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Lattice Points in Lattice Polytopes

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Abstract. If K is the underlying point-set of a simplicial complex of dimension at most d whose vertices are lattice points, and if $G(K)$ is the number of lattice points in K, then the lattice point enumerator $G(K,t) = 1 + \sum_{n \geqslant 1} G(nK)\, t^n$ takes the form $C(K,t)/(1-t)^{d+1}$, for some polynomial $C(K,t)$. Here, $C(K,t)$ is expressed as a sum of local terms, one for each face of K. When K is a polytope or its boundary, there result inequalities between the numbers $G_r(K)$, where $G(nK) = \sum_{r=0}^{d} n^r G_r(K)$.

1. Introduction

If K is any subset of k-dimensional euclidean space $\mathbb{E}^k$, and $\mathbb{Z}^k$ is the integer lattice in $\mathbb{E}^k$, consisting of those points with integer cartesian coordinates, then we denote by $G(K) = \operatorname{card}(K \cap \mathbb{Z}^d)$ the number of lattice points in K. Further,

$$nK = \{nx \mid x \in K\}$$

denotes the dilatate of K by the integer factor $n \geqslant 0$. We shall find it convenient right from the beginning to work with generating functions, which are formal power series in the indeterminate t. In particular, the *lattice point enumerator* $G(K,t)$ is defined by

$$G(K,t) = \sum_{n \geqslant 0} G(nK)\, t^n,$$

where the convention here is that $G(0K) = 1$.

This last-mentioned convention cannot be dismissed without comment. A natural convention would be to take $G(0K) = \chi(K)$, the Euler characteristic, because this fits in with the valuation property

$$G(K_1 \cup K_2) + G(K_1 \cap K_2) = G(K_1) + G(K_2)\,.$$

However, our approach is along the lines of the original papers of EHRHART (see, for example, [1]). Thus it is often helpful to think of K as embedded in the hyperplane

$$H_1 = \{(\xi_0, \ldots, \xi_k) \in \mathbb{E}^{k+1} \mid \xi_0 = 1\}$$

in $\mathbb{E}^{k+1}$, so that nK similarly lies in the hyperplane H_n with $\xi_0 = n$. We can then regard nK as the intersection $H_n \cap C$, where C is the cone with apex o in $\mathbb{E}^{k+1}$ generated by K; naturally, therefore, $0K = H_0 \cap C = \{o\}$.

We shall investigate in this paper the lattice point enumerator of certain special sets. By a *lattice complex* we shall mean a simplicial complex, whose vertices (0-cells) are lattice points in $\mathbb{E}^k$. We shall not distinguish between such a complex and its underlying point set, but no confusion will arise. A lattice complex is *pure*, of dimension d, if all its maximal simplices have the same dimension d.

For a (pure) lattice d-complex K, it is well known that $G(K, t)$ takes the form

$$G(K, t) = (1 - t)^{-(d+1)} C(K, t),$$

where $C(K, t)$ is a polynomial in t of degree at most $d + 1$ with integer coefficients (see [1]). Until recently little else was known about $G(K, t)$. STANLEY ([10, 11]) has shown, first by algebraic and then by more geometric techniques, that for a polytope K,

$$C(K, t) \geqslant 0,$$

where if $a(t)$ and $b(t)$ are series in t, by $a(t) \geqslant b(t)$ we mean $a_s \geqslant b_s$ for all s; here and subsequently, we write

$$p_s = p(t)|_s$$

for the coefficient of t^s in $p(t)$ (recall that we regard polynomials purely as generating functions).

Stanley obtains his results by studying polytopes with rational vertices. Here we take a different point of view. It is clear that the combinatorial structure of a lattice complex K and the lattice point enumerators of its component simplices determine the polynomial $C(K, t)$. In § 2, we use this approach to find an explicit formula for $C(K, t)$, in terms of its local structure. This formula is then applied to get further results about the coefficients $C_s(K)$ of $C(K, t)$ when K is a ball or a sphere; these include and generalize results of STANLEY ([11]).

In the last part of the paper, we consider d-dimensional lattice balls P in $\mathbb{E}^d$. The polynomiality of $C(P, t)$ implies that we have a polynomial expansion

$$G(nP) = \sum_{r=0}^{d} n^r G_r(P)$$

for integer $n \geqslant 0$ (this is also a consequence of the valuation property and integer translation invariance; see [6]). It is well known that

$$G_d(P) = V(P)$$

is just ordinary volume. The coefficient $G_{d-1}(P)$ is related to surface area:

$$G_{d-1}(P) = \tfrac{1}{2} \sum \tilde{S}(F),$$

where the lattice surface area $\tilde{S}(F)$ of the facet ($(d-1)$-face) F of P is its ordinary surface area divided by the determinant of the sublattice $\mathbb{Z}^d \cap \operatorname{aff} F$, and the sum extends over all facets F of P. Further

$$G_0(P) = 1.$$

For these facts, see, for example, [4].

Less seems to be known about the remaining $G_r(P)$ $(1 \leqslant r \leqslant d-2)$. Here we give one series of inequalities relating $V(P)$ and the $G_r(P)$ $(1 \leqslant r \leqslant d-1)$, the case $r = d-1$ being an improvement of a result of WILLS [12]; we also indicate the existence of a second series.

We end by obtaining some inequalities relating $G(nP)$ and $V(P)$.

2. The Lattice Point Enumerator

We begin this section by repeating some known results on lattice simplices.

Lemma 1. *Let $T = \operatorname{conv}\{x_0, \ldots, x_d\}$ be a lattice d-simplex. Then*

$$(1 - t)^{d+1} G(T, t) = C(T, t)$$

is the polynomial of degree at most d defined by

$$C_s(T) = \operatorname{card}\{x \in \mathbb{Z}^k \mid x = \sum_{i=0}^{d} \lambda_i x_i,\ 0 \leqslant \lambda_i < 1,\ \sum_{i=0}^{d} \lambda_i = s\}.$$

Lemma 2. *If T^0 is the relative interior of the lattice d-simplex T, then*

$$(1 - t)^{d+1} G(T^0, t) = t^{d+1} C(T, t^{-1}),$$

with $C(T, t)$ as in Lemma 1.

These two results are due to EHRHART [1].

We now define a new polynomial $C^*(T, t)$, associated with the lattice d-simplex T, by

$$C_s^*(T) = \text{card}\,\{x \in \mathbb{Z}^k \mid x = \sum_{i=0}^{d} \lambda_i x_i,\ 0 < \lambda_i < 1,\ \sum_{i=0}^{d} \lambda_i = s\}\ .$$

Thus $C^*(T, t)$ measures those lattice points which arise from T, but not any proper face of T. In the case $T = \emptyset$, we adopt the convention $C^*(\emptyset, t) = 1$; this is evidently in accord with the convention we introduced above. There clearly follows from the definition

Lemma 3. a) $C^*(T, t) = t^{d+1} C^*(T, t^{-1})$;

b) $C(T, t) = \sum_{S \subseteq T} C^*(S, t)$, *where the sum extends over all faces* S *of* T *(including* $\emptyset$ *and* T *itself).*

In what follows, K will be a pure simplicial complex (or its underlying point set). Denoting by $f_j(K)$ the number of j-simplices in K, with $f_{-1}(K) = 1$ and $f_j(K) = 0$ if $j < -1$ or $j > \dim K$, we write

$$f(K, t) = \sum_j f_j(K)(-t)^{j+1} = \sum_{T \in K} (-t)^{\dim T + 1}.$$

For historical reasons, we violate our usual conventions about the coefficients of generating functions here. Following [8], but with the later notation of, for example, [11], we define

$$h(K, t) = (1 - t)^{\dim K + 1} f\left(K, \frac{t}{t-1}\right).$$

It is clear that $h(K, t)$ is a polynomial in t of degree at most $\dim K + 1$, whose coefficients are integers. We shall later appeal to stronger results about these coefficients.

Let $T \in K$. The link of K, which by analogy to the notation for polytopes (see [7]) we denote by K/T, has its usual meaning; it is the subcomplex of K formed by those simplices which do not meet T, but which are sub-simplices of simplices which contain T. Note that $K/\emptyset = K$, and, more generally, that

$$\dim(K/T) = \dim K - \dim T - 1\ .$$

We now come to the general theorem, on which much of the rest of the paper is based.

Theorem 1. *Let* K *be a pure lattice complex, and define* $C(K, t)$ *by*

$$C(K, t) = (1 - t)^{\dim K + 1} G(K, t)\ .$$

Then $C(K, t)$ *is a polynomial in* t *of degree at most* $\dim K + 1$ *with integer coefficients, which admits the local expression*

$$C(K, t) = \sum_{T \in K} h(K/T, t)\, C^*(T, t)\ .$$

As in Lemma 2, we denote by S^0 the relative interior of a lattice simplex S. We have

$$G(K, t) = \sum_{S \in K} G(S^0, t)\ ,$$

where our conventions demand that $G(0S^0) = 0$, except that $G(\emptyset, t) = 1$. Lemma 2 then implies that

$$C(K, t) = (1 - t)^{\dim K + 1} \sum_{S \in K} G(S^0, t)$$
$$= \sum_{S \in K} (1 - t)^{\dim K - \dim S} t^{\dim S + 1} C(S, t^{-1})\ .$$

Using Lemma 3b, with t^{-1} for t, and reordering, we have

$$C(K, t) = \sum_{T \in K} \sum_{S \supseteq T} (1 - t)^{\dim K - \dim S} t^{\dim S + 1} C^*(T, t^{-1})$$
$$= \sum_{T \in K} \left\{ \sum_{S \supseteq T} (1 - t)^{\dim K - \dim S} t^{\dim S - \dim T} \right\} C^*(T, t),$$

by Lemma 3a. The expression in braces multiplying $C^*(T, t)$ can be written

$$(1 - t)^{\dim K - \dim T} \sum_{S \supseteq T} \left(\frac{t}{1-t}\right)^{\dim S - \dim T} =$$
$$= (1 - t)^{\dim K - \dim T} f\left(K/T, \frac{t}{t-1}\right) = h(K/T, t)\ ,$$

as we wanted. This completes the proof of Theorem 1.

For future reference, we remark at this point that if P is a lattice d-polytope, then the degree of $C(P, t)$ is only d; this is a consequence of Theorem 2 of [8] and the fact that $C_0(P) = 1$ (which is just Euler's theorem). It is also known that, if K is a ball or a sphere, then $h(K, t)$ has non-negative coefficients; for shellable K, this is shown in [7] or [8]; the more general result is in [9].

Recalling (compare the Introduction) that the lattice volume $\tilde{V}(T)$ of a lattice simplex T is its ordinary volume (of the appropriate dimension) divided by the determinant of the lattice $(\text{aff}\, T) \cap \mathbb{Z}^k$, we deduce from Theorem 1:

Theorem 2. *Let K be a (simplicial) lattice complex, whose underlying point set is a manifold (with or without boundary). Then*

$$C(K,t) \geqslant h(K,t),$$

with equality precisely when each simplex T in K is minimal, having lattice volume

$$V(T) = 1/(\dim T)!\,.$$

The proof is straightforward, since $C^*(T,t) \geqslant 0$ for all T, and $C^*(\emptyset, t) = 1$ by convention. Furthermore, for each $T \neq \emptyset$, the link K/T is a sphere if $T \notin \partial K$, the boundary complex of K, or a ball if $T \in \partial K$, so that $h(K/T, t) \geqslant 1$. For equality, we observe that, for any lattice simplex $T \neq \emptyset$, $C^*(S,t) = 0$ for all non-empty faces S of T if and only if T is minimal.

Note that, as particular cases of Theorem 2, we have Corollaries 2.5 and 2.7 of [11].

For a sphere K, we have $h(K,t) = t^{\dim K+1} h(K,t^{-1})$ (compare [8]), so that Theorem 1 and Lemma 3a yield

Theorem 3. *If K is a sphere, then*

$$C(K,t) = t^{\dim K+1} C(K,t^{-1})\,.$$

Theorem 3 shows that, if K is a d-sphere, then $C_s(K) = C_{d+1-s}(K)$ for $s = 0,\ldots,d+1$, and hence that

$$\sum_{s=0}^{d+1} s\,C_s(K) = \frac{d+1}{2}\sum_{s=0}^{d+1} C_s(K)\,.$$

For balls, there is no such equation, but there is an analogous inequality.

Theorem 4. *Let K be a lattice d-ball. Then*

$$\sum_{s=0}^{d} C_s(K) - 1 \leqslant \sum_{s=0}^{d} s\,C_s(K) \leqslant \frac{d+1}{2}\left(\sum_{s=0}^{d} C_s(K) - 1\right).$$

Note that $C_{d+1}(K) = 0$ for such lattice balls, as remarked above. We shall postpone the proof of Theorem 4 to the next section. In the special case of polytopes with (relatively) interior lattice points, however, we have a much stronger result than Theorem 4.

Theorem 5. *Let P be a lattice d-polytope with a relatively interior lattice point. Then there are polynomials $A(P,t)$ and $B(P,t)$, with integer coefficients, which satisfy*

$$\text{degree } A(P,t) = d,\ A(P,t) \geqslant 1,\ A(P,t) = t^d A(P,t^{-1})\,,$$

$$\text{degree } B(P,t) \leqslant d-1,\ B(P,t) \geqslant 0,\ B(P,t) = t^{d-1} B(P,t^{-1}),$$

such that

$$C(P,t) = A(P,t) + t\,B(P,t)\,.$$

To prove this, let e be a relatively interior lattice point of P, and let $T_1,\ldots,T_n$ be the simplices of a simplicial decomposition of the boundary bd P of P. Defining $S_j = \operatorname{conv}(\{e\} \cup T_j)$ $(j = 1,\ldots,n)$, then $T_1,\ldots,T_n, S_1,\ldots,S_n$ are the simplices of a simplicial decomposition of P itself. So, by Theorem 1,

$$C(P,t) = \sum_{j=1}^{n} h(P/S_j,t)\,C^*(S_j,t) + \sum_{j=1}^{n} h(P/T_j,t)\,C^*(T_j,t)\,.$$

Now, for each j, we have $P/S_j = \operatorname{bd} P/T_j$, while P/T_j is the join of bd P/T_j and the point e. Thus

$$h(P/S_j,t) = h(\operatorname{bd} P/T_j,t),$$

$$h(P/T_j,t) = h(\operatorname{bd} P/T_j,t),$$

the latter following from $f(P/T_j,t) = (1-t)f(\operatorname{bd} P/T_j,t)$, and the definition of h, which depends upon the dimension of the complex involved. Since bd P/T_j is a sphere, we have

$$h(\operatorname{bd} P/T_j,t) = t^{\dim P - \dim T_j - 1} h(\operatorname{bd} P/T_j,t^{-1})\,,$$

and hence

$$h(P/S_j,t) = t^{\dim P - \dim S_j} h(P/S_j,t^{-1})\,,$$

$$h(P/T_j,t) = t^{\dim P - \dim T_j - 1} h(P/T_j,t^{-1})\,.$$

Observing that $C_0^*(S_j) = 0$ for each j (since $S_j \neq \emptyset$), these relations yield Theorem 5, with

$$A(P,t) = \sum_{j=1}^{n} h(P/T_j,t)\,C^*(T_j,t)\,,$$

$$B(P,t) = t^{-1}\sum_{j=1}^{n} h(P/S_j,t)\,C^*(S_j,t).$$

The condition that P has a relatively interior lattice point is clearly necessary, since, for example, $C(T,t) = 1$ for any lattice simplex T with minimal volume $1/(\dim T)!$.

3. Lattice Points and Volume

We now examine the coefficient $G(K,t)|_n = G(nK)$ of t^n in the generating function $G(K,t)$ in more detail. For simplicity, we confine our attention here to the case of lattice d-balls in $\mathbb{E}^d$.

Comparing coefficients of t^n in the relation

$$G(P,t) = (1-t)^{-(d+1)} C(P,t)$$

of Theorem 1 for d-balls P, we obtain

$$G(nP) = \sum_{s=0}^{d} \binom{d+n-s}{d} C_s(P) = \sum_{r=0}^{d} n^r G_r(P),$$

say. Thus $G(nP)$ is a polynomial in the integer $n > 0$ of degree d. This result was first proved in [1]. The two formulae

$$G_d(P) = V(P),$$
$$G_{d-1}(P) = \tfrac{1}{2} \sum_F \tilde{S}(F),$$

which we have already mentioned, can now be made to yield more information about $C(P,t)$.

Comparison of coefficients of n^d in $G(nP)$ gives

$$d!\,V(P) = \sum_{s=0}^{d} C_s(P).$$

Similarly, from the coefficients of n^{d-1}, we obtain

$$\tfrac{1}{2} \sum_F \tilde{S}(F) = \frac{1}{(d-1)!} \left\{ \tfrac{1}{2}(d+1) \sum_{s=0}^{d} C_s(P) - \sum_{s=0}^{d} s\, C_s(P) \right\},$$

or, equivalently,

$$\sum_{s=0}^{d} s\, C_s(P) = \tfrac{1}{2}(d+1) \sum_{s=0}^{d} C_s(P) - \tfrac{1}{2}(d-1)! \sum_F \tilde{S}(F),$$

and since we have the trivial inequality

$$\sum_F \tilde{S}(F) \geqslant (d+1)/(d-1)!,$$

this leads directly to the right inequality of Theorem 4.

As the left inequality of Theorem 4 is a trivial consequence of Theorem 1 (since $C_0(P) = 1$ and $C_s(P) \geqslant 0$ for $s \geqslant 1$), we now only need examples to show that the inequalities cannot be improved. For this purpose, let $e_1, \ldots, e_d$ denote the standard unit vectors in $\mathbb{E}^d$. Define the simplex $S^{(m)}$ by

$$S^{(m)} = \operatorname{conv}\{o, e_1, \ldots, e_{d-1}, m e_d\}.$$

Using Lemma 1, we can easily compute

$$C(S^{(m)}, t) = 1 + (m-1)t.$$

Thus for $S^{(m)}$, the left inequality becomes an equation. Now for odd dimension d, let

$$T^{(m)} = \operatorname{conv}\{o, e_1, e_1+e_2, e_2+e_3, \ldots, e_{d-2}+e_{d-1}, e_{d-1}+m e_d\},$$

while for even d, let

$$T^{(2m+1)} = \operatorname{conv}\{o, e_1, e_1+e_2, e_2+e_3, \ldots, e_{d-2}+e_{d-1},$$
$$e_1 + e_{d-1} + (2m+1) e_d\}.$$

There again follow from Lemma 1

$$C(T^{(m)}, t) = 1 + (m-1) t^{(d+1)/2}, \; d \text{ odd},$$
$$C(T^{(2m+1)}, t) = 1 + m t^{d/2} + m t^{(d+2)/2}, \; d \text{ even},$$

and here we have equality on the right.

We now use Theorems 1 and 4 to bound the $G_r(P)$ and $G(nP)$ in terms of the volume $V(P) = G_d(P)$. We recall the following definition (compare [2]). The *Stirling number* $S_i(d)$ *of the first kind* is the coefficient of t^i in

$$S(d,t) = \prod_{j=0}^{d} (t-j).$$

Then we have:

Theorem 6. *Let P be lattice d-ball in $\mathbb{E}^d$. Then for $r = 1, \ldots, d-1$,*

$$G_r(P) \leqslant (-1)^{d-r} S_r(d)\, V(P) + (-1)^{d-r-1} S_{r+1}(d)/(d-1)!.$$

Equality holds if and only if

$$C(P,t) = 1 + (d!\,V(P) - 1)\,t.$$

From Theorem 1 and the following remark, we have $C(P,t) \geqslant 1$, $C_0(P) = 1$ and degree $C(P,t) \leqslant d$. Now, comparing coefficients of n^r, we have

$$G_r(P) = \left(\sum_{s=0}^{d} \binom{n+d-s}{d} C_s(P) \right)\Big|_r = \left(\sum_{s=0}^{d} \binom{n+d-s}{d}\Big|_r C_s(P) \right).$$

Since

$$\left.\binom{n+d-s}{d}\right|_r < \left.\binom{n+d-1}{d}\right|_r < \left.\binom{n+d}{d}\right|_r$$

for $s \geqslant 2$, we see that

$$G_r(P) \leqslant \left.\left\{\binom{n+d}{d} + \binom{n+d-1}{d}(d!\,V(P)-1)\right\}\right|_r$$

$$= \left.\left\{\binom{n+d-1}{d-1} + \binom{n+d-1}{d} d!\,V(P)\right\}\right|_r$$

$$= (-1)^{d-r-1} S_{r+1}(d)/(d-1)! + (-1)^{d-r} S_r(d)\,V(P),$$

as we wished to show.

Theorem 6 gives a family of linear inequalities of the form

$$G_r(P) \leqslant c_r G_d(P) + d_r.$$

This set of inequalities is certainly not complete, for by Ehrhart's reciprocity law (see [5] or [6], for example) and Theorem 3, it can easily be shown that there exists a second set of inequalities, of the form

$$G_{d-2r+1}(P) \leqslant \tilde{c}_r G_{d-1}(P) + \tilde{d}_r \quad \left(r = 1, \ldots, \left[\frac{d+1}{2}\right]\right),$$

though we cannot give the exact values of $\tilde{c}_r$ and $\tilde{d}_r$. We do not know if there are any further inequalities between the numbers $G_r(P)$, although there are examples which show that there is no inequality of the form

which

$$G_{d-2}(P) \leqslant \hat{c}\, G_{d-1}(P) + \hat{d}.$$

We end by giving two results relating $G(nP)$ and $V(P)$.

Theorem 7. *Let P be a lattice d-ball in $\mathbb{E}^d$. Then*

a) $G(nP) \leqslant \binom{n+d-1}{d} d!\,V(P) + \binom{n+d-1}{d-1}$;

b) $G(nP) \geqslant \binom{n+d}{d} + \binom{n+\frac{1}{2}(d-1)}{d}(d!\,V(P)-1)$, *$d$ odd,*

$G(nP) \geqslant \binom{n+d}{d} + \frac{1}{2}\left\{\binom{n+\frac{1}{2}d}{d} + \binom{n+\frac{1}{2}d-1}{d}\right\}(d!\,V(P)-1)$, *$d$ even.*

Moreover, all the inequalities are sharp.

The first part (a) is, for $n = 1$, a result of Blichfeldt (see [3]); its proof is just the same as that of Theorem 6.

For (b), we observe that, for non-negative integers n, d, i, j, with $i < j \leqslant n + d$,

$$\binom{n+d-i}{d} + \binom{n+d-j}{d} \geqslant \binom{n+d-i-1}{d} + \binom{n+d-j+1}{d};$$

this follows by induction from

$$\binom{n+d-i}{d} - \binom{n+d-i-1}{d} \geqslant \binom{n+d-i-1}{d} - \binom{n+d-i-2}{d}.$$

From our given sequence $\mathscr{C} = (C_1, \ldots, C_d)$, where $C_s = C_s(P)$, we define a new sequence $\mathscr{C}' = (C_1', \ldots, C_d')$ as follows (we have $C_0' = C_0 = 1$ fixed throughout). Let

$$s_* = \min\{s \mid C_s > 0\}, \quad s^* = \max\{s \mid C_s > 0\},$$

and if $s^* - s_* \geqslant 2$, and $D = \min\{C_{s_*}, C_{s^*}\}$, define

$$C_s' = \begin{cases} C_s - D, & s = s_*, s^*, \\ C_s + D, & s = s_* + 1, s^* - 1, \\ C_s, & \text{otherwise}, \end{cases}$$

except if $s^* - s_* = 2$, when the second relation is replaced by

$$C_s' = C_s + 2D, \ s = s_* + 1 = s^* - 1.$$

We see at once that

$$\sum_{s=1}^{d} C_s' = \sum_{s=1}^{d} C_s, \quad \sum_{s=1}^{d} s\,C_s' = \sum_{s=1}^{d} s\,C_s,$$

while $C_{s^*}' < C_{s^*}$ (possibly it is zero). The inequality noted above yields at once

$$G(nP) = \binom{n+d}{d} + \sum_{s=1}^{d}\binom{n+d-s}{d} C_s$$

$$\geqslant \binom{n+d}{d} + \sum_{s=1}^{d}\binom{n+d-s}{d} C_s'$$

We repeat the above process as many times as we can. After a finite number of steps, we eventually arrive at an inequality

$$G(nP) \geqslant \binom{n+d}{d} + \binom{n+d-\bar{s}}{d} \bar{C}_{\bar{s}} + \binom{n+d-\bar{s}-1}{d} \bar{C}_{\bar{s}+1},$$

for some $\bar{s}$, where possibly $\bar{C}_{\bar{s}+1} = 0$, and

$$\bar{C}_{\bar{s}} + \bar{C}_{\bar{s}+1} = \sum_{s=1}^{d} C_s,$$

$$\bar{s}\,\bar{C}_s + (\bar{s}+1)\,\bar{C}_{\bar{s}+1} = \sum_{s=1}^{d} s\,C_s.$$

In view of the second inequality of Theorem 4, which can be written

$$\sum_{s=1}^{d} (d+1-2s)\,C_s \geqslant 0,$$

we have

$$(d+1-2\bar{s})\,\bar{C}_{\bar{s}} + (d-1-2\bar{s})\,\bar{C}_{\bar{s}+1} \geqslant 0.$$

It follows that $\bar{s} \leqslant \frac{1}{2} d$, unless d is odd, when we can have $\bar{s} = \frac{1}{2}(d+1)$ and $\bar{C}_{\bar{s}+1} = 0$. Further, if d is even and $\bar{s} = \frac{1}{2} d$, then $\bar{C}_{\bar{s}+1} \leqslant \bar{C}_{\bar{s}}$. Since the binomial coefficients $\binom{n+d-s}{d}$ do not increase as s does, using

$$\sum_{s=1}^{d} C_s = d!\,V(P) - 1$$

the inequalities of part (b) follow by an easy induction argument.

We conclude by remarking that we have already shown the inequalities of Theorem 4 to be best possible.

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Integer Points on Curves and Surfaces[1]

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Abstract. Various upper bounds are given for the number of integer points on plane curves, on surfaces and hypersurfaces. We begin with a certain class of convex curves, we treat rather general surfaces in $\mathbb{R}^3$ which include algebraic surfaces with the exception of cylinders, and we go on to hypersurfaces in $\mathbb{R}^n$ with nonvanishing Gaussian curvature.

1. Introduction. It is well known (JARNIK [8]) that on a plane convex curve of length $l \geqslant 1$ there are $\ll l^{2/3}$ integer points. This estimate is best possible, and the constant in $\ll$ is absolute. The convex curve may be a closed curve or it may be a curve $y = f(x)$. In particular, if $f(x)$ is twice differentiable in some interval of length at most $N \geqslant 1$, with either $f'' > 0$ or $f'' < 0$ throughout, and if the range of f is contained in an interval of length N, then the number Z of integer points on the curve $y = f(x)$ satisfies

$$Z \ll N^{2/3}. \tag{1.1}$$

SWINNERTON-DYER [11] took up the question of what can be said if higher derivatives exist. Let $\mathfrak{C}$ be a fixed curve $y = f(x)$ where x runs through some finite closed interval, where f''' exists and is continuous, and where $f'' > 0$ or $f'' < 0$ throughout. Let Z_N be the number of integer points on the blown up curve $N\mathfrak{C}$, consisting of points (Nx, Ny) with (x, y) on $\mathfrak{C}$. Then according to Swinnerton-Dyer, we have

$$Z_N \leqslant c_1(\mathfrak{C}, \varepsilon)\, N^{(3/5)+\varepsilon} \tag{1.2}$$

for $N \geqslant 1$ and $\varepsilon > 0$.

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