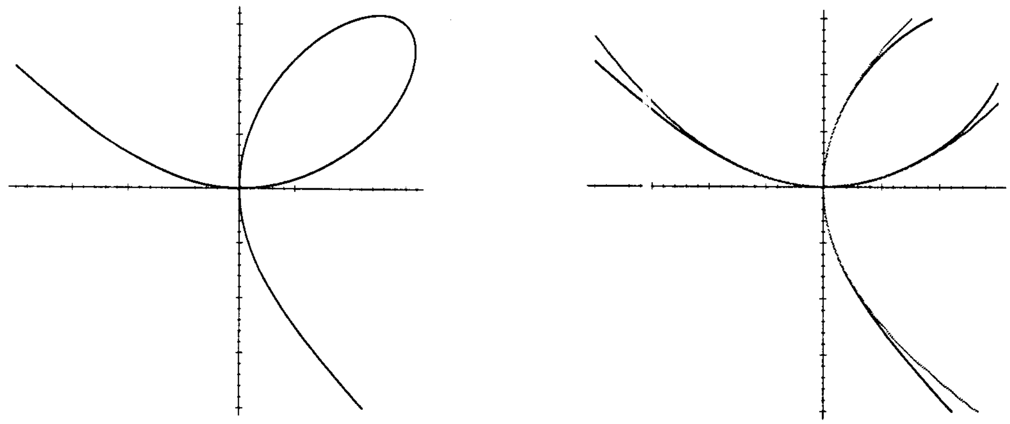

FIGURE (1.4). The Folium of Descartes and its first approximations.

This curve is pictured in the first graph of figure (1.4). Notice that two branches of the curve pass through $(0,0)$. The first branch, as illustrated by the second graph in figure (1.4), looks like the parabola, $y = (1/3)x^2$ The second, however, can be best approximated initially by a curve of the form $y = \pm\sqrt{3}x^{1/2}$. Also note that this "branch" is actually two branches when it comes to parameterizing the curve with respect to x. We will carry out Newton's construction for the case of the top half of this double branch.

The Newton polytope of the folium is the first polytope in figure (1.5). It has two bottom edges, one of slope -2 and one of slope $-1/2$, confirming our initial guess for the exponents of the first terms of the expansions. Moreover, the edge equation for the edge with slope $-1/2$ is

$$c^3 - 3c = 0.$$

Since we ignore the zero root, we obtain $c = \pm\sqrt{3}$, as we had guessed.

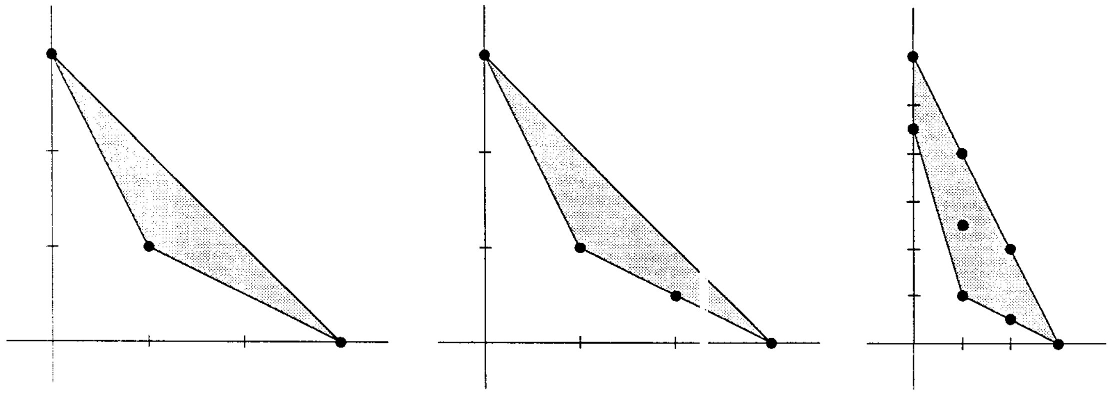

FIGURE (1.5). The polytopes in Newton's construction for the folium

Choosing $c = \sqrt{3}$ we obtain the first approximation for the desired branch

$$y = \sqrt{3}x^{1/2}.$$

Substituting $\sqrt{3}x^{1/2} + y$ into the original equation then yields

$$\begin{aligned} 0 &= x^3 + \left(\sqrt{3}x^{1/2} + y\right)^3 - 3x(\sqrt{3}x^{1/2} + y) \\ &= x^3 + 6xy + 3^{3/2}x^{1/2}y^2 + y^3. \end{aligned}$$

Whose Newton polytope is the second polytope pictured in figure (1.5).

We need to choose an edge on this new polytope and repeat the above actions. Consider the edge on the bottom of the polytope with slope strictly less than $-1/2$. (The next exponent must be strictly greater than the first.) This is the edge e containing the terms $6xy$ and x^3. Since, it has slope -2 and edge equation

$$6c + 1 = 0,$$

the next term in the expansion is $-1/6x^2$, yielding a second approximation for the expansion

$$y = \sqrt{3}x^{1/2} - \frac{1}{6}x^2.$$

Substituting $-1/6x^2 + y$ into the last equation yields

$$6xy + \frac{1}{3^{1/3}4}x^{9/2} - 3^{-1/3}x^{5/2}y + 3^{2/3}x^{1/2}y + y^3 - 6^{-3}x^6 + \frac{1}{12}x^4y - \frac{1}{2}x^2y^2,$$

whose Newton polytope is the last picture in figure (1.5). The slope of the desired edge is $-7/2$ and calculating the edge equation yields a third approximation

$$y = \sqrt{3}x^{1/2} - \frac{1}{6}x^2 - \frac{1}{3^{1/3}6^3}x^{7/2}.$$

Continuing in this fashion builds the series one step at a time.

1.3.3. Rings of fractional power series. The following definitions will facilitate our connections between the classical techniques in the previous section with series expansions in several variables.

If n is an integer greater than zero, and C is a strongly convex rational polyhedral cone in $\mathbb{R}^\ell$, then the set

$$C_n = C \cap \frac{1}{n}\mathbb{Z}^\ell$$

forms a semigroup under addition. From such a semigroup we can form the semigroup ring $\mathbb{C}[C_n]$, i.e. the ring of all finite formal sums of the form $\sum a_\alpha x^\alpha$ where $\alpha \in C_n$. We regard elements of $\mathbb{C}[C_n]$ as fractional Laurent polynomials in the variables $x_1, \ldots, x_\ell$. Let $\mathbb{C}[[C_n]]$ be the completion of the ring $\mathbb{C}[C_n]$. The ring of all formal fractional power series, $\sum_{\alpha \in C_n} a_\alpha x^\alpha$.

Definition 1.3.4. If C is a strongly convex rational polyhedral cone in $\mathbb{R}^\ell$, then the ring of fractional power series in the variables $x_1, \ldots, x_\ell$ with support in C is defined by

$$\mathbb{C}[[C_\mathbb{Q}]] = \bigcup_{n=1}^{\infty} \mathbb{C}[[C_n]].$$

More generally, the ring of fractional power series with support in some translate of C is

$$\mathbb{C}((C_{\mathbb{Q}})) = \bigcup_{\alpha \in \mathbb{Q}^\ell} x^\alpha \mathbb{C}[[C_{\mathbb{Q}}]]$$

It is essential to require that C be strongly convex, otherwise the set $\mathbb{C}[[C_{\mathbb{Q}}]]$ does not have a well defined multiplicative structure, since finding a coefficient when multiplying two generic series involves an infinite sum.

Let C be a strongly convex rational polyhedral cone. For any

$$f(x) = \sum_{\alpha \in \mathbb{Q}^\ell} a_\alpha x^\alpha$$

in $\mathbb{C}((C_{\mathbb{Q}}))$ we define the support of f to be the set of exponents which appear in f, i.e. $\text{Supp}(f) = \{\alpha \in \mathbb{Q}^\ell : a_\alpha \neq 0\}$. Since $f \in x^\alpha \mathbb{C}[[C_n]]$ for some n, the support of f must lie in some lattice $\frac{1}{n}\mathbb{Z}^\ell$.

For example, if $\ell = 1$ and $C = \mathbb{R}_+$, then $\mathbb{C}[[C_1]]$ is the usual ring of Laurent power series, $\mathbb{C}[[x]]$, over the complex numbers, and $\mathbb{C}((C_{\mathbb{Q}})) = \bigcup_{\alpha,n} x^\alpha \mathbb{C}((x^{\frac{1}{n}}))$ is the ring of fractional Laurent series in one variable as in [30] where it is denoted $\mathbb{C}(x)^*$. In [30] the Newton polygon construction is used to show that $\mathbb{C}(x)^*$ is an algebraically closed field.

1.3.5. Convergence and convex geometry. In order to speak of the convergence of fractional power series in $\mathbb{C}((C_{\mathbb{Q}}))$, we must define the manner in which these series act as functions on $\mathbb{C}^N$. More precisely, we must define the action of $x^\alpha = x_1^{\alpha_1} \cdots x_\ell^{\alpha_\ell}$ on $(\mathbb{C}^*)^\ell$. To do this we only need to choose, in each variable, a sector in $\mathbb{C}^*$ and define a branch of the logarithm in this sector, e.g. the principal branch of the log: Let $\mathbb{C} \setminus \mathbb{R}_-$ be the chosen sector and define

$$x_i^{\alpha_i} = e^{\alpha_i \log x_i}$$

for each variable x_i. We are primarily interested in the regions for which an $f \in \mathbb{C}[[C]]$ is absolutely convergent (i.e. where $\sum |a_\alpha||x^{\alpha}|$ converges).

DEFINITION 1.3.6. If C is a convex rational polyhedral cone, then $\mathbb{C}\{\{C_{\mathbb{Q}}\}\}$ will denote the subring of $\mathbb{C}((C_{\mathbb{Q}}))$ consisting of all series which are convergent at some point of $(\mathbb{C}^*)^\ell$, i.e. if for $f \in \mathbb{C}((C_{\mathbb{Q}}))$, D_f is the domain of convergence of f, then

$$\mathbb{C}\{\{C_{\mathbb{Q}}\}\} = \left\{f \in \mathbb{C}((C_{\mathbb{Q}})) : D_f \neq \emptyset\right\}$$

Note that $\mathbb{C}\{\{C_{\mathbb{Q}}\}\}$ consists only of convergent series whose exponents all lie in $\mathbb{Z}[1/n]$ for some n.

It is convenient to pass to the logarithms of the $|x_i|$ when considering convergence, therefore we introduce the space $\mathbb{R}^N_{\log}$ called the logarithmic space of $(\mathbb{C}^*)^\ell$. This space is associated to $(\mathbb{C}^*)^\ell$ via the map

$$\mathrm{Log} : (\mathbb{C}^*)^\ell \longrightarrow \mathbb{R}^\ell$$

given by

$$\mathrm{Log}(x_1, \ldots, x_\ell) = \Big(\log(|x_1|), \ldots, \log(|x_\ell|)\Big).$$

The usefulness of this notation is indicated by the following lemma.

LEMMA 1.3.7. *For each $f \in \mathbb{C}\{\{C_{\mathbb{Q}}\}\}$ the domain of convergence of f has the form $\mathrm{Log}^{-1}(U)$, for some convex set $U \subset \mathbb{R}^\ell_{\log}$.*

PROOF. For each such f there exists some $n \in \mathbb{Z}_+$ such that $f \in \mathbb{C}((C_n))$. Therefore, this lemma follows, by a change of variables, from the well known fact (see [19]) that this is true for power series with integer exponents. □

LEMMA 1.3.8. *Suppose $f = \sum a_\alpha x^\alpha$ is in $\mathbb{C}((C_{\mathbb{Q}}))$ and f has a nonempty domain of convergence D (i.e. $f \in \mathbb{C}\{\{C_{\mathbb{Q}}\}\}$), then there exists some $A \in (\mathbb{C}^*)^\ell$ such that*

$|a_\alpha| \leq |A^\alpha|$ *for almost all* α. *Moreover, if* x *is any point in* D, *and* C *is any cone which contains the Newton polytope* $P(f)$, *then* $C^* + \mathrm{Log}(x) \subset \mathrm{Log}(D)$.

PROOF. Suppose $x \in (\mathbb{C}^*)^\ell$ satisfies $\sum |a_\alpha||x|^\alpha \leq \infty$. We may assume that $|a_\alpha||x|^\alpha \leq 1$ for all α, since this must be true for all but a finite number of α. Also we can assume that $\mathrm{Supp}(f) \subset C$ since $f = x^\beta g$ for some β where g has support lying in C. Now if we rewrite the above inequality, we get

$$|a_\alpha| \leq \left|\frac{1}{x^\alpha}\right| = \left|\left(\frac{1}{x_1}, \ldots, \frac{1}{x_N}\right)^\alpha\right|.$$

Suppose $x' \in (\mathbb{C}^*)^\ell$ such that $\mathrm{Log}(x') \in C^* + \mathrm{Log}(x)$. Then since $\mathrm{Log}(x') = w + \mathrm{Log}(x)$ for some $w \in C^*$ and $\langle w, \alpha \rangle \leq 0$, we have that for each $\alpha \in C$

$$\Big\langle \mathrm{Log}(x'), \alpha \Big\rangle \leq \langle \mathrm{Log}(x), \alpha \rangle$$

and so

$$\alpha_1 \log(|x'_1|) + \cdots + \alpha_\ell \log(|x'_\ell|) \leq \alpha_1 \log(|x_1|) + \cdots + \alpha_\ell \log(|x_\ell|)$$

which implies that

$$\log(|{x'_1}^{\alpha_1} \cdots {x'_\ell}^{\alpha_\ell}|) \leq \log(|{x_1}^{\alpha_1} \cdots {x_\ell}^{\alpha_\ell}|).$$

Since, on $\mathbb{R}_+$, the function log is monotone increasing, we get

$$|{x'_1}^{\alpha_1} \cdots {x'_\ell}^{\alpha_\ell}| \leq |{x_1}^{\alpha_1} \cdots {x_\ell}^{\alpha_\ell}|.$$

Therefore, for every $\alpha \in C$, $|x'|^\alpha \leq |x|^\alpha$, yielding that $\sum |a_\alpha||x'|^\alpha \leq \sum |a_\alpha||x|^\alpha$. $\square$

We say that f converges at some point $y \in \mathbb{R}^N_{\log}$ if $\mathrm{Log}^{-1}(y) \subset D$ where D is the domain of convergence for f. The above lemma can be summarized by saying that if f converges at some point $y \in \mathbb{R}^N_{\log}$ then f converges on some translate of C^*. Using these two lemmas it is now possible for us to prove the following theorem.

THEOREM 1.3.9. *If C is a cone in $\mathbb{R}^\ell$ and if a series $f \in \mathbb{C}((C_\mathbb{Q}))$ is algebraic over $\mathbb{C}[x_1, \ldots, x_\ell]$ then there is some translate of C on which f is convergent.*

PROOF. The fact that there is a point at which f converges follows from [1]. Given this, the theorem then follows from Lemma 1.3.8. $\square$

CHAPTER 2

Expansions of a single equation

1. The main construction

For the case of a single equation $k = 1$, it is convenient to use a common letter for all the variables involved. As there is only one 'y' coordinate, we drop that notation and work with a single list of variables $x_1, \ldots, x_{N+1}$. We use the index N so that the notation will not be confused in the more general case.

We are now ready to turn to the question posed in the introduction: How does one construct a fractional power series $x_{N+1} = \phi(x_1, \ldots, x_N)$ which satisfies a given algebraic equation $F(x_1, \ldots, x_{N+1}) = 0$? To answer this we will need to closely investigate the Newton polytope of F.

2.1.1. Construction for generic equations. The following theorem is the main result of this chapter

THEOREM 2.1.2. *Let $F(x_1, \ldots, x_{N+1})$ be a polynomial in $N + 1$ variables. Let e be any admissible edge of the polytope $P(F)$. Let $C^* = C^*(e) \subset \mathbb{R}^N$ be its normal cone, and let k_e be the length of the projection of e onto the x_{N+1}-axis. Then*

a) *For each irrational $w \in C^*$ there exists some strongly convex rational polyhedral cone C_w such that $w \in C_w^*$ and such that the ring $\mathbb{C}((C_{w\mathbb{Q}}))$ contains* *at least k_e series $\{\phi_i\}_i$ counted with multiplicity such that for each i*

$$F(x_1, \ldots, x_N, \phi_i) = 0.$$

b) *In fact, $\mathbb{C}((C_{w\mathbb{Q}}))$ contains, up to multiplicity, exactly k_e series that correspond to e.*

PROOF. We will prove part a) and defer part b) until a later section. To prove a) we inductively build a series of the form

$$\sum_{n=1}^{\infty} c_n x_1^{\alpha_{n,1}} \cdots x_N^{\alpha_{n,N}} = \sum_{n=1}^{\infty} \psi_n.$$

So we first build ϕ_1 and then move on to ϕ_n for any n. In most respects, the constructions will be identical.

Let

$$F_1(x_1, \ldots, x_{N+1}) = F(x_1, \ldots, x_{N+1}) = \sum_{I \in S_1} a_I x^I$$

where $S_1 = \mathrm{Supp}(F)$. Let $e_1 = e$ be the edge of $P(F_1)$ chosen in the hypotheses, and let $W(e)$ and $B(e)$ be the barrier wedge and cone of e_1 respectively. The edge e_1 has a slope with respect to x_{N+1}, say $S(e_1) = (s_{1,1}, \ldots, s_{1,N})$. Define

$$\phi_1(x_1, \ldots, x_N) = c_1 x_1^{-s_{1,1}} \cdots x_N^{-s_{1,N}}$$

where c_1 is a solution of the equation

$$F_e(t) = \sum_{(i_1, \ldots, i_{N+1}) \in e_1 \cap S_1} a_{i_1, \ldots, i_{N+1}} t^{i_{N+1} - m(e)_{N+1}} = 0. \tag{2.1}$$

where $m(e)_{N+1}$ is the $(N+1)$-st coordinate of the minor vertex. We refer to this equation as the edge equation of e. This sum ranges over all of the terms of F_1 which correspond to points on the edge e_1. Such a solution exists since there must be at least two points of S on e_1.

The degree of equation (2.1) (discounting zero roots) is equal to the difference between the largest and the smallest i_{N+1} appearing in $e_1 \cap S$. Therefore, the number of non-zero solutions to equation (2.1), counting multiplicity, is equal to the length, k_e, of the projection of the edge $e = e_1$ onto the x_{N+1}-axis.

For the rest of this construction we will need to use the chosen element w of the normal cone of e_1. By assumption w is irrational. So, for any $\alpha \neq \alpha' \in \mathbb{Q}^N$ we have $\langle w, \alpha \rangle \neq \langle w, \alpha' \rangle$, and hence w induces a linear order on $\mathbb{Q}^N$. If $\psi = x_1^{i_1} \cdots x_N^{i_N}$ is a monomial in N variables, we will let $\langle w, \psi \rangle$ indicate the value $\langle w, (i_1, \ldots, i_N) \rangle$.

Assume that $F_{n-1}(x_1, \ldots, x_{N+1})$, e_{n-1} (an edge of $P(F_{n-1})$), and ϕ_{n-1} have been constructed. Let F_n be defined by

$$F_n(x_1, \ldots, x_{N+1}) = F_{n-1}(x_1, \ldots, x_N, \psi_{n-1} + x_{N+1}).$$

We assume that $\psi_n = 0$ is not a solution of $F_n = 0$. If it were we would have the desired solution of $F = 0$.

To construct ϕ_n we will choose an edge on the Newton polytope of F_n satisfying the conditions of the following lemma. For every $i = 1, \ldots, n-1$ we let k_i be the multiplicity of c_i as a root of the edge equation of e_i.

LEMMA 2.1.3. *On the Newton polytope $P(F_n)$ there is a unique coherent edge path*

$$E = e_{1,n}, \ldots, e_{k,n}$$

such that

a) *The major vertex $M(e_{k,n})$ lies on the line, L, through e_{n-1}.*

b) *The minor vertex $m(e_{1,n})$ lies on the null hyperplane.*

c) *The major vertex $M(e_{k,n})$ has x_{N+1} coordinate equal to k_{n-1}.*

d) $\langle w, s(e_{n-1})\rangle < \left\langle w, s(e_{k,n})\right\rangle < \cdots < \langle w, s(e_{1,n})\rangle$.

e) $w \in \bigcap_{i=1}^{k} C^*(e_{i,n})$.

The last condition assures us that for each edge $e_{i,n}$ the unique w-constant hyperplane containing $e_{i,n}$ is a supporting hyperplane for the polytope $P(F_n)$.

Let e_n be any edge on the edge path of Lemma 2.1.3, and define

$$\psi_n = c_n x_1^{-s_{n,1}} \cdots x_N^{-s_{n,N}}$$

where the N-tuple $(s_{n,1}, \ldots, s_{n,N})$ is the slope of the edge e_n and c_n satisfies the edge equation

$$(2.2) \qquad F_{e_n}(t) = \sum_{(i_1,\ldots,i_{N+1}) \in e_n \cap S_n} a_{i_1,\ldots,i_{N+1}} t^{i_{N+1} - m(e)_{N+1}} = 0.$$

where $S_n = \mathrm{Supp}(F_n)$.

The fact that

$$\langle w, s(e_{n+1})\rangle > \langle w, s(e_n)\rangle > \langle w, s(e_1)\rangle$$

assures us that the terms are linearly ordered under the order on $\mathbb{Q}^N$ induced by w. We will define $\phi_n = \phi_{n-1} + \psi_n$. Having inductively constructed ψ_n and ϕ_n for all n we define our candidate for a solution of $F = 0$ to be $\phi = \sum_{n=1}^{\infty} \psi_n$.

To complete this construction and show that ϕ satisfies the conditions of part a) of theorem 2.1.2, we must

(1) Prove Lemma 2.1.3.

(2) Show that $\mathrm{Supp}(\phi)$ lies in some proper cone C_w of $\mathbb{R}^N$

(3) Show that the exponents of ϕ lie in some lattice $\frac{1}{n}\mathbb{Z}$.

(4) Show that ϕ satisfies the equation $F(x_1, \ldots, x_{N+1}) = 0$.

(5) Show that the number of series, up to multiplicity, created by this process is greater than or equal to the length of the projection of e onto x_{N+1}.

2.1.4. Proof of Lemma 2.1.3. We will construct the required edge path and simultaneously prove parts $a)$, $b)$ and $e)$ of Lemma 2.1.3. Consider F_n and its Newton polytope, $P(F_n)$. We will investigate its relationship to F_{n-1} and $P(F_{n-1})$. As above, let $S_{n-1} = \text{Supp}(F_{n-1})$ and

$$F_{n-1}(x) = \sum_{I=(i_0,\dots,i_{N+1})\in S_{n-1}} a_I x^I.$$

Therefore

$$\begin{aligned} F_n(x_1,\dots,x_{N+1}) &= \sum_{I\in S_{n-1}} a_I x_1^{i_1}\cdots x_N^{i_N}(\psi_{n-1}+x_{N+1})^{i_{N+1}} \\ &= \sum_{I\in S_{n-1}} a_I x_1^{i_1}\cdots x_N^{i_N} \sum_{j=0}^{i_{N+1}} \binom{N+1}{j} \psi_{n-1}^j x_{N+1}^{i_{N+1}-j}. \end{aligned}$$

If we rearrange this second expression we get

(2.3)

$$F_n(x_1,\dots,x_{N+1}) = \sum_{I\in S_{n-1}} \sum_{j=0}^{i_{N+1}} \binom{N+1}{j} a_I c_{n-1}^j x_1^{i_1+j\alpha_{n-1,1}} \cdots x_N^{i_N+j\alpha_{n-1,N}} x_{N+1}^{i_{N+1}-j}.$$

Notice that, in these expressions, the exponent on x_{N+1} is always an integer. Examining these expressions, we can also see that for each term,

$$T = \binom{i_{N+1}}{j} a_I c_{n-1}^j x_1^{i_1+j\alpha_{n-1,1}} \cdots x_N^{i_N+\alpha_{n-1,N}} x_{N+1}^{i_{N+1}-j},$$

the point on the Newton polytope, $P(F_n)$, which corresponds to T, lies on the line through I (a point of $\text{Supp}(F_{n-1})$), with slope $(-\alpha_{n-1,1},\dots,-\alpha_{n-1,N})$. So each point on the Newton polytope $P(F_n)$ lies on a line through a point of $P(F_{n-1})$ and parallel to the edge e_{n-1}. One consequence of this is that the Newton polytope $P(F_n)$ is supported by the w-constant hyperplane P_w determined by e_{n-1}.

Consider the summand of F_n whose terms correspond to points on the line L_{n-1} determined by e_{n-1}. Let P_1 be the point $L_{n-1} \cap \{x_{N+1} = 0\}$, i.e. the point of intersection of L_{n-1} and the null-hyperplane. Let $P_2 = M(e_{n-1})$ be the major vertex of e_{n-1}. We will examine the coefficients of the monomials in F_n which correspond P_1 and P_2. Both monomials have the possibility of occurring in F_n with non-zero coefficients, since both appear in equation (2.3).

For P_1, the coefficient of the corresponding monomial is

$$(2.4) \qquad \sum_{I \in S_{n-1} \cap e_{n-1}} a_I c_{n-1}^{i_{N+1}},$$

since all of the terms in expression (2.3) which contribute to expression (2.4) must correspond to points on the edge e_{n-1} of $P(F_{n-1})$ and must also have a vanishing x_{N+1}-exponent. By the construction of c_n this sum is equal to 0. Therefore P_1 is not in $P(F_n)$.

The coefficient of the term corresponding to P_2 is unchanged from what it was in F_{n-1}. Any term $a_I x^{i_1} \cdots x^{i_{N+1}}$ other than that corresponding to P_2 contributes only terms corresponding to points lying on the line through I, parallel to e_{n-1}, and lying to the *left* of I (i.e. their x_{N+1} coordinates are less that that of I). Since there are no terms in F_{n-1} corresponding to points on L_{n-1} to the right of P_2, the only contribution to P_2 in F_n comes from P_2 itself. So j in expression (2.3) is 0, and hence the coefficient remains unchanged.

Lastly we need to note that there are terms in F_n which correspond to points on the null-hyperplane. Since $\psi_n = 0$ is not a root of $F_n = 0$, such points exist.

Putting all of this together, we get that there are points of $P(F_n)$ which lie on L_{n-1}. We know that all such points have strictly positive x_{N+1}-exponents, and that there are points of this polytope lying in the null-hyperplane, i.e. strictly to the left of all points in $P(F_n)$ lying on L_{n-1}. Hence conditions $a)$ and $b)$ of the lemma are satisfied. Since $P(F_n)$ is the convex hull of a set containing these points and lying

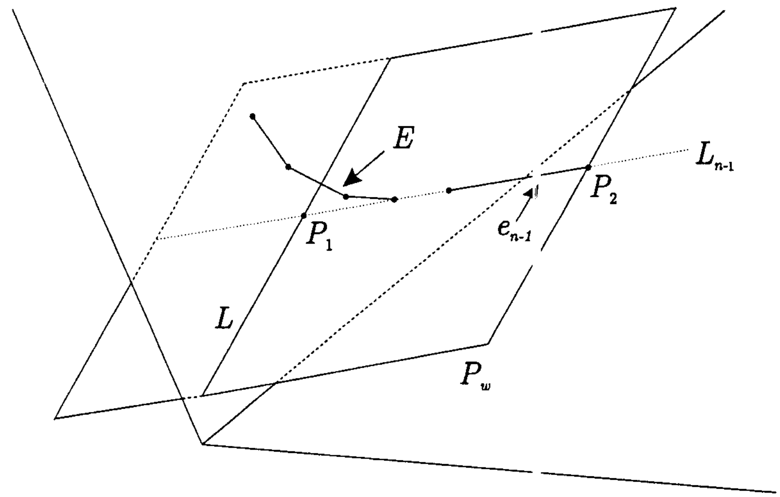

FIGURE (2.6). The edge path E and the previous edge e_{n-1}

on one side of the w-constant plane P_w, there must be an edge path, $e_{1,n}, \ldots, e_{k,n}$, on $P(F_n)$ such that $M(e_{k,n})$ lies on L_{n-1} and $m(e_{1,n})$ lies on the null-hyperplane, namely the edge path E that maximizes w on each vertical section (See figure (2.6)). Hence condition e) is satisfied. Since w is irrational this edge path is unique.

Next we will establish d) of Lemma 2.1.3 by showing that

$$\Big\langle w, s(e_{k,n})\Big\rangle > \langle w, s(e_{n-1})\rangle$$

The other inequalities follow from the same argument.

Let P_w be the w-constant hyperplane containing e_{n-1}. Note that P_w is a supporting hyperplane for $P(F_{n-1})$ and $P(F_n)$. Define the vectors $u := (u_1, \ldots, u_N, 1)$ and $v_i := (v_1, \ldots, v_N, 1)$ associated to e_{n-1} and $e_{k,n}$ respectively as follows. If $M = (M_1, \ldots, M_{N+1})$ and $m = (m_1, \ldots, m_{N+1})$ are the major and minor vertices of e_{n-1}, then

$$u_i = \frac{M_i - m_i}{M_{N+1} - m_{N+1}}$$

and likewise for $e_{k,n}$ and v. Note that u and v are tangent to e_n and $e_{k,n}$, and $(u_1, \ldots, u_N)$ and $(v_1, \ldots, v_N)$ are the slopes of e_n and $e_{k,n}$ with respect to x_{N+1}

respectively. If $I = (i_1, \ldots, i_{N+1})$ is the point of intersection of $e_{k,n}$ and L_{n-1}, and L_n is the line containing $e_{k,n}$, then the points of intersection of L_{n-1} and L_n with the null-hyperplane are

$$p_1 = (i_1 - i_{N+1}u_1, \ldots, i_N - i_{N+1}u_N, 0)$$

$$p_2 = (i_1 - i_{N+1}v_1, \ldots, i_N - i_{N+1}v_N, 0)$$

respectively. Let L be the line of intersection of the plane P_w with the null hyperplane. P_w contains L_{n-1}, so the point p_1 lies on L while P_2 lies in the interior of the half N-plane determined by P_w that contains $F(F_n) \cap \{x_{N+1} = 0\}$.

By the construction of the linear functional w and the edge path E, we have that $\langle w, p_1 \rangle > \langle w, p_2 \rangle$ and so

$$\langle w, -i_{N+1}u \rangle > \langle w, -i_{N+1}v \rangle$$

which implies, since w is linear, that $\langle w, u \rangle < \langle w, v \rangle$ and hence that $\left\langle w, s(e_{k,n}) \right\rangle > \langle w, s(e_{n-1}) \rangle$. This finishes the proof of condition $d)$ of Lemma 2.1.3.

Last, we need to prove condition $c)$. Let e_n be any edge on the edge path. Note that the normal cone of this edge does indeed intersect $C^*(e_{n-1})$, since in particular w is contained in both of these cones.

We need to show that the x_{N+1} coordinate of $M(e_{k,n})$ is k_n. Consider the derivatives with respect to x_{N+1} of both F_{n-1} and F_n. Recall that

$$F_n = F_{n-1}(x_1, \ldots, x_N, c_{n-1}x_1^{\alpha_{n-1,1}} \cdots x_N^{\alpha_{n-1,N}} + x_{N+1}).$$

Hence, $F'_n = F'_{n-1}(x_1, \ldots, x_N, \psi_n + x_{N+1})$, where F'_n denotes the derivative of F_n with respect to x_{N+1}. Let P and P' be the polytopes of F_{n-1} and F'_{n-1} respectively. We are taking the derivative of a polynomial in integer powers of x_{N+1}, so P' is obtained from P by removing the points of P lying on the null-hyperplane and then shifting the rest of the polytope by -1 in the x_{N+1} coordinate.

Let L_{n-1} be the line through e_{n-1}, and let L'_{n-1} be the translate $L_{n-1}-(0,\dots 1)$ of L_{n-1}. Finally, let Q_{n-1} and Q'_{n-1} be the intersection points of L_{n-1} and L'_{n-1} respectively with the null-hyperplane. We define the coefficient restriction of F_{n-1} to the edge e_{n-1} by

$$F_{n-1}|_{e_{n-1}} = \sum_{I \in e_{n-1}} a_I x^{\alpha_I}$$

Unless $(F_{n-1}|_{e_{n-1}})'$ is a constant, L'_{n-1} is the unique line which contains the terms of $(F_{n-1}|_{e_{n-1}})'$.

Let $\{a_i x^{\alpha_i}\}$ be the monomials of $F_{n-1}|_{e_{n-1}}$. By construction, c_{n-1} is a root of the edge equation

$$\sum a_i t^{\alpha_{N+1,i}-i_0} = 0 \tag{2.5}$$

So the coefficient in F_n of the monomial corresponding to Q_{n-1} is

$$\left(\sum a_i t^{\alpha_{N+1,i}-i_0}\right)\Big|_{t=c_{n-1}} = 0$$

To determine the coefficient on the monomial corresponding to Q'_{n-1} in F'_n, consider the terms of $\left(F_{n-1}|_{e_{n-1}}\right)'$. They are

$$\alpha_{N+1,i} a_i x_1^{\alpha_{1,i}} \cdots x_N^{\alpha_{N,i}} x_{N+1}^{\alpha_{N+1,i}-1} = \frac{d}{dx_{N+1}}\left(a_i x^{\alpha_i}\right)$$

By the chain rule we get

$$\frac{d}{dx_{N+1}}(F_n) = \left(\frac{d}{dx_{N+1}}(F_{n-1})\right)(x_1,\dots,x_N,\phi_{n-1}+x_{N+1})$$

So in the same way as we showed that (2.5) was the coefficient of the monomial corresponding to Q_{n-1} in F_n we see that

$$\sum \frac{d}{dt}\left(a_i t^{\alpha_{N+1,i}}\right)\Big|_{t=c_{n-1}} = \frac{d}{dt}\left(\sum a_i t^{\alpha_{N+1,i}}\right)\Big|_{t=c_{n-1}} \tag{2.6}$$

is the coefficient of the monomial corresponding to Q'_{n-1} in F'_n.

If $k_{n-1} = 1$ then c_{n-1} is not a root of equation (2.6). Therefore the coefficient of Q'_{n-1} in F'_n is non-zero, and so the x_{N+1} coordinate of $M(e_{k,n})$ is 1. The case where $k_{n-1} > 1$ follows by applying the same argument as above with higher order derivatives. This completes the proof of part c) and therefore the proof of Lemma 2.1.3.

2.1.5. The exponents of ϕ lie in a lattice. This argument is almost identical to its single variable counterpart (see [30]). For all n sufficiently large, c_n is a root of some multiplicity k_{n_0}, since the multiplicity of c_n as a root of the face equations can only drop a finite number of times. (Usually this stable multiplicity will be zero.)

Let $n > n_0$, and let (a_j, j) and $(a_0, 0)$ be the major and minor vertices of e_n respectively, where $a_i \in \mathbb{R}^N$, and $j = k_{n-1}$. Since we have only carried out a finite number of steps, F_n lies in some lattice $1/m\mathbb{Z}^N$ and so there are vectors m_0, m_j such that

$$\frac{m_0}{m} = a_0 \quad \text{and} \quad \frac{m_j}{m} = a_j.$$

Since e_i determines a line with slope $\alpha_n = (\alpha_{1,n}, \ldots, \alpha_{N,n})$

$$a_0 = a_j + j\alpha_n$$

and hence

$$\alpha_n = \frac{a_0 - a_j}{j} = \frac{m_0 - m_j}{mj} = \frac{p}{mq}$$

where $p \in \mathbb{Z}^N$ and $q \in \mathbb{Z}$ such that there is some i_0 for which p_{i_0} and q are relatively prime.

If $(a_h, h) \in e_n$ and $a_h = m_h/m$ then

$$\frac{p}{mq} = \alpha_n = \frac{a_0 - a_h}{h} = \frac{m_0 - m_h}{mh}.$$

Therefore $q(m_0 - m_h)_{i_0} = p_{i_0}h$, and since $(q, p_{i_0}) = 1$, we know that q divides h. Therefore the edge equation of e_n has the form

$$F_{en}(t) = g(t^q).$$

But by above for n sufficiently large c_n is a root of F_{en} of multiplicity $k_{n-1} = \deg_{x_{N+1}}(F_{en})$. Therefore

$$F_{en} = d(t - c_n)^{k_{n-1}}.$$

Since k_{n-1}, d and c_n are all non-zero, $F_{en}(t)$ must have a non-zero coefficient at t^1. Hence $q = 1$ which implies that $\alpha_n \in 1/m\mathbb{Z}^N$ for all n sufficiently large.

2.1.6. Proof that the support of ϕ lies in a cone.

LEMMA 2.1.7. *There exists some $n_0 > 0$ such that for all $n \geq n_0$ the major vertices of e_n are the same.*

PROOF. By Lemma 2.1.3 we know that the major vertex of e_n has x_{N+1}-coordinate equal to k_{n-1} which is a decreasing sequence. Since $k_n \geq 1$ for all n there must be some $n_0 > 0$ such that $k_{n_0-1} > k_{n_0}$ but $k_{n'} = k_{n_0}$ for all $n' > n_0$.

We now proceed by induction on $n \geq n_0$. Since the claim is obvious for e_{n_0}, let $n > n_0$. The length of e_n is equal to the length of e_{n-1}. Therefore the edge path constructed in Lemma 2.1.3 consists entirely of one edge, e_n. Therefore, the major vertex lies on the line L_{n-1} that contains e_{n-1} and has an x_{N+1} coordinate less than or equal to that of the major vertex of e_{n-1}. Hence these two points are equal. □

LEMMA 2.1.8. *The support of ϕ lies in some translate of $C(e_{n_0})$.*

PROOF. Let w be any element of $C^*(e_{n_0})$. Now, F_{n_0} lies in the barrier wedge of e_{n_0}, and since the minor vertex $m(e_{n_0+1})$ lies inside this barrier wedge, we can conclude that $P(F_{n_0+2})$ also lies in this barrier wedge. Using this argument recursively shows that for every $n > n_0$, $P(F_n)$ lies in the barrier wedge of e_{n_0}. This fact implies that the minor vertex of e_n lies in the barrier wedge of e_{n_0} but not on the line through e_{n_0}.

Using the same argument as was used in Lemma 2.1.3, we can show that

$$\langle w, s(e_n)\rangle > \left\langle w, s(e_{n_0})\right\rangle$$

Which shows, since w was arbitrary, that $\alpha_n = -S(e_n) \in C(e_{n_0})$ for all $n > n_0$. Since all but a finite terms are in this cone, the entire series must be contained in a translate of $C(e_{n_0})$. □

2.1.9. Proof that ϕ satisfies the equation. Consider F as an element of the the polynomial ring over $\mathbb{C}((C_w))$. Since ϕ is itself a member of this ring, we have a well defined notion of ϕ satisfying $F(\phi) = 0$, where $F(\phi) = F(x_1, \ldots, x_N, \phi)$. This result will follow from the following lemma.

LEMMA 2.1.10. *For each $r > 0$ there is a $n_0 > 0$ such that if $n > n_0$ and if $B(0; r)$ is the ball of radius r about 0, then*

$$\operatorname{Supp}\left(F(x_1, \ldots, x_N, \phi_n)\right) \subset \mathbb{R}^N \setminus B(0; r).$$

i.e. all the exponents of $\operatorname{Supp}\left(F(x_1, \ldots, x_N, \phi_n)\right)$ *lie beyond the ball of radius r.*

PROOF. Let v_1 be the intersection of the line l_1 containing e_1 with the null-hyper-plane. Since $\operatorname{Supp}(\phi)$ is contained in a half-plane determined by w, and w is a linear functional, it is sufficient to show the following. If, under the ordering induced by w, p_1 and p_2 are the largest points of $\operatorname{Supp}(F(\phi_n))$ and $\operatorname{Supp}(F(\phi_{n+1}))$

respectively, then $p_1 > p_2$. (Recall that w gets smaller as we move out along the terms of ϕ.)

We need to compare the two series $F(x_1, \ldots, x_N, \phi_n)$ and $F(x_1, \ldots, x_N, \phi_{n+1})$. To do this we will consider

$$F_n = F(x_1, \ldots, x_N, \phi_n + x_{N+1}) \text{ and } F_{n+1} = F(x_1, \ldots, x_N, \phi_{n+1} + x_{N+1}).$$

Assume that neither $F(x_1, \ldots, x_N, \phi_n)$ nor $F(x_1, \ldots, x_N, \phi_{n+1})$ is 0, since, if either were true, we would trivially have the desired result. By the discussion presented in the proof of Lemma 2.1.3, we know that $P(F_n)$ lies above the w constant plane determined by e_{n-1} and likewise $P(F_{n+1})$ lies above the plane determined by e_n. With this, the inequality,

$$\langle w, s(e_{n-1})\rangle < \langle w, s(e_n)\rangle$$

yields the desired result. □

2.1.11. Proof of the lower bound on the number of solutions. To conclude the proof of theorem 2.1.2 a), we need to show that the number of solutions obtained from an edge is at least the length of the projection of that edge onto the x_{N+1}-axis.

The first coefficient of any solution series corresponding to the chosen edge e is a root of a polynomial whose degree is equal to the length, k_e, of the projection of that edge. So there are, counting multiplicity, k_e possibilities for this first coefficient. For each distinct root we get a different solution series and hence we get at least as many series as the number of distinct roots of this equation.

For each multiple root we need to consider what happens with later coefficients. So assume that c_n is a multiple root of the edge equation of e_n. Since the length of the edge path constructed in Lemma 2.1.3 for e_{n+1} is equal to the multiplicity of c_n as a root of the edge equation of e_n, we see that the total number of possible choices for the $(n+1)$-st coefficient is at least equal to the multiplicity of c_n. Again, distinct coefficients will yield distinct series expansions.

Let ϕ be built as above. By Lemma 2.1.3, for all n sufficiently large, c_n has some fixed multiplicity k_ϕ as a root of the edge equation $F_{en}(t)$. We claim that the multiplicity of ϕ as a root of $F = 0$ is at least k_ϕ. Suppose $k_\phi > 1$ then by the proof of Lemma 2.1.3 each c_n is a root of $d^{(k)}F_{en}(t)/dt^k$ for all n and all $1 \leq k \leq k_\phi$. Therefore ϕ is a root of d^kF/dx_{N+1}^k and hence has multiplicity at least k_ϕ. Since the k_ϕ's must add up to at least the degree of the edge equation of e_1 we see that up to multiplicity the number of series solutions is greater than or equal to the degree of F_{e_1} and hence the length of the projection of e_1 onto the x_{N+1} axis.

Since all of these solutions lie in strongly convex rational cones whose duals contain w, and since w is irrational, there must be some strongly convex rational cone C_w containing the supports of all of these series. This finishes the proof of theorem 2.1.2. □

2.1.12. The case of simple roots. For a polynomial $F(x_1, \ldots, x_{N+1})$ we will let $\Delta_{x_{N+1}}(F)$ denote the classical discriminant of F with respect to x_{N+1}. Recall that the discriminant of a polynomial $p(x)$ in one variable is a polynomial in the coefficients of $p(x)$ defined only up to a non-zero constant multiple and which vanishes precisely when $p(x)$ has a multiple root. (For a thorough discussion of classical discriminants see [12] chapter 12.) Likewise for a polynomial $g(t)$ of one variable, $\Delta_t(g)$ will denote the discriminant of g with respect to t. In the next part, we will explore more general descriptions of discriminants and the related concept of the resultant.

COROLLARY 2.1.13. *If c_1 is a simple root of $F_e(t)$, and ϕ is the series built by the above algorithm, then* Supp(ϕ) $\subset C(e)$. *More generally, if the discriminant* $\Delta_t(F_e(t)) \neq 0$ *then all series generated from this edge have support in* $C(e)$.

PROOF. Since $k_1 = 1$, $n_0 = 1$ and so the result follows from the proof in section 2.1.6. □

REMARK 2.1.14. Recall that in section 1.2.20 we considered the case of a rational function

$$F(x) = x_{N+1} f(x_1, \ldots, x_N) - 1 = 0.$$

Note that the Newton polytope $P(F)$ is a cone over the Newton polytope of f. Therefore, the vertices of $P(f)$ correspond to the admissible edges of $P(F)$. For $(v_1, \ldots, v_N)$ a vertex of $P(f)$, the slope of the corresponding edge in $P(F)$ is $(-v_1, \ldots, -v_N)$.

2. Applications and comments

2.2.1. Fiber polytopes and full solution sets. If d is the degree of F in the variable x_{N+1}, then we would like to find rings of fractional power series which contain a full set of d solutions to the equation $F = 0$. We apply this construction to the following situation. Let $P = P(F)$ be the Newton polytope of F, let $Q = [0, d]$ where $d = \deg_{x_{N+1}}(F)$ and let ψ be the projection onto the last coordinate.

If E is a monotone edge path (see section 1.2.11) which is maximal, i.e. $\psi(E) = Q$, then E defines a section γ_E of ψ. From now on we consider only maximal edge paths. It was shown in [3] that the vertices of $\Sigma_\psi(P, Q)$ have the form $\int_{[0,d]} \gamma_E(x)\, dx$, where γ_E is a the section corresponding to a coherent edge path E. Moreover, if v is a vertex $\int \gamma_E(x)\,dx$ of $\Sigma_\psi(P,Q)$, then the barrier cone $C(v)$ is the union of the barrier cones of the edges in E.

We may now formulate the following corollary to theorem 2.1.2 (a).

COROLLARY 2.2.2. *Let $F(x_1,\ldots,x_{N+1})=0$ be an algebraic equation such that each edge of $P(F)$ satisfies $\Delta_t(F_e)\neq 0$. Let $E=\{e_1,\ldots,e_n\}$ be a coherent edge path on $P(F)$. Let $d=\deg_{x_{N+1}}(F)$ be the degree of F with respect to x_{N+1} and let*

$$C=\bigcup_{i=1}^{n} C(e_i)$$

be the union of the barrier cones of the edges in E. Then the ring $\mathbb{C}((C_{\mathbb{Q}}))$ contains d solutions to $F(x_1,\ldots,x_{N+1})=0$, counting multiplicity. Therefore complete systems of solutions are in one to one correspondence with the vertices of $\Sigma\,(P(F),[0,d])$.

PROOF. By the proof of theorem 2.1.2, if k_i is the length of the projection $\psi(e_i)$, then the number of solutions, counting multiplicity, that correspond to e_i, is at least k_i. Again, as in the proof of theorem 2.1.2, we can order $C_{\mathbb{Q}}$ via an element of ${C_{\mathbb{Q}}}^*$, to get that if ϕ corresponds to e_i and ϕ' corresponds to $e_{i'}$. then the lowest order monomial of ϕ and ϕ' must differ. Hence, in this case $\phi\neq\phi'$.

By these two facts we see that the number of solutions of $F(x_1,\ldots,x_{N+1})=0$ in $\mathbb{C}((C_{\mathbb{Q}}))$ is at least $k_1+\cdots+k_n=d$. Since this equation can have at most d solutions in any integral domain, we have that the number of such solutions is exactly d. □

REMARK 2.2.3. In the case where the edge discriminants are not necessarily 0 the same proof will apply to show that some cone C_w with $w\in C_w^*$ has a full set of d solutions, where w is chosen as in theorem 2.1.2.

2.2.4. Proof of theorem 2.1.2 part b). We can use remark 2.2.3 to prove theorem 2.1.2 (b). Let e be an edge of the polytope $P(F)$, and let k be the length of $\psi(e)$. Let $w \in C^*(e)$ be a linear functional with coordinates that are linearly independent over $\mathbb{Q}$. Then as described above, w defines a coherent edge path $E = \{e_1, \ldots, e_n\}$ on $P(F)$. Since w is maximized on e, we get that $e \in E$.

By remark 2.2.3, there is some cone, C, such that for each i, $C(e_i) \subset C$, and such that the ring $\mathbb{C}((C_{\mathbb{Q}}))$ contains d solutions, counting multiplicity, to

$$F(x_1, \ldots, x_{N+1}) = 0,$$

k_i of which correspond to each e_i. Since $\mathbb{C}((C_{\mathbb{Q}})$ can contain no more than d solutions and $k_1 + \cdots + k_n = d$, we have that the number of solutions corresponding to each edge, and e in particular, is exactly k_i, the length of $\psi(e)$. This completes the proof of theorem 2.1.2.

2.2.5. Estimate for the Ramification Locus. Let $X \subset (\mathbb{C}^*)^{N+1}$ be the variety defined by the equation $F(x_1, \ldots, x_{N+1}) = 0$. Assume that X is smooth and that F satisfies the discriminantal condition of corollary 2.2.2. Consider the projection $\Pi : X \longrightarrow (\mathbb{C}^*)^N$ defined by

$$\Pi(x_1, \ldots, x_{N+1}) = (x_1, \ldots, x_N).$$

A point $(\mathbb{C}^*)^N$ is called ramified under Π if the number of inverse images of y is less than the degree d of the mapping f, $d = \deg_{x_{N+1}}(F)$. A point that has n inverse images is called unramified. Note that the locus D thus defined also includes points where x_{N+1} becomes infinite. Therefore this locus is a subvariety given by the equation

$$P_d(x_1, \ldots, x_N)\Delta_{x_{N+1}}(F) = 0$$

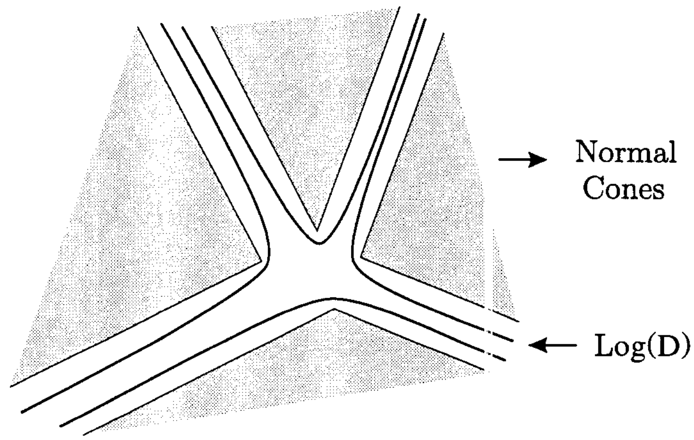

FIGURE (2.7). The normal fan and the ramification locus

where P_d is the coefficient of x_{N+1}^d in $F(x_1, \dots, x_{N+1})$ and $\Delta_{x_{N+1}}(F)$ is the discriminant of F with respect to the x_{N+1}.

Suppose that F has a complete set of d fractional power series expansions of x_{N+1} through $x_1, \dots, x_{N+1}$ in some ring $\mathbb{C}((C_{\mathbb{Q}}))$. There exists some translate $C_{\mathbb{Q}}{}'$ of $C_{\mathbb{Q}}$ such that the inverse image, under Π, of of $\mathrm{Log}^{-1}(C_{\mathbb{Q}}{}')$ is a union of graphs of analytic functions given by convergent fractional power series in $(\mathbb{C}^*)^{N+1}$. Since X is smooth, these graphs do not intersect. Therefore, the number of inverse images of any $y \in \mathrm{Log}^{-1}(C_{\mathbb{Q}})$ is precisely d. Hence, none of the points of this set are ramified.

Suppose that F is chosen so that $\Delta_t(F_e) \neq 0$ for all admissible edges $e \subset P(F)$. Let $\{C_i \subset \mathbb{R}^N\}$ be the normal cones of all vertices of the fiber polytope $\Sigma\,(P(F), [0, d])$. By Corollary 2.2.2 there are translates C_i' of C_i for each i such that all points of $\mathrm{Log}^{-1}(C_i')$ are unramified points of Π. Hence

$$\mathrm{Log}(D) \cap \bigcup_{i=1}^{m} (C_i') = \emptyset$$

where m is the number of vertices of $\Sigma\,(p(F), [0, d])$. So through the above methods we get a bound on the image, under the map Log, of the ramification locus of Π as indicated in figure (2.7).

EXAMPLE 2.2.6. Consider the general polynomial of degree n in one variable x. We consider the coefficients of this polynomial to be independent variables themselves, i.e.

$$F = x_0 + x_1 x + x_2 x^2 + \cdots - x_n x^n$$

The Newton polytope of F is a simplex, and the fiber polytope of its projection to the line segment $[0, n]$ is just the secondary polytope (See [12]) of the polytope $[0, n]$. Thus our results here agree with [12], wherein it was shown that the Newton polytope of the classical discriminant was precisely this secondary polytope. Note that, this secondary polytope has 2^{n-1} vertices corresponding to the various triangulations of $[0, n]$.

2.2.7. Fields other than $\mathbb{C}$. Note that the proof above works for any algebraically closed field of characteristic 0. As in the single variable case, a version of this construction works for fields of arbitrary characteristic. For fields of characteristic p, the inductive construction must be carried out transfinitely and so we may have solution series which have supports containing limit points, as in [25], [28] and [29]. Such solution series will always have well ordered supports in $\mathbb{Q}^N$ with respect to the chosen linear functional $-w$. The statement that the support of the solution series lies in a lattice and the proof that the constructed series lies in some cone no longer hold.

3. Necessity of the discriminantal condition

The following example will show that the conclusion of Corollary 2.1.13 is not necessarily true for edges that don't satisfy the discriminant condition of that corollary.

Let

$$F(x,y,z) = 1 + 2xz + 2yz + 2z^2 + x^2z^2 + y^2z^2.$$

We want to express z as a series in x and y. The Newton polytope of F is a simplex with vertices $(0,0,0)$, $(2,0,2)$, and $(0,2,2)$. Let e be the edge with end points $(0,0,0)$ and $(2,0,2)$. Then e itself is a maximal edge path and

$$F|_e = 1 + 2xz + x^2z^2 = (xz + 1)^2.$$

Therefore $F_e(t) = 1 + 2t + t^2$ and so $\Delta_t F_e(t) = 0$. Let $C(e)$ and $C^*(e)$ be the barrier and normal cones of e respectively. Assume that $\mathbb{C}((C(e)_{\mathbb{Q}}))$ contains a full set of 2 convergent series expansions for $F = 0$. There is then some translate $C^*(e) + \gamma$ of $C^*(e)$ such that there exist two series expansions on $C^*(e) + \gamma$. Note that $C^*(e)$ is the cone generated by $(0,-1)$ and $(1,1)$.

Calculating the discriminant $\Delta_z(F)$ we see that it is equal to $8(xy-1)$ and so the zero locus of $\Delta_z(F)$ is the irreducible variety $xy-1 = 0$. The ramification locus $R(F)$ of the projection $\pi : \{F = 0\} \to \mathbb{R}$ must be a subvariety of the zero locus of $\Delta_z(F)$. Therefore $R(F) = \{xy - 1 = 0\}$. Taking $\mathrm{Log}(R(F))$ we get $\log|y| = -\log|x|$. Therefore

$$R(F) = \left\{(u,v) \in \mathbb{R}^2_{\log} : u = -v\right\}$$

This implies that, regardless of what γ is, $R(F) \cap C^*(e) + \gamma \neq \emptyset$. By the remarks in section 2.2.5, this is a contradiction. Therefore $\mathbb{C}((C(e)_{\mathbb{Q}}))$ cannot contain 2 series solutions for $F = 0$.

Figure (2.8) shows two "slices" of the surface in this example. The first view contains the slice $y = x$, while the second is the slice $y = 2x$. Notice that the ramification happened in the first view at $x = 1$ and in the second at $x = 1/\sqrt{2}$. Just as our calculations indicated.

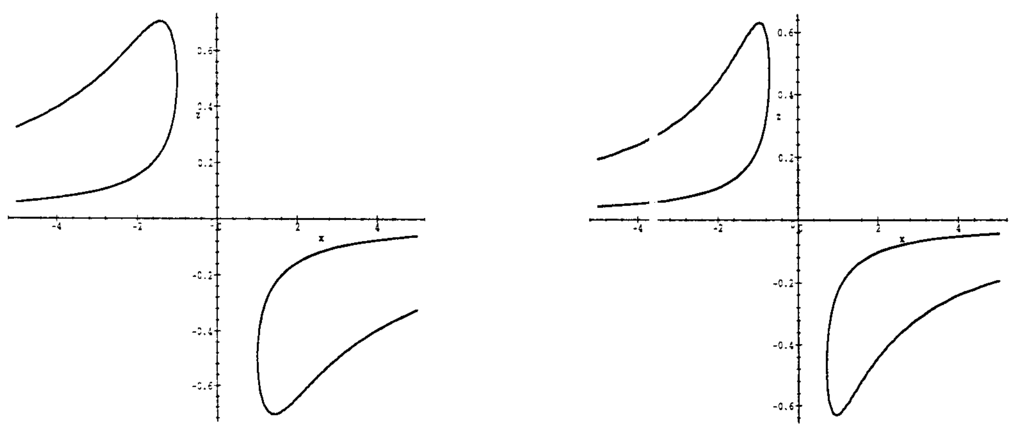

FIGURE (2.8). Two slices of t ıe surface

CHAPTER 3

Expansions of several equations

Suppose now that we have k algebraic equations in $k + \ell$ variables

$$F_i(x_1, \ldots, x_\ell, y_1, \ldots, y_k) = 0 \text{ for } i = 1, \ldots, k.$$

We wish to construct k fractional power series expansions, $y_i = \phi_i(x_1, \ldots, x_\ell)$, for $i = 1, \ldots, k$, such that $F_j(x_1, \ldots, x_\ell, \phi_1, \ldots, \phi_k) = 0$ for all j. Again, we wish to construct complete sets of fractional power series solutions to these equations that converge in some common region of $(\mathbb{C}^*)^\ell$.

In chapter 2 we demonstrated the relationship in a restricted setting ($k = 1$ and ℓ arbitrary). For computing such series expansions we gave an extension of Newton's construction, based on the Newton polytope of F In that case, however, only a special class of fiber polytopes appeared, namely those that arise from the projection of a polytope to a line segment.

As we are considering k equations, we will be using k Newton polytopes $P(F_i) \subset \mathbb{R}^{k+\ell}$, and the Minkowski sum P of the $P(F_i)$. The construction here will be based on k-faces of P.

In the most general case we cannot assure that the construction actually gives a series solution. However, under certain explicit conditions we can prove that the construction can be carried out and that the series built have common domains of convergence. These conditions are direct generalizations of the simple-root condition given for one equation in the last chapter.

We will see that, generically, the number of series solutions converging in a given cone is equal to the mixed volumes of the projections of $P(F_1), \ldots, P(F_k)$ to $\mathbb{R}^k$. This agrees with the theorem due to Bernstein [2] on the number of solutions to a system of equations.

1. Transfinities and transfinite induction

The construction of these fractional power series solutions will be based on a transfinite algorithm similar to the methods used in constructing series solutions for polynomials over fields of characteristic p, see [25], [28] and [29]. Therefore, a brief review of transfinites and transfinite induction is in order.[15]

Recall that a transfinite symbol, γ, is defined to be an equivalence class of well ordered sets, where the equivalence is given by order preserving bijection. Since bijections preserve the cardinality of a set, all sets in the equivalence class, γ, have the same cardinality. We call a transfinite countable if every set in its class is countable. Let Γ denote the set of all countable transfinites, and note that Γ is itself a well ordered but uncountable set.

In Γ there are two types of transfinites, those that arise as the immediate successor of a given $\gamma \in \Gamma$, and those that arise as the limit of the transfinites preceding it in the order on Γ. These two types are usually referred to as isolated and limit transfinites respectively and are written as

$$\gamma + 1 \text{ and } \lim_{\delta < \gamma} \delta.$$

For example, the empty set is a well ordered set represented by the symbol 0, it is the first ordinal number and is therefore the smallest symbol in Γ under its well ordering. Any other finite symbol n is the immediate successor of another symbol, namely $n-1$, and has $n+1$ as an immediate successor. Therefore all finite symbols are isolated transfinites. The first limit transfinite ω is the class which contains the well ordered set of positive integers under their usual order. So, ω can be defined as

$$\omega = \lim n$$

The isolated symbol $\omega + 1$ is represented by sets which increase to a limit point and then contain either the limit point or some element larger than the limit point. For example, the set

$$\left\{0, \frac{1}{2}, 1 - \frac{1}{3}, 1 - \frac{1}{4}, \ldots, 1\right\}$$

is a set in the equivalence class $\omega + 1$.

Transfinite induction is carried out in two steps that of proceeding from γ to $\gamma + 1$, and that of passing to a limit transfinite γ, once the process has been carried out for all $\delta < \gamma$. For any $\gamma \in \Gamma$ we will build a series ϕ_γ such that if $\gamma' < \gamma$, then $\phi_{\gamma'}$ is a summand of ϕ_γ. We will accomplish this by showing that if one has built ϕ_δ for all transfinites $\delta < \gamma$, then we can build ϕ_γ. In particular this means that for an isolated transfinite, γ, if we have constructed ϕ_γ, then we can construct $\phi_{\gamma+1}$. If γ is a limit transfinite we must show that if we have ϕ_δ for all $\delta < \gamma$ then we can construct $\phi_\gamma = \phi_{\lim \delta}$. Finally, we must show that we can build ϕ_0 and that the process will stop after some countable transfinite γ.

Transfinite induction will only be formally needed in the following construction. Once we have the general construction for the series expansions, we will be able to prove that its exponents never actually accumul ıte. in practice, therefore, the construction is never transfinite in nature, and the series thus constructed are just ordinary fractional power series in several variables.

2. The construction for generic systems

3.2.1. The first step. Let

$$F_i(x,y) = F_i(x_1,\ldots,x_\ell,y_1,\ldots,y_k) = 0, \;\; i = 1,\ldots,k$$

be a system of k equations in $k+\ell$ variables. We denote by $P_i \subset \mathbb{R}^m$ their Newton polytopes and will appeal to the notation of section 1.2.10 with respect to the decomposition of $\mathbb{R}^m = \mathbb{R}^{k+\ell} = \mathbb{R}^\ell \oplus \mathbb{R}^k$. We will continue to use the notation of α and β as the coordinates on $\mathbb{R}^\ell$ and $\mathbb{R}^k$ respectively. We assume, as before, that we have fixed an irrational linear functional $w \in (\mathbb{R}^\ell)^*$.

Suppose

$$F_i(x,y) = \sum_{(\alpha,\beta)\in S_i} a_{i,\alpha,\beta} x^{\alpha} y^{\beta}$$

where S_i is the support of F_i in $\mathbb{R}^m$. We need to build k series of the form

$$\phi_i = \sum_{\gamma\in\Gamma} c_{\gamma,i} x^{\delta_{\gamma,i}} = \sum_{\gamma\in\Gamma} \psi_{\gamma,i}$$

where Γ is some well ordered countable set, and for each i, the set $\{\delta_{\gamma,i}\}$, is a well ordered subset of $\mathbb{R}^\ell$ with respect to the order given by w. We will denote by $\phi_{\gamma,i}$ the γ-th partial sum of ϕ_i.

We will first build all of the $\phi_{1,i}$ and then we will consider the inductive step of moving the construction from γ to $\gamma+1$ and then define $\psi_{\gamma,i}$ for a limit transfinite γ. In many respects the first two parts will be identical.

Let $P_1^{(0)}, \ldots, P_k^{(0)} \subset \mathbb{R}^m$ be the Newton polytopes of $F_1, \ldots, F_k$ (i.e. $P_i^{(0)} = P_i$ for all i) and let $P^{(0)} = P_1^{(0)} + \cdots + P_k^{(0)}$ be their Minkowski sum. For the purposes of iterating the process described below, we will now write $F_i^{(0)}$ instead of F_i.

Let $\mathcal{S}$ be the section of $\pi : P^{(0)} \to \mathbb{R}^k$ determined by maximality with respect to w. We choose any admissible decomposable k-face $f^{(0)}$, of $\mathcal{S}$. Let $\{f_i^{(0)}\}$ be the faces of P_i respectively which sum to $f^{(0)}$. By assumption $\dim(f_i^{(0)}) > 0$ for all i. Let $Q^{(0)}$ be the k-plane determined by $f^{(0)}$ and let $Q_i^{(0)}$ be the translates of $Q^{(0)}$ which contain $f_i^{(0)}$ respectively, i.e. $P_i^{(0)} \cap Q_i^{(0)} = f_i^{(0)}$.

Since f is admissible, it has a matrix of slopes

$$\delta = \begin{bmatrix} \delta_1 \\ \delta_2 \\ \vdots \\ \delta_k \end{bmatrix} = \begin{bmatrix} \delta_{1,1} & \delta_{1,2} & \cdots & \delta_{1,\ell} \\ \delta_{2,1} & \delta_{2,2} & \cdots & \delta_{2,\ell} \\ \vdots & \vdots & \ddots & \vdots \\ \delta_{k,1} & \delta_{k,2} & \cdots & \delta_{k,\ell} \end{bmatrix}$$

For the first terms of our expansions we set

$$\psi_{1,i} = c_i x_1^{-\delta_{i,1}} \cdots x_\ell^{-\delta_{i,\ell}}$$

where $c = (c_1, \ldots, c_k)$ is a non-zero solution to the system of equations

$$\sum_{(\alpha,\beta) \in S_i \cap Q_i^{(0)}} a_{i,\alpha,\beta} c^\beta = 0$$

These equations are called the *face equations* of $f^{(0)}$. Note that by theorem 1.2.9 the number of such solutions counting multiplicity is, in general, equal to the mixed volume of the projections of $f_1^{(0)}, \ldots, f_k^{(0)}$ to $\mathbb{R}^k$.

A priori we have no guarantee that this system actually has a solution. For now, assume it does. Momentarily, we will give an explicit situation in which we can assure that it the above system of equations actually has a solution.

3.2.2. The inductive step. Next, we will look at the inductive step of proceeding from the transfinite γ to the transfinite $\gamma + 1$. We will consider the case of a limit transfinite at the end of this section. So, assume that we have constructed $\phi_{\gamma',i}$ for all i and for all $\gamma' \leq \gamma$ where γ is an isolated transfinite symbol.

Suppose that, in constructing the $\phi_{\gamma',i}$, we have constructed polynomials $F_i^{(\gamma')}$ and their polytopes $P_i^{(\gamma')}$. Let

$$F_i^{(\gamma')}(x,y) = \sum_{(\alpha,\beta)\in S_i^{(\gamma')}} a_{i,\alpha\,\beta}^{(\gamma'} x^\alpha y^\beta,$$

where $S_i^{(\gamma')}$ is the support of $F_i^{(\gamma')}$.

Suppose that we obtained faces $f_i^{(\gamma')}$ that sum to an admissible decomposable k-face $f^{(\gamma')}$ of their Minkowski sum $P^{(\gamma')}$. Also assume that

$$\psi_{\gamma',i} = c_i^{(\gamma')} x^{-\delta_i^{(\gamma')}}$$

where

$$\begin{bmatrix} \delta_1^{(\gamma')} \\ \delta_2^{(\gamma')} \\ \vdots \\ \delta_k^{(\gamma')} \end{bmatrix}$$

is the slope matrix of $f^{(\gamma')}$, and $c^{(\gamma')}$ is a non-zero solution of its face equations, which have the same form as the face equations given above for $f^{(0)}$. Notice that all of these criteria are met by the objects used in the first step of the construction.

To proceed from γ to $\gamma + 1$ we substitute

$$c_1^{(\gamma)} x^{-\delta_1^{(\gamma)}} + y_1, \ldots, c_k^{(\gamma)} x^{-\delta_k^{(\gamma)}} + y_k$$

in place of $y_1, \ldots, y_k$ in the polynomials $F_i^{(\gamma)}$ to obtain the polynomials $F_i^{(\gamma+1)}$. Let $P_i^{(\gamma+1)} = P\left(F_i^{(\gamma+1)}\right)$ be their Newton polytopes and let $P^{(\gamma+1)}$ be the Minkowski sum of these polytopes. Let $S_i^{(\gamma+1)}$ be the support of $F_i^{(\gamma+1)}$. To continue this construction we need to find a face $f^{(\gamma+1)}$ on $P^{(\gamma+1)}$ satisfying the following three conditions:

1) If

$$\begin{bmatrix} \delta_1^{(\gamma+1)} \\ \delta_2^{(\gamma+1)} \\ \vdots \\ \delta_k^{(\gamma+1)} \end{bmatrix}$$

is the matrix of slopes of $f^{(\gamma+1)}$ then $\left\langle w, \delta_i^{(\gamma+1)} \right\rangle > \left\langle w, \delta_i^{(\gamma)} \right\rangle$ for all i.

2) If $N\left(f^{(\gamma)}\right)$ is the normal cone of $f_i^{(\gamma)}$ (and likewise for $f^{(\gamma+1)}$) then

$$N\left(f^{(\gamma+1)}\right) \cap N\left(f^{(\gamma)}\right) \neq \emptyset$$

and in fact this intersection contains w.

3) The face equations of $f^{(\gamma+1)}$ have a non-zero solution.

We will find that it is not always possible to find such a face, and that, even when one exists, its face equations may have no solutions. In the the next section of this text we will explore a case in which we can always find a decomposable face satisfying all three of these criteria.

For the rest of this section assume that, at every isolated transfinite symbol $\gamma + 1$ such that the expansions $\phi_{\gamma,i} = \sum_{\gamma' \leq \gamma} \psi_{\gamma',i}$ is not a solution to the original system of equations, suppose we can find a face of $P^{(\gamma+1)}$ satisfying the above conditions. Then we set, as in the first step of the construction,

$$\psi_{\gamma+1,i} = c_i^{(\gamma+1)} x^{-\delta_i^{(\gamma+1)}}$$

where $c^{(\gamma+1)}$ is a non-zero solution of the face equations of $f^{(\gamma+1)}$. We set

$$\phi_{\gamma+1,i} = \sum_{\gamma' \leq \gamma+1} \psi_{\gamma'}.$$

For a limit transfinite γ, we set

$$\phi_{\gamma,i} = \sum_{\gamma' < \gamma} \psi_{\gamma'}.$$

This completes the description of the inductive process.

3.2.3. Properties of the series expansions. Here, we assume, for a series we are attempting to construct, that we were able to carry out the entire inductive process described above. So, at every step, we could find a decomposable face of the next Minkowski sum which satified the three critera on page 53. Later we will describe conditions under which we can assure that the above process works. We note, however, that these proofs are not interdependant. In other words, when we attatch the conditions we do not rely on any of the following properties in the construction. This discussion is more or less given as a general theorem on such series expansions. The series we construct in the next section will have much stronger properties.

THEOREM 3.2.4. *Assume that the series expansions ϕ_i have been constructed by the transfinitely inductive process described above. Then*

a) *The $\{\phi_i\}$ are formal roots of the original system equations $\{F_i = 0\}$.*
b) *The exponents of the ϕ_i lie in some strongly convex rational polyhedral cone C in $\mathbb{R}^\ell$ such that $w \in C^*$.*
c) *The exponents of the ϕ_i lie in some lattice $\frac{1}{\lambda}\mathbb{Z}^\ell$.*

Notice that b) and c) imply that the ϕ_i are elements of the ring $\mathbb{C}((C_{\mathbb{Q}}))$, the ring of power series with support in some translate of C. This gives us that the power series have convergence in some translate of the cone C^* (Lemma 1.3.9). Therefore, they have a common domain of convergence. Note that c) also implies that transfinite induction is not actually needed in this process. Unfortunately, the proof of c) depends heavily on part a).

PROOF. First we will prove part a). For this we need to show that the largest terms of

$$F_i(x_1, \ldots, x_\ell, \phi_{\gamma,1}, \ldots, \phi_{\gamma,k})$$

with respect to the linear functional w are decreasing as γ increases, and that they decrease without bound. Notice that

$$\begin{aligned} F_i^{(\gamma+1)}(x,y) &= F_i^{(\gamma)}(x, c^{(\gamma)}x^{-\delta^{(\gamma)}} + y) \\ &= F_i^{(\gamma-1)}(x, c^{(\gamma)}x^{-\delta^{(\gamma)}} + c^{(\gamma-1)}x^{-\delta^{(\gamma)}} + y) \\ &= \cdots = F_i\left(x, y + \sum_{\gamma' \leq \gamma} c^{(\gamma')}x^{-\delta^{(\gamma')}}\right) \\ &= F_i(x, \phi_\gamma + y) \end{aligned}$$

and so

$$F_i(x_1, \ldots, x_\ell, \phi_{\gamma,1}, \ldots, \phi_{\gamma,k}) = F_i^{(\gamma+1)}(x_1, \ldots, x_\ell, 0, \ldots, 0).$$

Therefore we can prove a) by showing that the maximal terms of $P_i^{(\gamma)} \cap \mathbb{R}^\ell$, with respect to w, decrease without bound.

3.2.5. Finding a supporting hyperplane. Assume that

$$F_i^{(\gamma)}(x,y) = \sum_{\alpha,\beta \in S_i^{(\gamma)}} a_{i,\alpha,\beta}^{(\gamma)} x^\alpha y^\beta$$

Explicitly carrying out the above substitution gives us

$$\begin{aligned} F_i^{(\gamma+1)}(x,y) &= \sum_{(\alpha,\beta)\in S_i^{(\gamma)}} a_{i,\alpha,\beta}^{(\gamma)} x^\alpha \left(c_1^{(\gamma)} x^{-\delta_1^{(\gamma)}} + y_1\right)^{\beta_1} \cdots \left(c_k^{(\gamma)} x^{-\delta_k^{(\gamma)}} + y_k\right)^{\beta_k} \\ &= \sum_{(\alpha,\beta)\in S_i^{(\gamma)}} a_{i,\alpha,\beta}^{(\gamma)} x^\alpha \prod_{j=1}^{k} \left(\sum_{\nu_j=0}^{\beta_j} \binom{\beta_j}{\nu_j} \left(c_j^{(\gamma)}\right)^{\nu_j} x^{-\nu_j \delta_j^{(\gamma)}} y_j^{\beta_j-\nu_j} \right) \\ &= \sum_{(\alpha,\beta)} \sum_{\nu} \binom{\beta}{\nu} a_{i,\alpha,\beta}^{(\gamma)} \left(c^{(\gamma)}\right)^{\nu} x^{\alpha-\nu_1\delta_1^{(\gamma)}-\cdots-\nu_k\delta_k^{(\gamma)}} y^{\beta-\nu} \end{aligned}$$

Where $\binom{\beta}{\nu} = \Pi\binom{\beta_i}{\nu_i}$.

We set $\left\langle \nu, \delta^{(\gamma)} \right\rangle = \nu_1\delta_1^{(\gamma)} + \cdots + \nu_k\delta_k^{(\gamma)}$ to simplify this expression, and we consolidate the coefficient, obtaining

(3.1)
$$F_i^{(\gamma)}(x,y) = \sum_{(\alpha,\beta),\nu} A_{i,\alpha,\beta}^{\gamma,\nu} \bullet x^{\alpha-\left\langle \nu, \delta^{(\gamma)} \right\rangle} \bullet y^{\beta-\nu}$$

Analyzing the exponents appearing in this equation yields that the points in the polytope, $P_i^{(\gamma+1)}$, arise from points on $P_i^{(\gamma)}$ and lie on k-planes through these points with slope $\delta^{(\gamma)}$ with respect to $\mathbb{R}^k$, i.e. k-planes parallel to the face $f^{(\gamma)}$.

Consider the points in $P_i^{(\gamma+1)}$ that arise from $(\alpha,\beta) \in S_i^{(\gamma)}$. Since ν in the above expression ranges over the set

$$\mathcal{Q}_{\alpha,\beta} = \{(\nu_1,\ldots,\nu_k) : \nu_i \in \mathbb{Z}, 0 \le \nu_i \le \beta_i \text{ for all } i\}$$

and the exponent of y is $\beta - \nu$, we see that points in $P_i^{(\gamma+1)}$ that arise from (α,β) project to points in the set $\mathcal{Q}_{\alpha,\beta}$ under the projection to the last k coordinates. See figure (3.9)

Let $H_i^{(\gamma)}$ and $H^{(\gamma)}$ be the unique w-constant hyperplanes through the k-planes $Q_i^{(\gamma)}$ and $Q^{(\gamma)}$ respectively. Note that these hyperplanes support their corresponding

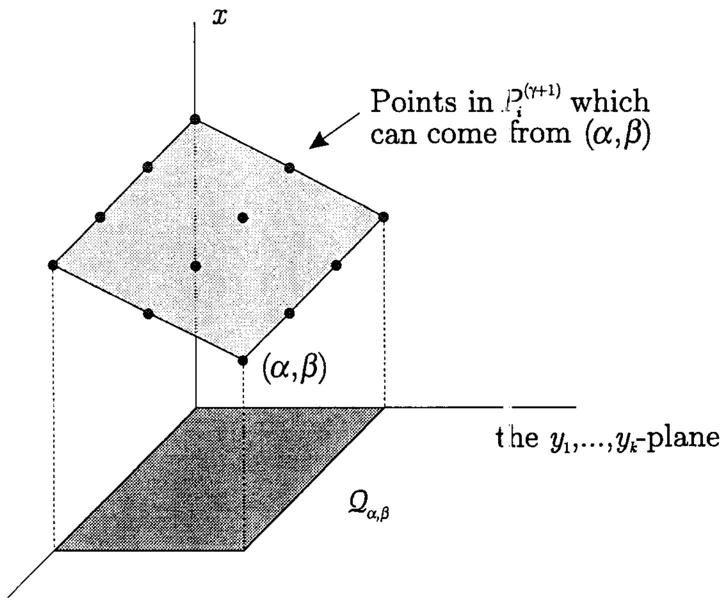

FIGURE (3.9). Points on $P_i^{(\gamma+1)}$

polytopes, because points on the new polytope must come from points on the old polytope translated along planes which are parallel to a supporting k-plane.

Recall that in the inductive hypothesis we assumed that we had obtained faces $f_i^{(\gamma)}$ of $P_i^{(\gamma)}$ of dimension > 0 and a k-face $f^{(\gamma)}$ of $P^{(\gamma)}$ such that

$$f_1^{(\gamma)} + \cdots + f_k^{(\gamma)} = f^{(\gamma)}.$$

Let $Q^{(\gamma)}$ be the k-plane containing $f^{(\gamma)}$ and let $Q_i^{(\gamma)}$ be the translate of $Q^{(\gamma)}$ which contains $f_i^{(\gamma)}$.

By the above discussion, we see that on the plane $Q_i^{(\gamma)}$ we get points in $P_i^{(\gamma+1)}$ whose coefficients are only affected by the terms of $f_i^{(\gamma)} = Q_i^{(\gamma)} \cap P_i^{(\gamma)}$. Moreover we get that these points must lie in the region of $Q_i^{(\gamma)}$ lying above the sets $\mathcal{Q}_{\alpha,\beta}$ for the various $(\alpha,\beta) \in f_i^{(\gamma)}$. Note also that, since every point of $P_i^{(\gamma+1)}$ lies on a plane through some point of $P_i^{(\gamma)}$ parallel to $Q_i^{(\gamma)}$ and since $H_i^{(\gamma)}$ was a supporting hyperplane for $P_i^{(\gamma)}$, we see that $H_i^{(\gamma)}$ is a supporting hyperplane of $P_i^{(\gamma+1)}$. (Actually, to conclude this we also need that $P_i^{(\gamma+1)} \cap Q_i^{(\gamma)} \neq \emptyset$ so that the polytope in question actually touches this hyperplane, but we will demonstrate this momentarily.)

3.2.6. Points in $\mathbb{R}^\ell$ lie strictly below the plane. In order to show that the upper bound (with respect to w) on the terms in the series decreases at every step, we must show that the points of $P_i^{(\gamma+1)}$ which lie in the null-hyperplane lie strictly below the supporting hyperplane we constructed above. (Note that, in our view, the hyperplane supports from above, with respect to w.)

Let $q_i = Q_i^{(\gamma)} \cap \mathbb{R}^\ell$. We need to determine the coefficient of the monomial corresponding to q_i in equation 3.1). Considering all the terms in equation 3.1) that contribute to the term corresponding to q_i, we get that this coefficient is equal to

$$\sum_{(\alpha,\beta)\in S_i^{(\gamma)}\cap Q_i^{(\gamma)}} a_{i,\alpha,\beta}^{(\gamma)} \left(c^{(\gamma)}\right)^{\beta} \tag{3.2}$$

Since this is precisely the ith face equation of $f^{(\gamma)}$, and $c^{(\gamma)}$ was assumed to be a root of the face equations, we see that this coefficient is zero for all i.

REMARK 3.2.7. There are several other points that we can explicitly determine on $P_i^{(\gamma+1)}$. Suppose q is a point on $f_i^{(\gamma)}$ such that for every other point q' on $f_i^{(\gamma)}$ there is some j such that $q_j > q_j'$. Then the only term in $F_i^{(\gamma)}$ that can contribute to the term corresponding to q in $F_i^{(\gamma+1)}$ is the term corresponding to q itself. Therefore the coefficient of q in $P_i^{(\gamma+1)}$ is unchanged from the previous step. Such vertices will be called extreme on the face f. Note that this proves that there are points of $P_i^{(\gamma+1)}$ on $Q_i^{(\gamma)}$, and so $H_i^{(\gamma)}$ is a supporting hyperplane for $P_i^{(\gamma+1)}$.

Hence, if there are points of $P_i^{(\gamma)}$ on $\mathbb{R}^\ell$ they must lie "below" the w-constant hyperplane $H_i^{(\gamma)}$ in the sense that they have smaller weight than the points on the intersection of the hyperplane $\mathbb{R}^\ell$. (all these points have the same weight by the definition of w-constant.)

3.2.8. The decreasing upper bound. Since the polytope $P_i^{(\gamma+1)}$ is supported by $H_i^{(\gamma)}$ and by $H_i^{(\gamma+1)}$, we see that the following are upper bounds on the the points of $P_i^{(\gamma+1)} \cap \mathbb{R}^\ell$.

$$\left\langle w, H_i^{(\gamma)} \cap \mathbb{R}^\ell \right\rangle \text{ and } \left\langle w, H_i^{(\gamma+1)} \cap \mathbb{R}^\ell \right\rangle .$$

Recall that w is constant on $H_i^{(\gamma)}$ over every point of $\mathbb{R}^k$. We will have a decreasing upper bound on the points of $P_i^{(\gamma+1)}$ in $\mathbb{R}^\ell$ if we can show that

$$\left\langle w, H_i^{(\gamma)} \cap \mathbb{R}^\ell \right\rangle > \left\langle w, H_i^{(\gamma+1} \cap \mathbb{R}^\ell \right\rangle .$$

Suppose that $\left\langle w, H_i^{(\gamma)} \cap \mathbb{R}^\ell \right\rangle \leq \left\langle w, H_i^{(\gamma+1)} \cap \mathbb{R}^\ell \right\rangle$. Since the slopes of $f^{(\gamma+1)}$ are all strictly greater than the corresponding slopes of $f^{(\gamma)}$, and $H_i^{(\gamma+1)}$ contains points of $P_i^{(\gamma+1)}$, we get that $H_i^{(\gamma+1)}$ lies strictly below $H_i^{(\gamma)}$ for all i. But this contradicts the known fact that each of these hyperplanes support $P_i^{(\gamma+1)}$.

The fact that this upper bound decreases without bound as γ increases, follows from the fact that we are using a transfinitely inductive process. This process does not stop until the system $\phi_{\gamma,i}$ satisfies the system of equations, i.e. until all the $P_i^{(\gamma)}$ have no terms left in $\mathbb{R}^\ell$. This proves that the series expansions obtained are formal roots of the initial system of equations.

3.2.9. The rest of the theorem. Both b) and c) can be proved using an application of the single equation construction detailed in [20]. We accomplish this by considering resultants. Namely, consider that the ϕ_i all have terms that are well ordered by the linear functional w. Therefore ϕ_i are elements of the ring $\mathbb{C}((w))$ for every i, where $\mathbb{C}((w))$ is the ring of transfinite power series whose terms are well ordered with respect to w.

Let $F(x,y)$ and $G(x,y)$ be any polynomials, we will denote by

$$\mathcal{R}_{x_j}(F,G)$$

the classical resultant of F and G with respect to x_j. This is a polytope on the coefficients of F and G as polynomials in x_j, which is 0 if and only if F and G have a common root. Note that in this case, the coefficients of F and G are themselves polynomials in several variables.

Since $\phi_1, \ldots, \phi_k$ simultaneously satisfy the system

$$F_i(x,y) = F_i(y_1, \ldots, y_k) = 0$$

we see that $\phi_1, \ldots, \phi_{k-1}$ satisfies the system of $k-1$ equations

$$\mathcal{R}_{x_k}(F_i, F_k).$$

By repeated applications of this argument we see that $y_1 = \phi_1$ satisfies a polynomial F in the variables $x_1, \ldots, x_\ell, y_1$. Therefore, since the terms of ϕ_1 are well ordered with respect to w we see that ϕ_1 must be one of the series solutions to F obtained by theorem 2.1.2. Therefore ϕ_1 has exponents lying in some lattice and contained in some strongly convex rational polyhedral cone C such that $w \in C^*$. The proofs for the other ϕ_i follow similarly. □

3. Conditions on the roots of the face equations

In this section we will explore one situation in which we can continue the above process. In this case we can determine a great deal about the structure of the series solutions generated by the procedure. We will require that the initial choice of roots of the face equations be a simple solution of the system, i.e. the Jacobian of the face equations is non-zero at c, and that all partial derivatives of the face equations of the initial face do not vanish at c.

THEOREM 3.3.1. *Let P be the Minkowski sum of the polytopes P_i of the polynomials F_i, and let δ be the matrix of slopes of f. Assume that the root $c = (c_1, \ldots, c_k)$ of the face equations f is a simple root, and assume that the first partial derivatives of these face equations are all non-zero at c. Then*

a) *The induction of section 4 can always be continued.*

b) *The support of ϕ_i lies in the translate of the barrier cone of f with vertex at $-\delta_i$ for all i and (Lemma 1.3.9) the ϕ_i converge in $\mathrm{Log}^{-1}(C^* + v)$ for some v in $(\mathbb{R}^\ell)^*$.*

PROOF. We will first analyze the second step of the induction for such systems and then move on to the inductive step.

Recall that $f^{(0)}$ was the chosen face of the Minkowski sum of the initial polytopes, and $f_1^{(0)}, \ldots, f_k^{(0)}$ were the faces that sum to $f^{(0)}$. We had

$$\sum_{(\alpha,\beta)\in S_i^{(0)}\cap Q_i^{(0)}} a_{i,\alpha,\beta}^{(0)} c^\beta = 0 \tag{3.1}$$

as the face equations of f, where $Q^{(0)}$ was the k-plane determined by $f^{(0)}$. Further, $Q_i^{(0)}$ was the translate of $Q^{(0)}$ containing $f_i^{(0)}$.

We will now replace c by $t = (t_1, \ldots, t_k)$ in equation 3.1, i.e. we consider the expressions on the left hand side of 3.1 as polynomials $E_i \in \mathbb{C}[t_1, \ldots, t_n]$.

$$E_i(t_1, \ldots, t_k) = \sum_{(\alpha,\beta)\in S_i\cap Q_i^{(0)}} a_{i,\alpha,\beta} t^\beta \tag{3.2}$$

The condition that c is a simple root of 3.1 means that the matrix

$$\mathcal{M} = \begin{bmatrix} \frac{\partial E_1}{\partial t_1}(c) & \cdots & \frac{\partial E_1}{\partial t_k}(c) \\ \vdots & \ddots & \vdots \\ \frac{\partial E_k}{\partial t_1}(c) & \cdots & \frac{\partial E_k}{\partial t_k}(c) \end{bmatrix} \tag{3.3}$$

is nonsingular, i.e. $\det(\mathcal{M})$ is non-zero.

Our second assumption gives that for all i and j

$$\frac{\partial E_i}{\partial t_j}(c) \neq 0$$

To draw conclusions from this second assumption we need introduce some notation.

DEFINITION 3.3.2. Let $P = \operatorname{conv}\{p_1, \ldots, p_n\}$ be any m-dimensional polytope. For each of these points, set $p_i = (p_{i,1}, \ldots, p_i, m)$. We define the partial derivative of the polytope P with respect to t_i by

$$\frac{\partial P}{\partial t_j} = \operatorname{conv}\{p_i - e_j : p_{i,j} \geq 1\}$$

where e_j is the i-th standard basis vector in $\mathbb{R}^k$.

So taking the derivative of a polytope with respect to t_j has the effect of shifting it by -1 in the t_j direction, and removing the parts that have negative t_j-coordinate. If F is a polynomial and $P(F)$ is its Newton polytope then we get the identity:

$$P\left(\frac{\partial F}{\partial t_j}\right) = \frac{\partial P(F)}{\partial t_j}.$$

3.3.3. Finding an appropriate decomposable face on $P^{(1)}$. We need to explicitly find a decomposable admissible face of the polytope $P^{(1)}$. So, let

$$q_i = Q_i^{(0)} \cap \mathbb{R}^\ell,$$

be the intersection points of these k-planes with $\mathbb{R}^\ell$. We noticed, in the second step of the construction, that since this coefficient is the same as the i^{th} face equation, the coefficient on the term corresponding to q_i in $P_i^{(1)}$ is 0 for all i. Let

$$q_i^{(j)} = \frac{\partial Q_i^{(0)}}{\partial t_j} \cap \mathbb{R}^\ell,$$

The Newton polytope of the partial derivative is just a translation of the original Newton polytope (with part cut off at the null hyperplane). So, showing that a point lying over e_i is actually a vertex on the bottom of the polytope is equivalent to showing that the coefficient on the term corresponding to $q_i^{(j)}$ in $\partial F_i^{(1)}/\partial y_j$ is non-zero. (Such a point would necessarily lie on the previous hyperplane, but points in the null-hyperplane lie below it with respect to w.) By the chain rule, we get that

$$\frac{\partial}{\partial y_j}\left(F_i^{(1)}\right) = \left(\frac{\partial}{\partial y_j}(F_i^{(0)})\right)(x, y + c^{(\gamma)}x^{-\delta^{(\gamma)}} + y_k)$$

Therefore, the partial derivatives of face equations E_i of $f^{(0)}$ are the face equations of the partial derivatives of $f^{(0)}$. So, in the same way that we showed that the coefficient on x^{q_i} in $F_i^{(1)}$ was zero, we can show that the coefficient on

$$(3.4) \qquad (x,y)^{q_i^{(j)}} \text{ in } \frac{\partial F_i^{(1)}}{\partial y_j} \text{ is } \frac{\partial E_i}{\partial t_j}(c).$$

Since

$$\frac{\partial E_i}{\partial t_j}(c) \neq 0$$

for all i and j, we see that the term in $\partial F_i^{(1)}/\partial y_j$ corresponding to $q_i^{(j)}$ has a non-zero coefficient. For all i, we let $s_i^{(j)} = (s_{i1}^{(j)}, \ldots, s_{ik}^{(j)}, e_j)$ be the point on $Q_i^{(0)}$ that projects to the point $(0, e_j)$ in $\mathbb{R}^k$. Taking the partial derivative of $P_i^{(1)}$ with respect to y_i takes $s_i^{(j)}$ to $q_i^{(j)}$. By equation 3.4, and the fact that we are taking the derivative of a monomial containing t_i^1, we see that the coefficient on

$$(3.5) \qquad (x,y)^{s_i^{(j)}} \text{ in } F_i^{(1)} \text{ is } \frac{\partial E}{\partial t_j}(c).$$

Hence the term in $F_i^{(1)}$ corresponding to $s_i^{(j)}$ has a non-zero coefficient for all j. So $s_i^{(j)}$ is contained in, and is in fact a vertex in $P_i^{(1)}$.

Let $r_i = (r_{i1}, \ldots, r_{i\ell}, 0, \ldots, 0)$ be the maximal point of $P_i^{(1)}$ in $\mathbb{R}^\ell$ (maximal with respect to w). If we assume that $\{y_i = c_i^{(0)} x^{\delta_i^{(0)}}\}$ is not a solution to all of the equations, then there must be points of $P_i^{(1)}$ on $\mathbb{R}^\ell$ for some i. If $P_i^{(1)}$ has no points in $\mathbb{R}^\ell$, then we formally set $r_i = \infty$. We add the finite r_i's to the $s_i^{(j)}$'s, and so get a set of points on $P^{(1)}$. To complete this construction we must use these points to find an explicit decomposable k face on $P^{(1)}$. The structure of the slopes of this face will give us some interesting results on the structure of the fractional power series that we generate from this process.

First, we notice that, on each of the $P_i^{(1)}$, the $(k-1)$-simplex formed by the points $\{s_i^{(j)}\}$ is a $(k-1)$-face. (Call this face $\tilde{f}_i$.) This is because this $(k-1)$-simplex is contained in $Q_i^{(0)}$, a supporting k-plane for $P_i^{(1)}$, and because each of these points lies over one of the coordinate axes. (There are no points of $P_i^{(1)}$ with negative y_j for any j.) Therefore all of the $P_i^{(1)}$ have at least one parallel $(k-1)$-face. Moreover, this face is maximized by w since $Q_i^{(0)}$ is a supporting hyperplane in the direction of w. Therefore, $\tilde{f}_1 + \cdots + \tilde{f}_k$ is a $(k-1)$-face of $P^{(1)}$. This face has vertices with α_i coordinate equal to ke_i for all i. See figure (3.10).

Consider, next, the points r_i. In the discussion above, we showed that $\langle w, r_i\rangle < \langle w, q_i\rangle$. (Recall that $q_i = Q_i^{(0)} \cap \mathbb{R}^\ell$.) Chose i_0 such that the expression

$$\left\langle w, q_{i_0}\right\rangle - \left\langle w, r_{i_0}\right\rangle \tag{3.6}$$

is minimal. The k-face formed by

$$\{s_{i_0}^{(1)}, \ldots, s_{i_0}^{(k)}, r_{i_0}\}$$

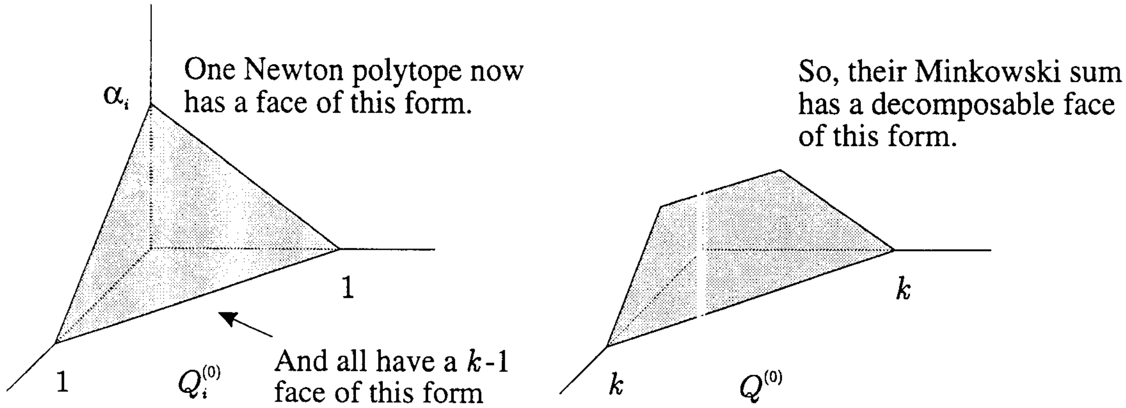

FIGURE (3.10). The face on the Minkowski sum of the new polytopes

is a w-maximal face of $P_{i_0}^{(0)}$. Call this face f_i. Reorder the polynomials (and hence the polytopes) so that the first i' polytopes satisfy the above minimality condition. i.e. for all $i > i'$ we have that

$$\langle w, q_i \rangle - \langle w, r_i \rangle > \langle w, q_{i'} \rangle - \langle w, r_{i'} \rangle . \tag{3.7}$$

Then $f^{(1)} := f_1 + \cdots + f_{i'} + \tilde{f}_{i'+1} + \cdots + \tilde{f}_k$ is a maximal admissible decomposable face of $P^{(1)}$. For the sake of notation rename its summands as $f_i^{(1)}$.

3.3.4. Slopes of the face. We need to show that all of the slopes for this decomposable face are actually increasing with respect to w. For this, we look at the parallel faces $f_1^{(1)}, \ldots, f_{i'}^{(1)}$. We see that $f_i^{(1)}$ is spanned by the $k+1$ affinely independent points

$$\begin{aligned}
r_i &= (r_{i1}, \ldots, r_{i\ell}, 0, \ldots, 0) \\
s_i^{(1)} &= (s_{i1}^{(1)}, \ldots, s_{i\ell}^{(1)}, 1, 0, \ldots, 0) \\
&\vdots \\
s_i^{(k)} &= (s_{i1}^{(k)}, \ldots, s_{i1}^{(k)}, 0, \ldots, 0, 1)
\end{aligned}$$

Moreover, these are the only points of $\mathrm{Supp}(F_i^{(1)})$ that $f_i^{(1)}$ contains. The same analysis holds for i's greater than i', but without the first point in the list. (They are $k-1$ faces, and each of the previous k-faces contains such a $k-1$ face.)

The k-faces summing to this face are all parallel and the $k-1$ faces all lie on translates of $Q^{(0)}$. Therefore, the slopes of $f^{(1)}$ with respect to x are the same as the slopes of $f_1^{(1)}$. So, consider the slope of $f_1^{(1)}$ in the direction of y_i. $f_1^{(1)}$ contains the point $s_1^{(i)}$, which lies over e_i. It also contains the point r_1 which lies over 0. These two points differ, over $\mathbb{R}^k$, by 1 in the y_i direction. Therefore the slope of $f_1^{(1)}$ in the y_i direction is

$$\delta_i^{(1)} := (s_{11}^{(i)} - r_{11}, \ldots, s_{1k}^{(i)} - r_{1k}) = s_1^{(i)} - r_1.$$

We set the exponents in the second terms of our series expansions to be $-\delta^{(1)}$

3.3.5. Solutions of the face equations. To finish the proof of part a), for this step, we need to show that the face equations of $f^{(1)}$ have a solution. Suppose that the coefficient on r_i is ρ_i. By equation 3.5 we get that the face equations of $f^{(1)}$ are

$$\begin{aligned} \frac{\partial E_1}{\partial t_1}(c)t_1 + \cdots + \frac{\partial E_1}{\partial t_k}(c)t_k &= \rho_1 \\ &\vdots \\ \frac{\partial E_k}{\partial t_1}(c)t_1 + \cdots + \frac{\partial E_k}{\partial t_k}(c)t_k &= \rho_k \end{aligned}$$

Since the matrix $\mathcal{M}$ in 3.3 is non-singular and not all of the ρ_i's are zero, this system of equations has a unique solution.

3.3.6. The series has support in the normal cone. To prove part b) of this theorem (at least for this step) we need to show that for every $w' \in B(f^{(0)})$ and every i, we get that $-\delta_i^{(1)}$ is in the translate of the barrier cone of $f = f^{(0)}$ with vertex at $-\delta_i^{(0)}$. For this we follow an argument similar to the one in lemma 2.1.8.

Recall that $Q_i^{(0)}$ has the same slopes as $f^{(0)}$. Since $s_1^{(i)}$ is on $Q_i^{(0)}$, these slopes are given by

$$\delta_i^{(0)} = (s_{11}^{(i)} - q_{11}, \ldots, s_{1k}^{(i)} - q_{1k}) = s_1^{(i)} - q_1.$$

By equation 3.6 we see that

$$\left\langle w, \delta_i^{(1)} \right\rangle = \left\langle w, s_1^{(i)} - r_1 \right\rangle > \left\langle w, s_1^{(i)} - q_1 \right\rangle = \left\langle w, \delta_i^{(0)} \right\rangle$$

Therefore we have that

$$\left\langle w, -\delta_i^{(1)} \right\rangle < \left\langle w, -\delta_i^{(0)} \right\rangle \tag{3.8}$$

implying that (at least at this term) the series is decreasing with respect to w.

To show that the series is actually in the normal cone, we need to show that the inequality in equation (3.8) holds for all $w' \in N(f^{(0)})$. For all $w' \in B(f^{(0)})$, let $H_{w'}$ be the unique w'-constant hyperplane through f_i. Let $H_{i,w'}$ be the translate of $H_{w'}$ which supports $P_i^{(0)}$ at $f_i^{(0)}$. Then $P_1^{(0)}$ also supports $P_i^{(1)}$ since all points of $P_i^{(1)}$ are translates of points of $P_i^{(0)}$ in the direction of $f^{(0)}$. Therefore the argument used above for w will also work for all w'.

This completes the second step of the construction. To finish this proof we need to show that the same analysis used in the second step will carry over into successive steps.

3.3.7. The inductive step. Suppose in previous steps we constructed $\delta^{(n-1)}$ and $c^{(n-1)}$. Also suppose that, in doing so, we have constructed $F_i^{(n-1)}$ and its polytope $P_i^{(n-1)}$ for all i. Suppose, further, that the $P_i^{(n-1)}$ satisfy the following condition.

- The points $s_j^{(i)}$, defined above for the second step of the construction, are vertices of $P_i^{(n-1)}$, and that they are extreme vertices of $f_i^{(n-1)}$. (Extreme in the sense of remark 3.2.7.)

Note that this condition holds for the case $n = 3$ by the above discussion.

Since the $s_j^{(i)}$ are on a supporting hyperplane for $P_i^{(n-1)}$, and are extreme on $f_i^{(n-1)}$, by remark 3.2.7 their coefficients are preserved in the transition from $P_i^{(n-1)}$ to $P_i^{(n)}$. Therefore we have, on the $P_i^{(n)}$, faces of the same form as above. Moreover, the face equations of these faces have the same coefficients as the system in equation 3.8. Since, by assumption, the approximation from the $(n-1)^{st}$ step of the induction didn't satisfy all the equations, some of the ρ_i are again non-zero. So we can continue the induction.

Since the faces we obtained for $P_i^{(n)}$ have the same form as those in the second step, the same arguments apply to show that the support of the series lies in a translate of the barrier cone of $f^{(0)}$ with vertex at $-\delta_i^{(0)}$. (Since $P_i^{(n)}$ lies in the barrier wedge of $f^{(0)}$, and the points of $P_i^{(n+1)}$ are obtained by translating points in $P_i^{(n)}$ in the direction of $f^{(n)}$, we see that $P_i^{(n+1)}$ is also in this barrier wedge.) □

3.3.8. Complete systems and normal cones. The following corollary relates the maximal collections of series solutions to the normal cones of the faces involved in the constructions.

COROLLARY 3.3.9. *Suppose that all of the decomposable faces in the w-maximal section of $P^{(0)}$ have face equation with maximal numbers of roots. (i.e. the number of roots is equal to the mixed volume of the projection to $\mathbb{R}^k$.) Also suppose that all the roots of these systems satisfy the conditions of theorem* 3.3.1. *Then the number* *of solutions to the system* $\{F_i\}$ *in* $\mathbb{C}((C_{\mathbb{Q}}))$, *where*

$$C = \bigcup_{\substack{f \text{ a face of } S \\ f \text{ decomposable}}} B(f) = \bigcup_{\substack{f \text{ a face of } S \\ f \text{ decomposable}}} N^*(f)$$

is equal to the mixed volume of the projection of the polytopes $P_1^{(0)}, \ldots, P_k^{(0)}$.

$$V = \mathrm{Vol}\left(\pi\left(P(F_1)\right), \ldots, \pi\left(P(F_k)\right)\right).$$

Therefore, the number of solutions converging in some translate of the intersection of the normal cones of these faces is equal to V.

Note that this intersection is non-empty since it contains w.

PROOF. We have, by theorem 3.3.1 at least one distinct series in this ring for each root. Therefore, this ring contains at least V solutions. By theorem 1.2.9 the number of series in this ring that satisfy the system can be no more than this mixed volume. □

Calculations that have been performed on the polytopes of more general systems of equations lead us to make the conjecture that the conditions of theorem 3.3.1 can be weakened slightly. It seems that the conclusions of this theorem should still hold true if c is a simple root of the face equations, i.e. eliminating the condition that all partial derivatives of the face equations are non-zero at c.

4. Fiber polytopes and the mixed fiber polytope

Suppose that all of the faces of P satisfy the conditions of Corollary 3.3.9. Then, this corollary tells us that complete systems of solutions correspond to maximal collections of decomposable, admissible faces whose normal cones have a non-trivial intersection. Consider that, by choosing a linear functional w, we get a coherent section of the projection

$$\pi : P = P_1 + \cdots + P_k \longrightarrow \mathbb{R}^k$$

of the Minkowski sum of the Newton polytopes of the original equations. By changing w, we change the section. Note, however, that if two sections differ by only indecomposable faces, then they yield the same set of series solutions in the above construction. If they differ by any decomposable faces, then they give (at least some) distinct series solutions. Therefore, the maximal sets of series solutions of $F_1, \ldots, F_k$ correspond to equivalence classes of coherent sections of π, where two sections are equivalent if they contain the same set of decomposable faces.

If $P_1 = \cdots = P_k$, then P is a dilation of P_i for all i. Therefore, every face on P is decomposable, and hence the maximal sets of series solutions correspond precisely to the coherent sections of π.

Let Δ be the fan in $\mathbb{R}^\ell$ whose cones are the domains of convergence of the maximal collections of series constructed above. Consider the fiber polytope [3] associated to the projection π. Let Q be the image of π, then this fiber polytope is denoted by

$$\Sigma_\pi(P) = \Sigma_\pi(P, Q) = \left\{ \int_Q \gamma(x)dx : \gamma \in \Gamma(P) \right\}.$$

Recall that the vertices of $\Sigma_\pi(P)$ are in one to one correspondence with the coherent sections of π, and that the barrier cone of the vertex v_w is equal to the union of the barrier cones of the faces that comprise the section Therefore we get the following corollary to theorem 3.3.1.

COROLLARY 3.4.1. *The normal fan* Δ_Σ *of* $\Sigma_\pi(F)$ *is a refinement of* Δ. *If* $P_1 = \cdots = P_k$, *then* $\Delta_\Sigma = \Delta$.

The properties of the fan Δ suggests the existence of a polytope $\Sigma_\pi(P_1, \ldots, P_k)$ called the mixed fiber polytope of the polytopes P_i associated to the projection π. This polytope should satisfy the following properties:

1) $\Sigma_\pi(P_1, \ldots, P_k)$ is a Minkowski summand of $\Sigma_\pi(P)$, and hence the normal fan of $\Sigma_\pi(P)$ is a refinement of the normal fan of $\Sigma_\pi(P_1, \ldots, P_k)$.
2) If $P_1 = P_2 = \cdots = P_k$, then $\Sigma_\pi(P_1, \ldots, P_k) = \Sigma_\pi(P)$.

5. The mixed discriminant and mixed fiber polytopes

As in chapter 2, these results indicate a relationship between the fiber polytope and a certain ramification locus. Namely, consider the ramification locus of the variety X defined by the equations $F_1 = \cdots = F_k = 0$. Assume that X is smooth. Then this locus corresponds to all multiple points in the projection to the $x_1, \ldots, x_\ell$-hyperplane, and since X cannot have singularities, must lie outside the domains where we have complete collections of series solutions.

Just as in the last chapter, the Log of this ramification locus is bounded by the cones which contain maximal series solution sets.

THEOREM 3.5.1. *If the Newton polytopes of the F_i coincide, then the* Log *of the ramification locus of the projection of x to the $x_1, \ldots, x_\ell$-hyperplane is bounded by translates of the normal cones of the fiber polytope associated to the projection of the Minkowski sum of the F_i to $\mathbb{R}^k$.*

In general, it will be necessary to construct the mixed fiber polytope to extend this relationship.

In chapter 2, this ramification locus corresponded to the zero locus of the discriminant of F with respect to x_{N+1}. Here, the locus is the zero locus of a generalization of this discriminant, called the mixed discriminant.

Intuitively, it is the locus where the equations $F_i = 0$ have a common multiple root. Generically, k equations in k variables intersect in a finite number of roots determined by the mixed volume of their Newton polytopes. We wish to make explicit the conditions on the F_i which determine when two of these roots merge into a common double root.

This happens when a translate of the $y_1, \ldots, y_k$-hyperplane is tangent to the variety X. But this is equivalent to saying that the tangent planes to the varieties $\{F_i = 0\}$ intersect in a ℓ-plane which lies parallel to the $x_1, \ldots, x_\ell$-hyperplane. This in turn is equivalent to the condition that the vectors

$$\det\left(\begin{bmatrix} \frac{\partial F_1}{\partial y_1}(c) & \cdots & \frac{\partial F_1}{\partial y_k}(c) \\ \vdots & \ddots & \vdots \\ \frac{\partial F_k}{\partial y_1}(c) & \cdots & \frac{\partial F_k}{\partial y_k}(c) \end{bmatrix}\right) \neq 0.$$

In other words, the F_i vanish, and thier first partial derivatives with respect to the y_i are linearly dependent. This gives $n+1$ algebraic conditions on the collection of polynomials, and therefore is a codimension 1 condition on the space of all such collections of sections.

We formulate the mixed resultant more generally as follows. Let X be an n dimensional algebraic variety over $\mathbb{C}$, and let $L_1, \ldots, L_n$ be very ample line bundles over X. (For the definition of very ample, see the discussion in chapter 4.) Let

$$V = H^0(X, L_1) \times \cdots \times H^0(X, L_k)$$

be the product of the spaces of global sections of these bundles. Consider the subvariety $\nabla \subset V$ defined by

$$\nabla = \left\{ (s_1, \ldots, s_n) : \begin{array}{c} \text{the } s_i \text{ vanish and their first} \\ \text{partial derivatives are linearly dependent} \end{array} \right\}$$

In order to completely construct the mixed discriminant, it is necessary to prove that ∇ is an irreducible hypersurface. (At the moment it is the irreducibility of this locus that is posing a slight problem.) We would then take the mixed discriminant to be the irreducible equation of ∇. The mixed discriminant thus defined would be a polynomial on the space V which is zero if and only if the collection of sections vanishes and simultaneously their first partial derivatives are linearly dependant.

The above results would then give the following result on the zero locus of the mixed discriminant of $F_1, \ldots, F_k$.

PROPOSITION 3.5.2. *Suppose that the Newton polytopes of the F_i coincide. then the zero locus of the mixed discriminant of the F_i with respect to the variables $y_1, \ldots, y_k$ is bounded by the normal cones of the fiber polytope associated to the projection of the Minkowski sum of the F_i to $\mathbb{R}^l$.*

Part 2

Resultants and Direct Images

CHAPTER 4

Introduction and Preliminaries on Resultants and Discriminants

1. Introduction

Let $X \xrightarrow{p} S$ be a smooth projective morphism of smooth varieties of relative dimension n. Suppose that $\dim(X) = n + m = r$, with $\dim(S) = m$. All varieties here will be complex unless otherwise stated.

Let $E_1, \ldots, E_k$ be a collection of vector bundles over X of rank r_i respectively. Assume that the E_i are relatively ample. That is, that they are ample over each fiber of p. Let $l_1, \ldots, l_k$ be a sequence of numbers which sum to $n + 1$. Consider the integral of the product of Chern classes of the E_i,

$$c = \int_{X/S} c_{l_1}(E_1) \cdots c_{l_k}(E_k) \in H^2(S, \mathbb{Z}) \tag{4.1}$$

where $\int_{X/S} : H^{2n+2}(X, \mathbb{Z}) \longrightarrow H^2(S, \mathbb{Z})$ is the direct image (integration over the fibers). In singular cohomology, this integral is given by the following composition of maps

$$H^{2(n+1)}(X;\mathbb{Z}) \xrightarrow{\delta_X} H_{2m-2}(X;\mathbb{Z}) \xrightarrow{p_*} H_{2m-2}(S;\mathbb{Z}) \xrightarrow{\delta_S^{-1}} H^2(S;\mathbb{Z}),$$

where, for an n-dimensional real manifold Y, δ_Y denotes the Poincaré duality isomorphism

$$\delta_Y : H^i(Y;\mathbb{Z}) \longrightarrow H_{n-i}(Y;\mathbb{Z})$$

and p_* is the induced map on homology.

In [7] Deligne proposed the problem of finding, functorially, an algebraic line bundle

$$I_{X/S}\left(c_{l_1}(E_1), \dots, c_{l_k}(E_k)\right)$$

over S such that

$$c_1\left(I_{X/S}\left(c_{l_1}(E_1), \dots, c_{l_k}(E_k)\right)\right) = c.$$

The difficult part is to build a bundle in the case $S = pt$ ($S = \operatorname{spec} k$, for k a field). That is, to construct a canonical 1-dimensional k-vector space $I_{X/S}$ for bundles E_i on a variety X with $\dim X = n$. Once this has been accomplished, then one can, in a very natural way, knit these spaces together to form a vector bundle over S.

Deligne executed the first step of this process, in [7], by introducing, for two line bundles L and M over a curve X, a vector space

$$\langle L, M \rangle .$$

Two meromorphic sections l and m, of L and M respectively, define a non-zero element $\langle l, m\rangle$ of $\langle L, M\rangle$ if and only if they have disjoint zero sets (divisors). Moreover, the $\langle l, m\rangle$ satisfy the following condition. If f is a rational function on X, then fl and fm are sections of L and M respectively, and

$$\langle fl, m\rangle = f(\operatorname{div} m)\,\langle l, m\rangle$$

where

$$f\left(\sum n_p p\right) := \prod f(p)^{n_p}.$$

The most important aspect of this construction, though, is that $\langle l, m \rangle$ is 0 if and only if l and m have a common zero over X. A determinantal representation for these spaces, which can be derived from the Koszul complex, was introduced by Moret-Bailly in [23].

In [10], Elkik considered a more general case using the norm of a bundle with respect to a projection. We will introduce a related interpretation of these integrals. In some cases we can show that our construction is isomorphic to the previous constructions. We construct "resultant" bundles with these cohomology classes as first Chern classes. By identifying these bundles as resultants, we are able to interpret them as the determinant of the cohomology of a certain complex, the Eagon-Northcott complex.

We will show that natural generalizations of the cube theorem also arise from these complexes. They turn out to be key tools for demonstrating the general properties of the spaces $I_{X/S}$. We end the discussion of applications with a few results concerning the relationships between discriminants and resultants.

2. Ample vector bundles

For our main construction, the essential property which we need from ample bundles is that they have enough sections to perform genericity arguments. The theory of ample line bundles is well developed in [14]. Let L be a line bundle over a variety X. Then L is called very ample if there is some embedding p of X into a projective space $\mathbb{P}^n$ such that $L \cong p_*\left(\mathcal{O}_{\mathbb{P}^n}(1)\right)$. In particular, this means that a very ample bundle is generated by its global sections.

A line bundle L is said to be ample if some positive tensor power of L is very ample. The following are some of the more important properties concerning ample and very ample line bundles. Proofs may be found in [14].

PROPOSITION 4.2.1. *If X is proper over k and L is an ample line bundle over X, then for every coherent sheaf F on X, there is some integer $n_0 > 0$ such that for all $n \geq n_0$ and all $i > 0$,*

$$H^i\left(X, F \otimes L^{\otimes n}\right) = 0.$$

The following proposition from [14] gives some information on generating new ample bundles from old.

PROPOSITION 4.2.2. *Let X be a scheme and let L and M be line bundles over X. Then*

1) *If $n > 0$ is an integer, then L is ample $\Leftrightarrow L^{\otimes n}$ is ample.*
2) *If L and M are ample, then so is $L \otimes M$.*
3) *If L is ample and M is arbitrary, then $L^{\otimes n} \otimes M$ is ample for large n.*

The definition for arbitrary vector bundles is a generalizations of these notions.

DEFINITION 4.2.3. A vector bundle E on X is ample if for every coherent sheaf F on X, there is an integer $n_0 > 0$ such that for every $n \geq n_0$ the sheaf $F \otimes S^n(E)$ is generated as an $\mathcal{O}_X$ module by its global sections. (Here $S^n(E)$ is the n^{th} symmetric power of E.) If $X \xrightarrow{p} S$ is a smooth projective morphism, then a bundle E over X is relatively ample if its restriction to any fiber X_s is ample.

THEOREM 4.2.4. *For every vector bundle E over an algebraic variety X, there is a line bundle L such that $E \otimes L$ is ample.*

3. Resultants

4.3.1. The classical setting. Let $f(x)$ and $g(x)$ be two polynomials in a single variable x over $\mathbb{C}$. Suppose

$$f(x) = a_0 + a_1 x + \cdots + a_n x^n$$

$$g(x) = b_0 + b_1 x + \cdots + b_m x^m$$

we wish to determine when f and g have a common root in $\mathbb{C}$. Polynomials have a common root if and only if their greatest common divisor is non-constant. But this is true if and only if their least common multiple has degree less than $m + n$. So, f and g have a common root if and only if there exist polynomials F and G with $\deg(F) < \deg(f)$ and $\deg(G) < \deg(g)$ and such that

$$fG = gF. \tag{4.1}$$

We write F and G in general form

$$F(x) = A_0 + A_1 x + \cdots + A_{n-1} x^{n-1}$$

$$G(x) = B_0 + B_1 x + \cdots + B_{m-1} x^{m-1}.$$

The equality 4.1 thus becomes a system of linear equations on the coefficients of F and G. Moreover, this is an $m + n$ by $m + n$ system, which has a solution if and only if its determinant is non-zero.

Therefore, the resultant is a polynomial on the coefficients of f and g which is zero if and only if f and g have a common root. Writing the determinant this way yields the classical Sylvester formula for the resultant. We will denote the resultant of f and g by either $R(f, g)$ or by $\langle f, g \rangle$ following Deligne's notation. See [16] or [12] for a more complete discussion of the classical resultant.

4.3.2. The modern setting. The modern methods for resultants put the above ideas into the following setting. (For a complete discussion of the ideas presented here see [12].) We consider $n+1$ line bundles $L_0, \ldots, L_n$ over an irreducible projective n-dimensional variety X. Certainly, we must consider sections of line bundles rather than polynomials when working over arbitrary X. (Even over $\mathbb{P}^1$ this is convenient, as $\mathcal{O}(n)$ is the line bundle whose sections are homogeneous polynomials of degree n.) Let

$$V = \prod H^0(X, L_i)$$

be the product of the spaces of global sections of the L_i, i.e. the space of global sections of the bundle $E = L_0 \oplus \cdots \oplus L_n$.

Consider the following subset of V

$$\nabla = \{(s_0, \ldots, s_n) \in V : s_0(x) = \cdots = s_n(x) = 0 \text{ for some } x \in X\}$$

Then, ∇ is the collection of all $n+1$-tuples of sections which vanish simultaneously somewhere on X.

PROPOSITION 4.3.3. *∇ is an irreducible homogeneous hypersurface in V.*

PROOF. That ∇ is an irreducible algebraic variety is easily seen from the fact that it is the projection of the incidence variety

$$I = \{(x, s_0, \ldots, s_n) \in X \times V : s_0(x) = \cdots = s_n(x) = 0\}$$

And that I is a vector bundle over the irreducible variety X assures us that I is irreducible. Hence, so is its projection, ∇.

The fact that ∇ is homogeneous can be easily seen by considering that if $s = (s_0, \ldots, s_n) \in \nabla$ is an $(n+1)$-tuple of sections, then the s_i are simultaneously 0 somewhere on X. Then the sections

$$\lambda s = (\lambda s_0, \ldots, \lambda s_n)$$

are simultaneously zero at the same points as s. So λs is in ∇ as well.

That ∇ has codimension one arises from a simple dimension count completely analogous to those given later, when we construct the generalized resultant spaces. For details see [12]. □

The resultant for these bundles is defined to be the irreducible equation of this hypersurface, and is denoted $R_{L_0,\ldots,L_n}$. This polynomial is defined, only up to a non-zero constant multiple, on tuples of sections and is zero if and only if the sections in question have a common root.

4. Discriminants

Classically, the the question answered by the discriminant is this: How can we determine when a polynomial of one variable $f(x)$ has a multiple root. The answer is simply that it has if and only if $f(x)$ and $f'(x)$ have a common root.

Therefore, we can define the discriminant of f to be

$$\Delta(f) = R(f, f').$$

The above discussion applied to this definition yields the classical Sylvester formula for the discriminant. Again, complete discussions can be found in either [16] or [12].

An exhaustive discussion of the more modern approach to discriminants can be found in [12]. Momentarily we will use the description of the discriminant in terms of jet bundles. We can, however, define the discriminant in a manner which is completely analogous to the definition of the resultant.

Consider an irreducible smooth algebraic variety X. For a line bundle L over X, we define the discriminant Δ_L as follows. Let $V = H^0(X, L)$. Then the collection of all sections in V which vanish somewhere on X along with their first partial derivatives (i.e. have a common root), form a hypersurface ∇. We are looking for the intersection of $n+1$ forms on X. Their intersection is thus a codimension one condition on V.

It can be shown (see [12]) that this hypersurface is irreducible and homogeneous. The discriminant is defined to be the irreducible homogeneous equation of this hypersurface, and is again defined only up to a non-zero scalar multiple.

For computational purposes, a more useful definition of the discriminant uses the theory of jets. We review some of the basic definitions and results for this theory here.

For a line bundle L we define the bundle $J(L)$ of first jets of sections of L as follows. The fiber of $J(L)$ at a point $x \in X$ is the quotient of the space of sections by the space of all sections defined near x which have a double root at x. Namely, if I_x is the ideal of functions which vanish at x, then

$$J(L)_x = L/I_x^2 L.$$

Therefore, an element of this fiber is defined by a section's value at x and the values of its n first partial derivatives. This implies that this fiber has dimension $n+1$.

Let f be a section of L. We can associate to f the section $j(f)$ which at x is the image of f in $J(L)_x$. Note that this correspondence is $\mathbb{C}$-linear, but not $\mathcal{O}_X$-linear.

PROPOSITION 4.4.1. *For any algebraic line bundle L, we have a short exact sequence*

$$0 \longrightarrow \Omega^1_X \otimes L \xrightarrow{\alpha} J(L) \xrightarrow{\beta} L \longrightarrow 0.$$

where Ω^1_X *is the sheaf of algebraic one forms on* X *and* β *takes any vector* $j(f)(x)$ *to* $f(x) \in L_x$.

With this, we can define the discriminantal space as follows. Consider the set $\nabla \subset V = H^0(J(L))$ consisting of all sections which vanish somewhere over X. Then one can show that ∇ is an irreducible homogeneous hypersurface in V. We define the discriminant for L to be the irreducible equation of this hypersurface (defined only up to a non-zero constant multiple).

5. The canonical vector space associated to a homogeneous hypersurface

Our development of $I_{X/S}$ will arise from the construction of a canonically defined vector space which is associated to a homogeneous hypersurface in an affine space. The ideas in this section will clean up some of the notation in the last several sections. Resultants and discriminants will no longer be defined only up to a non-zero constant multiple.

We work over an arbitrary base field k. Let X be a finite dimensional k-vector space (regarded as an algebraic variety over k). Let $H \subset X$ be a homogeneous hypersurface (a reduced subscheme of codimension one, such that if $x \in H$, then $\lambda x \in H$ for all λ). It is well known that H can be given by one homogeneous equation $\{f = 0\}$ which is unique up to a non-zero constant factor. Consider the set of all level hypersurfaces of H. If f is an equation for H, then H is unambiguously defined by $\{f = 0\}$, and the other points in the vector space are unambiguously defined by $\{f = d\}$ for various d. Call this set $\mathcal{L}(H)$.

This set forms vector space structure over $\mathbb{C}$ in the following way. Let f be a fixed equation for H. Then f has a well defined value on any level hypersurface of

h, namely d. We denote this by $h = H_d$, and say that f(h)=d. If H_d and H'_{d}, are elements of $\mathcal{L}(H)$, then the vector space structure on $\mathcal{L}(H)$ is given by

$$H_{d+d'} = H_d + H_{d'}$$

$$cH_d = H_{cd}.$$

These operations are well defined because any two equations of H differ by a non-zero constant multiple. Note, however, that $\mathcal{L}(H)$ is not canonically identified with $\mathbb{C}$. The defining equation f is only defined up to a scalar multiple, as there are many choices for the function f. As mentioned above, $\mathcal{L}(\nabla)$ is the canonical vector space wherein the "equation" f of H takes values. Namely, we have a canonical map

$$\nu : X \longrightarrow \mathcal{L}(\nabla)$$

which takes a section $x \in X$ to the level hypersurface of ∇ which contains it.

We can tighten the relationship between H and $\mathcal{L}(H)$ by considering the one dimensional space $\mathcal{E}(H)$ of equations of H. In fact, these two spaces are dual

$$\mathcal{L}(H) = \mathcal{E}(H)^{*}.$$

An equation for H has a well defined value on every level surface of H, and so defines a linear map from $\mathcal{L}(H)$ to $\mathbb{R}$.

PROPOSITION 4.5.1. *Suppose that*

$$E \xrightarrow{\pi} Y$$

is a vector bundle of rank n and that E_y is its fiber over the point y of Y. Let $H \subset Y$ such that H is algebraic and $H_y = H \cap E_y$ is a homogeneous hypersurface of degree d in E_y for all $y \in Y$. Then there exists a natural line bundle $\mathcal{L}(H)$ whose fiber over $y \in Y$ is $\mathcal{L}(H_y)$.

PROOF. Consider the space $\mathbb{P}_{d,y}$ of all hypersurfaces of degree d in the n-dimensional vector space E_y. Collectively, they form a projective bundle $\mathbb{P}_d \longrightarrow Y$. The map

$$\begin{aligned} i : Y &\longrightarrow \mathbb{P}_d \\ y &\longrightarrow H_y \end{aligned}$$

defines a section of this bundle.

Consider the bundle $\mathcal{O}_{\mathbb{P}_d}(-1)$. Its fiber over any (y, H') is $\mathcal{L}(H')$. Therefore,

$$L(H) = i^* (O(-1)).$$

□

CHAPTER 5

The determinant of the cohomology

1. Derived categories

In this section we outline the construction of the derived category of complexes of objects in an abelian category. Recall that this is the category of complexes of modules with quasi-isomorphisms formally inverted

Let $\mathcal{M}$ be an abelian category (we will primarily use the category of sheaves of modules over $\mathcal{O}_X$ over some variety X). Let $K^b(\mathcal{M})$ be the category whose objects are bounded (both above and below) complexes of objects of $\mathcal{M}$, and whose morphisms are the homotopy classes of maps between complexes. Since homotopic complexes have isomorphic homology, and since homotopic maps produce the same map on homology, we still have well defined homology functors $H^i : K^b(\mathcal{M}) \to \mathcal{M}$.

5.1.1. Properties of $K^b(\mathcal{M})$ and triangulated categories. While the category of complexes and actual morphisms of complexes is an abelian category, when we pass to the category whose morphisms are merely homotopy classes of morphisms we lose the ability to always define images of morphisms. For a proof of this, see [9].

Since the homotopy category of complexes is not abelian, we no longer have the notion of an exact sequence of objects. We do, however, have a notion which conveniently replaces exact sequences, namely triangles. A triangle in $K^b(\mathcal{M})$ is a sequence of morphisms

$$X \longrightarrow Y \longrightarrow Z \longrightarrow X[1],$$

where $X[1]$ is the complex X with the indexing shifted by 1.

Consider that every morphism in $K^b(\mathcal{M})$

$$f : X \longrightarrow Y$$

fits into a triangle with $Z = M(f) = X[1] \oplus Y$, the mapping cone of f. (Here $X[1]$ is the complex X with its grading shifted by one.) We say that a triangle is *distinguished* if it is isomorphic to a triangle of this form. We will also refer to such a triangle as exact.

The main reason to consider this structure on the category $K^b(\mathcal{M})$ is that distinguished triangles share many of the properties of exact sequences in abelian categories.

Any category with such a structure of triangles is called a triangulated category. Formally it means we have an automorphism T on the category and distinguished class of triangles

$$X \rightarrow Y \rightarrow Z \rightarrow T(X)$$

which satisfies a list of five axioms satisfied by the above triangles formed with mapping cones. For a list of the axioms and proofs of the following propositions, see [9].

Let $\mathcal{K}$ be a triangulated category, and let $\mathcal{A}$ be an abelian category. An additive functor

$$F : \mathcal{K} \rightarrow \mathcal{A}$$

is called cohomological if it takes distinguished triangles to exact sequences.

PROPOSITION 5.1.2. *For any $X \in \mathcal{K}$ the functors* $\hom_{\mathcal{K}}(X, \bullet)$ *and* $\hom_{\mathcal{K}}(\bullet, X)$ *are cohomological functors. Also, for an abelian category $\mathcal{M}$, the functor $H^0(\bullet)$: $K^b(\mathcal{M}) \to \mathcal{M}$ is cohomological.*

5.1.3. The derived category. The construction of the derived category of an abelian category is based on the notion of formally inverting a set of morphisms in the category.

As before let $\mathcal{M}$ be an abelian category, and let $K^b(\mathcal{M})$ be the homotopy category of complexes of objects in $\mathcal{M}$. Consider the collection S of quasi-isomorphisms in $K^b(\mathcal{M})$, i.e. morphisms which induce an isomorphism on cohomology. We then formally invert quasi-isomorphisms in $\mathcal{M}$. The derived category $D^b(\mathcal{M})$ is the category whose objects are complexes of objects of $\mathcal{M}$ and whose morphism are symbols of the form $\alpha^{-1}a$ where a is a morphism in $K^b(\mathcal{M})$ and α is a quasi-isomorphism.

One can check that, given a morphism $b\beta^{-1}$, we can rewrite it as $\alpha^{-1}a$ for some morphism a and quasi-isomorphism α. For this it is sufficient to show that given a morphism $a : Y \to Z$ and a quasi-isomorphism $\alpha : Y \to X$ that there exists some complex Y', a quasi-isomorphism $\beta : Z \to Y'$, and a map $b : X \to Y'$ such that $b\alpha = \beta a$. For details see [9].

The derived category thus constructed inherits the structure of a triangulated category by taking a distinguished triangle to be a diagram of complexes which is quasi-isomorphic to a distinguished triangle in $K^b(\mathcal{M})$. Often we will refer to distinguished triangles in $K^b(\mathcal{M})$ as exact.

5.1.4. Derived functors. Let $F : \mathcal{A} \longrightarrow \mathcal{B}$ be a left exact additive functor of abelian varieties. For any such functor, we obtain the right derived functor

$$RF : \mathcal{D}(\mathcal{A}) \longrightarrow \mathcal{D}(\mathcal{B})$$

As described in [9]. This construction is intimately related to the construction of the right derived functors R^iF usually given in homological algebra. In fact, for any object $A \in \mathcal{A}$, the objects $R^iF(A)$ are the cohomology of the complex $RF(A)$. We will spend a moment to explicitly work out the details of the derived functor in the case of the direct image on sheaves. We obtain the so-called derived direct image.

Let $p : X \longrightarrow S$ be a smooth projective morphism of smooth algebraic varieties. The derived direct image is the complex which calculates the higher direct images of sheaves on X (these can be thought of as the cohomology of $\mathcal{F}$ over the fibers of p). We define the higher direct images $R^ip_*\mathcal{F}$ as follows. Let $I^\bullet$ be an injective resolution of $\mathcal{F}$. We define the derived direct image to be $Rp_*\mathcal{F} = p_*(I^\bullet) \in \mathcal{D}^b(S)$. Then the higher direct images are

$$R^ip_*\mathcal{F} = H^i\left(Rp_*\mathcal{F}\right) := H^i\left(p_*\left(I^\bullet\right)\right).$$

Turning to the case of a complex $\mathcal{F}^\bullet$ of coherent sheaves over X, if we take an injective resolution of $\mathcal{F}^\bullet$, we obtain a double complex, $I^{\bullet\bullet}$. The derived direct image $Rp_*\mathcal{F}^\bullet$ is the direct image under p of the total complex of this double complex

$$Rp_*\mathcal{F}^\bullet = \left\{ \bigoplus_{m+n=i} p_*I^{mn} \right\}.$$

When we take the cohomology of this complex, we obtain the hyper-direct images (completely analogous to the hyper-cohomology) of the complex $\mathcal{F}^\bullet$,

$$\mathbb{R}^ip_*\mathcal{F}^\bullet = \mathbb{H}^i\left(Rp_*\mathcal{F}^\bullet\right).$$

Suppose that $\mathcal{F}^\bullet$ contains only one non-zero sheaf. Then the injective resolution above collapses to just a single injective resolution and $Rp_*\mathcal{F}^\bullet = p_*I^\bullet$. So, in this case $\mathbb{R}^ip_*\mathcal{F}^\bullet = R^ip_*\mathcal{F}^j$, where $\mathcal{F}^j$ is the non-zero term. Therefore, the hyper-direct image can be seen as a generalization of the higher direct images of a sheaf. For a more complete discussion of these ideas see [9].

2. Determinants

5.2.1. Determinants of vector spaces and complexes. Let W be a vector space of dimension n over an arbitrary base field k. We define

$$\det(W) = \bigwedge^{top}(W) = \bigwedge^{n}(W).$$

Therefore, $\det(W)$ is a one dimensional vector space. In fact, $\det(W)$ is the canonical one dimensional space wherein the determinants of maps

$$\phi : \mathbb{C}^n \longrightarrow W$$

lie. If $\phi : \mathbb{C} \longrightarrow W$ is an isomorphism, then its determinant is, by definition, the element of $\det(W)$ given as follows. Let $w_1, \ldots, w_n$ be the images of the standard basis elements in $\mathbb{C}^n$. Then $\det(\phi)$ is defined to be $w_1 \wedge \cdots \wedge w_n \in \det(W^\bullet)$.

The space $\det(W)$ is not canonically identified with k. In fact, choosing a basis for W defines an isomorphism between $\det(W)$ and k. As usual, though, $\det(\phi) = 0$ if and only if ϕ is not an isomorphism.

Let $W^\bullet$ be a complex of finite dimensional vector spaces over k. Define

$$\det(W^\bullet) = \bigotimes_i \det\left(W^i\right)^{\otimes(-1)^i},$$

where V^{-1} stands for the dual space $V^* = \mathrm{Hom}_k(V, k)$ for any vector space V over k.

Note that in the case where a map from $\mathbb{C}^n$ is regarded as a complex

$$W^\bullet = \left\{ \cdots \to 0 \to \mathbb{C}^n \to W^0 \to 0 \to \cdots \right\}$$

This definition of the determinant agrees with the former, because $\det(\mathbb{C}^n)$ is canonically isomorphic to $\mathbb{C}$. (We say it is canonically trivialized.)

Generally, the determinant acts on an isomorphism

$$V \xrightarrow{\phi} W$$

in the following way. ϕ induces a map on the top exterior powers

$$\phi_* : \det(V) \longrightarrow \det(W),$$

which can then be canonically considered to be an element of $\det(V)^* \otimes \det(W)$ which is the determinant of the complex

$$0 \longrightarrow V \xrightarrow{\phi} W \longrightarrow 0.$$

One of the key tools in working with determinants is the the manner in which the determinant behaves on exact complexes. For a complete discussion and proofs of these propositions, see [12]. First, we consider the effect of the determinant on a short exact sequence, the so-called Euler isomorphism.

PROPOSITION 5.2.2. *For an exact sequence of vector spaces*

$$0 \longrightarrow W \xrightarrow{\phi} V \xrightarrow{\psi} W' \longrightarrow 0$$

there is a natural isomorphism

$$i(\phi, \psi) : \det V \xrightarrow{\cong} \det W \otimes \det W'.$$

Using this proposition, one can derive the more general Euler isomorphism on complexes

PROPOSITION 5.2.3. *Let (W, d) be a complex of vector spaces with $d_i : W^i \to W^{i+1}$. There is a natural isomorphism*

$$i(d) : \det(W^\bullet) \longrightarrow \det(H^\bullet(W^\bullet)).$$

Note that, in particular, for an exact complex (W, d) this isomorphism defines a canonical isomorphism between the determinant of W and k.

Another useful property in our situation is the manner in which the determinant acts on triangles in the derived category of complexes of k-vector spaces. In stating this proposition, we consider complexes of R-modules for R an arbitrary Noetherian integral domain.

If R is coordinate ring of an algebraic variety (more generally, for rings with nilpotents, we must consider the scheme spec R), then vector bundles over the variety correspond to projective R-modules.

We can define the determinant of an R-module M to be the top exterior power of M, which is a projective R-module of rank 1. We define the determinant of a complex of modules in the same way as before.

Assume that R is regular (in the geometric case such rings correspond to smooth varieties). Since any complex of R-modules (for regular R) is quasi-isomorphic to a complex of projective R-modules, we have a functor defined on the derived category of complexes of R-modules.

$$\det : \mathcal{D}^b(R) \longrightarrow \mathrm{Inv}(R)$$

where $\mathrm{Inv}(R)$ is the collection of projective R-modules of rank 1, also called invertible R modules because of their correspondence to line bundles.

We now move to the case of sheaves over varieties. Let S be a smooth algebraic variety. Let $\mathcal{F}^\bullet \in D^b(S)$, be the derived category of complexes of coherent sheaves over S.

We wish to define a determinantal line bundle $\det(\mathcal{F}^\bullet) = \det_{S/S}(\mathcal{F}^\bullet)$ over S. Recall that every complex of coherent sheaves on S is quasi-isomorphic (and hence equal in the derived category) to a complex of vector bundles (locally free sheaves) over S. So, choose a quasi-isomorphism

$$\mathcal{F}^\bullet \xrightarrow{qis} \mathcal{G}^\bullet$$

and note that $\mathcal{F}^\bullet \cong \mathcal{G}^\bullet$ in the derived category.

DEFINITION 5.2.4. We define the determinant of this complex to be

$$\det(\mathcal{F}^\bullet) = \det_{S/S}(\mathcal{F}^\bullet) := \bigotimes \det\left(\mathcal{G}^i\right)^{\otimes(-1)^i}.$$

This is just the ordinary determinant of the complex $\mathcal{G}^\bullet$.

The fact that this is well defined is covered in [18], and follows from the Euler isomorphism on distinguished triangles.

PROPOSITION 5.2.5. *For any exact triangle in* $D^b(R)$

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1]$$

there is a natural isomorphism

$$i_R(u,v,w) : \det(X) \otimes_R \det(Z) \xrightarrow{\cong} \det(Y)$$

which is functorial with respect to isomorphisms of triangles, and is compatible with octahedrons.

For a proof of this and the following properties, see [18] as well, they all follow from the Euler isomorphim.

PROPOSITION 5.2.6. *Let $\mathcal{F}^\bullet$ be a complex of coherent sheaves on S.*

(1) *If $\mathcal{F}^\bullet$ is exact then* $\det(\mathcal{F}^\bullet)$ *is canonically trivial.*

(2) *If $\mathcal{F}_1^\bullet \xrightarrow{\phi} \mathcal{F}_2^\bullet$ is a quasi-isomorphism then* $\det(\mathcal{F}_2^\bullet) \cong \det(\mathcal{F}_2^\bullet)$.

(3) $\det(\mathcal{F}^\bullet) \cong \otimes \det \underline{H}^i(\mathcal{F}^\bullet)^{\otimes(-1)^i}$, *where $\underline{H}$ is the sheaf cohomology of the complex $H^\bullet$.*

(4) *If U is open in S, then* $\det_{S/S}(\mathcal{F}^\bullet)\big|_U = \det_{U/U}(\mathcal{F}^\bullet|_U)$.

5.2.7. The relative determinant. Let $X \longrightarrow S$ be a smooth projective morphism of smooth algebraic varieties. Let $\mathcal{F}$ be a sheaf on X.

The the relative determinant is defined to be the determinant of the derived direct image

$$\begin{aligned}\det_{X/S}(\mathcal{F}) := \det(Rp_*\mathcal{F}) &= \bigotimes \det\left(p_* I^i\right)^{\otimes(-1)^i} \\ &= \bigotimes \det\left(R^i p_* F\right)^{\otimes(-1)^i}\end{aligned}$$

This last equality is due to the Euler isomorphism.

In the more general case, let $\mathcal{F}^\bullet$ be a complex of sheaves on X. We define

$$\begin{aligned}\det_{X/S}(\mathcal{F}^\bullet) &= \bigotimes \left(\det Rp_*\mathcal{F}^i\right)^{\otimes(-1)^i} \\ &= \bigotimes_{i,j} \left(\det R^j p_* \mathcal{F}^i\right)^{\otimes(-1)^{i+j}},\end{aligned}$$

Where the last equality happens again by the Euler isomorphism. This construction agrees with the previous construction if $\mathcal{F}^\bullet$ has only one non-zero term.

Since $\det_{X/S}(\mathcal{F}^\bullet)$ is just $\det Rp_*\mathcal{F}^\bullet$, we have another Euler isomorphism

$$\det_{X/S}(\mathcal{F}^\bullet) = \det(Rp_*\mathcal{F}^\bullet) \cong \bigotimes_i \det\left(\mathbb{R}^i p_*\mathcal{F}^\bullet\right)^{\otimes(-1)^i}.$$

EXAMPLE 5.2.8. Consider the case of a projection from a variety X to a point, $S = *$. For a single coherent sheaf $\mathcal{F}$ the induced map p_* just associates to $*$ the vector space $H^0(X,\mathcal{F}) = \Gamma(\mathcal{F})$. So, the functor p_* is merely the global section functor, and so its right derived functors are just $R^i p_*(\mathcal{F}) = H^i(X,\mathcal{F})$. Thus, from the Euler isomorphism, we obtain,

$$\begin{aligned} \det_{X/*}(\mathcal{F}^\bullet) &= \det(Rp_*\mathcal{F}^\bullet) \\ &= \bigotimes_i \det(H^\bullet(X,\mathcal{F}^i))^{\otimes(-1)^i} \\ &= \bigotimes_{i,j} \det\left(H^j(X,\mathcal{F}^i)\right)^{\otimes(-1)^{i+j}} \end{aligned}$$

If the sheaves in $\mathcal{F}^\bullet$ have no higher cohomology, then

$$\det_{X/*}(\mathcal{F}^\bullet) = \bigotimes_i \det\left(H^0(X,\mathcal{F}^i)\right)^{\otimes(-1)^i}.$$

3. Koszul and Eagon-Northcott resolutions

5.3.1. The Koszul Complex. For ordinary resultants (the first case in the construction) the interpretation of the resultant as the determinant of the Koszul complex is well known. We will briefly review this construction here in preparation for what follows. For a complete discussion see [12]. This identification is obtained through the Cayley method, which was used by Gelfand, Kapranov and Zelevinski in [12] to derive formulas for the discriminant and resultant.

Let V be a rank n vector bundle over a variety X, and let s be a section of V. Consider the following complex $K^\bullet(V)$ of vector bundles

$$0 \longrightarrow \bigwedge^n V^* \xrightarrow{i_s} \bigwedge^{n-1} V^* \xrightarrow{i_s} \cdots \xrightarrow{i_s} V^* \xrightarrow{i_s} \mathcal{O}_X \longrightarrow 0$$

where

$$\begin{aligned} i_s : \bigwedge^i V^* &\longrightarrow \bigwedge^{i-1} V^* \\ v_1 \wedge \cdots \wedge v_i &\longmapsto \sum_{j=1}^{i} (-1)^j\, v_j(s)\, v_1 \wedge \cdots \wedge \widehat{v}_j \wedge \cdots \wedge v_i. \end{aligned}$$

Where the v_i are sections of V^*, and hence $v_j(s)$ is an element of $\mathcal{O}_y$. One can restate this simply by saying that the maps are given by contracting with s.

It is easy to see that this is a complex. If one composes α_i with α_{i-1}, the terms will cancel in pairs. For a complete discussion with proof of the following result concerning the cohomology of $K^\bullet$, see [12].

THEOREM 5.3.2. *This complex is exact if and only if s is nowhere zero on X. Moreover, if X is smooth and s vanishes along a smooth subvariety $Z \subset X$ of codimension exactly n and is transverse to the zero section, then $K(W)$ has only one non-trivial cohomology sheaf, $\mathcal{O}_Z$ regarded as a sheaf on X, in the last term.*

5.3.3. The Eagon-Northcott Complex. For the more general cases of $I_{X/S}$, we need to work with linear functions which are deficient in rank somewhere over Y. This is handled via a generalization of the Koszul complex, known as the Eagon-Northcott complex. This complex was originally introduced in [8], while a complete modern discussion of the complex can be found in [9].

Let V and W be vector bundles over a variety X, with $\operatorname{rk} V = k$, $\operatorname{rk} W = l$, and $l \leq k$. Let s be a morphism of vector bundles from W to V. We wish to construct a complex which will fail to be exact if and only if s fails to have full rank over some point of X. Note that this implies that we should consider only the case where the rank of W is less than or equal to the rank of V.

Consider the complex

$$0 \longrightarrow S^{k-l}W \otimes \bigwedge^{k} V^* \xrightarrow{i_s} S^{k-l-1}W \otimes \bigwedge^{k-1} V^* \xrightarrow{i_s} \cdots \xrightarrow{i_s}$$
$$\xrightarrow{i_s} S^1 W \otimes \bigwedge^{l+1} V^* \xrightarrow{i_s} \bigwedge^{l} V^* \xrightarrow{\bigwedge^l s} \bigwedge^{l} W^* \longrightarrow 0$$

where δ_i is defined as the following composition of maps:

$$S^iW \otimes \bigwedge^{l+i} V^* \longrightarrow S^{i-1}W \otimes W \otimes \bigwedge^{l+i} V^* \stackrel{id\otimes s\otimes id}{\longrightarrow}$$
$$\longrightarrow S^{i-1}W \otimes V \otimes \bigwedge^{l+i} V^* \longrightarrow S^{i-1}W \otimes \bigwedge^{l+i-1} V^*.$$

This composition is a generalization of the maps in the Koszul complex. Instead of contracting with a vector, we are contracting with a matrix. A proof of the exactness of this complex can be found in Eagon and Northcott's paper, [8]. We will denote this complex by $N^\bullet_s(W,V)$. The grading on this complex is usually counted with 0 being the degree of the right-most term. That is, all of the terms in this complex are in degrees ≤ 0.

This complex is generically a resolution of $\mathcal{O}_Z$ regarded as a sheaf on X where Z is the locus where s fails to have full rank.

THEOREM 5.3.4. *The above complex is a free resolution whenever*

$$\operatorname{codim}(V_l(s)) \geq k - l + 1$$

Where $V_l(s)$ is the subvariety defined by the vanishing of the $l \times l$ minors of s.

PROOF. This situation is dual to that in [9]. □

CHAPTER 6

Generalized Resultants

We continue to use the same notations as before. Namely, we consider a smooth projective morphism $X \longrightarrow S$ of relative dimension n, and let E be a relatively ample bundle of rank r on X.

In sections 1 to 3, we will work exclusively with $S = *$. Once we have the interpretation of $\int_{X/*}$ as the determinant of the cohomology of some complex, we will see that we can extend the definition to the relative case. Recall from section 5 that this is the main problem. As long as the degrees of the hypersurfaces involved remains constant, proposition 4.5.1 assures us that the spaces in question can be knit together to form a line bundle in the relative case.

We use the geometric interpretation of $c_{n+1}E$ and the direct image as a guide for our construction. It is this geometric viewpoint which enables us to construct vector spaces, which when pasted together over S in the more general case, yield the desired Chern class. Consider that the $(n+1)$-st Chern class of E is the Poincaré dual to the locus (if E is ample) where a generic collection of $r - n$ sections of E are linearly dependent, see [22]. Moreover, the direct image

$$\int_{X/S} c_{n+1}E,$$

is just the Poincaré dual of the projection of this locus in S.

One cautionary note. These geometric interpretations are a bit subtle in the case we will consider for the next several sections. If S is a point, then X has dimension n. Thus the $(n+1)-st$ Chern class lies in the $2n+2$ dimensional cohomology group, which is zero. Thus its direct image is also zero. (Every vector bundle over a point is trivial as it is just a vector space.) Therefore, we keep in mind this geometric goal for the more general construction in the case we now consider.

1. The definitions of the resultant spaces

6.1.1. The definition of $I_{X/S}(c_{n+1}E) = [E]$, with $\text{rank}(E) = n+1$. We first define the desired bundles in a very simple and classical case. This case is a direct generalization of the bundles which Deligne constructed (although in a slightly different form.) In fact, they are covered in [12] where some of their basic properties are investigated.

Let E be a very ample vector bundle over X of rank $n+1$. Let $V = H^0(X, E)$ be the space of all sections of E, and let $\nabla \subset V$ be the subset consisting of all sections which intersect the zero section of E. The proof of the following proposition can be found in [12]. The special case when E is a direct sum of line bundles is our proposition 4.3.3.

PROPOSITION 6.1.2. *∇ is an irreducible homogeneous hypersurface in V.*

We define the resultant space of E to be

$$[E] = I_{X/S}(c_{n+1}E) = \mathcal{L}(\nabla).$$

The hypersurface ∇ has a single defining equation called the E-resultant, i.e. the resultant of sections of E. It is denoted

$$R_E(s) \in [E]$$

This is a polynomial on V canonically taking values in $[E]$, which is zero precisely when a section has a root, since it is, by definition, zero precisely on ∇.

More generally, consider k bundles, $E_1, \ldots, E_k$ whose ranks sum to $n+1$. Then we define

$$\begin{aligned} [E_1, \ldots, E_k] = I_{X/S}(c_{\text{top}}(E_1), \ldots, c_{\text{top}}(E_k)) &= \int_{X/S} c_{\text{top}}(E_1) \ldots c_{\text{top}}(E_r) \\ &= \int_{X/S} c_{\text{top}}(E_1 \oplus \cdots \oplus E_k) = [E_1 \oplus \cdots \oplus E_r] \end{aligned}$$

So, $[E_1, \ldots, E_k]$ is the bundle where the resultants of sections of $E_1, \ldots, E_k$ lie. Such a resultant is zero precisely when the sections of the E_i have a common root. But a section s of $E_1 \oplus \cdots \oplus E_k$ is the same as a collection of sections $s_1, \ldots, s_k$ of $E_1, \ldots, E_k$. Moreover, s is 0 precisely when $s_1, \ldots, s_k$ have a common root. So the sections of this bundle are precisely the resultants of sections of $E_1, \ldots, E_k$.

REMARK. In section 4 we will show that for arbitrary S these spaces knit together to form a line bundle over S with the desired Chern class

$$c = \int_{X/S} c_{n+1} E.$$

6.1.3. Definition of $I_{X/S}(c_{n+1}E)$, with E of arbitrary rank. We take this intermediate generalization first, to get a feel for some of the pieces in the puzzle before delving into the notation of the general case. Let E be a very ample vector bundle over X of rank $r \geq n+1$. Let $V = H^0(X; E)^{r-n}$ be the space of all $(r-n)$-tuples of sections of E. Define $\nabla \subset V$ by

$$\nabla = \{(s_1, \ldots, s_{r-n}) : \text{ the } s_i \text{ are linearly dependent somewhere over } X\}$$

The following proposition is completely analogous to proposition 4.3.3.

PROPOSITION 6.1.4. *∇ is an irreducible homogeneous hypersurface in V.*

PROOF. Consider the space $V \times X$ and the incidence subvariety

$$\overline{\nabla} = \{(x, s_1, \ldots, s_{r-n}) : s_1(x), \ldots, s_{r-n}(x) \text{ are LD}\}.$$

Then ∇ is clearly a subvariety of $V \times X$ because it is given by the vanishing of various minors.

We have two naturally defined projections:

$$\nabla \xleftarrow{p_1} \overline{\nabla} \xrightarrow{p_2} X$$

Certainly, p_1 and p_2 are both surjective, which then immediately implies that ∇ is an affine variety in V. It is clear that $p_2^{-1}(x)$ is an irreducible subvariety of V_x, because it is the collection of $(r-n)$-tuples of vectors in E_x^{r-n} which are linearly dependent at x. This is precisely the subvariety of $M_{r,r-n}(\mathbb{C})$ consisting of matrices that are deficient in rank, an irreducible subvariety. (For a proof of this see [13].)

So, ∇_x is irreducible and has the same dimension for all x. $\overline{\nabla}$, being a fiber bundle over an irreducible variety whose fibers are irreducible and of the same dimension, is itself is irreducible. Thus, ∇, being the surjective image of an irreducible variety, is an irreducible subvariety of V.

The homogeneity of ∇ follows easily from the fact that if $s_1, \ldots, s_{r-n}$ are linearly dependent at some $x \in X$ then so are $\lambda s_1, \ldots, \lambda s_{r-n}$, for all $\lambda \in \mathbb{C}$.

To prove that ∇ is a hypersurface, we need to perform a dimension count. Since we assumed that E was very ample, a generic collection of $r-n$ sections in ∇ will have only one point of X over which they are linearly dependent. Hence, p_1 is generically one to one and is thus a birational isomorphism, yielding

$$\dim(\nabla) = \dim\left(\overline{\nabla}\right)$$

To determine the dimension of ∇, we calculate the dimension of $\overline{\nabla}$. We do this by finding its codimension in V. Consider a single $x \in X$ and consider its corresponding fiber F_x in $V \times X$. Consider also the evaluation map at x:

$$\begin{aligned} \phi_x : V = H^0(X,E)^{r-n} &\longrightarrow E_x^{r-n} \\ s_1, \ldots, s_{r-n} &\longmapsto (s_1(x), \ldots, s_{r-n}(x)) \end{aligned}$$

Note that ϕ_x is a surjection and that $\phi_x^{-1}(p)$ has the same dimension for any $p \in E_x^{r-n}$. So, if $Q \subset E_x^{r-n}$ is a subset of codimension k, then $\phi_x^{-1}(Q)$ also has codimension k.

Consider $\nabla_x \subset E_x^{r-n}$ defined as the collection of $(r-n)$-tuples which are linearly dependent. We know that, since E_x is a vector space of dimension r, that

$$\operatorname{codim}(\nabla_x) = r - (r-n) + 1 = n+1$$

And, so the codimension of $\phi_x^{-1}(\nabla_x)$ in $V \times \{x\}$ is $n+1$.

To finish off this proof, we notice that

$$\overline{\nabla} = \coprod_{x \in X} \phi_x^{-1}(\nabla_x)$$

and hence $\overline{\nabla}$ has codimension $n+1$. Since $\dim(X) = n$, we see that $\operatorname{codim}(\nabla) = 1$, and is hence a hypersurface. $\square$

We define

$$I_{X/S}(c_{n+1}(E)) = \mathcal{L}(\nabla).$$

The generalized resultant R_E^{n+1} of sections of E is the equation of ∇, note that this is again a polynomial on V canonically taking values in $\mathcal{L}(\nabla)$.

6.1.5. Definition of $I_{X/S}(c_1E_1,\ldots,c_kE_k)$ for E_i of arbitrary rank. Let $E_1,\ldots,E_k$ be very ample vector bundles over X such that the sum of their ranks $r_1,\ldots,r_k$ is greater than $n+1$. Let $l_1,\ldots,l_k$ be a sequence of integers which sum to $n+1$. Let

$$V = H^0(X;E_1)^{r_1-l_1+1}\times\cdots\times H^0(X;E_k)^{r_k-l_k+1} = V_1\times\cdots\times V_k$$

Define $\nabla\subset V$ by

$$\nabla = \left\{ \begin{array}{c} \left(s_{1,1},\ldots,s_{1,r_1-l_1+1},\ldots,s_{k,1},\ldots,s_{k,r_k-l_k+1}\right): \\ \text{The collections } s_i \text{ are linearly} \\ \text{dependent somewhere over } X \end{array} \right\}$$

PROPOSITION 6.1.6. *∇ is an irreducible homogeneous hypersurface in V.*

PROOF. Consider the space $V\times X$ and its subset

$$\overline{\nabla} = \left\{ \begin{array}{c} \left(x, s_{1,1},\ldots,s_{1,r_1-l_1+1},\ldots,s_{k,1},\ldots,s_{k,r_k-l_k+1}\right): \\ \text{The collections } s_i \text{ are all linearly dependent at } x \end{array} \right\}$$

Then ∇ is clearly Zariski closed in $V\times X$ because it is given by the vanishing of various minors. Therefore, we have two naturally defined projections:

$$\nabla \xleftarrow{p_1} \overline{\nabla} \xrightarrow{p_2} X$$

Certainly, p_1 and p_2 are both surjective, which implies that ∇ is a subvariety of V.

Again, $p_2^{-1}(x)$ is irreducible for every x, being the product of the irreducible varieties considered in the last case. Thus, ∇ is an irreducible subvariety of V. Moreover, $\overline{\nabla}$ is homogeneous by the same argument as before.

A generic set of collections of sections will have only one point of X over which they are simultaneously linearly dependent. Therefore, p_1 is generically one to one and is hence a birational isomorphism. Thus $\dim(\nabla) = \dim\left(\overline{\nabla}\right)$.

So, to determine the dimension of ∇, we again calculate the dimension of $\overline{\nabla}$. Let $x \in X$ and let F_x be the corresponding fiber in $V \times X$. Consider again the evaluation map at x:

$$\begin{aligned} \phi_{i,x} : H^0(X, E_i)^{r_i - l_i + 1} &\longrightarrow E_{i,x}^{r_i - l_i + 1} \\ s_{i,1}, \ldots, s_{i,r_i - l_i + 1} &\longmapsto \left(s_1(x), \ldots, s_{r_i - l_i + 1}(x)\right) \end{aligned}$$

Note that ϕ_x is a surjection and that $\phi_{i,x}^{-1}(p)$ has the same dimension for any $p \in E_{i,x}^{r_i - l_i + 1}$. So, if $Q \subset E_{i,x}^{r_i - l_i + 1}$ is a subset of codimension α, then $\phi_{i,x}^{-1}(Q)$ also has codimension α.

As before, we consider $\nabla_{i,x} \subset E_{i,x}^{r_i - l_i + 1}$ defined as the collection of $r_i - l_i + 1$-tuples which are linearly dependent. We know that, since $E_{i,x}$ is a vector space of dimension r_i, that

$$\operatorname{codim}(\nabla_x) = r_i - (r_i - l_i + 1) + 1 = l_i$$

And, so the codimension of $\phi_{i,x}^{-1}(\nabla_{i,x})$ in $V_i \times \{x\}$ is l_i.

Now, if we let

$$\nabla_{I,x} = \prod_{i=1}^{k} \nabla_{i,x}$$

we can see that the codimension of $\nabla_{I,x}$ in $V \times \{x\}$ is $l_1 + \cdots + l_k = n + 1$.

To finish off this proof, we notice that

$$\overline{\nabla} = \coprod_{x \in X} \phi_x^{-1}\left(\nabla_{I,x}\right)$$

and hence $\overline{\nabla}$ has codimension $n+1$. Since $\dim(X) := n$, we see that $\operatorname{codim}(\nabla) = 1$, and is hence a hypersurface. □

As before, we define

$$I_{X/S}\left(c_{l_1}(E_1), \ldots, c_{l_k}(E_k)\right) := \mathcal{L}(\nabla).$$

The generalized resultant

$$R^{l_1,\ldots,l_k}_{E_1,\ldots,E_k}(s),$$

is the equation of ∇, canonically taking values in this space, which defines a function on collections of sections

$$s = \left\{s_{1,1}, \ldots, s_{1,r_1-l_1+1}, \ldots, s_{k,1}, \ldots, s_{k,r_k-l_k+1}\right\} \in V$$

and is zero precisely if the sub-collections are simultaneously linearly dependent somewhere on X.

2. Determinantal representations

Our next task is to find ways to calculate formulas for the equations of ∇ and hence determine the one dimensional vector space

$$I_{X/S}\left(c_{l_1}(E_1), \ldots, c_{l_k}(E_k)\right)$$

in a functorial way. We will show that these spaces can be represented as the relative determinantal spaces of certain complexes of bundles. We continue to work over $S = *$.

6.2.1. The case of a single bundle. Our development here will mimic the proofs in [12] for the Koszul complex. Consider the case of a single bundle E over X of rank r. We wish to prove

THEOREM 6.2.2. *If X is an n-dimensional variety and E is a rank r bundle over X, then there is a natural isomorphism*

(6.1)

$$I_{X/S}(c_{n+1}E) \cong \bigotimes_{i=0}^{n} \det{}_{X/S}\left(\bigwedge^{r-n+i} E^* \otimes \mathcal{M}\right)^{\otimes(-1)^{i+1}\binom{r-n+i}{i}} \otimes \det{}_{X/S}(\mathcal{M}).$$

where $\mathcal{M}$ is any line bundle over X.

We will cover the more general case in section 6.2.5. The expression in that case will involve determinants of tensor products over all E_i of expression like 6.1.

PROOF. This proof will be carried out in three steps.

Step 1: Interpreting the vanishing of the resultant as the non-exactness of a complex of bundles. Let F be the trivial bundle of rank $r-n$. A morphism $\phi : F \longrightarrow E$ is precisely a collection of $r-n$ sections of E. Let $N_\phi^\bullet(F,E) = N_\phi^\bullet(E)$ be the Eagon-Northcott complex. This complex fails to be exact precisely when ϕ is deficient in rank somewhere on X, i.e. $\phi = \{v_1, \ldots, v_n\} \in \nabla$ if and only if $N_\phi^\bullet(E)$ is not exact. Therefore,

$$R_E^{n+1}(\phi) = 0$$

if and only if $N_\phi^\bullet(E)$ fails to be exact.

Since the determinantal *space* doesn't depend on the maps involved in the complex, we will drop the ϕ when considering determinants in the following developments. Also, for the sake of convenience for calculations we will tensor this complex with an ample line bundle $\mathcal{M}$ over X obtaining a complex $N_{\mathcal{M}}^\bullet$. The resulting complex is exact if and only if $N^\bullet$ is.

Step 2: Interpreting the non-exactness of the complex of vector bundles as the non-exactness of a spectral sequence. A spectral sequence is called exact if it converges to 0. Let $I^{\bullet\bullet}$ be an injective resolution for the Eagon-Northcott complex $N_{\phi}^{\bullet}(E)$.

Consider a map $f : Y \longrightarrow Z$ of topological spaces. Let $\mathcal{F}^{\bullet}$ be any complex of sheaves on Y. There are two associated spectral sequences converging to the hyperdirect image $\mathbb{R}^{\bullet}f_{*}\mathcal{F}^{\bullet}$ of $\mathcal{F}^{\bullet}$. Both are derived from the sequence associated to a double complex (the first page of one is simply the first page of the other with indices interchanged)

$$E_1^{i,j} = R^j f_* \mathcal{F}^i \implies \mathbb{R}^{i+j} f_* \mathcal{F}^{\bullet}$$

$$'E_2^{i,j} = R^i f_* \underline{H}^j (\mathcal{F}^{\bullet}) \implies \mathbb{R}^{i+j} f_* \mathcal{F}^{\bullet}$$

where $\underline{H}$ is the sheaf cohomology of the complex. If Z is a point, then the hyperdirect image is just the hypercohomology of Y with coefficients in the complex $\mathcal{F}^{\bullet}$.

In the present case, we take $Y = X$, $Z = S = pt$ and $f = p$ the projection to a point. The hyperdirect image spectral sequences take the form

$$E_1^{i,j} = R^j p_* N_{\mathcal{M}}^i (E) \implies \mathbb{R}^{i+j} p_* (N_{\mathcal{M}}^{\bullet}(E)),$$

$$'E_2^{i,j} = R^j p_* H^i (N_{\mathcal{M}}^{\bullet} E) \implies \mathbb{R}^{i+j} p_* (N_{\mathcal{M}}^{\bullet}(E)).$$

Moreover, this sequence is exact if and only if the $\mathbb{R}^{i+j} p_* (N_{\mathcal{M}}^{\bullet}(E))$ are all 0. The determinant of either of these spectral sequences is

$$\det_{X/S}(N_{\mathcal{M}}^{\bullet}(E)).$$

Step 3: Passing to the determinant of this sequence. To finish, we take the determinant of the complex constructed in step 2 and show that the resulting bundle is in fact $I_{X/S}$. Then the determinant defines a rational function on the space $V = H^0(X, E)^{r-n}$ canonically taking values in the determinantal space of the spectral sequence

$$\begin{aligned} \det_{X/S}(N^{\bullet}_{\mathcal{M}}) &= \det Rp_* N^{\bullet}_{\mathcal{M}} \\ &= \bigotimes_i \left(\det Rp_* N^i_{\mathcal{M}}\right)^{\otimes(-1)^i}, \end{aligned}$$

where $N^i_{\mathcal{M}}$ is the i^{th} term in the $\mathcal{M}$-twisted Eagon-Northcott complex. Note that the above are, since $S = pt$, vector spaces, and in general are line bundles over S.

Let $\Delta_{E,\mathcal{M}}$ denote this function. The proof of the theorem follows from the following lemma.

LEMMA 6.2.3. *$\Delta_{E,\mathcal{M}}$ and R^{n+1}_E vanish on the same locus (namely ∇) with the same multiplicity (namely 1).*

Proof of lemma. In order to analyze R^{n+1}_E and $\Delta_{E,\mathcal{M}}$ as functions, instead of analyzing their effects on specific ϕ, we turn to the universal complex over the symmetric algebra $\mathcal{S} = S^{\bullet}(V^*)$, where $V = H^0(X, E)$:

$$\mathcal{N}^{\bullet}(E) = \left\{ S^n F \otimes \bigwedge^r E^* \otimes \mathcal{S} \longrightarrow \cdots \longrightarrow \bigwedge^{r-n} E^* \otimes \mathcal{S} \longrightarrow \mathcal{S} \right\}. \tag{6.2}$$

The individual complexes considered above are the fibers of this complex. This complex is a special case of the above Eagon-Northcott complex, where all bundles lie over $\mathbb{P}(V)$, and the morphism which defines the complex is the universal morphism σ given by

$$(\phi, w) \xrightarrow{\sigma} (\phi, \phi(w))$$

When we tensor this complex with $\mathcal{M}$ we obtain the complex,

$$\mathcal{N}^{\bullet}_{\mathcal{M}}(E) = \left\{ S^r(F)\mathcal{M}\mathcal{S} \bigwedge^n E^* \longrightarrow \cdots \longrightarrow \mathcal{M}\mathcal{S} \bigwedge^{r-n} E^* \longrightarrow \mathcal{M}\mathcal{S} \right\} \tag{6.3}$$

This complex is exact on the same locus as the complex 6.2 because the sheaves in question are locally free.

Consider the incidence variety $W \subset X \times \mathbb{P}(V)$ of all tuples $(x, f_1, \ldots, f_{r-n})$ such that the vectors

$$\{f_1(x), \ldots, f_{r-n}(x)\}$$

are linearly dependent. (Note that $W = \mathbb{P}\left(\overline{\nabla}\right)$ from section 6.1.3.) Let

$$X \xleftarrow{p_1} W \xrightarrow{p_2} P(V)$$

be the projections. Then $\mathbb{P}(\nabla) = p_2(W)$, and p_1 is surjective. Now, $p_2^{-1}(f_1, \ldots, f_n)$ is the collection of all x over which the sections $f_1, \ldots, f_n$ are linearly dependent. As was mentioned in section 6.1.3, since E is very ample, this is generically a single point. Therefore, p_2 is generically one to one and is hence a birational isomorphism between $\mathbb{P}(\nabla)$ and W. Let p_X and p_V be the projections of $X \times P(V)$ to each factor respectively.

The complex $\mathcal{N}^{\bullet}_{\mathcal{M}}(E)$ is a resolution of the sheaf $\mathcal{O}_W \otimes p_X^* \mathcal{M}$ by theorem 5.3.4 (i.e. $p_X^* \mathcal{M}$ restricted to the incidence variety W.) To see this, notice that σ vanishes precisely on W. But W has codimension $n+1$ in $X \times P(V)$ (W is the projectivization of $\overline{\nabla} \subset X \times V$ which has codimension $n+1$ by the argument in section 6.1.3). Since $\operatorname{rank} F = r - n$, and $\operatorname{rank}(E) = r$, theorem 5.3.4 tells us that since

$$\operatorname{codim} W = \operatorname{rank} E - \operatorname{rank} W + 1 = r - (r - n) + 1 = n + 1,$$

the complex $\mathcal{N}^{\bullet}_{\mathcal{M}}$ is a free resolution of $\mathcal{O}_W$.

From this and the two spectral sequences we see that

$$R^i p_{V*}\left(\mathcal{N}^{\bullet}_{\mathcal{M}}(E) \otimes p_x^* \mathcal{M}\right) = R^i p_{V*}\left(\mathcal{O}_W \otimes p_X^* \mathcal{M}\right).$$

We need to calculate the π-adic order of the determinant of this sequence for any irreducible homogeneous form π on V. But by theorem 30 in appendix A of [12], this order is equal to the alternating sum of the orders of the terms of the limit of the sequence along H_π, the hypersurface associated to π, i.e.

$$ord_\pi\left(R^i p_{V*}\left(\mathcal{O}_W \otimes p_X^* \mathcal{M}\right)\right).$$

Since $p_2 : W \longrightarrow \nabla$ is a birational isomorphism, the only hypersurface along which these sheaves can have support is ∇, and the multiplicity of this support is 1.

To finish the proof of this theorem we need to use the above results to construct an isomorphism between the spaces

$$I_{X/S}(E)_{n+1} \quad \text{and} \quad \det_{X/S}\left(N^\bullet_{\mathcal{M}}(E)\right).$$

Let $v \in I_{X/S}(E)_{n+1}$. Since R_E^{n+1} is surjective there is some $\phi_v \in V$ such that $v = R_E^{n+1}(\phi_v)$. Indeed, any v in the appropriate level hypersurface of ∇ will do. Define

$$\psi(v) = \Delta_{E,M}(\phi_v).$$

By lemma 6.2.3 this map is well defined. In fact it is a linear map of one-dimensional vector spaces which doesn't vanish entirely, and therefore is an isomorphism.

With this isomorphism between $\det(N)$ and $\mathcal{L}(\nabla)$, we see that, by the above discussion, both $\Delta_{E,M}$ and R_E^{n+1} are polynomials vanishing on the same locus with the same multiplicity. Therefore, they must be equal up to a non-zero constant multiple.

The last step of this proof is to demonstrate the form of the right hand side of the equation in this theorem.

$$\begin{aligned}
\Delta_{E,M} &= \det\left(N^\bullet(E, \mathcal{M})\right) \\
&= \det\left(S^n(F) \otimes \bigwedge^r E^* \otimes \mathcal{M} \longrightarrow \cdots \longrightarrow \bigwedge^{r-n} E^* \otimes \mathcal{M} \longrightarrow \mathcal{O}_X \otimes \mathcal{M}\right)
\end{aligned}$$

But, by the definition of the determinant of a complex, this is then

$$= \bigotimes_{i=0}^{n} \det\left(S^i(F) \otimes \bigwedge^{r-n+i} E^* \otimes \mathcal{M}\right)^{\otimes(-1)^{i+1}\binom{r-n+i}{i}}$$

$$= \bigotimes_{i=0}^{n} \det\left(\bigwedge^{r-n+i} E^* \otimes \mathcal{M}\right)^{\otimes(-1)^{i+1}\binom{r-n+i}{i}} \otimes \det\left(\mathcal{O}_X \otimes \mathcal{M}\right)$$

As desired. The last equality holds because F is trivial and so

$$S^i(F) \otimes \bigwedge^{r-n+i} E^* \otimes \mathcal{M} = \bigoplus^{\dim S^i(F)} \left(\bigwedge^{r-n+i} E^* \otimes \mathcal{M}\right)$$

$$= \binom{r-n+i}{i} \left(\bigwedge^{r-n+i} E^* \otimes \mathcal{M}\right)$$

The result follows because the determinant of a direct sum is the tensor product of the determinants. Note that the last term in the sequence is $\bigwedge^{r-n} F^* = \mathcal{O}_X$ since F is a trivial bundle. □

EXAMPLE 6.2.4. Let $\mathcal{M}$ be sufficiently ample that the twisted Eagon-Northcott complex $\mathcal{N}^\bullet_{\mathcal{M}}$ has no higher cohomology. Then the spectral sequence above collapses, and the relative determinant of this sequence is just the determinant of the complex of global sections.

6.2.5. The construction for several bundles For the general case of finding a determinantal representation of $I_{X/S}\left(c_{l_1}E_1, \ldots, c_{l_k}E_k\right)$, we must find a resolution for the intersection of the loci where l_i generic sections of E_i fail to be linearly independent. We accomplish this by taking the tensor products of the Eagon-Northcott complexes associated to each of the E_i, as described above. For notations sake here and in the rest of this paper we will define, for a bundle E of rank r over

X

$$C_m^\bullet(E) = \left\{ \cdots \longrightarrow \left(\bigwedge^{r-m+1+i} E^* \right)^{\oplus \binom{r-m+1+i}{i}} \longrightarrow \left(\bigwedge^{r-m+i} E^* \right)^{\oplus \binom{r-m+i}{i-1}} \longrightarrow \cdots \right\}$$

the Eagon-Northcott complex obtained from $V = E$ and $F = \mathcal{O}_X^m$. Conveniently, we can rewrite the natural isomorphism in theorem 6.2.2 as

$$I_{X/S}(c_{n+1}E) \cong \det_{X/S}\left(C_{n+1}^\bullet E\right).$$

In a similar proof to that in the last section we can prove, for the more general case

THEOREM 6.2.6. *Let $E_1, \ldots, E_k$ be vector bundles of rank $r_1, \ldots, r_k$ over $X \longrightarrow S$, a relative variety of relative dimension n. Let $l_1, \ldots, l_k$ be positive integers which sum to $n+1$. Then there is a natural isomorphism*

$$I_{X/S}\left(c_{l_1}E_1, \ldots, c_{l_k}E_k\right) \cong \det_{X/S}\left(C_{l_1}^\bullet(E_1) \otimes \cdots \otimes C_{l_k}^\bullet(E_k)\right).$$

Note that $C_{l_i}^\bullet(E_i)$ is a resolution of the locus where $r_i - l_i + 1$ generic sections of E_i are simultaneously linearly dependent. Therefore the tensor product of these complexes is a resolution of the intersection of these loci, which is precisely the locus needed in the construction of $I_{X/S}\left(c_{l_1}E_1, \ldots, c_{l_k}E_k\right)$.

3. The degree of the generalized resultant

Since we have an interpretation of these resultants as determinants, the following is a direct consequence of Corollary 15 from appendix A of [12]. There it is shown that the degree of the determinant of a complex is a weighted sum of the dimensions of the terms involved. Recall that if S is a point then $\det_{X/S}$ is just $\det H^\bullet$. Therefore we have

LEMMA 6.3.1. *The degree of the $(E, n+1)$-resultant R_E^{n+1} is*

$$\begin{aligned}
\deg(R_{E,n+1}) &= \sum_{i,j}(-1)^{i+j+1}\cdot i\cdot \dim H^j\left(\left(\bigwedge^{r-n+i} E^*\otimes\mathcal{M}\right)^{\oplus\binom{r-n+i}{i}}\right)+\\
&\quad +\sum_j(-1)^j\dim H^j(\mathcal{M})\\
&= \sum_{i=0}^{n}(-1)^{i+1}\cdot i\sum_j(-1)^j\dim H^j\left(\left(\bigwedge^{r-n+i} E^*\otimes\mathcal{M}\right)^{\oplus\binom{r-n+i}{i}}\right)+\\
&\quad +\chi(\mathcal{M})\\
&= \sum_{i=1}^{n}(-1)^{i+1}\cdot i\chi\left(\left(\bigwedge^{r-n+i} E^*\otimes\mathcal{M}\right)^{\oplus\binom{r-n+i}{i}}\right)+\chi(\mathcal{M})
\end{aligned}$$

That is, the degree is just a weighted sum of the Euler characteristics of the terms in the Eagon-Northcott complex.

If $\mathcal{M}$ is sufficiently ample so that the terms in the Eagon-Northcott resolution have no higher cohomology, then

$$\sum_{i=1}^{n}(-1)^{i+1}\cdot i\cdot\dim H^0\left(\left(\bigwedge^{r-n+i} E^*\otimes\mathcal{M}\right)^{\oplus\binom{r-n+i}{i}}\right)+\dim H^0(\mathcal{M})$$

A similar result holds for the more general case. The degree of the resultant is a weighted alternating sum of the Euler characteristics of the terms in the associated resolution.

4. Arbitrary base spaces

Suppose now that $p : X \longrightarrow S$ is a smooth projective morphism of smooth varieties of relative dimension n. Let $E_1, \ldots, E_k$ be relatively very ample bundles on X of rank $l_1, \ldots, l_k$ respectively. Suppose that $\sum l_i = n+1$. We wish to construct a resultant bundle

$$I_{X/S}(c_{l_1}E_1, \ldots, c_{l_k}E_k)$$

over S with the property that

$$c_1\left(I_{X/S}(c_{l_1}E_1, \ldots, c_{l_k}E_k)\right) = \int_{X/S} c_{l_1}E_1 \cdots c_{l_k}E_k.$$

We consider the case of a single bundle E of rank r over X. The extension to the case of several bundles is a completely analogous construction. We perform the above construction, for the case of X over a point, for every fiber of the projection p. For any $y \in S$, let E_y be the fiber of p over y. Over every point of $y \in S$ we have a hypersurface in $V_y = H^0(E_y)$ given by $E \longrightarrow S$ over y. Also, since $p : X \longrightarrow S$ is smooth, and all sheaves in the complex are locally free, the Euler characteristics in the degree formula are constant over S. Therefore, the degree of the $(E_y, n+1)$-resultant is itself constant over S.

By proposition 4.5.1 we can therefore construct a vector bundle L over S, whose fibers are the resultant spaces constructed above. Let $s_1, \ldots, s_{r-n}$ be a tuple of sections of E. If we apply the E-resultant to the tuple $\{s_i|_{p^{-1}(y)}\}$ for every $y \in S$, then we get an element of the fiber of this new bundle at y. This defines a section $R_E(s)$ of $I_{X/S}(c_{n+1}E)$ which is zero at any point y such that the tuple $\{s_i\}$ is linearly dependent somewhere over the fiber $p^{-1}(y)$. It follows from 4.5.1 that this is actually a regular section of L. This allows us to prove

Lemma 6.4.1.

$$c_1 I_{X/S}\left(c_{l_1}E_1, \ldots, c_{l_k}E_k\right) = \int_{X/S} c_{l_1}(E_1) \cdots c_{l_k}(E_k).$$

PROOF. Consider the Poincaré duals to the Chern classes in question:

$$c = \delta^*(c_{l_1}E_1 \cdots c_{l_k}E_k) = \left\{ \begin{array}{c} \left[\left\{s_{1,1}, \ldots, s_{k,r_k-l_k+1}\right\}\right] \\ s_{i,j} \text{ is a generic section of } E_i. \\ \text{the } s_{i,j} \text{ are simultaneously lin. dep.} \end{array} \right\}$$

$$\gamma_2 = \delta^*\left(c_1 I_{X/S}\left(c_{l_1}E_k \cdots c_{l_k}E_k\right)\right) = \left\{ \begin{array}{c} [\{t=0\}] : t \text{ is a generic section} \\ \text{of } I_{X/S}\left(c_{l_1}E_k \cdots c_{l_k}E_k\right). \end{array} \right\}$$

Where [ ] denotes the cohomology class of a locus. We need to show that $\gamma_2 = c$.

By the definition of $\int_{X/S}$, c is the Poincaré dual to the projection of the set where generic collections of sections of $E_1, \ldots, E_k$ are simultaneously linearly dependent. Such collections define a section $R_{X/S}(s)$ of $I_{X/S}\left(c_{l_1}E_k \cdots c_{l_k}E_k\right)$, but $R_{X/S}(s)$ vanishes whenever the collections are simultaneously linearly dependent somewhere in the fiber. So, the line bundle defined by the class c has sections which generically vanish in loci which are homologous to the zero loci of the polynomials $R_{X/S}(s)$. □

To complete this discussion, we need to extend the determinantal formulas obtained above to the relative case. In fact, we have a more general isomorphism between the resultant bundles and the determinantal bundles of the Eagon-Northcott complexes.

THEOREM 6.4.2. *If $X \longrightarrow S$ is a smooth projective morphism of smooth varieties of relative dimension n, and $E_1, \ldots, E_k$ are relatively ample bundles on X of rank $l_1, \ldots, l_k$ with $\sum l_i = n+1$, then there is a natural isomorphism*

$$I_{X/S}\left(c_{l_1}E_1, \ldots, c_{l_k}E_k\right) \cong \det_{X/S}\left(C^\bullet_{l_1}(E_1) \otimes \cdots \otimes C^\bullet_{l_k}(E_k)\right).$$

PROOF. Consider the fibers of

$$\det_{X/S}\left(C^\bullet_{l_1}(E_1) \otimes \cdots \otimes C^\bullet_{l_k}(E_k)\right).$$

Let $y \in S$. Then the fiber of this bundle over y is

$$\det_{X_y/y}\left(C^{\bullet}_{l_1}(E_1) \otimes \cdots \otimes C^{\bullet}_{l_k}(E_k)\right)$$

where the bundles are restricted to the fiber X_y. The proof of theorem 6.4 extends more or less word for word to this relative case. Recall that the fibers of $\det_{X/S}$ are just the determinants of the cohomology of the fibers. So the fibers here are just the spaces we encountered in the previous proof. [ ]

CHAPTER 7
Applications, formulas and comparisons

1. Cube theorems

In this section, we consider some generalizations of the classical theorem of the cube.

7.1.1. The classical theorem. Two good sources for information on the ideas in this section are [4] and [5]. Let G be an abelian variety defined over a field k. Let L be a line bundle on G. Consider the variety $G^3 = G \times G \times G$, and let

$$\begin{aligned} m_{123}(g_1, g_2, g_3) &= g_1 + g_2 + g_3 \\ m_{ij}(g_1, g_2, g_3) &= g_i + g_j \\ p_i(g_1, g_2, g_3) &= g_i \end{aligned}$$

be the maps $G \times G \times G \longrightarrow G$ given by addition and projection on G^3. Then the classical cube theorem states that the "second difference" bundle

$$\Theta(L) = \frac{m_{123}^* L \otimes p_1^* L \otimes p_2^* L \otimes p_3^* L}{m_{12}^* L \otimes m_{13}^* L \otimes m_{23}^* L}$$

on G^3 is trivial.

7.1.2. Determinantal bundles. Let X be an algebraic variety and let $P = \mathrm{Pic}(X)$ be the Picard group of X, i.e. the variety consisting of line bundles over

X. This is a disconnected variety whose components consist of line bundles with a common degree (first Chern class). The degree zero component is an abelian variety. Then $\det_{X/*} = \det(H^0)$, is a functor from line bundles on X to one dimensional vector spaces. Therefore $\det_{X/*}$ defines a line bundle on P. For simplicity of notation, we will let det be $\det(H^\bullet)$ for the remainder of this section. Moreover, for convenience in this and the following sections, we will omit the tensor product, $\otimes$, in the formulas, so for bundles L and M

$$LM = L \otimes M.$$

Consider the Poincaré line bundle L_p on $X \times P$. This is the line bundle whose restriction to $X \times \zeta$ is just the line bundle ζ. Consider $\det_{X\times P/P}(L_p)$. Its fiber over any $\zeta \in P$ is $\det_{X/*}(\zeta) = \det H^\bullet(x, \zeta)$. The cube theorem for the determinantal bundle is

THEOREM 7.1.3. *Let X be an n dimensional variety, and let $L_1, \ldots, L_{n+2} \in P$ be very ample line bundles on X. Then the line bundle*

$$\det(K^\bullet(L_1, \ldots, L_{n+2})) = \det \mathcal{O}_X \otimes \left(\bigotimes_{j=1}^{n+2} \left(\bigotimes_{i_1 < \cdots < i_j} \det(L_{i_1} \cdots L_{i_j}) \right)^{\otimes(-1)^j} \right)$$

on $P^{n+2} = P \times \cdots \times P$ is a trivial bundle.

PROOF. Consider the Koszul complex $K^\bullet(L_1, \ldots, L_{n+2})$ associated to the L_i. The determinant of this complex defines a rational function from $V = \prod H^0(X, L_i)$ to $\det(K^\bullet)$. This complex fails to be exact only on a subvariety of V of codimension greater than two. Therefore, the function det is a rational function on V which is nonzero only on a locus of codimension greater than one. Since a rational function can only be non-zero on a locus of codimension one, this function must be non-zero everywhere. Thus det is a constant function, and therefore canonically trivializes $\det(K^\bullet)$. □

This theorem is mentioned in [4].

Notice that, while the classical cube theorem holds for any line bundle L on an arbitrary abelian variety, this theorem concerns a very specific bundle on the determinantal bundle of a very specific variety, the picard group of X.

7.1.4. Cube structures arising from Eagon-Northcott complexes. Consider the category $\mathcal{B}^{\text{amp}}$ whose objects are very ample vector bundles of all ranks over X, and whose morphisms are isomorphisms. On this category we have several operations (bifunctors), namely $\oplus, \otimes$, and $\bigwedge$. Let $\mathcal{B}_r^{\text{amp}}$ be the sub-category of $\mathcal{B}^{\text{amp}}$ consisting of bundles over X of rank r.

We restrict our view once more to the case where S is a point. In this case, $\det_{X/*}$ determines a functor $\mathcal{B}_r^{\text{amp}} \longrightarrow \text{Vect}$, the category of one dimensional vector spaces over $\mathbb{C}$. We can use the above methods to obtain relations among these functors.

These relations are obtained by considering a locus of codimension higher than 1 in the spaces of tuples of sections. We will explain this first for the case of a single vector bundle E of rank r. The case of several bundles is similar and is covered later in this section. Let X be a variety of dimension n. Consider $V = H^0(X, E)^{n-r-k}$ with $k > 0$. Then the collection of $(n-r-k)$-tuples which are linearly dependent somewhere over X form an algebraic subvariety of V of codimension $k+1$ (by a dimension count similar to that above.) When turning to the determinant of the cohomology we consider here $C^\bullet_{n+k+1}(E)$.

THEOREM 7.1.5. *If* $\dim X = n$ *and* $k > 0$, *then we have a natural identification* *of functors on* $\mathcal{B}_r^{\mathrm{amp}}$

$$\det_{X/*}\left(C^{\bullet}_{n+k+1}E\right) \cong \mathbb{C} \quad \textit{(The trivial functor).}$$

PROOF. Consider that the calculations carried out for the determinant of the cohomology can be carried out in this situation. When we reach the final step, though, we see that all of the cohomology sheaves of the Eagon-Northcott complex are supported on varieties of codimension greater than or equal to 2. The determinant of this complex defines a rational map from V to $\det_{X/*}(C^{\bullet}_{n+k+1}(E))$. This map is non-zero where the complex is exact. But this happens on a locus of codimension greater than 1. Therefore this function must be constant, and thus the determinant is canonically trivialized. □

This discussion can easily be extended to the case of several line bundles,

$$C^{\bullet}_{l_1}(E_1) \otimes \cdots \otimes C^{\bullet}_{l_m}(E_m),$$

if $l_1 + \cdots + l_m > n+1$. We obtain more generally

THEOREM 7.1.6. *If* $\dim X = n$ *and* $l_1 + \cdots + l_m > n+1$, *then, for integers* $r_1, \ldots, r_m > 0$ *there is a natural identification of functors on* $\mathcal{B}_{r_1}^{\mathrm{amp}} \times \cdots \times \mathcal{B}_{rm}^{\mathrm{amp}}$

$$\det_{X/*}\left(C^{\bullet}_{l_1}E_1 \otimes \cdots \otimes C^{\bullet}_{l_m}E_m\right) \cong \mathbb{C}.$$

We now consider what happens in the case where the collection of bundles can be knit together to form a variety. Let $\mathcal{B}_{r,s}^{\mathrm{amp}}$ be the category of stable bundles of rank r. Recall that the collection of stable bundles of rank r over X actually form an algebraic variety M_r^{amp} (the moduli space of stable bundles of rank r.)

For each bundle over X, the relative determinant $\det_{X/*}E$ defines a one dimensional vector space, and therefore a line bundle on M_r^{amp}. Likewise, $\det_{X/*}\left(\bigwedge^i E\right)$

defines a line bundle on M_r^{amp}. The above results actually give us relations between these bundles on M_r^{amp}.

THEOREM 7.1.7. *If* $\dim X = n$ *and* $k > 0$, *then we have a natural trivialization*

$$\det_{X/*}\left(C^{\bullet}_{n+k+1}\right) \cong \mathcal{O}_{M_r^{\mathrm{amp}}} \quad \textit{(The trivial bundle on } M_r^{\mathrm{amp}}\textit{)}.$$

A similar theorem holds for the case of several bundles.

2. Properties of generalized resultants

Here, we explore some of the main properties of these bundles. First we will work with some relations with respect to tensor products.

PROPOSITION 7.2.1. *Let*

$$0 \longrightarrow F \xrightarrow{\gamma} E \xrightarrow{\delta} F' \dashrightarrow 0 \tag{7.1}$$

be a short exact sequence of vector bundles on a relative variety $X \longrightarrow S$ *of relative dimension* n *with* E *of rank* r. *Then*

$$I_{X/S}(E)_{n+1} \cong \bigotimes_{\alpha+\beta=n+1} I_{X/S}\left(c_\alpha F, c_\beta F'\right). \tag{7.2}$$

PROOF. We will use the determinant of the cohomology, and the fact that this determinant is multiplicative on filtrations of bundles. The main fact used here is the following lemma

LEMMA 7.2.2. *For a short exact sequence 7.1, there is a filtration of* $\bigwedge^i E$ *whose* j^{th} *quotient is*

$$\bigwedge^{i-j} F' \otimes \bigwedge^{j} F.$$

PROOF. (of lemma) Consider, for every j, the map

$$\bigwedge^{i-j} F \otimes \bigwedge^{j} E \xrightarrow{\psi_j} \bigwedge^{i} E.$$

We set

$$\phi_j = \operatorname{im}(\psi_j).$$

It is fairly easy then to see that

$$\frac{\phi_j}{\phi_{j-1}} = \frac{\operatorname{im}\left(\bigwedge^{i-j} F \otimes \bigwedge^{j} E\right)}{\operatorname{im}\left(\bigwedge^{i-j+1} F \otimes \bigwedge^{j-1} E\right)} \cong \bigwedge^{i-j} F \otimes \bigwedge^{j} F'.$$

□

Given this lemma, the rest of the proof amounts to applying this to the left hand side of equation 7.2 and then re-ordering the terms □

In the following propositions and proofs we will let det $=$ $\det_{X/S}$ for the sake of convenience. Also we will let $\mathcal{O} = \mathcal{O}_X$ be the structure sheaf (the trivial line bundle) over X.

PROPOSITION 7.2.3. *Let L and M be line bundles over $X \longrightarrow S$ of relative dimension n, and let E be a rank n bundle over X. Then*

$$I_{X/S}(L \otimes M, E) = I_{X/S}(L, E) \otimes I_{X/S}(M, E)$$

PROOF. Using the determinantal formulas we obtain

$$\begin{aligned}
I_{X/S}(LM, E) &= \det \mathcal{O} \bigotimes_{i=1}^{n+1} \bigwedge^{i} (LM \oplus E)^{\otimes(-1)^i} \\
&= \det \mathcal{O} \bigotimes_{i=1}^{n+1} \left(\det\left(LM \bigwedge^{i-1} E\right) \det\left(\bigwedge^{i} E\right)\right)^{\otimes(-1)^i} \\
&= \det \mathcal{O} \bigotimes_{i=1}^{n} \left(\det\left(LM \bigwedge^{i-1} E\right) \det\left(\bigwedge^{i} E\right)\right)^{\otimes(-1)^i} \otimes \\
&\quad \otimes \det\left(LM \bigwedge^{n} E\right)^{\otimes(-1)^{n+1}}.
\end{aligned}$$

But by the cube theorem for $C^\bullet_{n+2}(L, M, E)$ we have

$$\begin{aligned}
\mathbb{C} &\cong \det \mathcal{O} \bigotimes_{j=1}^{n+2} \bigwedge^{j} (L \oplus M \oplus E)^{\otimes(-1)^j} \\
&= \det \mathcal{O} \bigotimes_{j=1}^{n+2} \left(\det\left(\bigwedge^{j} E\right) \det\left(L \bigwedge^{j-1} E\right) \det\left(M \bigwedge^{j-1} E\right) \det\left(LM \bigwedge^{j-2} E\right)\right)^{\otimes(-1)^j} \\
&= \det \mathcal{O} \bigotimes_{j=1}^{n+1} \left(\det\left(\bigwedge^{j} E\right) \det\left(L \bigwedge^{j-1} E\right) \det\left(M \bigwedge^{j-1} E\right) \det\left(LM \bigwedge^{j-2} E\right)\right)^{\otimes(-1)^j} \otimes \\
&\quad \otimes \det\left(LM \bigwedge^{n} E\right)^{\otimes(-1)^{n+2}}
\end{aligned}$$

So

$$\begin{aligned}
&\det\left(LM \bigwedge^{n} E\right)^{\otimes(-1)^{n+1}} \cong \\
&\det \mathcal{O} \bigotimes_{j=1}^{n+1} \left(\det\left(\bigwedge^{j} E\right) \det\left(L \bigwedge^{j-1} E\right) \det\left(M \bigwedge^{j-1} E\right) \det\left(LM \bigwedge^{j-2} E\right)\right)^{\otimes(-1)^j} .
\end{aligned}$$

Substituting this into the above expression for $I_{X/S}(LM, E)$ yields

$$\begin{aligned}
I_{X/S}(LM, E) &\cong \det^2 \mathcal{O} \bigotimes_{i=1}^{n} \left(\det\left(\bigwedge^{i} E\right) \det\left(LM \bigwedge^{i-1} E\right)\right)^{\otimes(-1)^i} \otimes \\
&\quad \otimes \bigotimes_{j=1}^{n+1} \left(\det\left(\bigwedge^{j} E\right) \det\left(L \bigwedge^{j-1} E\right) \det\left(M \bigwedge^{j-1} E\right) \det\left(LM \bigwedge^{j-2} E\right)\right)^{\otimes(-1)^j} \\
&= \det^2 \mathcal{O} \bigotimes_{i=1}^{n} \det^2\left(\bigwedge^{i} E\right)^{\otimes(-1)^i} \det\left(L \bigwedge^{i-1} E\right)^{\otimes(-1)^i} \det\left(M \bigwedge^{i-1} E\right)^{\otimes(-1)^i} \otimes \\
&\quad \otimes \det\left(LM \bigwedge^{i-1} E\right)^{\otimes(-1)^i} \det\left(LM \bigwedge^{i-1} E\right)^{\otimes(-1)^{i+1}} \otimes \\
&\quad \otimes \det\left(L \bigwedge^{n} E\right)^{\otimes(-1)^{n+1}} \det\left(M \bigwedge^{n} E\right)^{\otimes(-1)^{n+1}} .
\end{aligned}$$

Obviously the two terms on the fourth line of this calculation cancel to yield

$$\begin{aligned} I_{X/S}(LM,E) &\cong \det\mathcal{O}\bigotimes_{i=1}^{n+1}\left(\left(\bigwedge^{i}E\right)^{\otimes(-1)^i}\det\left(L\bigwedge^{i-1}E\right)^{\otimes(-1)^i}\right)\otimes \\ &\quad\otimes\det\mathcal{O}\bigotimes_{i=1}^{n+1}\left(\left(\bigwedge^{i}E\right)^{\otimes(-1)}\det\left(M\bigwedge^{i-1}E\right)^{\otimes(-1)^i}\right) \\ &= I_{X/S}(L,E)I_{X/S}(M,E) \end{aligned}$$

Completing the proof of this proposition. □

PROPOSITION 7.2.4. *Let L and M be line bundles over a relative surface $X \longrightarrow S$, and let E be a rank 2 vector bundle over X. Then*

$$\begin{aligned} I_{X/S}(L,E\otimes M) &= I_{X/S}(L,E)\otimes I_{X/S}(L,M,M)\otimes I_{X/S}(L,\det E,M) \\ I_{X/S}(L,E\otimes M) &= I_{X/S}(L,E)\otimes I_{X/S}(L,M,M)\otimes I_{X/S}(c_1L,c_1E,c_1M). \end{aligned}$$

PROOF. The second of these equations follows from the the first and the fact that $c_1\left(\bigwedge^2 E\right) = c_1E$.

We prove the first in a similar manner to the last proposition. If we write all of these spaces in terms of the determinant of the cohomology, we get

$$\begin{aligned} \frac{\det\mathcal{O}\det\left(\bigwedge^2 L\oplus EM\right)}{\det L\det EM\det\left(\bigwedge^3 L\oplus EM\right)} &\cong \frac{\det\mathcal{O}\det\left(\bigwedge^2 L\oplus E\right)}{\det L\det E\det\left(\bigwedge^3 L\oplus E\right)}\bullet \\ &\bullet\frac{\det\mathcal{O}\det(LM)^2\det\left(M^2\right)}{\det L(\det M)^2\det\left(LM^2\right)}\bullet\frac{\det\mathcal{O}\det(LM)\det\left(L\bigwedge^2 E\right)\det\left(M\bigwedge^2 E\right)}{\det L\det M\det\left(\bigwedge^2 E\right)\det(LM\bigwedge^2 E)}. \end{aligned}$$

We can rewrite the terms in this equation to obtain

$$\begin{aligned} &\frac{\det(LEM)\det\left(M^2\bigwedge^2 E\right)}{\det(EM)\det\left(LM^2\bigwedge^2 E\right)}\cong \\ &\frac{(\det\mathcal{O})^2\det\left(\bigwedge^2 E\right)\det(LE)(\det(LM))^2\det M^2}{(\det L)^2(\det M)^3\det E\det\left(L\bigwedge^2 E\right)\det\left(LM^2\right)}\bullet\frac{\det LM)\det\left(L\bigwedge^2 E\right)\det\left(M\bigwedge^2 E\right)}{\det\left(\bigwedge^2 E\right)\det\left(LM\bigwedge^2 E\right)} \end{aligned}$$

Notice that between the first and second lines we cancelled a $\det \mathcal{O}$. If we rewrite $\det\left(LM^2 \bigwedge^2 E\right)$ using the cube theorem for $I_{X/S}\left(C_1^{\bullet}\left(L, M, M. \bigwedge^2 E\right)\right)$, we obtain the following expression for the right hand side

$$\frac{\det(LEM)\det\left(M^2 \bigwedge^2 E\right)\det\mathcal{O}\det(LM)\det\left(L\bigwedge^2 E\right)\det(LE)\det(LM)}{\det(EM)\det L\,(\det M)^2\det\left(\bigwedge^2 E\right)\det\left(LM^2\right)\det\left(LM\bigwedge^2 E\right)\det\left(M^2\bigwedge^2 E\right)}$$

Now, cancelling terms in these expressions yields

$$\frac{\det(LEM)\det\left(M^2\bigwedge^2 E\right)}{\det(EM)\det\left(LM\bigwedge^2 E\right)} \cong \frac{\det\mathcal{O}\det\left(\bigwedge^2 E\right)\det(LE)\det(LM)}{\det L\det M\det E\det\left(L\bigwedge^2 E\right)}$$

or

$$\det(LEM) \cong \frac{\det\mathcal{O}\det(LE)\det(LM)\det(EM)\det\left(\bigwedge^2 E\right)\det\left(LM\bigwedge^2 E\right)}{\det L\det M\det E\det\left(L\bigwedge^2 E\right)\det\left(M\bigwedge^2 E\right)}$$

which is true by the triangle inequality for $I_{X/S}\left(C_4^{\bullet}(L, E, M)\right)$. □

3. Comparison with other constructions

In this section we will compare the above construction of $I_{X/S}$ to the constructions given by both Deligne and Elkik. In some cases we will be able to show that the constructions are naturally isomorphic. However, in the general case, more study is needed for the isomorphism between the present spaces and Elkik's.

Suppose E and F are line bundles over a curve C. Then $I_{X/S}(c_1(E)\,c_1(F)) =: [E, F] = [E \oplus F]$ is the vector space in which the resultants of sections of $E \oplus F$ lie. Consider, though, that a section of $E \oplus F$ is merely a section e of E and a section f of F. Moreover s is zero precisely where e and f have a common zero. Therefore, $[E, F]$ is the one dimensional vector space where the resultants of sections of E and F lie.

We wish to show that this is isomorphic to Deligne's space $\langle E, F\rangle$. We have two regular maps

$$[E,F] \xleftarrow{R} V \xrightarrow{\phi} \langle E, F\rangle$$

where ϕ is given by evaluating a section of E on the roots of a section of F. (This is well defined by Weil reciprocity) The point is, they are both given by polynomials that vanish on the same locus with the same multiplicity. So, as before, we can use them to define a natural isomorphism between $[E,F]$ and $\langle E,F\rangle$.

If we construct the desired isomorphism in a different way, we will be more readily able to extend the ideas to the case of Elkik's work. Consider the following commutative diagram.

$$\begin{array}{ccccc}
 & & & & 0 \\
 & & & & \downarrow \\
\mathcal{O} & \xrightarrow{e} & E & \xrightarrow{\psi} & \bigoplus_{x\in \text{div} e} E_x \\
{\scriptstyle f}\downarrow & & \downarrow{\scriptstyle 1_E \otimes f} & & \downarrow \\
F & \xrightarrow[e\otimes 1_F]{} & E\otimes F & \xrightarrow[\psi']{} & \bigoplus_{x\in \text{div} e} F_x \otimes E_x \\
 & & & & \downarrow \\
 & & & & 0
\end{array}$$

The right-hand column is formed by taking the cokernels of the horizontal maps in the left-hand square. Note that the maps ϕ and ϕ' are quasi-isomorphisms of complexes. Therefore their determinants are naturally isomorphic.

The left-hand square is simply the Koszul complex for L and M written as a square instead of a complex, therefore, its determinant is $[E,F]$. Moreover, the determinant of the right hand side is

$$\frac{\det\left(\bigoplus_{x\in \text{div}\, e} E_x \otimes F_x\right)}{\det\left(\bigoplus_{x\in \text{div}\, e} E_x\right)} = \bigotimes_{x\in \text{div}\, e} \frac{E_x \otimes F_x}{E_x} = \bigotimes_{x\in \text{div}\, e} F_x = \langle E,F\rangle\, .$$

Therefore, these two spaces are naturally isomorphic.

7.3.1. Elkik's Construction for line bundles. Next, we turn to Elkik's construction of $I_{X/S}$. In the case of line bundles, she makes the following definition. Let $L_1, \ldots, L_r$ be line bundles over a relative variety $X \xrightarrow{p} S$ of relative dimension n. Let $l_1, \ldots, l_r$ be sections of these bundles respectively. Let Z be the set of common zeros of $l_2, \ldots, l_r$. Then

$$I_{X/S}(L_1, \ldots, L_r) = N_{X/S}(L_1|_Z).$$

Over each fiber of $s \in S$, let $Z_s = Z|_{p^{-1}(s)}$, then the fiber of $I_{X/S}(L_1, \ldots, L_r)$ is

$$N_{X/S}(L_1|_Z)_x = \bigotimes_{x \in Z_s} (L_1)_x. \tag{7.1}$$

We will use a diagram similar to the one used in the last section In that diagram, the top row of the left hand square is the Koszul complex of E, and the bottom row is this complex tensored with F. Consider the diagram.

$$\begin{array}{ccc}
 & & 0 \\
 & & \downarrow \\
K^\bullet(L_2, \ldots, L_r) & \longrightarrow & \bigoplus_{x \in Z}(L_2 \otimes \cdots \otimes L_r)_x \\
\downarrow & & \downarrow \\
L_1 \otimes K^\bullet(L_2, \ldots, L_r) & \longrightarrow & \bigoplus_{x \in Z}(L_1 \otimes \cdots \otimes L_r)_x \\
 & & \downarrow \\
 & & 0
\end{array}$$

The horizontal maps are again quasi isomorphisms of complexes, and so the determinant of the left hand column (which is our $I_{X/S}$) is isomorphic to the determinant of the right hand column (which is Elkik's $I_{X/S}$).

7.3.2. Elkik's General Construction. At the moment we are unable to prove that our general construction is isomorphic to Elkik's. This section contains some conjectures along this line though, and raises some questions about relationships between certain complexes.

Having constructed $I_{X/S}(c_1L_1, \ldots, c_1L_{n+1})$ for line bundles over X, she proceeds as follows. We will describe her construction for the case of a single vector bundle. So, let E be a bundle of rank r over $X \longrightarrow S$ of relative dimension n.

Elkik uses the Segre classes of E to extend the line bundle definition. The total Segre class $\sum s_i$ is defined as the inverse, in the cohomology ring of X, of the total Chern class $\sum c_i$, i.e.

$$\left(\sum_{i=0}^{r} s_i\right)\left(\sum_{j=0}^{r} c_i\right) = 1$$

Using this formula, we can recursively determine the classes s_i uniquely as homogeneous polynomials in the Chern classes (considering s_i to have degree i). In the same way, we can determine the Chern classes as homogeneous polynomials in the Segre classes.

The procedures is to define $I_{X/S}\left(s_{l_1}E \cdots s_{l_k}E\right)$ and then to use the description of $c_{n+1}E$ as a polynomial in the Segre classes to define $I_{X/S}(c_{n+1}E)$. To define $I_{X/S}(s_{n+1}E)$, she uses the fact that

$$s_{n+1}E = \int_{P(E)/X} c_1\left(\mathcal{O}_{P(E)}(1)^{\oplus(r+n)}\right),$$

and so

$$\int_{X/S} s_{n+1}E = \int_{P(E)/S} c_1\left(\mathcal{O}_{P(E)}(1)^{\oplus(r+n)}\right).$$

We can use the developments in the above section on the determinant of the cohomology to interpret this bundle as the determinant of a complex. Namely, consider the Koszul complex associated to $c_1(\mathcal{O}(1))^{\oplus(r+n)}$,

$$\begin{aligned} 0 &\longrightarrow \bigwedge^{r+n} \mathcal{O}(1)^{r+n} \longrightarrow \cdots \longrightarrow \mathcal{O}(1)^{r+n} \longrightarrow \mathcal{C}_{P(E)} \longrightarrow 0 \\ &= \quad 0 \longrightarrow \mathcal{O}(r+n) \longrightarrow \mathcal{O}(r+n-1)^{\oplus\binom{r+n}{r+n-1}} \longrightarrow \cdots \longrightarrow \mathcal{O}_{P(E)} \longrightarrow 0 \end{aligned}$$

Let $p : P(E) \longrightarrow X$ be the projection. When we pull this complex down to X via p, and use the fact that $p_*O(k) = S^k E^*$, we obtain the complex

$$S^\bullet_{n+1}E = \left\{0 \longrightarrow S^{r+n}E^* \longrightarrow \left(S^{r+n-1}E^*\right)^{\oplus\binom{r+r}{r+n-1}} \longrightarrow \cdots \longrightarrow \mathcal{O}_X \longrightarrow 0\right\}.$$

From this we then obtain that

$$I_{X/S}(s_{n+1}E) \cong \det\left(S^\bullet_{n+1}E\right).$$

It is hoped that there are relationships between the determinants of the complexes $S^\bullet_i E$ and $C^\bullet_j E$ which will allow us to find the necessary isomorphism between these two constructions. In particular, we are hoping that it may be possible to demonstrate an isomorphism analogous to the relationship between Segre classes and Chern classes, namely we know that

$$\left(\sum_{i=0}^{r} s_i\right)\left(\sum_{j=0}^{r} c_i\right) = 1$$

and we are hopeful that there is an isomorphism

$$\bigotimes_{i=0}^{r} \det\left(S^\bullet_i \otimes C^\bullet_{n-i}\right) \cong \mathbb{C}.$$

4. Polarization formulas for discriminants and resultants

It is interesting to note that the above constructions can be used in determining the structure of various discriminantal bundles as well. We recall the connection between the classical discriminant and resultants for polynomials of one variable.

Let $f(x)$ and $g(x)$ be two polynomials of a single variable. Let $R(f,g)$ and $\Delta(f)$ denote the resultant and discriminant respectively. Then

$$R(f,g)^2 = \frac{\Delta(fg)}{\Delta(f)\Delta(g)}.$$

We wish to generalize this relationship between discriminants and resultants.

In [12] Gelfand, Kapranov, and Zelevinski demonstrate that

$$\Delta(L) = \det\left(K^{\bullet}\left(J\left(L\right)\right)\right),$$

where $J(L)$ is the jet bundle of L, and $K^{\bullet}$ is the ordinary Koszul complex. From this we see that

$$\Delta(L) = I_{X/S}\left(c_{n+1}\left(J\right)\right).$$

By proposition 7.2.1 and the fact that there is a short exact sequence

$$0 \longrightarrow \Omega^1 \otimes L \longrightarrow J \longrightarrow I \longrightarrow 0$$

we obtain the useful formula

PROPOSITION 7.4.1. *Let L be a line bundle over a variety X of dimension n. Then*

$$\Delta(L) = I_{X/*}\left(c_1(L)\, c_n\left(\Omega^1 \otimes L\right)\right)$$

where Ω^1 is the bundle of 1-forms on X, note that this bundle has rank n, for a smooth variety X.

7.4.2. Over a curve or a relative curve. First we will look at the classical situation. We can now use the above results to prove the polarization formula for resultants on a curve X. Note that our bundle $\Delta(L)$ for a curve is denoted in Deligne's work by $\langle L\rangle$.

PROPOSITION 7.4.3. *For all ample line bundles L and M on a smooth relative curve $X \longrightarrow S$,*

(1) We have an isomorphism of line bundles

$$[L,M]^{\otimes 2} \cong \frac{\Delta(L \otimes M)}{\Delta(L)\Delta(M)}.$$

(2) For $f \in H^0 L$ and $g \in H^0 M$, let $R(f,g) \in [L,M]$ be their resultant, and $\Delta(f) \in \Delta(L)$ be the discriminant of f. Then, under the identification from (1) we have

$$R(f,g) = \frac{\Delta(fg)}{\Delta(f)\Delta(g)}.$$

This is an analog of the above classical polarization formula, but for any line bundle over any curve X.

PROOF. Given part (1), part (2) is obvious. For (1) we use proposition 7.4.1 to obtain

$$\begin{aligned} \frac{\Delta(L \otimes M)}{\Delta(L)\Delta(M)} &\cong \frac{\left[L \otimes M, L \otimes M \otimes \Omega^1\right]}{[L, L \otimes \Omega^1][M, M \otimes \Omega^1]} \\ &= \frac{[L,L][L,M]\left[L,\Omega^1\right][M,L][M,M]\left[M,\Omega^1\right]}{[L,L][L,\Omega^1][M,M][M,\Omega^1]}. \end{aligned}$$

Cancelling terms yields the desired result. □

7.4.4. Over a surface. We turn now to the situation over a surface X. For a line bundle L over a surface, the discriminant of a section l is zero if and only if the intersection of l with the zero section is a singular curve.

THEOREM 7.4.5. *For line bundles L, M, and N on a smooth surface S,*

$$I_{X/S}(L,M,N)^{\otimes 6} = \frac{\Delta(L \otimes M \otimes N)\Delta(L)\Delta(N)\Delta(M)}{\Delta(L \otimes M)\Delta(L \otimes N)\Delta(M \otimes N)}.$$

PROOF. The proof of this follows by rewriting the right hand side of this expression using proposition 7.4.1, and then applying the formulas in section 2.of convenience, we let $[E, L] = I_{X/S}(c_2E, c_1L)$ where E is a rank two bundle over X and L is a line bundle.

$$\frac{\Delta(LMN)\Delta(L)\Delta(N)\Delta(M)}{\Delta(LM)\Delta(LN)\Delta(MN)} =$$
$$= \frac{[LMN, \Omega^1 LMN][L, \Omega^1 L][M, \Omega^1 M][N, \Omega^1 N]}{[LM, \Omega^1 LM][LN, \Omega^1 LN][MN, \Omega^1 MN]}$$
$$= \frac{[L, \Omega^1 LMN][M, \Omega^1 LMN][N, \Omega^1 LMN][L, \Omega^1 L][M, \Omega^1 M][N, \Omega^1 N]}{[L, \Omega^1 LM][M, \Omega^1 LM][L, \Omega^1 LN][N, \Omega^1 LN][M, \Omega^1 MN][N, \Omega^1 MN]}$$

Expanding these using propositions 7.2.4 and 7.2.3, and cancelling terms leaves only six copies of $[L, M, N]$ in the numerator. □

This theorem poses an interesting question. It gives an analog, for the surface, to part (1) of theorem 7.4.3. But, if we attempted to write down the analog of (2), both the numerator and denominator in this expression would be identically 0, an indeterminate expression. We hope that a suitable interpretation of this expression exists. Perhaps one can perturb the system slightly to eliminate this indeterminacy.

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