Remarks + Basic definitions (Assuming basic knowledge f-vectors, polytopes)

(1) The h-vector of a simplicial complex of dimension $(d-1)$ a change of basis

$$\sum_{i=0}^{d} h_i x^{d-i} = \sum_{i=0}^{d} f_{i-1}(x-1)^{d-i}$$

The g-vector is $(g_0, \dots, g_{\lfloor d/2 \rfloor})$ defined by $g_0 = 1$, $g_i = h_i - h_{i-1}$ for $i \geq 1$

(2) Note that the h-vectors entries form an M-sequence already!!! This is needed for the Upperbound theorem!!

<u>Theorem</u> $(h_0, h_1, \dots, h_d)$ is the h-vector of a $(d-1)$-dimensional Cohen-Macaulay complex $\Longleftrightarrow$

Given $a \geq r$ there is a unique expression

$$a = \binom{a_r}{r} + \binom{a_{r-1}}{r-1} + \dots + \binom{a_i}{i}$$

$$a_r > a_{r-1} > \dots > a_i \geq i \geq 1$$

Given such representation

$$a^{\langle r \rangle} = \binom{a_r + 1}{r+1} + \binom{a_{r-1} + 1}{r} + \dots + \binom{a_i + 1}{i+1}$$

set $0^{\langle r \rangle} = 0$.

$(h_0, h_1, h_2, \dots) \Leftarrow$ this is an M-sequence

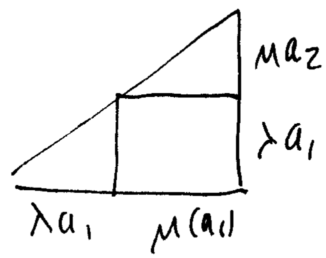
$\Leftrightarrow$

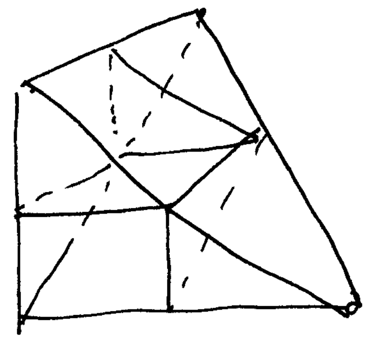
$$h_0 = 1$$

$$0 \leq h_{r+1} \leq h_r^{\langle r+1 \rangle}$$

Macaulay's Fundamental lemma: $(h_0, h_1, \dots h_i \dots)$ is an M-sequence iff $\exists$ a graded commutative algebra

Figure for Lemma 10





Thm (g-theorem) A non-negative integer vector $(h_0, h_1, \ldots, h_d)$ is the h-vector of a simplicial convex polytope $\Leftrightarrow$

1) $h_i = h_{d-i}$
2) $(g_0, \ldots, g_{\lfloor d/2 \rfloor})$ is an M-sequence $\left\{ \begin{array}{l} g_K \geq 0 \\ g_{K+1} \leq g_K^{\langle K+1 \rangle} \end{array} \right.$

Goal: To construct a graded commutative algebra over $\mathbb{R}$. This is the polytope algebra....

Polytope Algebra (comes from Jessen-Thorup) group.

$\mathcal{P}$ convex polytopes in $\mathbb{R}^d$. A valuation $\phi : \mathcal{P} \to$ Abelian group.

Π polytope algebra, free abelian group $[P]$ $P \in \mathcal{P}$

$0 = [\phi]$ $\quad 1 = [\cdot]$, then add relations

(T) $[P+t] = P$

(V) $[P \cup Q] + [P \cap Q] = [P] + [Q]$.

MULTIPLICATION: $[P] \cdot [Q] = [P + Q]$

DILATION $\Delta(\lambda)[P] = [\lambda P]$

To verify that it is a commutative ring one needs

$P \oplus (Q_1 \cup Q_2) = (P \oplus Q_1) \cup (P \oplus Q_2)$

$P \oplus (Q_1 \cap Q_2) = (P \oplus Q_1) \cap (P \oplus Q_2)$ provided $Q_1 \cup Q_2$ is a convex polytope.

Structure thm $\Pi = \bigoplus_{r=0}^{d} \Xi_r$, Ξ_r real vector space, scalar multiple compatible with addition for $r > 0$

$\Xi_0 \simeq \mathbb{Z}$, $\Xi_d \simeq \mathbb{R}$ (volume of P)

$\Xi_r \cdot \Xi_s = \Xi_{r+s}$

$x, y \in Z_1 = \bigoplus_{r=1}^{d} \Xi_r$ $(\lambda x) y = x (\lambda y)$

$\Delta(\lambda)$ induces an endomorphism.

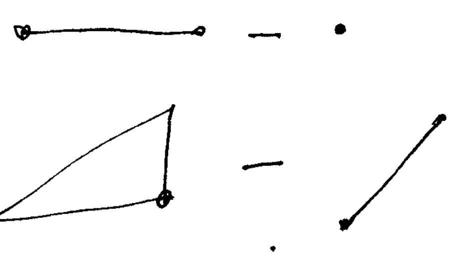
Sect. 6 of the "Polytope Algebra" paper there is a proof of structure thm for $\mathbb{Q}$ instead of $\mathbb{R}$.

Canonical dissection of simplices

$S(a_1, a_2, \ldots, a_K)$ are linear ind. vectors in $\mathbb{R}^d$

$$\mathrm{conv}(0, a_1, a_1+a_2, a_1+a_2+a_3, a_1+\cdots+a_k)$$
$$-\,\mathrm{conv}(0, a_1, \ldots + a_j + a_{K-1})$$

Example:



Generate Π !!

Lemma 10: $\Delta(\lambda+\mu)(S(a_1, \ldots, a_K)) = \Delta(\mu)\, S(a_1 \ldots a_K) +$
$\Delta(\lambda)\, S(a_1)\, \Delta(\mu)\, s(a_2 \ldots a_K) + (\Delta(\lambda)\, S(a_1 a_2)\, \Delta(\mu)\, S(a_3, \ldots a_K))$
$+ \cdots + \Delta(\lambda)\, S(a_1, \ldots, a_K)$ (see figure in front

Lemma 11 Decomposing $\Delta(n)\, x$ $\quad x \in \Pi$
we get for

$$\Delta(n)\, S(a_1, a_2, \ldots, a_K) = \sum_{r=1}^{K} \binom{n}{r} Z_r$$

$$Z_r = \sum S(a_1 \ldots)\, S(a_{i+1} \ldots a_\cdot)\, S(\ldots)$$

$[\quad][\quad][\quad]$ r pieces.

Lemma 12 (Fixed tile types) for $X \in \Pi$

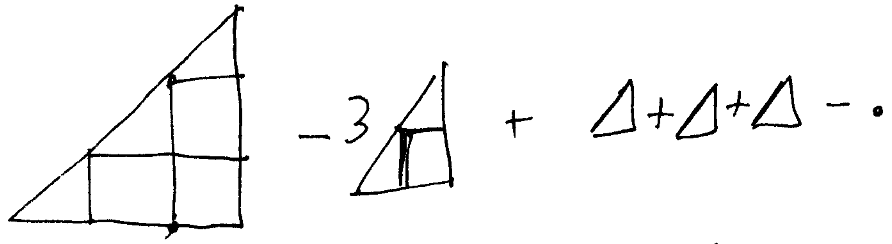
$$\Delta(n)\, X = \sum_{r=0}^{d} \binom{n}{r} y_r \qquad y_r \text{ dont dep on } n$$

Unique for given X.

$$y_r = \sum^{r} (-1)^{r-i} \binom{r}{i} \Delta(i)\, X$$

Example $\quad \Delta(3)x - 3\Delta(2)\,x + 3x - \cdot = 0$

For a triangle.



Essentially exclusion - inclusion.

$$\Delta(n)[P] = [nP] = [P]^n = \left(1+([P]-1)\right)^n$$
$$= 1 + \sum_{r=1}^{n} \binom{n}{r} ([P]-1)^r$$
$$\Rightarrow \quad ([P]-1)^r = 0 \quad \text{for } r>d. \ \forall P \in \mathcal{P}\setminus\{0\}$$

$$Z_r := \langle ([P]-1)^j : j \geq r \quad P \in \mathcal{P}\setminus\{\phi\} \rangle$$
$$Z_r = \bigoplus_{j=r}^{d} \Xi_j$$

Multiplication by rational numbers

we want to know what is $\frac{1}{m}X$ it comes from the fact that $X = my$ has a unique solution for $(m \in \{1,2,\dots\})$

From this we get what the misterious Ξ_r is!

$$X = (X_0, X_1, \dots, X_d)$$

Formaly $\quad \Delta(n)\,X = \sum_{r=0}^{d} \binom{n}{r} y_r = \sum_{r=0}^{d} n^r \tilde{y}_r$

r entry in

we rewrote everything as a polynomial in n

Another way to Ξ_r

Z_1 nilpotent

$$\log(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \cdots$$

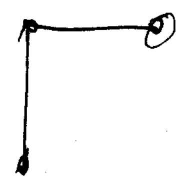
$$\exp(z) = 1 + \frac{z}{1!} + \frac{z^2}{2!} + \cdots$$

$\log(X_1 X_2) = \log X_1 + \log X_2$ if $\Delta(0)\,X_1 = \Delta(0)X_2 = [0] = 1$

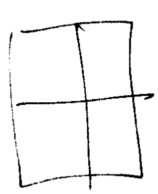
$\log[P]$ = coef of n^1 in expansion $\Delta(n)[P]$

$\mathrm{Log}\ \square = \log\ — + \log I$ but we know.

$$= \left[(I - \cdot) + \frac{(\vdash\!\dashv - \cdot)^2}{2} \right] = (\vdash\!\dashv - \cdot) + (I - \cdot)$$

$=$ 

$$\log(\square) = (\square - \cdot) - \frac{1}{2}(\square - \cdot)^2$$

 $- 2\square + \cdot = 2(\square - I - \vdash\!\dashv)$

<u>Remark</u> $\Xi_1 = \langle \log[P] \rangle$

$\Xi_r = \langle (\log[P])^r \rangle$

Log[P] is a 1-dimensional thing.

g- theorem

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Def: An r-weight on P is a function $a : \mathcal{F}^r(P) \longrightarrow \mathbb{R}$ ~~for~~ which for each $G \in \mathcal{F}^{r+1}(P)$ satisfies

$$\sum a(F)\, u_{F,G} = 0.$$

Let $\Omega_r(P)$ = real vector space of r-weights

and $\Omega(P) = \bigoplus_{r=0} \Omega_r(P)$

Examples a) If $r=0$ a takes same value on each vertex.

b) If $r=d-1$, $\Omega_{d-1}(P)$ has dimension $n-d$. (3-dim for cube) $n = \#$ of facets

why? choose a basis for $\mathbb{R}^d$ from outer normals of facets of P, then remaining weights can be assigned arbitrarily.

c) $\Omega_d(P) \cong \mathbb{R}$, (d) $\Omega(P) = \Omega(P+P) = \Omega(P+P+P,)$

Note: $a \in \Omega_r(P)$ is a row vector with $f_j(P)$ entries

Let $U_{r,r+1}$ be the $f_r(P) \times d f_{r+1}(P)$ matrix whose entries are $1 \times d$ vectors $u_{F,G}$.

$$\begin{bmatrix} (\quad)(\quad) \ldots 0 \ldots \\ \\ \\ \end{bmatrix}$$

THEN $a \in \Omega_r(P) \iff a \in \ker U_{r,r+1}$

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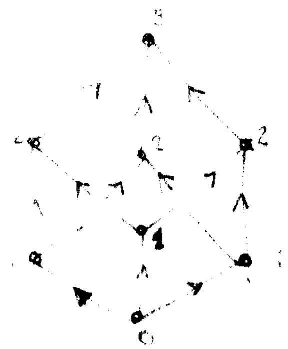
Thm 8.1 If $0 \le r \le d$ $\dim \Omega_r(P) = h_r(P)$, **P** simple

proof: (Coming from original "**On simple polytopes**")

Recall from the usual shelling procedure that $h_r(P) = \#$ of vertices of type **r** (with respect to an orientation vector v "generic")

IDEA at each vertex of type r we can assign an r-weight to the corresponding r-face of P... It doesn't match my intuition for case **$r = d-1$** !!)



h-vector **(1,3,3,1)**

Discuss why?

A very important component of the argument is the first weight space:

Lemma 8.1 Suppose $P \approx P'$ (normal equivalence) then each edge corresponds to an element $p' \in \Omega_1(P)$ with weight $p'(E) =$ length of the corresponding edge of P'.

Moreover: For each positive weight $p' \in \Omega_1(P)$ $\exists\ P' \approx P$ unique up to translation.

IDEA OF PROOF Fix an origin v' from the polytope P'.

if w is adjacent to v

$$W' = V' + \frac{p'(E)}{p(E)}(w - v)$$

iterate the process.

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KEY steps (1) Define a product for weights

(2) For P polytope p_r be its r-class

$\Rightarrow P_r = \left(\frac{1}{r!}\right) P_1^r$ call $P_1 = \log P$!!!

(3) Thm Let P be a simple polytope $p = \log(P)$. Then $p^{d-2r}\Omega_r(P) = \Omega_{d-r}(P)$ for $0 \le r \le \frac{1}{2}d$.

Define $\tilde{\Omega}_r(P) = \{ x \in \Omega_r(P) \mid p^{d-2r+1}x = 0 \}$

Claim $\tilde{\Omega}_r \approx \Omega_r / p\Omega_{r-1}$

$\forall x \in \Omega_r \ \exists! \ y \in \Omega_{r-1}$

$x + py \in \tilde{\Omega}_r$

$p^{d-2r+1}(x+py) = 0$

$p^{d-2r+1}x + p^{d-2(r-1)}y = 0$.

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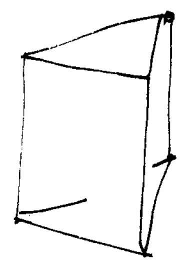

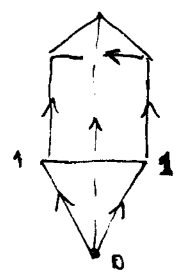

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