Lattice invariant valuations on rational polytopes

By

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Abstract. Let Λ be a lattice in d-dimensional euclidean space $\mathbb{E}^d$, and $\overline{\Lambda}$ the rational vector space it generates. If φ is a valuation invariant under Λ, and P is a polytope with vertices in $\overline{\Lambda}$, then for non-negative integers n there is an expression $\varphi(nP) = \sum_{r=0}^{d} n^r \varphi_r(P, n)$, where the coefficient $\varphi_r(P, n)$ depends only on the congruence class of n modulo the smallest positive integer k such that the affine hull of each r-face of kP is spanned by points of Λ. Moreover, φ_r satisfies the Euler-type relation $\sum_F (-1)^{\dim F} \varphi_r(F, n) = (-1)^r \varphi_r(-P, -n)$, where the sum extends over all non-empty faces F of P. The proof involves a specific representation of simple such valuations, analogous to Hadwiger's representation of weakly continuous valuations on all d-polytopes. An example of particular interest is the lattice-point enumerator G, where $G(P) = \operatorname{card}(P \cap \Lambda)$; the results of this paper confirm conjectures of Ehrhart concerning G.

1. Introduction. In an earlier paper [4], we investigated the following situation. Let Λ be an additive subgroup of d-dimensional euclidean space $\mathbb{E}^d$, and let $\mathscr{P}(\Lambda)$ denote the class of (convex) polytopes whose vertices lie in Λ. A *valuation* on $\mathscr{P}(\Lambda)$ is a real valued function φ such that, if $P, Q, P \cup Q \in \mathscr{P}(\Lambda)$, then

$$\varphi(P \cup Q) + \varphi(P \cap Q) = \varphi(P) + \varphi(Q).$$

(The conditions imply that $(P \cup Q) + (P \cap Q) = P + Q$, and so ensure that $P \cap Q \in \mathscr{P}(\Lambda)$ also.) We call φ a *Λ-valuation* if $\varphi(P + t) = \varphi(P)$ whenever $P \in \mathscr{P}(\Lambda)$ and $t \in \Lambda$. Among other things, we proved that, under certain reasonable conditions on Λ or φ, if φ is a Λ-valuation on $\mathscr{P}(\Lambda)$, if $P_1, \ldots, P_k \in \mathscr{P}(\Lambda)$ and $n_1, \ldots, n_k$ are non-negative integers, then $\varphi(n_1 P_1 + \cdots + n_k P_k)$ is a polynomial in the n_i of total degree at most d. In particular, $\varphi(nP) = \sum_{r=0}^{d} n^r \varphi_r(P)$, and φ_r is a homogeneous Λ-valuation of degree r. Moreover, φ_r satisfies the Euler-type relation $\sum_F (-1)^{\dim F} \varphi_r(F) = (-1)^r \varphi_r(P)$, where the sum on the left extends over all non-empty faces F of P.

A particular example of such a valuation is the lattice-point enumerator G, where Λ is a lattice (discrete additive subgroup), and $G(P) = \operatorname{card}(P \cap \Lambda)$. Now, in several papers Ehrhart has investigated $G(P)$ in case $P \in \mathscr{P}(\bar{\Lambda})$, where $\bar{\Lambda}$ is the rational vector space generated by Λ (see [1]). In a more general context, we began investigating in [4] valuations on $\mathscr{P}(\bar{\Lambda})$ which are only invariant under Λ. We obtained an analogue of the polynomial expansion, and conjectured an Euler-type relation, these generalizing Ehrhart's results on G.

In this paper, we shall extend these results, strengthening that about the polynomial expansion (and, incidentally, establishing a conjecture of Ehrhart), and proving the Euler-type relation. In the course of the proofs, we generalize a characterization of [2] of certain simple valuations (which vanish on polytopes of smaller dimension than d).

Though the basic approach is quite different, many of our results, particularly in § 3, repeat those of [4]; we shall therefore omit most of their proofs, referring the reader to the earlier paper.

2. Simple valuations. We begin our account by discussing the behaviour of a *simple* valuation ψ; that is, $\psi(P) = 0$ if $\dim P < d$. More generally, if A is a $\bar{\Lambda}$-*flat*, that is, if A is a flat (affine subspace) spanned by points of $\bar{\Lambda}$, we write $\mathscr{P}(A) = \mathscr{P}(A \cap \bar{\Lambda})$, and if $\mathscr{A}$ is a translation class (under $\bar{\Lambda}$) of $\bar{\Lambda}$-flats, we write $\mathscr{P}(\mathscr{A}) = \bigcup\{\mathscr{P}(A) \mid A \in \mathscr{A}\}$. We then say that a Λ-valuation ψ on $\mathscr{P}(\mathscr{A})$ is *simple* if $\psi(P) = 0$ whenever $P \in \mathscr{P}(\mathscr{A})$ with $\dim P < \dim \mathscr{A}$ $(= \dim A$ for any $A \in \mathscr{A})$.

We say a function $\varkappa$ of k variable (unit) vectors $u_1, \ldots, u_k$ is *odd* if

$$\varkappa(\varepsilon_1 u_1, \ldots, \varepsilon_k u_k) = \varepsilon_1 \ldots \varepsilon_k \varkappa(u_1, \ldots, u_k)$$

for all $\varepsilon_i = \pm 1$ $(i = 1, \ldots, k)$. If u is a unit vector, $\mathscr{A}_u$ denotes the translation class (under $\bar{\Lambda}$) of $\bar{\Lambda}$-hyperplanes with normal u (so that $\mathscr{A}_u = \mathscr{A}_{-u}$; observe that $\mathscr{A}_u = \emptyset$ unless u is normal to some linear Λ-hyperplane). If $P \in \mathscr{P}(\bar{\Lambda})$, we denote by P_u the face of P in direction u, that is, the intersection of P with its support hyperplane with outer normal u.

Our first two results generalize their analogues of [2].

Theorem 1. *Let ψ be a simple Λ-valuation on $\mathscr{P}(\Lambda)$. Then for all $\mathscr{P} \in \mathscr{P}(\bar{\Lambda})$; there is an expression*

$$\psi(P) = \mu V(P) + \sum_u \varkappa(u) \psi_u(P_u),$$

where $\mu \in \mathbb{R}$, V is ordinary volume, $\varkappa$ is an odd function, $\psi_u = \psi_{-u}$ is a simple Λ-valuation on $\mathscr{P}(\mathscr{A}_u)$, and the sum extends over all unit vectors u. Conversely, any such expression defines a simple Λ-valuation on $\mathscr{P}(\bar{\Lambda})$.

Theorem 2. *Let $\{v_1, \ldots, v_d\}$ be a basis of Λ, let $H = \operatorname{lin}\{v_1, \ldots, v_{d-1}\}$, and write $I(\eta) = \operatorname{conv}\{0, \eta v_d\}$ for $0 \leqq \eta \in \mathbb{Q}$. If $\tilde{\psi}$ is a simple Λ-valuation on $\mathscr{P}(H)$, then there is a simple Λ-valuation ψ' on $\mathscr{P}(\bar{\Lambda})$, such that*

$$\psi'(Q + I(\eta)) = \eta \tilde{\psi}(Q).$$

Before proceeding with the proofs, let us make a remark about valuations. If A is a $\bar{\Lambda}$-flat, and φ is a Λ-valuation on $\mathscr{P}(A)$, we can define an associated Λ-valuation

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ϑ on $\mathscr{P}(L)$, where $L = A - a$ is the parallel linear subspace with $a \in A \cap \overline{\Lambda}$ any point, by $\vartheta(Q) = \varphi(Q + a)$ for $Q \in \mathscr{P}(L)$. Moreover, the appropriate group of translations in L is $L \cap \Lambda$, which is just the subgroup of Λ which preserves A. We note that L is spanned by $L \cap \Lambda$, so that L is a linear Λ-flat. (More generally, a $\overline{\Lambda}$-flat A is a Λ-flat if and only if $A \cap \Lambda \neq \emptyset$.) Thus our results do not merely apply to linear Λ-flats, as might superficially appear.

We shall prove Theorems 1 and 2 in parallel; our proof closely follows that of [2], with appropriate modifications to take account of the fact that our valuations are only invariant under Λ. We first observe that the converse statement in Theorem 1 is clear. The remainder of the proof proceeds by induction on d.

We begin with the case $d = 0$. A (simple) Λ-valuation on the points of $\overline{\Lambda}$ (regarded as 0-polytopes) just assigns to each point a value, which depends (in general) only on its equivalence class modulo Λ. Theorem 1 is then clear (we conventionally take the 0-volume of a point to be 1), and μ is the appropriate value. Theorem 2 is, of course, vacuous.

So, let us now assume that $d \geqq 1$, and that the theorems hold for $d - 1$ dimensions. The proof of Theorem 2 will be incorporated as a step in that of Theorem 1. We take $\{v_1, \ldots, v_d\}$ to be the given basis of Theorem 2, and use the notation of that theorem. We define a function $\tilde{\psi}$ on $\mathscr{P}(H)$ by $\tilde{\psi}(Q) = \psi(Q + I(1))$ for $Q \in \mathscr{P}(H)$. Clearly $\tilde{\psi}$ is a simple Λ-valuation. By Theorem 1 for $d - 1$ dimensions, we have an expression

$$\tilde{\psi}(Q) = \nu V_{d-1}(Q) + \sum_u' \varkappa(u) \tilde{\psi}_u(Q_u),$$

where the sum extends over all unit vectors u with $\langle u, v_d \rangle = 0$, and $\tilde{\psi}_u = \tilde{\psi}_{-u}$ is a simple Λ-valuation on $\mathscr{P}(\mathscr{A}_u(H))$, with $\mathscr{A}_u(H)$ the family of $(d-2)$-dimensional $\overline{\Lambda}$-flats in H with normal u.

Now, firstly there is a constant μ such that $\mu V(Q + I(1)) = \nu V_{d-1}(Q)$. Secondly, by Theorem 2 for $d - 1$ dimensions, for each unit vector u with $\langle u, v_d \rangle = 0$, there is a simple Λ-valuation $\psi_u = \psi_{-u}$ on $\mathscr{P}(\mathscr{A}_u)$ such that $\psi_u(Q' + I(\eta)) = \tilde{\psi}_u(Q')$ for $Q' \in \mathscr{P}(\mathscr{A}_u(H))$ and $0 \leqq \eta \in \mathbb{Q}$. Now define the function ψ' on $\mathscr{P}(\overline{\Lambda})$ by

$$\psi'(P) = \mu V(P) + \sum_u' \varkappa(u) \psi_u(P_u).$$

Then ψ' clearly has the properties required in Theorem 2.

We now set $\psi'' = \psi - \psi'$. Then ψ'' is a simple Λ-valuation on $\mathscr{P}(\overline{\Lambda})$, such that $\psi''(Q + I(n)) = 0$ for $Q \in \mathscr{P}(H)$ and non-negative integers n. Let u be any unit vector normal to a Λ-hyperplane. Replacing u by $-u$ if necessary, we may suppose that $\langle u, v_d \rangle > 0$ (the case $\langle u, v_d \rangle = 0$ will not be relevant). We now define a simple Λ-valuation $\psi_u = \psi_{-u}$ on $\mathscr{P}(\mathscr{A}_u)$. Let $F \in \mathscr{P}(\mathscr{A}_u)$. We may translate F under Λ so that it lies in the half-space H^+ bounded by H which contains v_d. Let F' be the image of F under projection on to H in direction v_d, and let $\bar{F} = \operatorname{conv}(F \cup F')$. We define $\psi_u = \psi_{-u}$ by $\psi_u(F) = \psi''(\bar{F})$.

Now ψ_u is clearly invariant under Λ. This is obvious for $t \in \Lambda$ of the form $t = \sum_{i=1}^{d-1} m_i v_i$, and follows for $t = m_d v_d$ (with $m_d \geqq 0$) since

$$\overline{F + m_d v_d} = (\bar{F} + m_d v_d) \,\dot{\cup}\, (F' + I(m_d)),$$

where $\dot\cup$ denotes union with disjoint interiors (or relative interiors, as appropriate). This translation invariance clearly allows us to make a consistant definition of $\psi_u(F)$ if $F \in \mathscr{P}(\mathscr{A}_u)$ but $F \nsubseteq H^+$. That ψ_u is a simple valuation follows from $\overline{F \dot\cup G} = \bar{F} \dot\cup \bar{G}$, if $F, G, F \cup G \in \bigcap(A)$ for some $A \in \mathscr{A}_u$, with $\dim(F \cap G) \leqq d-2$.

If we now translate P under Λ so that $P \subseteqq H^+$, we see that

$$P \dot\cup \dot\bigcup\{\bar{P}_u \mid \langle u, v_d\rangle < 0\} = \dot\bigcup\{\bar{P}_u \mid \langle u, v_d\rangle > 0\},$$

where the unions are over the facets P_u of P. Thus

$$\begin{aligned}\psi''(P) &= \sum_{\langle u, v_d\rangle > 0} \psi''(\bar{P}_u) - \sum_{\langle u, v_d\rangle < 0} \psi''(\bar{P}_u) \\ &= \sum\nolimits_u'' \varkappa(u)\, \psi_u(P_u),\end{aligned}$$

where $\varkappa(u) = 1$ or -1 as $\langle u, v_d\rangle > 0$ or < 0, and the sum extends over all unit vectors u with $\langle u, v_d\rangle \neq 0$. Combining this with the expression for ψ, and using $\psi = \psi' + \psi''$, we have the required expression for ψ. This completes the proof of Theorems 1 and 2.

There is an immediate consequence of Theorem 1. Let $\mathscr{U}_k$ denote the family of ordered orthogonal sets $U = (u_1, \ldots, u_k)$, and for a polytope P, define by induction $P_U = (P_{(u_1, \ldots, u_{k-1})})_{u_k}$. We denote by $U^\perp$ the linear subspace of E^d completely orthogonal to U. Then we have:

Theorem 3. *Let ψ be a simple Λ-valuation on $\mathscr{P}(\bar{\Lambda})$. Then there is an expression*

$$\psi(P) = \sum_{\substack{U \in \mathscr{U}_{d-r} \\ r=0}}^{d} \varkappa(U, P_U)\, V_r(P_U),$$

where $\varkappa(U, F)$ is odd as a function of U, and depends only upon the translation class modulo Λ of the translate of $U^\perp$ containing F, and V_r is r-dimensional volume.

We note that the sum is, in fact, finite, since we have cancellation of the terms involving $(\pm u_1, \ldots, \pm u_k)$, unless $P_{(\varepsilon_1 u_1, \ldots, \varepsilon_j u_j)}$ is a $(d-j)$-face for $j = 1, \ldots, k$ and some $\varepsilon_i = \pm 1\,(i = 1, \ldots, k)$.

There follows in turn from Theorem 3 the analogue for simple valuations of the polynomial expansion formula of the introduction. We write $\operatorname{ind}_r(P)$, called the *r-index* of P, for the smallest positive integer m such that each r-face of mP spans a Λ-flat. If $\dim P = d' < d$, we naturally take $\operatorname{ind}_r(P) = 1$ for $d' < r \leqq d$. Then:

Theorem 4. *Let ψ be a simple Λ-valuation on $\mathscr{P}(\bar{\Lambda})$. Then for $P \in \mathscr{P}(\bar{\Lambda})$ and integer $n \geqq 0$, there is an expression*

$$\psi(nP) = \sum_{r=0}^{d} n^r \psi_r(P, n),$$

where $\psi_r(P, n)$ is a simple Λ-valuation in P on $\mathscr{P}(\Lambda)$ which depends only on the congruence class of n modulo $\operatorname{ind}_r(P)$.

For, if $F = P_U\,(U \in \mathscr{U}_{d-r})$ is an r-face of P, then

$$\varkappa(U, nF)\, V_r(nF) = n^r \varkappa(U, nF)\, V_r(F),$$

where we employ the notation of Theorem 3, and $\varkappa(U, nF)$ depends only on the congruence class of n modulo $\operatorname{ind}_r(P)$. This gives the desired expression, which we shall call a *near-polynomial*. If now $P, Q \in \mathscr{P}(\bar{\Lambda})$, with $P \cup Q$ convex and $\dim(P \cap Q) < d$, then $\psi(n(P \cup Q)) = \psi(nP) + \psi(nQ)$. Comparing coefficients of n^r in the near-polynomial expansion, for n in a fixed congruence class modulo the lowest common multiple of $\operatorname{ind}_r(P)$ and $\operatorname{ind}_r(Q)$, we see that $\psi_r(P \cup Q, n) = \psi_r(P, n) + \psi_r(Q, n)$, which is the simple valuation property. Since the invariance under Λ is obvious, we have proved the theorem.

In fact, these coefficients are *near-homogeneous*, in the sense that

$$\psi_r(mP, n) = m^r \psi_r(P, mn),$$

as may be seen by comparing the coefficients of n^r in

$$\sum_{r=0}^{d} n^r \psi_r(mP, n) = \psi(mnP) = \sum_{r=0}^{d} (mn)^r \psi_r(P, mn).$$

We note also that, since $\psi_r(P, n)$ depends only on the congruence class of n modulo $\operatorname{ind}_r(P)$, we may replace n by any other number in the same congruence class, and, in particular, by a negative integer, without confusion.

Theorem 5. *Let ψ be a simple Λ-valuation on $\mathscr{P}(\bar{\Lambda})$, which is near-homogeneous of degree r. Then $\psi(-P, -n) = (-1)^{d-r} \psi(P, n)$.*

For,

$$\psi(P) = \psi(P, 1) = \sum_{U \in \mathscr{U}_{d-r}} \varkappa(U, P_U)\, V_r(P_U).$$

If $\operatorname{ind}_r(P) = k$, then for any (suitably large) integer s,

$$\varkappa(U, (sk - n)(-P_U)) = \varkappa(U, nP_U).$$

Further, $-P_U = (-P)_{-U}$, and $\varkappa(-U, .) = (-1)^{d-r} \varkappa(U, .)$. Thus

$$\begin{aligned}(sk - n)^r \psi(-P, -n) &= \psi((sk - n)(-P)) \\ &= (sk - n)^r \textstyle\sum_U (-1)^{d-r} \varkappa(U, nP_U)\, V_r(P_U),\end{aligned}$$

and

$$n^r \psi(P, n) = \psi(nP) = n^r \textstyle\sum_U \varkappa(U, nP_U)\, V_r(P_U).$$

The result is now clear.

Theorem 5 is at the basis of the Euler-type relation of Theorem 8 below.

3. General valuations. A number of the arguments we use in this section are exactly the same as those of [4], so we shall refer the reader to that paper for the proofs.

We let $\beta(F, P)$ and $\gamma(F, P)$ be the normalized internal and external angles of the polytype P at its face F, always measured intrinsically. Then we have [3]:

Lemma 1. *The relations*

$$\psi(P) = \sum_F (-1)^{\dim P - \dim F} \beta(F, P)\, \varphi(F)$$
$$\varphi(P) = \sum_F \gamma(F, P)\, \psi(F)$$

between functions φ and ψ defined on all polytypes are equivalent. The sums extend over all non-empty faces F of the polytope P.

Lemma 2. *Let $\mathscr{A}$ be a translation class of $\bar{\Lambda}$-flats. If φ is a Λ-valuation on $\mathscr{P}(\bar{\Lambda})$, let ψ be defined as in Lemma 1, and define $\psi_{\mathscr{A}}(P) = \psi(P)$ if* aff $P \in \mathscr{A}$ *and* 0 *otherwise. Then $\psi_{\mathscr{A}}$ is a simple Λ-valuation on $\mathscr{P}(\mathscr{A})$.*

Lemma 3. *For each translation class $\mathscr{A}$ of $\bar{\Lambda}$-flats, let $\psi_{\mathscr{A}}$ be a simple Λ-evaluation on $\mathscr{P}(\mathscr{A})$. For $P \in \mathscr{P}(\bar{\Lambda})$, write $\psi(P) = \psi_{\mathscr{A}}(P)$ if* aff $P \in \mathscr{A}$. *If φ is defined as in Lemma 1, then φ is a Λ-valuation on $\mathscr{P}(\bar{\Lambda})$.*

There immediately follows the analogue of Theorem 4.

Theorem 6. *Let φ be a Λ-valuation on $\mathscr{P}(\bar{\Lambda})$. Then there is a near-polynomial expression $\varphi(nP) = \sum_{r=0}^{d} n^r \varphi_r(P, n)$ for $P \in \mathscr{P}(\bar{\Lambda})$ and non-negative integer n, where $\varphi_r(P, n)$ is a near-homogeneous Λ-valuation of degree r in P, which depends only on the congruence class of n modulo* $\mathrm{ind}_r(P)$.

Concerning the proof of this theorem, we only remark that $\mathrm{ind}_r(F)$ is a divisor of $\mathrm{ind}_r(P)$, for each face F of P. We easily extend this result to combinations $n_1 P_1 + \cdots + n_k P_k$ by means of the following

Lemma 4. *Let φ be a Λ-valuation on $\mathscr{P}(\bar{\Lambda})$, let $Q \in \mathscr{P}(\bar{\Lambda})$ be fixed, and define ϑ by $\vartheta(P) = \varphi(P + Q)$. Then ϑ is a Λ-valuation on $\mathscr{P}(\bar{\Lambda})$.*

An easy induction argument on k now yields

Theorem 7. *Let φ be a Λ-valuation on $\mathscr{P}(\bar{\Lambda})$. Then for $P_1, \ldots, P_k \in \mathscr{P}(\bar{\Lambda})$ and non-negative integers $n_1, \ldots, n_k$, $\varphi(n_1 P_1 + \cdots + n_k P_k)$ is a near-polynomial in $n_1, \ldots, n_k$ of total degree at most d, whose coefficient of $n_1^{r_1} \ldots n_k^{r_k}$ is a near-homogeneous Λ-valuation of degree r_i in P_i, which depends only on the congruence class of n_i modulo* $\mathrm{ind}_{r_i}(P_i)$.

The assertion about the total degree follows from expanding

$$\varphi(mn_1 P_1 + \cdots + mn_k P_k) = \varphi(m(n_1 P_1 + \cdots + n_k P_k))$$

as near-polynomials in $mn_1, \ldots, mn_k$, and m and $n_1, \ldots, n_k$, and comparing coefficients. In analogy with the mixed volumes, we may call the coefficients *mixed Λ-valuations*. The rest of the proof is clear, on hand of Theorem 6 and Lemma 4.

Finally, we establish the appropriate Euler-type relations. If φ is a Λ-valuation, we write $\varphi^*(P) = \sum_F (-1)^{\dim F} \varphi(F)$.

Theorem 8. *Let φ be a near-homogeneous Λ-valuation of degree r on $\mathscr{P}(\overline{\Lambda})$. Then for each $P \in \mathscr{P}(\overline{\Lambda})$ and integer n, $\varphi^*(P, n) = (-1)^r \varphi(-P, -n)$.*

We shall just sketch the proof here; the complete proof is analogous to that of Theorems 11 and 12 of [4]. From [6], it follows that, for each face G of P,

$$\sum_{F \supseteq G} (-1)^{\dim F} \beta(F, P) = (-1)^{\dim P} \beta(G, P).$$

For each translation class $\mathscr{A}$ of $\overline{\Lambda}$-flats, let $\psi^*_{\mathscr{A}}$ be the simple valuation corresponding to φ^* (that φ^* is, in fact, a valuation is a consequence of what follows). Then

$$\begin{aligned}
\psi^*_{\mathscr{A}}(P) &= \sum\nolimits_F (-1)^{\dim P - \dim F} \beta(F, P) \varphi^*(F) \\
&= \sum\nolimits_F (-1)^{\dim P - \dim F} \beta(F, P) \sum_{G \subseteq F} (-1)^{\dim G} \varphi(G) \\
&= \sum\nolimits_G (-1)^{\dim G} \beta(G, P) \varphi(G) \\
&= (-1)^{\dim \mathscr{A}} \psi_{\mathscr{A}}(P),
\end{aligned}$$

since $\dim \mathscr{A} = \dim P$. Hence, by Theorem 5,

$$\begin{aligned}
\varphi^*(P, n) &= \sum\nolimits_F \gamma(F, P) \psi^*(F, n) \\
&= \sum\nolimits_F \gamma(F, P)(-1)^{\dim F} \psi(F, n) \\
&= \sum\nolimits_F \gamma(F, P)(-1)^{\dim F}(-1)^{\dim F - r} \psi(-F, -n) \\
&= (-1)^r \varphi(-P, -n),
\end{aligned}$$

since $\gamma(-F, -P) = \gamma(F, P)$. This proves the theorem.

4. An application. The investigation of this paper was prompted by work of Ehrhart (see [1]) on the lattice point enumerator G, which is defined by $G(P) = \operatorname{card}(P \cap \Lambda)$. We shall consider a generalization $G(.\,; t)$ of G, where $t \in \mathbb{E}^d$, which is defined by $G(P; t) = G(P + t)$. We observe that $G(.\,; t)$ is a Λ-valuation on $\mathscr{P}(\overline{\Lambda})$, so the results we have obtained above all apply.

So, we first note that we have near-polynomial expansions

$$G(nP; t) = \sum_{r=0}^{d} n^r G_r(P, n; t),$$

where $G_r(P, n; t)$ depends on the congruence class of n modulo $\operatorname{ind}_r(P)$. In particular, if $\operatorname{ind}_r(P) = 1$, so that the affine hull of each r-face of P is a Λ-flat, then $G_r(P, n; t) = G_r(P; t)$ is independent of n. In case $t = o$, this confirms a conjecture of Ehrhart.

Further, we have the Euler-type relation

$$\begin{aligned}
G^*_r(P, n; t) &= (-1)^r G_r(-P, -n; t) \\
&= (-1)^r G_r(P, -n; -t),
\end{aligned}$$

the latter equation following from $G(-Q) = G(Q)$. Now, the number of lattice points in relint P is

$$\begin{aligned}
G^0(P) &= G(\operatorname{relint} P) \\
&= \sum\nolimits_F (-1)^{\dim P - \dim F} G(F) \\
&= (-1)^{\dim P} G^*(P),
\end{aligned}$$

by the Möbius inversion formula ([5]; see also [3]). Hence

$$\begin{aligned} G^0(nP;t) &= (-1)^{\dim P} G^*(nP;t) \\ &= (-1)^{\dim P} \sum_{r=0}^{d} n^r G_r^*(P, n; t) \\ &= (-1)^{\dim P} \sum_{r=0}^{d} (-n)^r G_r(P, -n; -t). \end{aligned}$$

In the particular case $t = o$, this result is due to [1], though his proof seems not to be generalizable to other valuations. The result is known as the *reciprocity law.*

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