everything we have said in the paper carries over to general ordered fields; in particular, the Hodge-Riemann-Minkowski inequalities and generalized Aleksandrov-Fenchel inequalities remain valid.

In a different direction, the techniques developed in this paper also lead to settling questions left open in [16] about syzygies between the frame functionals, and their ranges. However, since this would take us away from the topic of simple polytopes, we shall present this material in a later paper [17].

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Invent. math. 113, 445 (1993)

Inventiones mathematicae

Erratum

The number of reducible hypersurfaces in a pencil

Angelo Vistoli

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Due to a most unfortunate error, pages 249 and 250 were wrongly paginated and should have been placed between pages 258 and 259. We apologize for any inconvenience caused.

Theorem 14.1 *The ring R is isomorphic to $\Pi(P)$.*

By taking $o \in \operatorname{int} P$, and scaling the normal vectors u_j appropriately, we can write our polytope P in the form

$$P = \{x \in \mathbb{E}^d \mid \langle x, u_j \rangle \leqq 1 \text{ for } j = 1, \ldots, n\} .$$

Then the dual simplicial polytope P^* has vertex-set $U := \{u_1, \ldots, u_n\}$. Now consider the r-volume of an r-face G of a general polytope Q which is strongly isomorphic to P, with corresponding support parameter vector $y = (\eta_1, \ldots, \eta_n)$, say. By direct calculation, we can see that this volume is a homogeneous polynomial of degree r in $\eta_1, \ldots, \eta_n$, which involves exactly those η_j corresponding to facets G_j of Q which meet G. Indeed, a monomial $\eta_{j(1)} \ldots \eta_{j(r)}$ does not occur as a term in such a volume when $G_{j(1)} \cap \ldots \cap G_{j(r)} = \emptyset$.

These r-volumes do not change when G is translated, that is, when we add to the support vector y a vector of the form $(\langle a, u_1 \rangle, \ldots, \langle a, u_n \rangle)$ for some $a \in \mathbb{E}^d$. It is these translations which give rise to the h.s.o.p.; we take

$$\lambda_j := \langle a, u_j \rangle \quad \text{for } j = 1, \ldots, n ,$$

with a ranging over a linear basis of $\mathbb{E}^d$ in the definition above. We are now ready to describe the isomorphism. Each basis vector e_j of $\mathbb{E}^n$, thought of as a support vector, corresponds to an element of $\Xi_1(P)$; actually, if we identify $\Xi_1(P)$ with the representation space $\mathbb{E}^{n-d}$, then e_j corresponds to $\bar{u}_j$. We therefore define $\varphi: \mathbb{R}[x_1, \ldots, x_n] \to \Pi(P)$ by $\varphi(x_j) := \bar{u}_j$ $(j = 1, \ldots, n)$, and extend by polynomiality. The image of φ is obviously $\Pi(P)$. Its kernel clearly contains the h.s.o.p. and the non-faces, and so φ induces a homomorphism on $R(P^*)$; counting dimensions now shows that this latter homomorphism must be an isomorphism.

One curiosity which will not have escaped the reader's notice is that the minimal square-free monomials $\eta_{j(1)} \ldots \eta_{j(r)}$ which are absent in r-volume terms for r-faces of the polytopes Q correspond to missing $(r-1)$-faces of the dual P^*, which under the duality correspond to empty $(d-r)$-faces of P (or of Q), regarded as intersections of r facets. What this really appears to mean is that the r-grade term R_r of R is better identified with $\Xi_r(P)$ as the dual space of $\Xi_{d-r}(P)$ (see Theorem 5.2).

Before considering the general case, let us first take $r = 1$. We can identify an element $\sum_{j=1}^n \eta_j x_j \in R_1$ with a support vector $(\eta_1, \ldots, \eta_n)$, or rather with $y := \sum_{j=1}^n \eta_j \bar{u}_j \in \Xi_1(P)$, bearing in mind the fact that we have factored out the h.s.o.p. The multiplication with an element $a := (\alpha_1, \ldots, \alpha_n) \in \Xi_{d-1}(P)$, thought of as a vector of (signed) facet areas, is then $ay = \sum_{j=1}^n \alpha_j \eta_j$, as expected.

The general case is only a little more complicated. An element $w \in \Xi_{d-r}(P)$ can be identified with a $(d-r)$-weight ω. Multiplying w by q^r, where $q := \log Q$ with Q strongly isomorphic to P, yields a sum $\sum_F \omega(F)\sigma(F)$, where each term $\sigma(F)$ is a homogeneous polynomial in the support parameters η_j of Q of degree r. We can replace q^r by a general element of $\Xi_r(P)$ by substituting real numbers for the monomials in the η_j. Monomials corresponding to non-faces do not occur, of course, and the Minkowski relations on the weight ω force the factoring out of the h.s.o.p.

Let us end by tying Gale diagrams into our discussion. A few years ago, we informally conjectured the following

Theorem 14.2 *Let $\bar{V} = \{\bar{v}_1, \ldots, \bar{v}_n\} \subseteq \mathbb{E}^{n-d-1}$ be a Gale diagram of the dual of a simple d-polytope with n facets. Define the polynomial ring $R := \bigoplus_{r \geqq 0} R_r$ to be $\mathbb{R}[x_1, \ldots, x_n]$, factored out by the ideal generated by all elements of the form*

(a) $\sum_{j=1}^n \lambda_j x_j$, whenever $\sum_{j=1}^n \lambda_j \bar{v}_j = o$,
(b) $\prod_{j \in J} x_j$, whenever $\{\bar{v}_j \mid j \in J\} = \bar{V} \cap H$ for some open half space H in $\mathbb{E}^{n-d-1}$ bounded by a hyperplane through o.
Then $\dim R_r = g_r(P)$ for $0 \leqq r \leqq \frac{1}{2}d$.

We can see how this relates to what has gone before, when we observe (after [14]) that we obtain a Gale diagram of the dual P^* of P by letting $\bar{v}_j$ be the image of $\bar{u}_j$ in the representation space under the linear mapping with kernel lin $\{p\}$, where p is the representative of P. Together with the linear relations on $\bar{U}$, this gives (a), and (b) is just the condition for non-faces.

We may also observe that Theorem 14.2 has an alternative interpretation. The ring R is equivalently obtained by adjoining to the h.s.o.p. ideal H the extra element $\sum_{j=1}^n x_j$, that is, taking $\lambda_j = 1$ for each j. This corresponds to taking the values given by affine, rather than linear, functionals on U.

15 Final remarks

So far, we have confined our attention to simple polytopes. However, many of the questions we have addressed can equally be directed at general (not necessarily simple) polytopes. In particular, we can ask about the structure of the subalgebra $\Pi(P)$ of Π generated by the classes of summands of an arbitrary d-polytope P.

Let us consider some examples. If P is indecomposable with respect to Minkowski summation, then its type cone $\mathscr{K}(P)$ in the representation space is one-dimensional, and it follows that $\dim \Xi_r(P) = 1$ for each $r = 0, \ldots, d$. We should recall in this context that simplicial d-polytopes are indecomposable when $d \geqq 3$.

When P is decomposable, but non-simple, the situation changes dramatically. For example, consider the polyhedron P obtained from the regular octahedron in $\mathbb{E}^3$ (with 8 facets) by fixing one vertex, and moving the opposite vertex with the facets which contain it into general position. Now $\dim \Xi_1(P) = \dim \mathscr{K}(P) = 3$, but it is straightforward to see that $\dim \Xi_2(P) = 5 (= 8 - 3)$, its maximal possible value. Indeed, it is easy to see that the strict convexity of volume on type cones implies that $\dim \Xi_1(P) \leqq \dim \Xi_{d-1}(P)$ in all cases, and one should generally expect to have strict inequality.

Nevertheless, we may conjecture that Theorems 8.1 and 8.2 remain valid; now, though, multiplication by p^{d-2r} would only induce a monomorphism from $\Xi_r(P)$ into $\Xi_{d-r}(P)$. We should observe that Minkowski's second inequality leads to a proof of this when $r = 1$. However, even in the case of the transition polytopes T, we know as yet little about the structure of the subalgebra $\Pi(T)$.

In a different direction, we may consider the extent to which the results of this paper can be extended to the polytope algebra over a field $\mathbb{F}$ other than $\mathbb{R}$. Careful analysis of the arguments shows we have used no properties of $\mathbb{R}$ which do not apply to a general ordered field (even the continuity arguments are very mild). The only problem, as exemplified by [16], is expressing things such as the Minkowski relations, when one cannot necessarily take square roots, but this merely complicates the formulation of results, without affecting their validity. Thus, in fact, Similarly, the assertions of Theorems 12.1 and 12.2 involving Q instead of P are also straightforward – all that is needed is a calculation of dimensions, with the additional fact that $ts = 0$, and the sign of any extra eigenvalue. Note that the cases $r < m$ of the theorems are trivial. First, $t^{d-2r}y = t^{d-2r}x$ with our usual convention (because $r < \frac{1}{2}d$), and hence we have

$$t^{d-2r}\Xi_r(Q) = t^{d-2r}\Xi_r(P) = \Xi_{d-r}(P) = \Xi_{d-r}(Q)\,,$$

using Theorem 11.3(c), which gives the first part of Theorem 12.1(b). We also have $t^{d-2r}y^2 = t^{d-2r}x^2$, and the first part of Theorem 12.2(b) follows using the analogue of Theorem 8.6. The only remaining case is $r = (m =)\frac{1}{2}d$. For Theorem 12.1(b), there is nothing to prove. For Theorem 12.2(b), since $ts = 0$, we see that s gives an additional component in the "primitive" space of $\Xi_r(Q)$. But $(-1)^r s^2 = (-1)^m s^2 > 0$, so that the extra eigenvalue has the correct sign, and the required assertion follows.

In conclusion, we have established all the inductive arguments, and so have shown the main Theorems 7.3 and 8.2. In particular, Theorem 7.2 has been proved.

13 Further quadratic inequalities

Just as the Aleksandrov-Fenchel inequalities generalize the second Minkowski inequality, so we have generalizations of Theorems 7.3 and 8.2. A *monomial* is an element of $\Pi(P)$ of the form $c = p_1 \dots p_k$, for some $p_1, \dots, p_k \in \mathscr{K}(P)$.

Theorem 13.1 (Strong Lefschetz decomposition) *Let* $0 \leqq r \leqq \frac{1}{2}d$, *and let* $c \in \Xi_{d-2r}(P)$ *be a monomial. Then* $c\Xi_r(P) = \Xi_{d-r}(P)$.

The name employed here for this result is a convenient misnomer.

Theorem 13.2 (Generalized Aleksandrov-Fenchel inequalities) *Let* $0 \leqq r \leqq \frac{1}{2}d$, *let* $c \in \Xi_{d-2r}(P)$ *be monomial, and let* $p \in \mathscr{K}(P)$. *Then the quadratic form* $(-1)^r cx^2$ *is positive definite on* $\{x \in \Xi_r(P) \mid pcx = 0\}$.

We shall write SLD(d) to mean that there is a strong Lefschetz decomposition as in Theorem 13.1, and GAF(d) to mean that the generalized Aleksandrov-Fenchel inequalities of Theorem 13.2 hold. Then there is an exactly analogous argument to that of Lemma 8.3 (which we shall leave to the reader; compare the original proof of the Aleksandrov-Fenchel inequalities in [1]), which shows

Lemma 13.3 *If* $d \geqq 1$, *then* GAF($d-1$) *implies* SLD(d).

We may now use continuity arguments to deduce the generalized Aleksandrov-Fenchel inequalities from the Hodge-Riemain-Minkowski inequalities.

Theorem 13.4 HRM(d) (*with* GAF($d-1$)) *implies* GAF((d).

We shall show that the analogue of Theorem 8.6 holds, in that the form cx^2 on $\Xi_r(P)$ has the same number of positive and negative eigenvalues as $p^{d-2r}x^2$. Let $p_{r+1}, \dots, p_{d-r} \in \mathscr{K}(P)$, and let $c := p_{r+1} \dots p_{d-r}$. Our inductive assumption GAF($d-1$) implies SLD(d), and, as in Theorem 8.1, we can say equivalently that the form cx^2 is non-singular on $\Xi_r(P)$. If we now replace p_i by $q_i := (1-\lambda)p_i + \lambda p$ ($i = r+1, \dots, d-r; 0 \leqq \lambda \leqq 1$), the same remains true. Thus, as λ varies, the form keeps the same rank and signature (relative to any positive definite form on $\Xi_r(P)$),

that is, the same numbers of positive and negative eigenvalues, namely those given by Theorem 8.6.

We now need to obtain the decomposition of the form cx^2 which will yield Theorem 13.2. With $p \in \mathscr{K}(P)$ as in the statement of Theorem 13.2, we have the analogue of Theorem 8.5, namely

$$\Xi_r(P) = p\Xi_{r-1}(P) \oplus \{z \in \Xi_r(P) \mid pcx = 0\}$$

if $r \geqq 1$, and, if $x \in \Xi_r(P)$ is written as $x = py + z$ under this direct sum decomposition, then $cx^2 = cp^2y^2 + cz^2$. We now appeal to induction on r, the assertion being trivial if $r = 0$. Counting signs of eigenvalues of the form cx^2 and of the subform cp^2y^2 (which is also non-singular), we deduce that the form $(-1)^r cz^2$ is indeed positive definite, as claimed.

It is worth ending this section with a remark. Theorem 13.2 was proved using Theorem 8.2, and this, or rather its equivalent Theorem 8.6, was proved by tracing how the quadratic forms changed with the combinatorial type. In particular, the classical Brunn-Minkowski theorem for volumes of linear combinations has not been employed. In fact, there is more. It is well-known that the usual Aleksandrov-Fenchel inequalities (in effect, the case $r = 1$ of Theorem 13.2; compare [1]) actually imply the Brunn-Minkowski theorem in a purely algebraic way. One does, however, lose the characterization of the cases of equality in the theorem.

The Aleksandrov-Fenchel inequalities apply to arbitrary convex bodies. It would be interesting to know whether the generalized Aleksandrov-Fenchel inequalities also have analogues for arbitrary convex bodies; probably, however, this is not the case.

14 The face ring

In this section, we discuss the connexions between $\Pi(P)$ and the face ring (or Stanley-Reisner ring) of the dual simplicial polytope P^*. We shall also tie Gale diagrams into the exposition.

Traditionally (compare, among other references, [22, 19]) it is assumed that the vertex-set $V := \{v_1, \dots, v_n\}$ of P^* spans $\mathbb{E}^d$ affinely, with each v_i a rational vector. However, we shall drop the assumption of rationality. With each v_i is associated an indeterminate x_i. The *face ring* is then (for our purposes) the quotient of the polynomial ring $\mathbb{R}[x_1, \dots, x_n]$ by the homogeneous ideal $N := \mathrm{N}(P^*)$ generated by the *non-faces* of P^*, that is, by the elements $x_{j(1)} \dots x_{j(r)}$ such that $\mathrm{conv}\{v_{j(1)}, \dots, v_{j(r)}\}$ is not a face of P^*.

Actually, we are not so much interested in $\mathbb{R}[x_1, \dots, x_n]/N$ itself, as in a further quotient. We now assume that $o \in \mathrm{int\,conv}\, V$; we then additionally factor out by the ideal $H := H(P^*)$ generated by a suitable family of d elements $\sum_i \lambda_i x_i$ (how we find them will be discussed later); these form a *homogeneous system of parameters* (h.s.o.p.). This new ring $\mathbb{R}[x_1, \dots, x_n]/(N+H)$ we shall call the *reduced face ring* of P^*, and we shall denote it by $R := R(P^*)$ (Oda [19] uses the term *Chow ring*).

We see that R is a polynomial ring, and so is graded by degree; if we denote the grading by $R := \bigoplus_{r=0}^{d} R_r$, then Stanley [20] showed that $\dim R_r = h_r(P)$. A perspicuous proof of this result was given in [9].

We shall now prove

working in $\Xi_{2r}(\bar{S})$, except that we must change signs as appropriate. Let $\bar{K}$ be a $(2r)$-face of $\bar{S}$ of kind k. However we perform the volume calculation, we are taking the product of r-volumes of a face of kind i and one of kind $k-i$; that is, the weight on the corresponding face K of $P+Q$ will be $(-1)^k\sigma_{2r}$. In case $d=2r$, when $k=m$ is the only choice, this shows that $(-1)^m s^2>0$.

More generally, we again proceed by induction. If $2r>d$, in evaluating the restriction of $q^{d-2r-1}s^2$ to a facet G_j, we need only consider $j=m+1,\dots,d+1$. But now G_j was obtained from F_j by an m-flip; the obvious inductive assumption $(-1)^m(q^{d-2r-1}s^2)|_{G_j}>0$ leads at once to

Lemma 11.7 *If s is the r-evert, with $m\leqq 2r\leqq d$, then $(-1)^m q^{d-2r}s^2>0$.*

The lower restriction on r ensures that the product is indeed non-zero, but it will automatically be satisfied when we use the lemma.

12 Transition polytopes

The core of our proof consists in showing that a transition polytope T shares all the properties of the simple polytope P itself, as far as multiplication of elements of $\Pi(P)$ by $t:=\log T$ are concerned. We set up an exactly analogous inductive structure, which is given by the following results. For completeness (and because at one point we need one part of the result), we also describe some of its effects on $\Pi(Q)$. The same conventions as before apply, namely that Q is obtained from P by an m-flip, where $1\leqq m\leqq \frac{1}{2}(d+1)$.

Theorem 12.1 (a) *If $0\leqq r\leqq\frac{1}{2}d$, then $t^{d-2r}\Xi_r(P)=\Xi_{d-r}(P)$;*
(b) *if $0\leqq r<m$, or if $r=\frac{1}{2}d$, then $t^{d-2r}\Xi_r(Q)=\Xi_{d-r}(Q)$.*

Theorem 12.2 (a) *If $0\leqq r\leqq\frac{1}{2}d$, then the quadratic form $(-1)^r t^{d-2r}x^2$ is positive definite on $\{x\in\Xi_r(P)\,|\,t^{d-2r+1}x=0\}$;*
(b) *if $0\leqq r<m$, or if $r=m=\frac{1}{2}d$, then the quadratic form $(-1)^r t^{d-2r}y^2$ is positive definite on $\{y\in\Xi_r(Q)\,|\,t^{d-2r+1}y=0\}$.*

We begin by showing that Theorem 12.2 in dimension $d-1$ implies Theorem 12.1(a) (we shall leave part (b) until later). The proof is exactly analogous to that of Lemma 8.3, which showed that HRM$(d-1)\Rightarrow$ LD(d). First, we can suppose that $m>1$ (and hence $d\geqq 3$), since if $m=1$ we have $T=P$, and there is nothing new to prove. The cases $r=0,\frac{1}{2}d$ being trivial, suppose that $1\leqq r<\frac{1}{2}d$, and that $x\in\Xi_r(P)$ is such that $t^{d-2r}x=0$; we must show that $x=0$. As before, we look at a facet K of P, and use the fact that $t|_K^{d-2r}x|_K=0$ implies that $(-1)^r t|_K^{d-2r-1}x|_K^2\geqq 0$, with equality if and only if $x|_K=0$. Then we deduce that $(-1)^r t^{d-2r}x^2>0$ if $x\neq 0$, contradicting $t^{d-2r}x=0$.

The facets K are of two kinds. If K does not contain the special face F, then $t|_K\in\mathscr{K}(K)$, and we can appeal to HRM$(d-1)$ for K. If, on the other hand, K does contain F, then $t|_K\in\mathrm{cl}\,\mathscr{K}(K)$ corresponds to a transition polytope. There is now a split into two subcases.

If $m<\frac{1}{2}(d+1)$, then in fact $m\leqq\frac{1}{2}(d-1)$, and we can use Theorem 12.2(a), since the flip leading from K to the corresponding facet of Q is of type $m-1$ or m. If $m=\frac{1}{2}(d+1)$ (with d odd), then for those facets K for which the flip is of type m the rôles of P and Q for K are reversed, since an m-flip on K is the inverse of an $(m-1)$-flip; now, however, we can employ the last part of Theorem 12.2(b). This establishes the inductive step.

As a consequence, for each r the quadratic form $t^{d-2r}x^2$ is non-singular on $\Xi_r(P)$. If we now replace p by $p_\lambda:=(1-\lambda)t+\lambda p$ with $0<\lambda\leqq 1$, and let $\lambda\to 0$, the forms $p_\lambda^{d-2r}x^2$ and $t^{d-2r}x^2$ are non-singular of the same rank, and so have the same signature.

We now need to compare the form $t^{d-2r}x^2$ with $q^{d-2r}y^2$, with $y\in\Xi_r(Q)$. For this, the interaction between t and the evert $s:=s_r$ is crucial. The product ts is supported on the $(r+1)$-faces of $P+Q$ which meet $F+G$. In any such face, since we can choose the special vertex of T to be o, the induced support function of T is 0; thus $ts=0$ (compare the calculations in §11 above).

Our discussion now splits into two cases. Recall that, after Theorem 11.2, each $y\in\Xi_r(Q)$ can be written as $y=x+vs$, where $x\in\Xi_r(P)$ and $v\in\mathbb{R}$ are unique. If $r<m$ (and hence with our assumption $r<\frac{1}{2}d$), it follows that $t^{d-2r}y^2=t^{d-2r}x^2$; further, $\Xi_r(P)$ and $\Xi_r(Q)$ are isomorphic (Theorem 11.3). Replacing q by $q_\lambda:=(1-\lambda)t+\lambda q$ with $0<\lambda\leqq 1$, and taking the limit as $\lambda\to 0$, shows that the forms $t^{d-2r}y^2$ and $q_\lambda^{d-2r}y^2$ on $\Xi_r(Q)$, being non-singular of the same rank, have the same signature. That is,

Lemma 12.3 *If $r<m$, then the forms $p^{d-2r}x^2$ and $q^{d-2r}y^2$ have the same rank and signature.*

There remains to consider the case $m\leqq r\leqq\frac{1}{2}d$. By Theorem 11.3(b), $\Xi_r(P)$ is a subspace of $\Xi_r(Q)$ of codimension 1. Now $p^r s=0$ for all $p\in\mathscr{K}(P)$ by Lemma 11.5, and so we have $\Xi_r(P)s=\{0\}$, since these elements p^r generate $\Xi_r(P)$. Of course, the subspaces lin$s$ and $\Xi_r(P)$ are complementary in $\Xi_r(Q)$. With y, x and v as in our convention, we thus have

$$q^{d-2r}y^2=q^{d-2r}(x+vs)^2=q^{d-2r}x^2+v^2q^{d-2r}s^2\,.$$

The form $q^{d-2r}y^2$ is non-singular, and since $q^{d-2r}s^2\neq 0$ by Lemma 11.7, the form $q^{d-2r}x^2$ must also be non-singular on $\Xi_r(P)$. Replacing q by q_λ as before for $0<\lambda\leqq 1$, and letting $\lambda\to 0$, we conclude that, since the forms $t^{d-2r}x^2$ and $q_\lambda^{d-2r}x^2$ are non-singular with the same rank, they also have the same signature. In other words, using Lemma 11.7 again, we have

Lemma 12.4 *If $m\leqq r\leqq\frac{1}{2}d$, the form $q^{d-2r}y^2$ has rank 1 greater than that of $p^{d-2r}x^2$, with additional eigenvalue having the sign of $(-1)^m$.*

These two lemmas complete the argument, apart from tying up some loose ends. Recalling the first paragraph of §11, our proof above shows that, if Theorem 8.6 holds for P, then it holds for Q, and vice versa. We then obtain our given polytope from a simplex by a sequence of m-flips (with no restriction on m); since Theorem 8.6 is true for simplices, it holds generally.

However, we must also establish the remaining parts of Theorems 12.1 and 12.2. We have dealt with Theorem 12.1(a) above: note that it yields a Lefschetz decomposition of $\Pi(P)$ under multiplication by t. We have also shown that the forms $t^{d-2r}x^2$ on $\Xi_r(P)$ involving the transition polytope T have the same ranks and signatures as the forms $p^{d-2r}x^2$. Then Theorem 8.6 attaches the correct signs of the eigenvalues to the primitive subspaces of $\Pi(P)$ under multiplication by t instead of p, so that Theorem 12.2(a) holds.

We first consider the relationship between the spaces $\Xi_r(P)$ and $\Xi_r(Q)$. As might be expected, there is a fairly close connexion between them. Of course, we can regard them both as subspaces of $\Xi_r(P+Q)$, noting that $P+Q$ is also a simple d-polytope; if $m=1$ it is isomorphic to Q itself, while if $m \geqq 2$ it has $n+1$ facets, the extra facet normal u_0, say, being that to its facet $F+G$, which is a direct sum of an $(m-1)$- and a $(d-m)$-simplex.

We can show fairly directly that, if $m \leqq r \leqq d$, then $\Xi_r(P)$ is (in a natural way) a subspace of $\Xi_r(Q)$. However, we shall set this result in a more general context. The $d+1$ hyperplanes $H_j := \operatorname{aff} G_j$ (with $j=1,\ldots,d+1$) bound a d-simplex $\bar{S}$, say. (Observe that $\bar{S}$ is on the same side of $H_{m+1},\ldots,H_{d+1}$ as P and Q, and on the opposite side of $H_1,\ldots,H_m$.) For future convenience, we perform an affinity, if necessary, so that $\bar{S}$ is a regular simplex with unit edge-lengths (this simplifies some numerical calculations). If we write $\bar{s} := \log \bar{S}$, then $\Xi_r(\bar{S}) = \operatorname{lin} \bar{s}^r$ is 1-dimensional for each $r=0,\ldots,d$. The r-volume (or weight) of each r-face of $\bar{S}$ is then a constant, which we shall write as $(r!)^{-1}\sigma_r$. (Thus σ_r actually gives the weight on an r-face of $\bar{s}^r$.)

Now, we may observe that each face $\bar{K}$ of $\bar{S}$ corresponds to a parallel face K of $P+Q$, or even of P and Q, except for G and the opposite $(m-1)$-face, which is a scalar multiple of $-F$. Again, there is no loss of generality in supposing that this scalar multiple is 1. We say that K is of *kind* k if precisely k of the vertices of $\bar{K}$ lie outside G, and if K is an r-face, we give to K the weight $s_r(K) := (-1)^k \sigma_r$. We shall call s_r the *r-evert* (of P and Q); in some sense, we think of turning the simplex $\bar{S}$ inside out. Then

Lemma 11.1 *For each r, the evert s_r lies in $\Xi_r(P+Q)$.*

The calculation is straightforward, once it is noted that there are just k hyperplanes H_j containing K for which $\bar{S}$ is on the oppsoite side of H_j to P or Q. Then k induces a corresponding change of sign of the outer normal vector at $\bar{K}$ to an $(r+1)$-face of $\bar{S}$ which contains it.

We shall write $s_r =: s$, if the number r can be understood from the context (recall that m is always fixed). Then we have the following

Theorem 11.2 *Each element of $\Xi_r(Q)$ differs from one of $\Xi_r(P)$ by a unique scalar multiple of the r-evert.*

To see this, we section $P+Q$ in a direction almost parallel to u_0, so that the vertices of $F+G$ are those which are encountered last. The elements of the corresponding section bases of $\Xi_r(P)$ and $\Xi_r(Q)$ clearly coincide on all r-faces which do not meet F or G. Subtracting one from the other, and changing signs as appropriate, gives an r-weight on $\bar{S}$, which is just a multiple of $\bar{s}^r$. The result now follows.

To avoid constant repetition, we introduce a convention we shall follow hereafter. If $y \in \Xi_r(Q)$, we shall write $y = x + vs$, with $x \in \Xi_r(P)$ and $v \in \mathbb{R}$ specified uniquely by Theorem 11.2.

Theorem 11.2 now leads to our earlier claim, since if $r \geqq m$, the r-evert already belongs to $\Xi_r(Q)$ (the evert can have no component in F, and so is supported entirely by faces of Q). In fact, counting dimensions, we have more.

Theorem 11.3 (a) *If $0 \leqq r < m$, then $\Xi_r(P) \cong \Xi_r(Q)$;*
(b) *if $m \leqq r \leqq d-m$, then $\Xi_r(P)$ is a subspace of $\Xi_r(Q)$ of codimension* 1;
(c) *if $d-m < r \leqq d$, then $\Xi_r(P) = \Xi_r(Q)$.*

It will help future calculations if we make the following observation. Define

$$E(P,Q) := \{x \in \Pi(P+Q) \mid x|_J = 0 \text{ if } J \text{ is a face such that } J \cap (F+G) = \emptyset\};$$

that is, regarded as weights, the elements of $E(P,Q)$ are supported by the faces of $P+Q$ which do meet $F+G$.

Theorem 11.4 *$E(P,Q)$ is an ideal in $\Pi(P+Q)$.*

The proof is easy: if $e \in E(P,Q)$ and $x \in \Pi(P+Q)$, then for each face J of $P+Q$ which does not meet $F+G$, we have

$$(ex)|_J = e|_J x|_J = 0,$$

so that $ex \in E(P,Q)$ also. The remaining properties of an ideal are obvious.

We call $E(P,Q)$ the *evert ideal* of $\Pi(P+Q)$, and write $E_r(P,Q) := E(P,Q) \cap \Xi_r(P+Q)$. Clearly, each evert lies in $E(P,Q)$; it is not hard to see that $E(P,Q)$ is actually generated by the everts. The advantage of Theorem 11.4 is that it enables us to calculate its elements by their restrictions to the hyperplanes normal to $u_0,\ldots,u_{d+1}$.

Let us now make a further observation. Let $z \in E_{d-1}(P,Q)$. The $z|_{F_j} = 0$ for each $j = d+2,\ldots,n$, so that the product of z by anything in Ξ_1 depends only on its (generalized) support parameters in directions $u_0,\ldots,u_{d+1}$. In particular, such a product by an element of $\Xi_1(P+Q)$ depends effectively on its support parameters in the restriction to $F+G$.

We can now set up an inductive scheme. As we saw in Lemma 10.3, if $j=1,\ldots,m$, then F_j is turned into G_j by an $(m-1)$-flip. Further, the rôle of F relative to F_j is now played by one of its facets ($(m-2)$-faces), whereas $G \subseteq G_j$ plays the same rôle as it did before. For $j = m+1,\ldots,d+1$, of course, the situation is reversed. We are now in the position to perform some calculations.

Lemma 11.5 *Let s be the r-evert, and let $p \in \mathscr{K}(P)$. If $k \geqq m$, then $p^k s = 0$.*

To prove this, we bear in mind the observation above. Since our calculations depend on F rather than on P, and since the class $[F]$ of F has no k-component if $k \geqq m$, the mixed volume calculations for $p^k s$ must yield 0, as required.

If that proof seems too quick, we can proceed alternatively as follows. Let $z \in E_{d-k}(P,Q)$, with $k \geqq m$ as before. As usual, we are taking $m \geqq 1$. We can suppose that the origin $o \in F$; thus the support function of P in any direction $u_0, u_{m+1},\ldots,u_{d+1}$ is 0, and in calculating $p^k z$, we need only evaluate $(p^{k-1}z)|_{F_j}$ for $j=1,\ldots,m$. In such an F_j, we see that z restricts to an element of an evert ideal of type $m-1$. We can now appeal to induction (we have lowered both k and m by 1), to say that $(p^{k-1}z)|_{F_j} = 0$; if $m=1$, then the support function of P at F is identically 0. Thus $p^k z = 0$.

More generally, let $z \in E_r(P,Q)$, with $r < d-k$. If $p^k z \neq 0$, then we can find some $w \in \Xi_{d-k-r}(P+Q)$, such that $p^k z w \neq 0$. But $zw \in E_{d-k}(P,Q)$, contradicting what we have just shown. We conclude that we have proved

Lemma 11.6 *Let $z \in E(P,Q)$, and let $k \geqq m$. Then $p^k z = 0$.*

A similar argument would apply, to show that $q^k z = 0$ whenever $k > d-m$; however, we shall not need such a case. What we shall need to find is $q^{d-2r}s^2$, when s is the r-evert. We begin with s^2. We can perform this calculation as if we were

10 Flips

We shall prove Theorem 7.3, which leads to the Lefschetz decomposition of $\Pi(P)$ and hence to a proof of the g-theorem, through the stronger Riemann-Hodge-Minkowski inequalities which imply it. We shall take the latter as stated equivalently in Theorem 8.6, and study how the quadratic form $p^{d-2r}x^2$ on $\Xi_r(P)$ changes as the combinatorial type of P changes.

We shall once again take P to be a simple d-polytope with n facets, except that now we shall assume that the (unit) normal vectors $u_1, \ldots, u_n$ to the facets of P are in linearly general position (that is, no d of them lie in any linear hyperplane of $\mathbb{E}^d$). As we saw in §9, if we can prove our results for such a polytope, then we can prove them for any simple polytope.

Our assumption implies that the corresponding linear transform $\bar{U} = (\bar{u}_1, \ldots, \bar{u}_n)$ of $U = (u_1, \ldots, u_n)$ also has its points in linearly general position in $\mathbb{E}^{n-d}$ (see [14, 15]). We shall see what happens as p moves from the original full-dimensional type cone $\mathscr{K}(P)$ to an adjacent one.

Let us first describe the way the combinatorial type of P changes. Suppose that our labelling of the normal vectors is chosen so that initially $\mathscr{K}(P)$ is bounded by (among others) the hyperplane $H := \operatorname{lin}(\bar{u}_{d+2}, \ldots, \bar{u}_n\}$, and that $\bar{u}_1, \ldots, \bar{u}_m$ lie on the same side of H as the representative p of P (recall that we identify this with $\log P$), with $\bar{u}_{m+1}, \ldots, \bar{u}_{d+1}$ on the other. If we can pass from $\mathscr{K}(P)$ to $\mathscr{K}(Q)$ (where $Q \in \mathscr{P}(U)$ is also a simple polytope) across $C := \operatorname{relint} \operatorname{pos}\{\bar{u}_{d+2}, \ldots, \bar{u}_n\}$, then we say that Q is obtained from P by an *m-flip*. The inverse operation which yields P from Q is clearly a $(d-m+1)$-flip.

The cases $m = 0$ and 1 are somewhat special. A 0-flip just brings into being a d-simplex (from nothing), and thus a $(d+1)$-flip destroys one; we shall usually ignore this case, since simplices are well understood (in our context; see §11 below). A 1-flip introduces a new facet, since $\bar{U} \setminus \{\bar{u}_1\}$ is not a cofacial set for P. This causes a small notational problem, since in order to keep to the above notation, we shall have to assume that here P only has $n-1$ facets. In any discussion of the case $m = 1$, the appropriate notational changes are tacitly assumed.

A straightforward calculation (compare [13, 18]) yields

Lemma 10.1 *If Q is obtained from P by an m-flip, then*

$$g_r(Q) = g_r(P) + \delta_{rm} - \delta_{r,d-m+1}\,.$$

Here, δ_{rm} is the usual Kronecker delta.

An equivalent formulation of this result is

Corollary 10.2 *If $0 \leqq m \leqq \frac{1}{2}d$ and Q is obtained from P by an m-flip, then* $\dim \tilde{\Xi}_r(Q) = \dim \tilde{\Xi}_r(P) + \delta_{rm}$.

Note that a $\frac{1}{2}(d+1)$-flip changes the combinatorial type of P, but does not change the dimension of any of the weight spaces.

Let us now describe the effect of an m-flip in direct combinatorial terms. With the conventions above, the incidence relationships involving any facets F_j of P (or G_j of Q) with $j \geqq d+2$ are not affected by the flip. In P, the intersection $F := F_{m+1} \cap \ldots \cap F_{d+1}$ is an $(m-1)$-simplex, bounded in its affine hull by the facets $F_1, \ldots, F_m$. After the flip, in Q the pattern is reversed: the corresponding intersection $G := G_1 \cap \ldots \cap G_m$ is a $(d-m)$-simplex, bounded in its affine hull by $G_{m+1}, \ldots, G_{d+1}$. We shall refer to these faces F of P and G of Q as the *special faces*.

At the transition between the two combinatorial types, all these $d+1$ facets meet in a single point; we denote such a corresponding polytope by T, and call it a *transition polytope*. When $m = 1$, the transition polytope is P itself. We may remark that, in the dual context, the simplicial polytope Q^* is obtained from P^* by a bistellar operation.

The line segment between $p \in \mathscr{K}(P)$ and $q \in \mathscr{K}(Q)$ will always cross the relatively open cone C defined above, and so some transition polytope T has a representative of the form $t = (1-\lambda)p + \lambda q$ with $0 < \lambda < 1$. However, the linear extensions of $\mathscr{K}(P)$ and $\mathscr{K}(Q)$ in Ξ_1 to $\mathbb{E}^{n-d}$ are different; they coincide only on the hyperplane $\operatorname{lin}(\operatorname{cl}\mathscr{K}(P) \cap \operatorname{cl}\mathscr{K}(Q))$. Thus, while $t = \log T$ in $\Pi(P)$ and $\Pi(Q)$, it is not expressible as $t = (1-\lambda)p + \lambda q$ in Ξ_1 (this would be $\log((1-\lambda)P + \lambda Q)$, with the latter polytope strongly isomorphic to $P + Q$).

An alternative picture is also helpful. Making a possibly different choice of $p \in \mathscr{K}(P)$ and $q \in \mathscr{K}(Q)$, we can suppose that $q - p$ is in a direction $\bar{u}_0$ in general position with respect to $\bar{U}$, such that $\operatorname{pos}(\bar{U} \cup \{\bar{u}_0\})$ is still a pointed cone. We now take $(w_0, \ldots, w_n) \subset \mathbb{E}^{d+1}$ to be a linear transform of $(\bar{u}_0, \ldots, \bar{u}_n)$. Then p and q represent different parallel sections perpendicular to the direction w_0 of a simple $(d+1)$-polytope with facet normals $w_1, \ldots, w_n$; these sections are (affinely equivalent to) P and Q. At the transition polytope T, the section passes a vertex of type m.

We remark that we may clearly choose a vector $\bar{u}_0$ so that the ray from p in direction $-\bar{u}_0$ does not meet the intersection of any two hyperplanes in $\mathbb{E}^{n-d}$ which are spanned by vectors in $\bar{U}$, and eventually leaves $\operatorname{pos}\bar{U}$. If we then proceed along this ray in the opposite direction $\bar{u}_0$ starting outside $\operatorname{pos}\bar{U}$, we see that P can be obtained (even in $\mathscr{P}(U)$) from nothing (or a simplex) by a sequence of flips.

Under an m-flip, the faces of P which do not meet the special face F keep the same combinatorial type. On the other hand, the faces of P which do meet F are themselves flipped. Let K be such a k-face. Then K is the intersection of $d-k$ facets of P. If say, l of the normals to these facets come from $u_1, \ldots, u_m$ (and the rest from $u_{m+1}, \ldots, u_{d+1}$), then we see from the representation that the flip acts as an $(m-l)$-flip on K. In particular, in the notation introduced above,

Lemma 10.3 *Let Q be obtained from P by an m-flip, with $1 \leqq m \leqq d$. Then*

(a) *if $j = 1, \ldots, m$, then G_j is obtained from F_j by an $(m-1)$-flip;*
(b) *if $j = m+1, \ldots, d+1$, then G_j is obtained from F_j by an m-flip.*

11 Everts

In the discussion in this section and the next, we shall suppose that Q is obtained from P by an m-flip, and we henceforth make the blanket assumption that $1 \leqq m \leqq \frac{1}{2}(d+1)$ (we can clearly ignore the case $m = 0$). We wish to compare the quadratic forms $p^{d-2r}x^2$ and $q^{d-2r}y^2$ for $x \in \Xi_r(P)$ and $y \in \Xi_r(Q)$, where $0 \leqq r \leqq \frac{1}{2}d$. We shall show that the changes in the rank and signature of the form depend only on the type of the flip, and not on the polytope to which it is applied. That is, we shall show that the rank and signature are preserved if $0 \leqq r < m$, and that the form has rank 1 greater, with additional eigenvalue having the sign of $(-1)^m$, when $m \leqq r \leqq \frac{1}{2}d$. We then obtain our given polytope from a simplex (or nothing) by a sequence of flips, and the quadratic forms change in the correct way to yield Theorem 8.6. We remark that, if P is a d-simplex, then $h_r(P) = 1$ for each $r = 0, \ldots, d$, and so the quadratic forms $p^{d-2r}x^2$ are all positive definite of rank 1.

this covers the cases $d \leq 1$. Further, for $r = \frac{1}{2}d$ there is nothing to prove. If $1 \leq r < \frac{1}{2}d$, let F be any facet of P, and write $f := p|_F$ and $y := x|_F$. Then $f^{d-2r}y = (p^{d-2r}x)|_F = 0$. By HRM$(d-1)$ with F instead of P, and noting that $f = \log F$, we have $(-1)^r f^{d-2r-1}y^2 \geq 0$, with strict inequality unless $y = 0$. Multiplying by p (using the mixed volume calculation, and noting that the support parameters of P are all positive after a suitable translation), we deduce that $(-1)^r p^{d-2r}x^2 = p \cdot (-1)^r p^{d-2r-1}x^2 \geq 0$, with strict inequality unless $x|_F = 0$ for each facet F of P, that is, unless $x = 0$. Since $(-1)^r p^{d-2r}x^2 > 0$ contradicts $p^{d-2r}x = 0$, we conclude that $x = 0$, as required.

In the rest of this section, we shall assume that HRM$(d-1)$ holds, and hence that we have the Lefschetz decomposition of $\Pi(P)$. We next recall Theorem 5.2, which says that the vector spaces $S\Xi_r(P)$ and $\Xi_{d-r}(P)$ are in duality. If $0 \leq s \leq d$, and A is a subspace of $\Xi_s(P)$, we write

$$A^{\perp} := \{y \in \Xi_{d-s}(P) \mid yA = 0\},$$

which we call the *annihilator* of A.

Lemma 8.4 *If $0 \leq r \leq \frac{1}{2}d$, then*

$$\tilde{\Xi}_r(P) = (p^{d-2r+1}\Xi_{r-1}(P))^{\perp}.$$

The proof is straightforward. If $x \in \tilde{\Xi}_r(P)$, so that $x \in \Xi_r(P)$ satisfies $p^{d-2r+1}x = 0$, then $p^{d-2r+1}xy = 0$ for all $y \in \Xi_{r-1}(P)$, and hence $x \in (p^{d-2r+1}\Xi_{r-1}(P))^{\perp}$. The argument is reversible, and the lemma follows.

For our purposes, there is an important consequence.

Theorem 8.5 *If $1 \leq r \leq \frac{1}{2}d$, then $\Xi_r(P) = p\Xi_{r-1}(P) \oplus \tilde{\Xi}_r(P)$. Moreover, if $x \in \Xi_r(P)$ is expressed as $x = py + z$ with $y \in \Xi_{r-1}(P)$ and $z \in \tilde{\Xi}_r(P)$, then*

$$p^{d-2r}x^2 = p^{d-2r+2}y^2 + p^{d-2r}z^2.$$

Since we are assuming LD(d), we know that the subspaces $p\Xi_{r-1}(P)$ of $\Xi_r(P)$ and $p^{d-2r+1}\Xi_{r-1}(P)$ of $\Xi_{d-r}(P)$ both have dimension $h_{r-1}(P) = \dim \Xi_{r-1}(P)$, and so the subspaces of the theorem have complementary dimensions in $\Xi_r(P)$. Suppose then that

$$x \in p\Xi_{r-1}(P) \cap (p^{d-2r+1}\Xi_{r-1}(P))^{\perp}$$

(here we are using Lemma 8.4). Since $x = py$ for some $y \in \Xi_{r-1}(P)$, it follows that

$$p^{d-2r+2}y\Xi_{r-1}(P) = (py)(p^{d-2r+1}\Xi_{r-1}(P)) = xp^{d-2r+1}\Xi_{r-1}(P) = 0,$$

so that $y = 0$, since $p^{d-2r+2}\Xi_{r-1}(P) = \Xi_{d-r+1}(P)$ separates $\Xi_{r-1}(P)$, and hence $x = 0$. Thus the subspace are indeed complementary. The decomposition of the quadratic form is an immediate consequence.

Thoerem 8.5 can be thought of as saying that the quadratic form $p^{d-2r+2}y^2$ with $y \in \Xi_{r-1}(P)$ is a subform of $p^{d-2r}x^2$ on $\Xi_r(P)$. In fact, it is usually more convenient to state Theorem 8.2 in an equivalent form.

Thoerem 8.6 *If $0 \leq r \leq \frac{1}{2}d$, then the quadratic form $(-1)^r p^{d-2r}x^2$ on $\Xi_r(P)$ has $\sum_{i=0}^{r}(-1)^i h_{r-i}(P)$ positive eigenvalues, and $\sum_{i=0}^{r-1}(-1)^i h_{r-i-1}(P)$ negative ones, relative to any positive definite form on $\Xi_r(P)$.*

It is clear that the statement of the theorem is implied by Theorem 8.2, when we bear in mind the remark above, about the embedding of one form in the other. For the converse, we again use the remark, and Theorem 8.1, which states that the form is non-singular. The subform $p^{d-2r+2}y^2$ accounts for $h_{r-1}(P)$ of the eigenvalues with appropriate signs given by the theorem with $r-1$ in place of r (the initial case $r = 0$ is trivial). There remain $g_r(P) = h_r(P) - h_{r-1}(P)$ eigenvalues, which all have the sign of $(-1)^r$. These must be attached to the complementary form $p^{d-2r}x^2$ with $x \in \tilde{\Xi}_r(P)$, and Theorem 8.2 thus follows.

9 Continuity arguments

It is sometimes helpful to be able to prove a result in a somewhat special case, and the use continuity arguments to establish it in general. We shall discuss such continuity arguments in this section. Since, as before, we are assuming an inductive framework, we shall suppose that the Hodge-Riemann-Minkowski inequalities HRM$(d-1)$ hold, and hence that we have the Lefschetz decomposition LD(d) (Lemma 8.3).

We shall prove a very general result.

Theorem 9.1 *If Theorem 8.6 holds for one choice of simple d-polytope P, then it holds for all polytopes Q in the same connected component of its isomorphism class.*

This means that, if Theorem 8.6 holds (and hence Theorems 7.3 and 8.2 do also) for one particular polytope P, and we can obtain Q from P through a continuous family of isomorphic polytopes, then the corollary holds for Q as well. As a special case, we can thus vary P freely within its strong isomorphism class (or, equivalently, vary $p := \log P$ freely within $\mathscr{K}(P)$).

We prove the theorem as follows. There is a technical problem, because we are changing the subalgebra $\Pi(P)$ with P, and so the subspace $\Xi_r(P)$ also changes, particularly if we think of it as the $h_r(P)$-dimensional subspace of a real vector space of dimension $f_r(P)$ (with coordinates indexed by the r-faces of P) determined by the Minkowski relations, since these relations also vary as P varies, although obviously in a continuous way (and so the subspace itself varies continuously as a coordinate subspace). However, it will then follow that the primitive subspace $\tilde{\Xi}_r(P)$ also varies continuously with P.

We may, though, regard $\Xi_r(P)$ as an $h_r(P)$-dimensional space, coordinatized relative to a section basis constructed as in §6. If Q is combinatorially isomorphic to P, and close enough to it in the Hausdorff metric, then the basic r-faces of Q will be those corresponding to the basic r-faces of P, since vertices of the two polytopes will be of the same type with respect to the moving section. The induced weights on the remaining r-faces will clearly change continuously with Q, and thus so will the quadratic form $q^{d-2r}y^2$ with $q := \log Q$, since we can regard $y \in \Xi_r(Q)$ as a vector in a fixed space. The form is always non-singular with a fixed rank (we use the induction HRM$(d-1) \Rightarrow$ LD(d) as before) and so has the same signature within some neighbourhood of the original P, yielding Theorem 8.6 for Q.

We may remark that any simple d-polytope P can be approximated as closely as we wish by isomorphic polytopes with rational vertices. Thus any strictures apparently imposed by rationality can already be removed. We may also tilt the facets of P slightly, and so approximate P by isomorphic polytopes whose facet normals are in linearly general position.

equations for P are therefore equivalent to

$$g_r(P) = -g_{d+1-r}(P)$$

for each r.

If a, r are positive integers, then the *r-canonical representation* of a is its unique expression in the form

$$a = \binom{a_r}{r} + \binom{a_{r-1}}{r-1} + \cdots + \binom{a_i}{i},$$

with

$$a_r > a_{r-1} > \ldots > a_i \geqq i \geqq 1 .$$

If s is another positive integer, then the *partial power* $a^{\langle s|r \rangle}$ is defined by

$$a^{\langle s|r \rangle} := \binom{a_r + s - r}{s} + \binom{a_{r-1} + s - r}{s-1} + \ldots + \binom{a_i + s - r}{i + s - r},$$

with the usual conventions for binomial coefficients (which are not needed if $s > r$). We also define $0^{\langle s|r \rangle} := 0$ for all r and s. Finally, we call a sequence $(h_0, h_1, \ldots)$ of integers an *M-sequence* if $h_0 = 1$, and $0 \leqq h_{r+1} \leqq h_r^{\langle r+1|r \rangle}$ for $r \geqq 0$ (the upper inequality has no force if $r = 0$).

It is well known (see [11, 21]) that we have

Lemma 7.1 *There is a graded (commutative) algebra $R = \bigoplus_{r \geqq 0} R_r$ over a field $\mathbb{F}$ generated by the finite dimensional R_1, with $R_0 \cong \mathbb{F}$, if and only if the sequence given by $h_r := \dim_{\mathbb{F}}(R_r)$ is an M-sequence.*

The inequalities $h_{r+1}(P) \leqq h_r(P)^{\langle r+1|r \rangle}$ resulting from applying Theorem 6.1 and Lemma 7.1 to the subalgebra $\Pi(P)$ yield Stanley's strong form of the upper bound theorem in [20].

In [13] (in the dual formulation for simplicial polytopes), McMullen conjectured the following, which was proved by Billera and Lee [3] (sufficiency) and Stanley [22] (necessity).

Theorem 7.2 (*g*-theorem) *For $g = (g_0, \ldots, g_{d+1})$ to be the g-vector of some simple d-polytope, it is necessary and sufficient that g satisfy*

(a) $g_r = -g_{d+1-r}$ *for each r,*
(b) $(g_0, \ldots, g_{\lfloor d/2 \rfloor})$ *is an M-sequence.*

The conditions of the *g*-theorem are often called *McMullen's conditions.* The sufficiency of the conditions will not concern us here. Bearing in mind Lemma 7.1, we see that the necessity of the *g*-theorem for a simple *d*-polytope P follows immediately if we can construct a polynomial algebra $R = \bigoplus_{r \leqq \lfloor d/2 \rfloor} R_r$ with $\dim R_r = g_r(P)$.

The approach which we adopt here mimics (in a sense) that of Stanley, except that it works within the polytope algebra, avoiding the need to use deep techniques of algebraic geometry, namely the hard Lefschetz theorem applied to the cohomology ring of the toric variety associated with a simple (or simplicial) polytope with rational vertices. However, although our methods are quite different, the initial motivation for them came from [22].

The central result is

Theorem 7.3 *Let P be a simple d-polytope, let $p := \log P$, and let $0 \leqq r \leqq \frac{1}{2}d$. Then $p^{d-2r}\Xi_r(P) = \Xi_{d-r}(P)$.*

In other words, multiplication by p^{d-2r} induces an isomorphism between $\Xi_r(P)$ and $\Xi_{d-r}(P)$. In analogy to [22], the quotient algebra $R := \Pi(P)/\langle p \rangle$ (where $\langle p \rangle$ is the ideal of $\Pi(P)$ generated by p) is that which satisfies the condition above, and so gives rise to Theorem 7.2(b)

Theorem 7.3 leads at once to the *Lefschetz decomposition* of $\Pi(P)$ (compare [6, p. 122]). For $0 \leqq r \leqq \frac{1}{2}d$, we define the *r-th primitive space* of $\Pi(P)$ to be

$$\tilde{\Xi}_r(P) := \{x \in \Xi_r(P) \mid p^{d-2r+1}x = 0\} ;$$

then

$$\Xi_s(P) = \bigoplus_{r=0}^{s} p^{s-r}\tilde{\Xi}_r(P) ,$$

where we set $\tilde{\Xi}_r(P) = 0$ if $r > \lfloor \frac{1}{2}d \rfloor$.

The results we shall prove are actually stronger than can be deduced from algebraic geometry. For example, we can dispense with the artifical assumption that P has rational vertices. Further, from later analysis of Stanley's proof, it appeared that some kind of genericity might be necessary in the choice of hyperplane section (the analogue of p); however, we shall see that this is not the case.

8 Quadratic inequalities

The approach we follow will use induction on the dimension d, and will involve investigating certain quadratic forms on the weight space $\Xi_r(P)$.

In view of the fact that $\Xi_r(P)$ separates $\Xi_{d-r}(P)$, Theorem 7.3 has an equivalent formulation.

Theorem 8.1 *Let P be a simple d-polytope, let $p := \log P$, and let $0 \leqq r \leqq \frac{1}{2}d$. Then the quadratic form $p^{d-2r}x^2$, with $x \in \Xi_r(P)$, is non-singular.*

In fact, more than this is true. In analogy to the Hodge-Riemann bilinear inequalities on the cohomology ring (compare [6, p. 123]), we have

Theorem 8.2 (Hodge-Riemann-Minkowski inequalities) *Let $0 \leqq r \leqq \frac{1}{2}d$. Then the quadratic form $(-1)^r p^{d-2r}x^2$ is positive definite on the primitive space $\tilde{\Xi}_r(P)$.*

These inequalities were named by Jonathan Fine, who also drew attention to their counterparts in algebraic geometry (private communication). The case $r = 1$ is essentially Minkowski's second inequality for mixed volumes, which completes the account of the nomenclature.

We first consider the relationship between Theorems 7.3 and 8.2; the polytope P will remain fixed during the discussion. One step in the proof of the theorems is inductive, so we shall write LD(d) to stand for the existence of the Lefschetz decomposition, and HRM(d) to mean that the Hodge-Riemann-Minkowski inequalities hold, in a given dimension d. We first show

Lemma 8.3 *If $d \geqq 1$, then* HRM$(d-1)$ *implies* LD(d).

Since $\dim \Xi_r(P) = \dim \Xi_{d-r}(P) = h_r(P)$, it suffices to prove that, if $x \in \Xi_r(P)$ is such that $p^{d-2r}x = 0$, then $x = 0$. If $r = 0$, then $x = 0$ is clear, since $p^d > 0$ is a number;

Lemma 5.4 *Let* $0 \leqq r \leqq d$, *and let* $x \in \Omega_r(P)$ *be such that* $x \neq 0$. *Then there exists* $y \in \Xi_{d-r}(P)$ *such that* $xy \neq 0$.

The extreme case $r = d$ is trivial. The crucial case is $r = d - 1$; we shall prove that first. If $x \in \Omega_{d-1}(P)$ with $x \neq 0$, then there is some facet F of P for which the restriction $x|_F$ of x to F does not vanish. If $F = F_j$ and $\bar{u} := \bar{u}_j \in \Xi_1(P)$ corresponds to F_j (that is, we choose $\bar{u}$ with support parameters $\eta_j = 1$ and $\eta_k = 0$ if $k \neq j$), then, by direct calculation, $x|_F = x\bar{u}$ (these are both numbers). That is, $x\bar{u} \neq 0$, as required.

Now suppose that $r < d - 1$; we proceed by induction on d. Since $x \neq 0$, we can find some r-face G of P on which the weight $x|_G$ induced by x does not vanish. Let F be any facet of P which contains G; then $x|_F \neq 0$. By the inductive assumption, and using Theorem 2.4 (which says that $x \mapsto x|_F$ maps $\Pi(P)$ onto $\Pi(F)$), we can find a $z \in \Xi_{d-r-1}(P)$, such that $(xz)|_F = x|_F z|_F \neq 0$. With $\bar{u}$ as above, and $y = z\bar{u}$, we then have $xy = xz\bar{u} \neq 0$, and we have completed the proof.

Lemma 5.4 says that $\Xi_{d-r}(P)$ separates $\Omega_r(P)$. There follows at once

$$\dim \Xi_r(P) \leqq \dim \Omega_r(P) \leqq \dim \Xi_{d-r}(P) .$$

Interchanging the rôles of r and $d - r$ shows that we have equality throughout, and the two theorems are immediate consequences.

We may note that Theorem 5.2 already shows that $\Xi_r(P)$ and $\Xi_{d-r}(P)$ have the same dimension; we shall identify this dimension in §6 below. The argument of the first part of Lemma 5.4 can be extended to yield

Theorem 5.5 *Let* $x \in \Xi_r(P)$, *and let* G *be an* r-*face of* P. *If* $F_1, \ldots, F_{d-r}$ *are the facets of* P *which contain* G, *and if* $\bar{u}_1, \ldots, u_{d-r}$ *are the corresponding elements of* $\Xi_1(P)$, *then*

$$x|_G = \det U x\bar{u}_1 \ldots \bar{u}_{d-r} ,$$

where $\det U := \sqrt{\det UU^T}$, *and* U *is the* $(d - r) \times d$ *matrix whose rows* $u_1, \ldots, u_{d-r}$ *are the unit normal vectors to* P *at* $F_1, \ldots, F_{d-r}$.

Actually, it is not necessary here to assume that the u_j are unit vectors, since scaling u_j (and hence $\det U$) by $\lambda_j > 0$ scales $\bar{u}_j$ by λ_j^{-1}.

6 The dimensions of the weight spaces

We now establish a basic result of the paper. As usual, P is our fixed simple d-polytope with n facets.

Theorem 6.1 *If* $0 \leqq r \leqq d$, *then* $\dim \Xi_r(P) = h_r(P)$.

In fact, we shall use Theorem 5.1, and work with the spaces $\Omega_r(P)$. The idea of the proof is to show that at each vertex of type r acquired by the moving half-space of §4, we can freely assign an r-weight to the corresponding r-face of P.

It is easy to see that we have no more freedom than this. Obviously, we can make no new assignment of r-weight at a verttex of type s with $s < r$. At a vertex of type s with $s > r$, we similarly have no freedom, because the Minkowski relations completely determine the r-weights on the newly acquired r-faces through the vertex.

The only problem is that of consistency, that is, that the assignment of weight on the new r-faces determined by the Minkowski relations on the $(r + 1)$-face through the vertex (in the latter case above) is independent of the particula $(r + 1)$-faces from which they are calculated. There is nothing to prove if $s = r + 1$ and it is clear that the general case $s \geqq r + 2$ will follow from the case $s = r + 2$

So, we change the notation, and write $r = d - 2$. Suppose that we have assigned a weight $\omega(F')$ to each $(d - 2)$-face F' of a simple d-polytope P which does no contain a fixed vertex v of P, in such a way that the weights satisfy the Minkowsk relations where appropriate. A $(d - 2)$-face F of P which contains v also contain $d - 2$ of the d edges of P through v; let L be the plane spanned by the remainin 2 edges, and fix an orientation of L. The section of P by a general translate o L (which does not contain a vertex of P) is a polygon, whose vertices correspond t $(d - 2)$-faces of P, and whose edges correspond to facets. With respect to th orientation, the $(d - 2)$-faces of a given facet G of P can be called *lower* or *upper* according as they are encountered first or second in going around such a polygona section.

For each $(d - 2)$-face F', there is a constant $\gamma(F') \geqq 0$, which is the factor b which $(d - 2)$-volume is multiplied under the projection on a $(d - 2)$-space ortho gonal to L; in fact, $\gamma(F') = 0$ precisely when the affine hull of F' contains a translat of L. The Minkowski relations say that, for each facet G of P,

$$\sum_{\text{lower } F' \subset G} \gamma(F')\omega(F') = \sum_{\text{upper } F' \subset G} \gamma(F')\omega(F') .$$

If $F \subset G$, this then assigns $\omega(F)$ (with respect to G), since F is the only $(d - 2)$-fac of P through v whose affine hull does not contain a translate of L. Now, a $(d - 2)$ face which is upper for one facet is lower for the next in the orientation. Hence, if w sum up the above relations over all facets G, the terms $\gamma(F')\omega(F')$ all cancel, excep possibly for $F' = F$. But then they must cancel for $F' = F$ also, and the assignmen of $\omega(F)$ is unique.

Finally, we remark that there are $h_r(P)$ vertices of P of type r, and this prove the theorem.

There is an obvious way to construct a basis of $\Xi_r(P)$ using Theorem 6.1 A typical member of the basis is obtained by assigning the weight 1 to on particular r-face acquired at a vertex of P of type r (we call this a *basic* face), an 0 to all the other such r-faces. What results is the *section basis* of $\Xi_r(P)$, with respec to the given direction v.

7 The g-theorem

As stated in §1, the main aim of the present investigation is to present a proof o what is called the *g-theorem* entirely within convexity. We shall now describe thi theorem.

The *g-vector* $(g_0(P), \ldots, g_{d+1}(P))$ of the simple d-polytope P is given in term of $g(P, \tau) := \sum_{r=0}^{d+1} g_r(P)\tau^r$ by

$$g(P, \tau) := (1 - \tau)h(P, \tau) = (1 - \tau)f(P, \tau - 1) .$$

Thus $g_r(P) = h_r(P) - h_{r-1}(P)$ for each r. We can recover $h_r(P)$ from the $g_r(P)$ b $h(P, \tau) = (1 - \tau)^{-1}g(P, \tau)$, or $h_r(P) = \sum_{s \leqq r} g_s(P)$. Note that the Dehn-Sommervill nough to multiply the r-component $[P]_r$ of the class of a polytope P by the -component $[Q]_s$ of that of Q. Now the lifting theorem of [24], applied to the $+ s)$-faces of $P + Q$, shows that such an $(r + s)$-face can be dissected into a union f direct sums $F + G$ of faces F of P and G of Q, with $\dim F + \dim G = r + s$. If im $F = r$ and $\dim G = s$, then

$$\mathrm{vol}_{r+s}(F + G) = \alpha_{F,G}\,\mathrm{vol}_r(F)\,\mathrm{vol}_s(G)\,,$$

here the constant $\alpha_{F,G}$ depends only on the "angle" between the affine hulls of and G (it is what is obtained if F and G are replaced by unit cubes with the same ffine hulls). The sum of these volume terms (over the given $(r + s)$-face) now gives he appropriate contribution, thought of as a weight, to Ξ_{r+s}. (Note that there are o binomial coefficients as factors, since we are not calculating mixed volumes, but roducts in $\Pi(P)$.)

If we now mechanically replace the terms $\mathrm{vol}_r(F)$ and $\mathrm{vol}_s(G)$ by r-and -weights, we appear to extend the multiplication from $\Xi_r(P)\otimes\Xi_s(Q)$ to $_r(P)\otimes\Omega_s(Q)$. However, two things are far from clear: first, that the resulting xpression is independent of the dissection of the faces, and second, that it is in $_{r+s}(P + Q)$, in other words, that the Minkowski relations still hold. In [17], we hall show this in general; it is just an extension of the argument we give below.

Since we are concerned here with $\Pi(P)$ when P is a simple d-polytope, we shall onfine our attention to this case. The main result is then:

heorem 5.1 *For each $r = 0, \ldots, d$, the embedding of $\Xi_r(P)$ in $\Omega_r(P)$ is an iso- orphism.*

n fact, this result will also involve our establishing

heorem 5.2 *For each $r = 0, \ldots, d$, the weight spaces $\Xi_r(P)$ and $\Xi_{d-r}(P)$ are in uality under the multiplication on $\Pi(P)$.*

f course, this implies that these spaces have the same dimension.

We proceed in a different way from that above, which utilizes the alternative escription of $\Xi_1(P)$ and of multiplication by its elements.

emma 5.3 *Multiplying an element of $\Omega_r(P)$ by one of $\Xi_1(P)$ yields an element of $_{r+1}(P)$.*

he alternative description of the multiplication is (apart from the omission of constant factor) the *mixed volume calculation*, as we shall call it. We first recall hat, if $P_1, \ldots, P_d$ are d-polytopes in $\mathbb{E}^d$, if $u_1, \ldots, u_n$ are the distinct unit normal ectors to the facets of the sum $P_2 + \cdots + P_d$, and if η_{1j} is the support parameter of $_1$ in direction u_j $(j = 1, \ldots, n)$, then the mixed volume of $P_1, \ldots, P_d$ is recursively iven by

$$V(P_1, \ldots, P_d) = \frac{1}{d}\sum_{j=1}^{n} \eta_{1j} A(F_{2j}, \ldots, F_{dj})\,,$$

here $F_{ij} := (P_i)_{u_j}$ is the face of P_i in direction u_j, and A denotes mixed area $(d - 1)$-volume). It is easily seen that, in fact, the mixed volume so defined is ymmetric in its arguments, and coincides with the usual definition. The scaling actor $\frac{1}{d}$ is dropped below, since we shall calculate products in Π.

Now suppose that $y \in \Xi_1(P)$. In view of Theorem 3.3 and the discussion of §3, ve can identify y with a generalized vector $(\eta_1, \ldots, \eta_n)$ of support parameters,

bearing mind that there is a free choice up to a vector of the form $(\langle t, u_1\rangle, \ldots, \langle t, u_n\rangle)$ with $t \in \mathbb{E}^d$. If $a \in \Omega_{d-1}(P)$ has wieght α_j on F_j, then the natural generalization of the mixed volume calculation (which accounts for the name) gives

$$ya = \sum_{j=1}^{n} \eta_j \alpha_j\,.$$

The Minkowski relations for a ensure that the expression above is well-defined, that is, independent of the choice of the η_j for the given y.

The calculations for $\Omega_r(P)$ with $r < d - 1$ are similar, except that they are performed on $(r + 1)$-faces, and the vector of η's is replaced by the induced (generalized) support parameters on those faces (we give a particular example immediately below).

In order to check the Minkowski relations on the product, it suffices to consider the case $r = d - 2$. The unit normal vector v_k to the (non-empty) $(d - 2)$-face $G_{jk} := F_j \cap F_k$ of the facet F_j of P is

$$v_k = \operatorname{cosec}(\vartheta_{jk})u_k - \cot(\vartheta_{jk})u_j\,,$$

where ϑ_{jk} is the angle between the unit normals u_j to F_j and u_k to F_k. The support parameter corresponding to G_{jk} in F_j is thus

$$\eta_{jk} = \eta_k \operatorname{cosec}(\vartheta_{jk}) - \eta_j \cot(\vartheta_{jk})\,,$$

so that, if $\omega_{jk} := \omega(G_{jk})$, the product weights $\psi_j := \psi(F_j)$ satisfy

$$\begin{aligned}\psi_j &= \sum_k^{*} \eta_{jk}\omega_{jk} \\ &= \sum_k^{*} \eta_k\omega_{jk}\operatorname{cosec}(\vartheta_{jk}) - \eta_j\left(\sum_k^{*} \omega_{jk}\cot(\vartheta_{jk})\right),\end{aligned}$$

where such sums extend over those k for which $G_{jk} \neq \emptyset$.

Now the Minkowski relation on F_j says that

$$\begin{aligned}o &= \sum_k^{*} \omega_{jk} v_k \\ &= \sum_k^{*} \omega_{jk}\operatorname{cosec}(\vartheta_{jk})u_k - \left(\sum_k^{*} \omega_{jk}\cot(\vartheta_{jk})\right)u_j\,,\end{aligned}$$

so that

$$\begin{aligned}\sum_{j=1}^{n} \psi_j u_j &= \sum_{j=1}^{n}\left\{\sum_k^{*} \eta_k\omega_{jk}\operatorname{cosec}(\vartheta_{jk}) - \eta_j\left(\sum_k^{*} \omega_{jk}\cot(\vartheta_{jk})\right)\right\}u_j \\ &= \sum_{j=1}^{n} \eta_j\left\{\sum_k^{*} \omega_{jk}\operatorname{cosec}(\vartheta_{jk})u_k - \left(\sum_k^{*} \omega_{jk}\cot(\vartheta_{jk})\right)u_j\right\} \\ &= o\,,\end{aligned}$$

by the above, where we have used the symmetry between j and k when $G_{jk} \neq \emptyset$. This establishes the lemma.

The core of Theorem 5.2 (and hence Theorem 5.1) is the analogue of Theorem 11 of [16] (or, rather, of its consequence), namely

The crucial connexion is that which determines the facial structure of Q from its representative q. We write

$$G_j := \{x \in Q \mid \langle x, u_j \rangle = \eta_j\},$$

which is (in general) a facet of Q. We then call a subset $V \subseteq U$ *facial* (for Q) if $\{G_j \mid u_j \in V\}$ is exactly the set of facets which contains a (non-empty) face of Q. We call a subset $\tilde{V} \subseteq \tilde{U}$ *cofacial* (again, for Q) if $q \in$ relint pos $\tilde{V}$. Then we have from [14]:

Theorem 3.2 *If $Q \in \mathscr{P}(U)$, then $V \subseteq U$ is a facial subset for Q if and only if $\tilde{V} := \{\bar{u}_j \mid u_j \notin V\}$ is a cofacial set.*

The polytope Q, through its representative q, determines a *type cone* $\mathscr{K}(Q)$, which is the intersection of the sets relint pos $\tilde{V}$ which contain q. Every polytope in $\mathscr{P}(U)$ whose representative lies in $\mathscr{K}(Q)$ is then strongly isomorphic to Q. The type cones have another characterization (see [14]): they are the maximal relatively open convex subsets of pos $\bar{U}$ on which the representation is (non-negative) linear, so that

$$\lambda_1 Q_1 + \lambda_2 Q_2 \mapsto \lambda_1 q_1 + \lambda_2 q_2,$$

with q_i the representative of $Q_i \in \mathscr{P}(U)$ and $\lambda_i \geqq 0$ for $i = 1, 2$.

A particularly important case arises from our original simple polytope P. Here, the type cone $\mathscr{K}(P)$ is full-dimensional, and hence its linear hull is $\mathbb{E}^{n-d}$ itself. Theorems 3.1 and 3.2 together then yield

Theorem 3.3 *Let P be a simple d-polytope with n facets. Then the first weight space $\Xi_1(P)$ of $\Pi(P)$ is isomorphic to $\mathbb{E}^{n-d}$.*

Under this isomorphism, if Q is a summand of P, then Q has a unique representative $q \in \mathrm{cl}\, \mathscr{K}(P)$, which is then identified with $\log Q$. Note, though, that if Q has fewer facets than P, or is lower dimensional, it may well have further representatives in the cone pos $\bar{U}$.

For future reference, we repeat the observation of [14] that the representative q of $Q \in \mathscr{P}(U)$ also represents each face G of Q. However, it must be borne in mind that some of the normal vectors u_j may now be *redundant* for G, meaning that the corresponding hyperplane $\{x \in \mathbb{E}^d \mid \langle x, u_j \rangle = \eta_j\}$ does not meet G, or supports it in a face of dimension at most $\dim G - 2$, so that G has other representatives than q.

We have identified $\Xi_1(P)$ with $\mathbb{E}^{n-d}$ in a natural way above. In this context, the multiplication by an element of $\Xi_1(P)$ is performed by means of the mixed volume calculation (see §6), bearing in mind the separation Theorem 2.2 of §2. However, we can also identify an element of $\Xi_1(P)$ with the values taken by frame functionals of type 1; when we are concerned with a given strong isomorphism class, this just corresponds to the lengths (which may be signed) of the edges of polytopes in the class. In this context, it is worth recalling [7, 15.1.2], which shows (in effect) how the edge-lengths determine the support parameters of the facets.

4 The Dehn-Sommerville equations need

Let P be a simple d-polytope with n facets. If τ is an indeterminate, we write $f(P, \tau) := \sum_{j=0}^{d} f_j(P)\tau^j = \sum_F \tau^{\dim F}$, where $f_j(P)$ is the number of j-faces of P, so that $(f_0(P), \ldots, f_{d-1}(P))$ is the *f-vector* of P, and the second sum extends over all non-empty faces F of P. (P is taken as a face of itself, but we usually omit mention of $f_d(P) = 1$ in the f-vector; the empty set is not counted as a face.) Then the *h-vector* $(h_0(P), \ldots, h_d(P))$ of P is defined by $h(P, \tau) := f(P, \tau - 1)$, with $h(P, \tau) =: \sum_{r=0}^{d} h_r(P)\tau^r$. The central relationship involving the h-vector is:

Theorem 4.1 (Dehn-Sommerville equations) *If P is a simple d-polytope and $0 \leqq r \leqq d$, then $h_r(P) = h_{d-r}(P)$.*

Because this theorem is so important, we shall give its proof. In fact, the method of proof is equally important for what follows. Let v be a (unit) vector, which does not lies in any of the finitely many hyperplanes through o orthogonal to an edge or diagonal of P. Consider the faces of P which lie in the variable half-space $H^-(v, \beta) := \{x \in \mathbb{E}^d \mid \langle x, v \rangle \leqq \beta\}$. As β increases, so that $H^-(v, \beta)$ moves through P, it acquires one vertex of P at a time. We say such a vertex is of *type r* (with respect to v) if precisely r of the d edges of P which contain it lie in $H^-(v, \beta)$ (and the remaining $d - r$ do not). In acquiring this vertex, $H^-(v, \beta)$ also acquires an r-face of P, and all its s-faces (with $s < r$) which contain the vertex; there are just $\binom{r}{s}$ such s-faces. Thus, the total contribution to $f(P, \tau)$ from this acquisition is $\Delta f(P, \tau) = \sum_{s=0}^{r} \binom{r}{s} \tau^s = (\tau + 1)^r$, whence the corresponding contribution to $h(P, \tau)$ is $\Delta h(P, \tau) = \tau^r$. It follows that $h_r(P)$ is the (necessarily non-negative) number of vertices of P of type r.

The proof is now easily completed. If we replace v by $-v$, then a vertex of type r with respect to v becomes one of type $d - r$ with respect to $-v$. Thus $h_r(P) = h_{d-r}(P)$, which is the assertion of the theorem.

This approach to the Dehn-Sommerville equations was first given in a brief note at the end of [12], where the proof of the upper bound theorem for the numbers of faces of polytopes with a given number of facets was also sketched.

5 Weights

We now introduce a crucial definition for our investigations. This coincides (as we understand from Carl Lee and Jonathan Fine — private communication) with the notion of *linear stress* on the dual polytope. Denote by $\mathscr{F}_r(P)$ the family of r-faces of a polytope P. Then an *r-weight* on P is a mapping $\omega : \mathscr{F}_r(P) \to \mathbb{R}$ which satisfies the Minkowski relations, namely that

$$\sum_{F_j \subset G} \omega(F_j) v_j = 0,$$

where such a sum runs over all the r-faces F_j of an $(r + 1)$-face G of P, with v_j the unit outer normal vector (parallel to aff G) to G at F_j. The real vector space of r-weights on P is denoted by $\Omega_r(P)$.

The separation Theorem 2.2 associates each element $x \in \Xi_r(P)$ with a unique r-weight ξ, so that there is a natural embedding $\Xi_r(P) \subsetneq \Omega_r(P)$. For this reason, we often use a symbol such a w for a weight, and write $w|_F$ for its restriction to a face F. (If w is an r-weight, and F is an r-face, then $w|_F$ is the value of w on F.) Our aim in this section is to prove that, for a simple polytope, this embedding is a isomorphism. We may remark that this will not remain true for a general polytope; for example, if P is a simplicial d-polytope with n facets, then $\Xi_r(P)$ is 1-dimensioanl for each $r = 0, \ldots, d$; however, we clearly have $\dim \Omega_{d-1}(P) = n - d$.

We begin, however, with an attempt to give a little insight into how multiplication on Π works, and how it might be extended to multiplication of weights. It is where W is a $(d-r)$-frame; this induces a corresponding homomorphism (also denoted f_W) on Π. The natural convention is to take the frame functional of type d (with empty frame) to be ordinary volume. We then have

Theorem 2.2 *The frame functionals separate Π.*

That is, if $x \in \Pi$ is such that $f_W(x) = 0$ for every frame W, then $x = 0$.

The frame functionals are not independent; relationships between them are called *syzygies*. There are two kinds of *syzygy*, of which one can be thought of as trivial; it just says that, if two adjacent vectors in a frame are varied in a fixed plane with a fixed orientation, then faces determined by the frames are encountered twice. The non-trivial syzygies arise from Minkowski's theorem on facet areas. Let Q be a d-polytope, whose n facets have unit outer normal vectors $u_1, \ldots, u_n$ and corresponding areas ($(d-1)$-volumes) $\alpha_1, \ldots, \alpha_n$. Then

$$\sum_{j=1}^{n} \alpha_j u_j = o \,.$$

The corresponding relation for the r-faces of an $(r+1)$-face of a general polytope leads to a syzygy between frame functionals of type r. We shall not write it down explicitly, because we work directly with Minkowski's relation itself; for more details in the general context, we refer to [16].

We now turn to the subalgebra $\Pi(P)$. We shall usually take P to be a simple d-polytope (by which we mean, of course, that each of its vertices lies in exactly d facets), but the definition holds for general polytopes. As mentioned in §1, $\Pi(P)$ is generated by the classes $[Q]$ of summands Q of P, where Q being a *summand of* P means that $P = Q + Q'$ for some polytope Q'. In fact, $\Pi(P)$ is generated by the classes of polytopes which are *strongly (combinatorially) isomorphic* to P, in the sense that parallel support hyperplanes determine faces of the same dimension; such polytopes are clearly (combinatorially) isomorphic in the usual sense. This claim is made clear by the discussion in §3 below; actually, an even stronger result holds.

Lemma 2.3 *The subalgebra $\Pi(P)$ is generated by the classes of polytopes in any neighbourhood of P in its strong isomorphism class.*

The separation Theorem 2.2 has the following implication. Each face F of P is of the form $F = P_W$ for some frame W. We then write $x|_F := x_W$ for $x \in \Pi(P)$, which is an algebra homomorphism from $\Pi(P)$ to $\Pi(F)$, called a *face map*. (Note that, if Q is a summand of P, then its face G corresponding to F is a summand of F.) We then deduce from Lemma 2.3:

Theorem 2.4 *If P is a simple polytope and F a face of P, then the face map $x \mapsto x|_F$ from $\Pi(P)$ to $\Pi(F)$ is onto.*

The reason is clear: since we can translate the facets of P freely within some neighbourhood of P and stay in the strong isomorphism class, we see that among the faces corresponding to F we can obtain a neighbourhood of F in its strong isomorphism class. The result then follows.

Observe that we are not claiming that every polytope strongly isomorphic to F can occur as a face of a polytope strongly isomorphic to P in this way; generally, this is untrue.

3 The first weight space

In [16], we constructed an isomorphism between the first weight space Ξ_1 and a space $\mathscr{P}_T$, which is the abelian group (actually, in a fairly obvious way a vector space) consisting of the pairs (P, Q) with P, Q non-empty polytopes, factored out by the equivalence relation

$$(P, Q) \sim (P', Q') \text{ if and only if } P + Q' = P' + Q + t \text{ for some } t \in \mathbb{E}^d \,,$$

with addition induced by Minkowski addition, and given by

$$(P, Q) + (P', Q') := (P + P', Q + Q') \,.$$

Note that the identity is $(\{o\}, \{o\})$, and that the additive inverse of (P, Q) is (Q, P); we recall that the property

$$Q = \{x \in \mathbb{E}^d \mid P + x \in P + Q\}$$

implies the cancellation law in the semigroup $\langle \mathscr{P}, + \rangle$. Summarizing, we have

Theorem 3.1 $\Xi_1 \cong \mathscr{P}_T$.

If P is a non-empty polytope, we write $[P] := \sum_{r=0}^{d} [P]_r$, with $[P]_r \in \Xi_r$ its *r-component*. We always have $[P]_0 = 1$, and since $Z_1 := \bigoplus_{r=1}^{d} \Xi_r$ is nilpotent, $\log P := \log[P]$ is well-defined; then $[P]_1 = \log P$, and we usually employ this notation. The logarithm obeys the familiar rules, and its inverse is given by $[P] = \exp p$, with $p := \log P$.

When we further restrict to $\Pi(P)$, where P is a fixed simple d-polytope with n facets, we can use the representation theory of [14] to find another isomorphism. We need a brief outline of the representation theory of [14] (see also [15]). Suppose that $U := (u_1, \ldots, u_n)$ is a fixed (ordered) set of non-zero vectors which positively spans $\mathbb{E}^d$. (We shall usually assume that the u_j are distinct unit vectors.) Following [19], a *linear transform* of U is a set $\bar{U} := (\bar{u}_1, \ldots, \bar{u}_n) \subseteq \mathbb{E}^{n-d}$, which is universal for the property

$$\sum_{j=1}^{n} u_j \otimes_{\mathbb{R}} \bar{u}_j = o \,.$$

A more prosaic definition (see [14]) is the following. A *linear dependence* of U is a vector $a = (\alpha_1, \ldots, \alpha_n) \in \mathbb{E}^n$, such that $\sum_{i=1}^{n} \alpha_i u_i = o$. If any basis (or even spanning set) $\{a_i = (\alpha_{i1}, \ldots, \alpha_{in}) \mid i = 1, \ldots, r\}$ of the space of linear dependences is chosen, then the vectors $\bar{u}_j := (\alpha_{1j}, \ldots, \alpha_{rj}) \in \mathbb{E}^r$ form a linear transform $\bar{U}$ of U. (A rather more abstract definition is given in [15].) The linear transform is defined exactly up to linear equivalence, and that of $\bar{U}$ is again U. The important feature of the linear transform we need is that $a = (\alpha_1, \ldots, \alpha_n) \in \mathbb{E}^n$ is linear dependence of U if and only if $\alpha_j = \langle \bar{a}, \bar{u}_j \rangle$ $(j = 1, \ldots, n)$ for some vector $\bar{a} \in \mathbb{E}^{n-d}$.

We denote by $\mathscr{P}(U)$ the family of polytopes in $\mathbb{E}^d$ which are intersections of half-spaces with outer normal vectors in U. If

$$Q := \{x \in \mathbb{E}^d \mid \langle x, u_j \rangle \leq \eta_j \quad (j = 1, \ldots, n)\} \in \mathscr{P}(U) \,,$$

then we call $\eta_1, \ldots, \eta_n$ the *support parameters* of Q, and $q := \sum_{j=1}^{n} \eta_j \bar{u}_j \in \mathbb{E}^{n-d}$ the *representative* of Q. In fact, q represents precisely the translates of Q; however, if Q has fewer than n facets, or is lower dimensional, then other points may also represent Q.

this necessity which avoids such heavy machinery, preferably entirely within convexity.

Such a proof is presented in this paper. It uses the polytope algebra Π, which was originally devised to investigate problems of translation decomposability of polytopes (that is, the analogue of Hilbert's third problem with lower dimensional components not discarded; see [16]). However, a notion of stress on polytopes, investigated by, for example, Kalai [8], Lee [10], and others, and applied to combinatorial problems, suggested connexions with the polytope algebra. Motivated by details of the first proof in [22], and further parallels with algebraic geometry, the exploration of these relationships yields the new proof.

The basic object of our study is the subalgebra $\Pi(P)$ of Π generated by the classes of summands of a fixed simple d-polytope P. Associated with P is its h-vector $(h_0(P), \ldots, h_d(P))$, which encapsulates the information about the numbers of its faces of each dimension. The h-vector has had a number of interpretations, and was crucial in proving the upper bound theorem for convex polytopes and its extensions. We now give it another interpretation – it turns out that $h_r(P)$ is the dimension of the r-th weight space $\Xi_r(P)$ of $\Pi(P)$. The upper bound theorem for convex polytopes is an immediate consequence, using the argument of Stanley in [20].

In fact, more can be shown; the necessity of McMullen's conditions results from the existence of a Lefschetz decomposition of $\Pi(P)$ (the analogue of the hard Lefschetz theorem). However, the method of proof now no longer parallels that from algebraic geometry, since it involves demonstrating even stronger results: there is a family of Hodge-Riemann-Minkowski quadratic inequalities on $\Pi(P)$ (analogous to the Hodge-Riemann inequalities), which generalize Minkowski's second inequality. These can be used to provide far-reaching generalizations of the Aleksandrov-Fenchel inequalities for mixed volumes. A striking feature of the proof is that it does not rely on Brunn-Minkowski theory; indeed, the Brunn-Minkowski theorem can be deduced from it, although the characterization of the cases of equality in it is lost.

The paper may be briefly outlined as follows. After a discussion of general background material on the polytope algebra, the basic properties of the subalgebra $\Pi(P)$ are established, in particular the fact that the weight space $\Xi_r(P)$ has dimension $h_r(P)$. One main plank in the proof of the g-theorem is that the Hodge-Riemann-Minkowski inequalities in one dimension imply the existence of the Lefschetz decomposition in the next. The proof of this parallels an argument in Aleksandrov [1], and is also reminiscent of the proof in [5] of the lower bound theorem for simple polytopes (see also [2, 23]), but actually establishes the g-theorem in that dimension, including the generalized lower bound theorem of [18] (although with no insight into when equality occurs). The other plank consists in keeping track of how the quadratic forms change under flips (the duals to bistellar operations); the changes turn out to be local, and depend on the Hodge-Riemann-Minkowski inequalities holding (as far as is needed) for the transition polytope between the two combinatorial types.

2 The polytope algebra

In this section, we give a brief description of the polytope algebra Π, and of its salient features which we shall need in the remainder of the paper. For the general terminology and notation for convex polytopes, we refer to [7].

We shall work here only over the real field $\mathbb{R}$, in contrast to [16]; in §15, we shall make some remarks about more general fields.

The *polytope algebra* Π is initially an abelian group, with a generator $[P]$ for each $P \in \mathscr{P}$, the family of convex polytopes in $\mathbb{E}^d$; we define $[\emptyset] := 0$. We call $[P]$ the *class* of P. These generators satisfy the relations (V): $[P \cup Q] + [P \cap Q] = [P] + [Q]$ whenever $P, Q \in \mathscr{P}$ are such that $P \cup Q \in \mathscr{P}$ also (this corresponds to the *valuation property*), and (T): $[P + t] = [P]$ when $P \in \mathscr{P}$ and $t \in \mathbb{E}^d$ is a translation vector (this is *translation invariance*). Next, the (commutative) *multiplication* on Π is given by (M): $[P] \cdot [Q] = [P + Q]$, and extended to Π by linearity. Finally, we have the *dilatation*, defined on the generators by (D): $\Delta(\lambda)[P] = [\lambda P]$ for $P \in \mathscr{P}$ and $\lambda \in \mathbb{R}$.

In this context, we recall that the *vector* (or *Minkowski*) *sum* of P and Q is

$$P + Q := \{x + y \mid x \in P, y \in Q\} ;$$

we also write $P + t := P + \{t\}$ if $t \in \mathbb{E}^d$. Similarly, the *scalar multiple* of P by $\lambda \in \mathbb{R}$ is

$$\lambda P := \{\lambda x \mid \in P\} .$$

The main structure theorem for Π is the following.

Theorem 2.1 *The polytope algebra is almost a graded (commutative) algebra, in the following sense:*

(a) *there is a direct sum decomposition $\Pi = \bigoplus_{r=0}^{d} \Xi_r$, such that $\Xi_0 \cong \mathbb{Z}$, and Ξ_r is a real vector space for $r = 1, \ldots, d$ (with $\Xi_d \cong \mathbb{R}$);*
(b) *$\Xi_r \cdot \Xi_s = \Xi_{r+s}$ for each r, s (with $\Xi_r = \{0\}$ for $r > d$);*
(c) *if $x, y \in Z_1 := \bigoplus_{r=1}^{d} \Xi_r$ and $\lambda \in \mathbb{R}$, then $(\lambda x)y = x(\lambda y) = \lambda(xy)$;*
(d) *if $x \in \Xi_r$ and $\lambda \geqq 0$, then $\Delta(\lambda)x = \lambda^r x$ (with $\lambda^0 = 1$).*

We call Ξ_r the *r-th weight space of* Π. The two extreme cases need special mention. First, Ξ_0 is generated by the class $[o] = [t]$ of a point (we write $[t] := [\{t\}]$ for $t \in \mathbb{E}^d$; o denotes the zero vector); we actually write $1 := [o]$, and identify Ξ_0 with $\mathbb{Z}$ in the obvious way. In some respects, it is inconvenient not to have the full algebra properties; however, we can easily impose these, if we replace $\Xi_0 \cong \mathbb{Z}$ by the tensor product $\mathbb{R} \otimes \Xi_0 \cong \mathbb{R}$ (all tensor products are over $\mathbb{Z}$, unless specified otherwise). While this is more satisfying from the algebraic point of view (and it is the convention we shall henceforth adopt), it is perhaps less so from the geometric. Second, Ξ_d is just volume. Moreover, if L is a linear subspace of $\mathbb{E}^d$, then we can define the subalgebra $\Pi(L)$ to be generated by the classes $[P]$, such that $P \subseteq L + t$ for some $t \in \mathbb{E}^d$ (we only use this and related notation in this paragraph). If $\dim L = k$, then $\Xi_k(L) \cong \mathbb{R}$ is just k-dimensional volume (in translates of L), which we denote by vol_k, or vol if no confusion about the dimension is likely.

We shall make much use of the separation criterion for Π. A *k-frame* is an orthogonal set $W = (w_1, \ldots, w_k)$ of unit vectors. If we denote by Q_w the face of the polytope Q in direction w, that is, the intersection of Q with its support hyperplane whose outer normal vector is w, and define recursively

$$Q_W := (Q_{(w_1, \ldots, w_{k-1})})_{w_k} ,$$

with W as above, then the mapping $Q \mapsto Q_W$ induces an algebra endomorphism $x \mapsto x_W$ of Π. A *frame functional of type r* is then a mapping f_W defined by

$$f_W(Q) := \mathrm{vol}_r Q_W ,$$

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On simple polytopes

Peter McMullen

Department of Mathematics, University College London, Gower Street, London WC1E 6BT, UK

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Summary. Let P be a simple d-polytope in d-dimensional euclidean space $\mathbb{E}^d$, and let $\Pi(P)$ be the subalgebra of the polytope algebra Π generated by the classes of summands of P. It is shown that the dimensions of the weight spaces $\Xi_r(P)$ of $\Pi(P)$ are the h-numbers of P, which describe the Dehn-Sommerville equations between the numbers of faces of P, and reflect the duality between $\Xi_r(P)$ and $\Xi_{d-r}(P)$. Moreover, $\Pi(P)$ admits a Lefschetz decomposition under multiplication by the element of $\Xi_1(P)$ corresponding to P itself, which yields a proof of the necessity of McMullen's conditions in the g-theorem on the f-vectors of simple polytopes. The Lefschetz decomposition is closely connected with the new Hodge-Riemann-Minkowski quadratic inequalities between mixed volumes, which generalize Minkowski's second inequality; also proved are analogous generalizations of the Aleksandrov-Fenchel inequalities. A striking feature is that these are obtained without using Brunn-Minkowski theory; indeed, the Brunn-Minkowski theorem (without characterization of the cases of equality) can be deduced from them. The connexion found between $\Pi(P)$ and the face ring of the dual simplicial polytope P^* enables this ring to be looked at in two ways, and a conjectured formulation of the g-theorem in terms of a Gale diagram of P^* is also established.

1 Introduction

The g-theorem, which describes the possible f-vectors, or sequences of numbers of faces, of simple (or simplicial) polytopes, was formulated as a conjecture in 1970 (see [13]), and proved by Billera and Lee (sufficiency, [3]) and Stanley (necessity, [22]) about ten years later. Both before and since the proof, the g-theorem has provided a focus for many investigations into polytopes and related topics. However, while the proof of the sufficiency of the conditions (often called McMullen's conditions) of the theorem was fairly direct (if rather ingenious), that of the necessity involved deep techniques of algebraic geometry — the hard Lefschetz theorem applied to the cohomology ring of the toric variety associated with a rational simplicial polytope. One main aim has therefore been to find a proof of