A GENERALIZED LOWER-BOUND CONJECTURE FOR SIMPLICIAL POLYTOPES

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Abstract. Let P be a simplicial d-polytope, and, for $-1 \leqslant j < d$, let $f_j(P)$ denote the number of j-faces of P (with $f_{-1}(P) = 1$). For $k = 0, \ldots, [\frac{1}{2}d] - 1$, we define

$$g_k^{(d+1)}(P) = \sum_{j=-1}^{k} (-1)^{k-j} \binom{d-j}{d-k} f_j(P),$$

and conjecture that

$$g_k^{(d+1)}(P) \geqslant 0,$$

with equality in the k-th relation if and only if P can be subdivided into a simplicial complex, all of whose simplices of dimension at most $d-k-1$ are faces of P. This conjecture is compared with the usual lower-bound conjecture, evidence in support of the conjecture is given, and it is proved that any linear inequality satisfied by the numbers $f_j(P)$ is a consequence of the linear inequalities given above.

1. *Introduction.* Of considerable interest in the combinatorial theory of convex polytopes are the problems of determining the maximum and minimum possible numbers of faces of a polytope of a given dimension with a given number of vertices. Motzkin [1957] put forward the Upper-bound Conjecture (actually in categorical terms) for the answer to the maximal problem, and research into this conjecture culminated recently in its proof by the first author of this paper [McMullen, 1970]. The corresponding minimal problem has proved less tractable, however, and even now reasonable conjectures as to the form of the answer have been proposed in only a few cases (see Grünbaum [1967, §10.2], McMullen [1971a]).

If we restrict our attention to simplicial polytopes, the minimal problem seems more likely of solution. Following Grünbaum [1967] (as we shall largely do in matters of terminology and notation), we let $f_j(P)$ denote the number of j-faces of a d-polytope P, for $j = 0, \ldots, d-1$. Of long standing is the

Lower-Bound Conjecture. *Let P be a simplicial d-polytope. Then for* $j = 1, \ldots, d-1$,

$$f_j(P) \geqslant \binom{d}{j} f_0(P) - \binom{d+1}{j+1} j - (f_0(P) - d - 1)\delta_j^{d-1}.$$

Moreover, if $d \geqslant 4$ then equality holds if and only if P is the union of d-simplices, each $(d-2)$-face of which is a face of P.

The polytopes described in the statement of the conjecture have been called *stacked polytopes*. The class of stacked polytopes may be defined alternatively as containing every d-simplex and each simplicial polytope obtained from a stacked polytope with one fewer vertex by adding a pyramid over some facet (i.e., $(d-1)$-face). This inductive definition suggests the possibility of proving the Lower-bound

Conjecture by some inductive argument on the number of vertices; however, no such proof has yet been found. Recently, Barnette [1971] has proved the conjecture for the case $j = d-1$ (in the dual formulation for simple polytopes). It is also reported by Grünbaum [1970, p. 1154] that M. A. Perles has a proof for the cases $f_0(P) \leqslant d + 11$, but no details of his proof have been published. Relatives of the Lower-bound Conjecture for triangulated 3- and 4-manifolds have been established by Walkup [1970].

The principal object of this paper is to formulate a generalization of the Lower-bound Conjecture for simplicial polytopes. Let P be a simplicial d-polytope, and, for integers $k \geqslant -1$ and $e \geqslant d$, let

$$g_k^{(e)}(P) = \sum_{j=-1}^{k} (-1)^{k-j} \binom{e-j-1}{e-k-1} f_j(P),$$

where we adopt the conventions $f_{-1}(P) = 1$ and $f_j(P) = 0$ if $j < -1$ or $j \geqslant d$. (We will adhere to these conventions throughout the paper. They indicate that, in many respects, we are more interested in the boundary complex of P then in P itself.) We shall say that P is a *k-stacked polytope* if P admits a subdivision into a simplicial complex, every $(d-k-1)$-face of which is a face of P. (Thus a stacked polytope is just a 1-stacked polytope.) We propose

Generalized Lower-Bound Conjecture. *Let P be a simplicial d-polytope. Then, for $k = 0, \ldots, [\frac{1}{2}d] - 1$,*

$$g_k^{(d+1)}(P) \geqslant 0.$$

Moreover if $d \geqslant 4$ then equality holds in the k-th *relation if and only if P is a k-stacked polytope.*

In the sections which follow we shall discuss this conjecture and present some evidence in its favour. In §2 we obtain a class of particularly simple reformulations of the well-known Dehn–Sommerville equations using the quantities $g_k^{(e)}(P)$ and develop some useful relationships involving applications of $g_k^{(e)}(.)$ to more general simplicial complexes, including triangulations of open and closed d-cells and $(d-1)$-spheres. In §3 we introduce classes Q_k^d of simplicial d-polytopes which are simultaneously k-neighbourly and k-stacked. In §4 we observe that the closed convex hull of the f-vectors $(f_0(P), \ldots, f_{d-1}(P))$ of the polytopes in the classes Q_k^d, $1 \leqslant k \leqslant [\frac{1}{2}d]$, is the closed convex cone C determined by the inequalities $g_k^{(d+1)}(P) \geqslant 0$ and the Dehn–Sommerville equations. If the Generalized Lower-bound Conjecture is true, then C is in fact the closed convex hull of the f-vectors of all simplicial d-polytopes. §5 contains some further remarks on the lower-bound conjectures and some possible methods of attacking them. It is shown that the generalized conjecture implies the usual one and is strictly stronger for $d \geqslant 6$. Also sketched is a proof of the generalized conjecture for polytopes admitting a certain type of shellable triangulation.

2. *Reformulations of the Dehn–Sommerville equations.* Let P be a simplicial d-polytope. The numbers $f_j(P)$ of its faces satisfy a number of linear equations, known as

The Dehn–Sommerville Equations. *For $k = -1, \ldots, d-1$,*

$$\sum_{j=k}^{d-1} (-1)^j \binom{j+1}{k+1} f_j(P) = (-1)^{d-1} f_k(P).$$

Since we have defined $f_j(P) = 0$ for $j \leqslant -2$ and $j \geqslant d$, it is actually unnecessary to put any restrictions on the range of k; the additional equations are all trivial. For proofs of these equations, see Dehn [1905] (in case $d = 4$ or 5), Sommerville [1927], or more recently, Klee [1964], Grünbaum [1967, §9.2], or McMullen and Shephard [1971]. It is known that these equations determine a flat of dimension $[\frac{1}{2}d]$ and this flat is spanned by the f-vectors of simplicial d-polytopes. It has been variously observed (Vaccaro [1956], Klee [1964], Grünbaum [1967, p. 152], Walkup [1970, §2]) that the Dehn–Sommerville equations actually hold for a large class of simplicial complexes which include, in order of increasing generality: boundary complexes of simplicial d-polytopes, triangulations of topological $(d-1)$-spheres, triangulations of homology $(d-1)$-spheres, and the Eulerian $(d-1)$-spheres described by Klee.

More generally, let M be any finite simplicial complex (closed or not) of dimension at most $d-1$, and let $f_j(M)$ be the number of j-simplices of M. As before, put $f_{-1}(M) = 1$, $f_j(M) = 0$ if $j \leqslant -2$ or $j \geqslant d$. We introduce the polynomial generating function

$$f(M, t) = \sum_j (-1)^{j+1} f_j(M)\, t^{j+1}.$$

It is readily seen, by multiplying the k-th Dehn–Sommerville Equation by $(-t)^{k+1}$ and summing, that these equations are equivalent to the single relation

$$f(P, 1-t) = (-1)^d f(P, t).$$

Now for any integer $e \geqslant d$ we introduce a new function

$$g^{(e)}(M, t) = (1-t)^e f\left(M, \frac{t}{t-1}\right).$$

It can be verified by direct substitution that the reciprocal relation has the same form,

$$f(M, t) = (1-t)^e g^{(e)}\left(M, \frac{t}{t-1}\right).$$

It is readily seen that $g^{(e)}(M, t)$ is a polynomial of degree at most e, specifically,

$$g^{(e)}(M, t) = \sum_k g_k^{(e)}(M)\, t^{k+1},$$

where $g_k^{(e)}(M) = 0$ if $k \leqslant -2$ or $k \geqslant e$. The remaining coefficients $g_k^{(e)}(M)$ may be determined by noting that

$$g^{(e)}(M, t) = (1-t)^e f\left(M, \frac{t}{t-1}\right)$$
$$= \sum_{j \geqslant -1} f_j(M)(1-t)^{e-j-1}\, t^{j+1},$$

whence,

$$g_k^{(e)}(M) = \sum_{j=-1}^{k} (-1)^{k-j} \binom{e-j-1}{e-k-1} f_j(M).$$

The reciprocal formula

$$f_j(M) = \sum_{k=-1}^{j} \binom{e-k-1}{e-j-1} g_k^{(e)}(M),$$

may be derived in similar fashion.

The Dehn–Sommerville equations now have the following reformulation.

THEOREM 1. *Let P be a simplicial d-polytope, and let $e \geqslant d$ be any integer. Then, for $k = -1, \ldots, [\frac{1}{2}e] - 1$,*

$$g_k^{(e)}(P) = (-1)^{e-d} g_{e-k-2}^{(e)}(P).$$

In fact, we shall show that these equations are equivalent to the Dehn–Sommerville equations, in the following way. We have

$$g^{(e)}(P, t) = (1-t)^e f\left(P, \frac{t}{t-1}\right) \qquad \text{(definition of } g^{(e)}\text{)}$$
$$= (1-t)^e f\left(P, 1 - \frac{1}{1-t}\right)$$
$$= (1-t)^e (-1)^d f\left(P, \frac{1}{1-t}\right) \qquad \text{(Dehn–Sommerville Equations)}$$
$$= (-1)^{e-d} t^e (1-t^{-1})^e f\left(P, \frac{t^{-1}}{t^{-1}-1}\right)$$
$$= (-1)^{e-d} t^e g^{(e)}(P, t^{-1}) \qquad \text{(definition of } g^{(e)}\text{)}.$$

Comparing coefficients, we at once deduce the statement of the theorem. (Note that the equations of the theorem are each given twice, except in the case $k = [\frac{1}{2}e] - 1$, e even, in which case the equation is trivial if $e - d$ is even.)

Conversely, if the statement of the theorem holds, then

$$g^{(e)}(P, t) = (-1)^{e-d} t^e g^{(e)}(P, t^{-1}),$$

and so

$$f(P, 1-t) = t^e g^{(e)}\left(P, \frac{1-t}{-t}\right)$$
$$= t^e (-1)^{e-d} \left(\frac{t-1}{t}\right)^e g^{(e)}\left(P, \frac{t}{t-1}\right)$$
$$= (-1)^d (1-t)^e g^{(e)}\left(P, \frac{t}{t-1}\right)$$
$$= (-1)^d f(P, t),$$

which implies the Dehn–Sommerville equations.

In case $e = d$, the equations of Theorem 1 are (apart from a change of signs) just the reformulated Dehn–Sommerville Equations of Sommerville [1927]. (See also Grünbaum [1967, §9.2.2].)

An obvious consequence of the definition of the numbers $g_k^{(e)}(M)$ and the simple relation

$$g^{(e+1)}(M, t) = (1-t)\, g^{(e)}(M, t),$$

is

LEMMA 1. *Let M be a simplicial complex of dimension at most $d-1$, and let $e \geqslant d$. Then, for $k = -1, \ldots, e$,*

$$g_k^{(e+1)}(M) = g_k^{(e)}(M) - g_{k-1}^{(e)}(M),$$

(*where, of course,* $g_{-2}^{(e)}(M) = 0$), *and, for* $k = -1, \ldots, e-1$,

$$g_k^{(e)}(M) = \sum_{j=-1}^{k} g_j^{(e+1)}(M).$$

Now let K be a triangulation of a closed d-cell, let ∂K denote the induced triangulation of the boundary of K, and let K^0 denote the open complex $K - \partial K$. With respect to K^0 *only* we will adopt the convention $f_{-1}(K^0) = 0$, so that $f_j(K) = f_j(\partial K) + f_j(K^0)$ holds for all j.

LEMMA 2. *Let K be a triangulation of a closed d-cell, and let $e \geqslant d+1$. Then, subject to the above convention,*

$$g^{(e)}(K, t) = g^{(e)}(\partial K, t) + g^{(e)}(K^0, t).$$

This is an immediate consequence of $f_j(K) = f_j(\partial K) + f_j(K^0)$.

THEOREM 2. *Let K be a triangulation of a closed d-cell, and let $e \geqslant d+1$. Then, subject to the above convention, for $k = -1, \ldots, e$,*

$$g_k^{(e)}(\partial K) = g_k^{(e)}(K) - (-1)^{e-d-1} g_{e-k-2}^{(e)}(K)$$
$$= -g_k^{(e)}(K^0) + (-1)^{e-d-1} g_{e-k-2}^{(e)}(K^0).$$

For, let L be the triangulated d-sphere consisting of K, the joins of the simplices of ∂K to a new vertex v, and v itself. Then, for $j = -1, \ldots, d$,

$$f_j(L) = f_j(K) + f_{j-1}(\partial K),$$

and so

$$f(L, t) = f(K, t) - tf(\partial K, t).$$

Now L is a triangulation of a d-sphere, and ∂K a triangulat'on of a $(d-1)$-sphere, for which the Dehn–Sommerville Equations apply. Thus

$$\begin{aligned} f(K, 1-t) - (-1)^d(1-t)f(\partial K, t) &= f(K, 1-t) - (1-t)f(\partial K, 1-t) \\ &= f(L, 1-t) \\ &= (-1)^{d+1} f(L, t) \\ &= (-1)^{d+1}\{f(K, t) - tf(\partial K, t)\}, \end{aligned}$$

or, rearranging terms,

$$f(\partial K, t) = f(K, t) + (-1)^d f(K, 1-t).$$

Replacing t by $t/(t-1)$ and multiplying through by $(1-t)^e$, we obtain

$$g^{(e)}(\partial K, t) = g^{(e)}(K, t) - (-1)^{e-d-1} t^e g^{(e)}(K, t^{-1}),$$

which leads to the first statement of the theorem. Using Lemma 2 to eliminate the terms $g^{(e)}(K, .)$ in this expression and cancelling the Dehn–Sommerville equations

$$g^{(e)}(\partial K, t) = (-1)^{e-d} t^e g^{(e)}(\partial K, t^{-1}),$$

we obtain

$$g^{(e)}(\partial K, t) = (-1)^{e-d-1} t^e g^{(e)}(K^0, t^{-1}) - g^{(e)}(K^0, t),$$

from which the second statement of the theorem follows.

An immediate consequence of the second part of the theorem and the convention $f_{-1}(K^0) = 0$ is

COROLLARY. *Let $0 \leqslant k \leqslant [\frac{1}{2}(d-1)]$, and let P be a k-stacked d-polytope. Then $g_k^{(d+1)}(P) = 0$.*

Of course, the case $k = [\frac{1}{2}(d-1)]$ (d odd) is a trivial consequence of Theorem 1.

3. *Neighbourly stacked polytopes.* For any d and k satisfying $d \geqslant 2$ and $1 \leqslant k \leqslant [\frac{1}{2}d]$, let Q_k^d denote the class of simplicial d-polytopes P with the following two properties:

(a) P is k-neighbourly (that is, any subset of k vertices of P is the set of vertices of a $(k-1)$-face of P).

(b) P is a k-stacked polytope.

Observe that Q_1^d is just the class of stacked polytopes, since every polytope is 1-neighbourly. At the end of this section we shall see that the definition of Q_k^d is as restrictive as possible in the sense that the only d-polytopes which are simultaneously k'-neighbourly and k-stacked for $k < k'$ are d-simplices.

We shall demonstrate the existence of a member P of Q_k^d with v vertices for any d, k, and v satisfying $2 \leqslant 2k \leqslant d < v$. Let d, k, and v be such numbers and let Q be any k-neighbourly $2k$-polytope with $v - d + 2k$ vertices. (For a discussion of neighbourly polytopes, including a proof of the existence of k-neighbourly $2k$-polytopes with any number of vertices greater than $2k$ and a proof that such polytopes must be simplicial, see Grünbaum [1967, §4.7 and §7].) We can construct a simplicial subdivision L of Q as follows. Let x be any vertex of Q. Then the $2k$-simplices of L are the convex hulls of x and the facets of Q which do not contain x. Because Q is k-neighbourly every $(k-1)$-simplex of L is necessarily a face of Q, and so Q is a k-stacked polytope.

Now we may suppose that Q lies in d-dimensional space E^d. Let T be any $(d-2k)$-simplex in E^d whose relative interior meets aff Q (the affine hull of Q) in the point x alone. Then $P = \operatorname{conv}(Q \cup T)$ is a simplicial d-polytope with v vertices. The faces of P are of two types: the convex hull of T and a face of Q which contains x, and the convex hull of a face of T and a face of Q which does not contain x. We see at once that P is k-neighbourly.

The simplicial subdivision L of Q induces a simplicial subdivision K of P whose d-simplices are the convex hulls of T and the $2k$-simplices of L. From this it follows that P is a k-stacked polytope. For let R be any $(d-k-1)$-simplex of K. From the description of K, R is the convex hull of a face of T (possibly T itself) and a face G of Q which does not contain x. In case the face of T is proper, R is clearly a face of P from the classification of the faces of P given above. In case $R = \operatorname{conv}(T \cup G)$, the face G of Q has at most $d - k - (d - 2k + 1) = k - 1$ vertices, and since Q is k-neighbourly, this implies that conv $(\{x\} \cup G)$ is a face of Q, so that again R is a face of P. This completes the proof that P is k-stacked and hence is a member of Q_k^d.

Of course since P is k-neighbourly

$$f_i(P) = \binom{v}{i+1},$$

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for $-1 \leqslant i < k$. Consequently

$$g_j^{(d+1)}(P) = \sum_{i=-1}^{j} (-1)^{j-i} \binom{d-i}{d-j} \binom{v}{i+1}$$

$$= \binom{v+j-d-1}{j+1}$$

for $-1 \leqslant j < k$. Moreover, since P is k-stacked, it is also j-stacked for $j \geqslant k$, and it follows from the final corollary of the previous section that

$$g_j^{(d+1)}(P) = 0,$$

for $k \leqslant j < [\frac{1}{2}d]$. These calculations, among other things, establish the remark made at the beginning of this section. For, if $k < k'$ and P is simultaneously k'-neighbourly and k-stacked, then

$$\binom{v+k-d-1}{k+1} = 0.$$

But this is impossible unless $v = d+1$, i.e., unless P is a d-simplex.

4. *The f-vectors of simplicial polytopes.* Let P be a simplicial d-polytope. The sequence $f(P) = (f_0(P), f_1(P), \ldots, f_{d-1}(P))$ is known as the *f-vector* of P. In this section we shall investigate the closed convex hull of the f-vectors of all simplicial polytopes.

The Dehn–Sommerville equations

$$g_k^{(d+1)}(P) = -g_{d-k-1}^{(d+1)}(P), \qquad -1 \leqslant k \leqslant [\tfrac{1}{2}(d-1)],$$

and the inequalities

$$g_k^{(d+1)}(P) \geqslant 0, \qquad 0 \leqslant k \leqslant [\tfrac{1}{2}d]-1,$$

with the usual convention $f_{-1}(P) = 1$, determine a simplicial cone C of dimension $[\frac{1}{2}d]$ in the space of all sequences $(f_0, \ldots, f_{d-1})$. The Generalized Lower-bound Conjecture would imply that the f-vector of every simplicial d-polytope lay in C.

We shall now show that the conjecture, if true, is the strongest possible conjecture involving linear inequalities. Specifically, we shall prove

THEOREM 3. *The closed convex hull of the f-vectors $f(P)$, $P \in Q_k^d$, $1 \leqslant k \leqslant [\frac{1}{2}d]$ is the cone C.*

We first observe that the d-simplex T^d is in each class Q_k^d and that $g_k^{(d+1)}(T^d) = 0$ for $0 \leqslant k < [\frac{1}{2}d]$, so that $f(T^d)$ is the apex of C. From the last section we also see that each $f(P)$, $P \in Q_k^d$, $1 \leqslant k \leqslant [\frac{1}{2}d]$, lies in C. Finally, if we denote by $P_k^d(v)$ a member of Q_k^d with v vertices, it is easily computed that

$$\lim_{v\to\infty} \frac{g_j(P_{k+1}^d(v))}{\binom{v+k-d-1}{k+1}} = \delta_j^k,$$

for each $j, k = 0, \ldots, [\frac{1}{2}d]-1$, and so C is contained in the closed convex hull of the $f(P)$. This proves the theorem.

5. *Further remarks.* We first justify an assertion made in the introduction.

THEOREM 4. *The Generalized Lower-bound Conjecture implies the Lower-bound Conjecture for $d \geqslant 2$. Moreover, for $d \geqslant 6$ the lower-bound inequalities and the Dehn–Sommerville equations together fail to imply the generalized lower-bound inequalities.*

Let P be a simplicial d-polytope, $d \geqslant 2$. Combining the expressions for the $f_j(P)$ in terms of the $g_k^{(d+1)}(P)$ with the Dehn–Sommerville equations of Theorem 1, we obtain

$$f_j(P) = A_j + B_j,$$

for $1 \leqslant j \leqslant d-1$, where

$$A_j = \sum_{k=-1}^{0} \left\{ \binom{d-k}{d-j} - \binom{k+1}{d-j} \right\} g_k^{(d+1)}(P),$$

$$B_j = \sum_{k=1}^{[\frac{1}{2}d]-1} \left\{ \binom{d-k}{d-j} - \binom{k+1}{d-j} \right\} g_k^{(d+1)}(P).$$

The term for $k = \frac{1}{2}(d-1)$ (d odd) is automatically zero, and so can be omitted. Resubstituting $g_{-1}^{(d+1)}(P) = 1$ and $g_0^{(d+1)}(P) = f_0(P) - d - 1$ into the expression for A_j and rearranging, we obtain

$$A_j = \binom{d}{j} f_0(P) - \binom{d+1}{j+1} j - (f_0(P) - d - 1)\delta_j^{d-1},$$

which is just the right-hand side of the inequality in the Lower-bound Conjecture. Finally, we note that each of the bracketed factors in the expression for B_j is non-negative.

From the above observations it follows immediately that the generalized lower-bound inequalities $g_k^{(d+1)}(P) \geqslant 0$ imply the usual inequalities $f_j(P) \geqslant A_j$. Moreover, if equality holds in the usual inequality with $1 \leqslant j \leqslant d-1$, then $B_j = 0$. And if $d \geqslant 4$ the first term of B_j is present and the coefficient of $g_1^{(d+1)}(P)$ is non-zero. Thus $g_1^{(d+1)}(P) = 0$, and by the Generalized Lower-bound Conjecture P is 1-stacked. This establishes the first part of the theorem.

For the second part of the theorem consider the set of integers $g_{-1}^*, \ldots, g_{d-1}^*$, $d \geqslant 6$, given by

$$g_k^* = \begin{cases} 1, & \text{if } k = -1, \\ 0, & \text{if } 0 \leqslant k \leqslant [\frac{1}{2}d]-3, \\ 1, & \text{if } k = [\frac{1}{2}d]-2, \\ -1, & \text{if } k = [\frac{1}{2}d]-1, \\ -g_{d-k-1}^*, & \text{if } [\frac{1}{2}d] \leqslant k \leqslant d-1, \end{cases}$$

and let $f^* = (f_{-1}^*, \ldots, f_{d-1}^*)$ be the f-vector derived from the g_k^* by the formulae of §2. By Theorem 1 and the definition of the g_k^*, the f_j^* satisfy the Dehn–Sommerville equations. Now write $f_j^* = A_j^* + B_j^*$ as above. Since $d \geqslant 6$, it follows that $[\frac{1}{2}d] - 2 \geqslant 1$, and hence

$$B_j^* = \binom{d+2-[\frac{1}{2}d]}{d-j} - \binom{[\frac{1}{2}d]-1}{d-j} - \binom{d+1-[\frac{1}{2}d]}{d-j} + \binom{[\frac{1}{2}d]}{d-j}$$

$$= \binom{d+1-[\frac{1}{2}d]}{d-j-1} + \binom{[\frac{1}{2}d]-1}{d-j-1} \geqslant 0.$$

Thus $f_j^* \geqslant A_j^*$, i.e., f^* satisfies the lower-bound inequalities. But by construction f^* does not satisfy the generalized lower-bound inequalities $g_k^* \geqslant 0$. This completes the proof of the theorem.

We also mention here the following interesting fact, the independent discovery of which originally prompted our collaboration on this paper. The inequalities of the Lower-bound Conjecture include the inequality

$$g_1^{(d+1)}(P) = f_1(P) - df_0(P) + \binom{d+1}{2} \geqslant 0.$$

Assuming this inequality for all $d \leqslant d^*$, observing that the vertex figure of a simplicial polytope is again a simplicial polytope, and applying an inductive argument, it is possible to derive the complete set of inequalities of the Lower-bound Conjecture for polytopes of dimension d^*. We omit the details of the proof since we have not seen how to apply the same kind of arguments in the setting of the Generalized Lower-bound Conjecture.

The first author of this paper has proved the Generalized Lower-bound Conjecture in case $f_0(P) \leqslant d+3$ in the wider context of the complete classification of the f-vectors of simplicial d-polytopes with at most $d+3$ vertices (McMullen [1971b]). The proof of the most interesting case of exactly $d+3$ vertices uses the technique of Gale diagrams (Grünbaum [1967, §5.4 and §6.3], McMullen–Shephard [1970, §3.4]).

If P has a triangulation K which is shellable, that is, if the d-simplices of K can be labelled $S_1, \ldots, S_m$ in such a way that, for $j = 2, \ldots, m$,

$$B_j = S_j \cap \left(\bigcup_{i=1}^{j-1} S_i\right),$$

is topologically a $(d-1)$-ball, then arguments analogous to those of McMullen [1970] show that, for $-1 \leqslant k \leqslant d$,

$$g_k^{(d+1)}(K) \geqslant 0.$$

In fact, $g_k^{(d+1)}(K)$ is just the number of j for which B_j is the union of the $k+1$ facets of S_j which contain some $(d-k-1)$-face. This face must, of course, be an interior face of K. Using Theorem 2 (with $e = d+1$), we see that if $k \leqslant [\frac{1}{2}d] - 1$, and K has no interior k-faces, then $g_{d-k-1}^{(d+1)}(K) = 0$, and so

$$g_k^{(d+1)}(P) = g_k^{(d+1)}(K) \geqslant 0.$$

Now an application of the theorem of Tverberg [1966] to the Gale diagram of a simplicial d-polytope P with

$$v \leqslant \frac{(k+1)d-1}{k},$$

vertices (which may be assumed to be in sufficiently general position) shows that P has a shellable triangulation with no interior k-faces, so that, for $0 \leqslant j \leqslant k$,

$$g_j^{(d+1)}(P) \geqslant 0.$$

It should be noted that this range of possible values of v is not large. In case $k = [\frac{1}{2}d] - 1$, we see that we must have $v \leqslant d+2$, when the conjecture is already known to hold. In the case $k = 1$ we obtain the usual lower-bound inequality for $f_1(P)$ provided $v \leqslant 2d-1$. By the inductive argument using vertex figures mentioned

above it can be shown that the usual lower-bound inequality for $f_j(P)$ holds provided $v \leqslant 2d-j$. We again omit the details of the proof.

We have remarked that the Dehn–Sommerville Equations apply to triangulations of $(d-1)$-spheres and more general complexes as well as boundary complexes of simplicial d-polytopes. With at most minor adaptations in the proofs, all the positive results of this paper hold equally well for triangulated $(d-1)$-spheres. The one possible exception occurs in the paragraph immediately above, which relies on the use of Gale diagrams. Mani [1972] has shown that any triangulation of a $(d-1)$-sphere with at most $d+3$ vertices is isomorphic to the boundary complex of some d-polytope; hence the reliance on Gale diagrams three paragraphs above is only a matter of convenience. Nevertheless, there are real differences as well as deep theoretical questions to be met with in extending results on simplicial polytopes to triangulated spheres (see Grünbaum [1970]). We have therefore satisfied ourselves with venturing the Generalized Lower-bound Conjecture for polytopes only.

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We next prove that for $0 \le j \le d$ we have

$$\sum_{i=0}^{d-j} \binom{d-i}{j}\binom{p-d+i-1}{i} = \binom{p}{d-j}. \tag{3}$$

The validity of (3) is proved using identities from Appendix 3 as indicated:

$$\begin{aligned}\sum_{i=0}^{d-j} \binom{d-i}{j}\binom{p-d+i-1}{i} &\overset{(3)}{=} \sum_{i=0}^{d-j}(-1)^{d-i-j}\binom{-j-1}{d-i-j}\binom{p-d+i-1}{i}\\ &\overset{(2)}{=} \sum_{i=0}^{d-j}(-1)^{d-i-j}\binom{-j-1}{d-i-j}(-1)^i\binom{-p+d}{i}\\ &= (-1)^{d-j}\sum_{i=0}^{d-j}\binom{-p+d}{i}\binom{-j-1}{(d-j)-i}\\ &\overset{(7)}{=} (-1)^{d-j}\binom{-p+d-j-1}{d-j}\\ &\overset{(2)}{=} \binom{p}{d-j},\end{aligned}$$

as desired.

Using (3) we can now rewrite the second sum in (2):

$$\begin{aligned}\sum_{i=0}^{m}\binom{d-i}{j}\binom{p-d+i-1}{i} &= \sum_{i=0}^{d-j}\binom{d-i}{j}\binom{p-d+i-1}{i} - \sum_{i=m+1}^{d-j}\binom{d-i}{j}\binom{p-d+i-1}{i}\\ &= \binom{p}{d-j} - \sum_{i=m+1}^{d-j}\binom{d-i}{j}\binom{p-d+i-1}{i}.\end{aligned}$$

Hence, we have the two remaining terms in the desired expression.

(c) Although we already know that the statement is true, we would like to give a direct proof. For $j \ge n+1$, each term in the first sum in (2) has the value 0. In the second sum, all terms corresponding to values of i that are $> d-j$ also have the value 0. Therefore,

$$\Phi_j(d,p) = \sum_{i=0}^{d-j}\binom{d-i}{j}\binom{p-d+i-1}{i}.$$

Combining with (3) above, we then get

$$\Phi_j(d,p) = \binom{p}{d-j}.$$

When $m = n$, this completes the proof. When $m = n-1$, it remains to consider the value $j = m+1 = n$. However, this is easily handled by returning to the expression for $\Phi_j(d,p)$ in case (b). The details are left to the reader. □

By duality, we also have an Upper Bound Theorem for the simplicial d-polytopes. It may be stated as follows:

Corollary 18.3. *For any simplicial d-polytope P with p vertices we have*

$$f_j(P) \le \Phi_{d-1-j}(d,p), \qquad j = 1,\ldots,d-1.$$

If P is neighbourly, then

$$f_j(P) = \Phi_{d-1-j}(d,p), \qquad j = 1,\ldots,d-1.$$

If P is not neighbourly, then

$$f_j(P) < \Phi_{d-1-j}(d,p), \qquad j = n-1,\ldots,d-1,$$

(and possibly also for smaller values of j).

Finally, it is interesting to note that (f) and (h) in the proof of Theorem 18.1 show that

$$\sum_{j=0}^{d}(-1)^j\binom{j}{i}f_j(P) = \sum_{j=0}^{d}(-1)^{d+j}\binom{j}{d-i}f_j(P), \qquad i = 0,\ldots,d,$$

i.e. $(f_0(P),\ldots,f_{d-1}(P))$ satisfies the Dehn–Sommerville System of Theorem 17.5. Hence, we have an independent proof of the Dehn–Sommerville Relations which does not rely on Euler's Relation.

§19. The Lower Bound Theorem

In the preceding section we determined the largest number of vertices, edges, etc. of a simple d-polytope, $d \ge 3$, with a given number of facets. In this section we shall find the smallest number of vertices, edges, etc. The result which is known as the Lower Bound Theorem was proved by Barnette in 1971–73. Like the Upper Bound Theorem, it is a main achievement in the modern theory of convex polytopes.

As we saw at the beginning of Section 18, all simple 3-polytopes with a given number of facets have the same number of vertices and the same number of edges. So, as in the case of the Upper Bound Theorem, the problem is only of significance for $d \ge 4$.

We define

$$\varphi_j(d,p) := \begin{cases}(d-1)p - (d+1)(d-2), & j = 0;\\ \binom{d}{j+1}p - \binom{d+1}{j+1}(d-1-j), & j = 1,\ldots,d-2.\end{cases}$$

Note that

$$\varphi_{d-2}(d, p) = dp - \binom{d+1}{d-1}$$
$$= dp - (d^2 + d)/2.$$

With this notation the *Lower Bound Theorem* may be stated as follows:

Theorem 19.1. *For any simple d-polytope P with p facets we have*

$$f_j(P) \geq \varphi_j(d, p), \qquad j = 0, \ldots, d-2.$$

Moreover, there are simple d-polytopes P with p facets such that

$$f_j(P) = \varphi_j(d, p), \qquad j = 0, \ldots, d-2.$$

Since $\varphi_0(3, p) = 2p - 4$ and $\varphi_1(3, p) = 3p - 6$, we see immediately as in the case of the Upper Bound Theorem that the theorem is true for $d = 3$, in fact, with equality for all simple polytopes.

Before proving Theorem 19.1 we need some notation and some preparatory lemmas.

We remind the reader that a facet system in a polytope P is a non-empty set $\mathscr{S}$ of facets of P. When $\mathscr{S}$ is a facet system in P, we denote by $\mathscr{G}(\mathscr{S})$ the union of the subgraphs $\mathscr{G}(F)$, $F \in \mathscr{S}$, of $\mathscr{G}(P)$, and we say that $\mathscr{S}$ is connected if $\mathscr{G}(\mathscr{S})$ is a connected graph. These concepts were introduced in Section 15, where we also proved some important results about connectedness properties of $\mathscr{G}(\mathscr{S})$.

When $\mathscr{S}$ is a facet system in P and G is a face of P, then we shall say that G is in $\mathscr{S}$ or G is a face of $\mathscr{S}$, if G is a face of some facet F belonging to $\mathscr{S}$. In particular, the vertices of $\mathscr{S}$ are the vertices of the facets in $\mathscr{S}$.

In the following, we shall restrict our attention to facet systems in simple polytopes. Let $\mathscr{S}$ be a facet system in a simple d-polytope P, and let x be a vertex of $\mathscr{S}$. Then x is a vertex of at least one member F of $\mathscr{S}$. Therefore, the $d - 1$ edges of F incident to x are edges of $\mathscr{S}$. If the remaining edge of P incident to x is also in $\mathscr{S}$, we shall say that x is *internal* in $\mathscr{S}$ or that x is an internal vertex of $\mathscr{S}$. If, on the other hand, the remaining edge of P incident to x is not in $\mathscr{S}$, we shall say that x is *external* in $\mathscr{S}$ or that x is an external vertex of $\mathscr{S}$. In other words, a vertex x of $\mathscr{S}$ is external if and only if it is a vertex of only one member of $\mathscr{S}$.

The first lemma ensures the existence of external vertices under an obvious condition. (In the following, we actually need only the existence of just one external vertex.)

Lemma 19.2. *Let $\mathscr{S}$ be a facet system in a simple d-polytope P such that at least one vertex of P is not in $\mathscr{S}$. Then $\mathscr{S}$ has at least d external vertices.*

Proof. If all vertices of $\mathscr{S}$ are external, then each member of $\mathscr{S}$ contributes at least d external vertices. Suppose that some vertex z of $\mathscr{S}$ is internal. By the assumption we also have a vertex y not in $\mathscr{S}$. We then use the d-connectedness of $\mathscr{G}(P)$, cf. Theorem 15.6, to get d independent paths joining y and z. Traversing the ith path from y to z, let x_i be the first vertex which is in $\mathscr{S}$. Then the preceding edge is not in $\mathscr{S}$, and therefore x_i is external in $\mathscr{S}$. Since the x_i's are distinct, we have the desired conclusion. □

During the proof of Theorem 15.7 it was shown that if $\mathscr{S}$ is a connected facet system in P and $\mathscr{S}$ has at least two members, then there is a member F_0 of $\mathscr{S}$ such that $\mathscr{S}\backslash\{F_0\}$ is again connected. When P is simple, we have the following much stronger result:

Lemma 19.3. *Let $\mathscr{S}$ be a connected facet system in a simple d-polytope P. Assume that at least one vertex of P is not in $\mathscr{S}$, and that $\mathscr{S}$ has at least two members. Then there is a pair (x_0, F_0) formed by an external vertex x_0 of $\mathscr{S}$ and the unique member F_0 of $\mathscr{S}$ containing x_0 such that the facet system $\mathscr{S}\backslash\{F_0\}$ is again connected.*

Proof. We know from Lemma 19.2 that $\mathscr{S}$ has external vertices. Let (x_1, F_1) be a pair formed by an external vertex x_1 of $\mathscr{S}$ and the unique member F_1 of $\mathscr{S}$ containing x_1. Suppose that $\mathscr{S}\backslash\{F_1\}$ is not connected. Let $\mathscr{S}_1$ be a maximal connected subsystem of $\mathscr{S}\backslash\{F_1\}$. We shall prove that then there is another pair (x_2, F_2) such that $\mathscr{S}_1 \cup \{F_1\}$ is a connected subsystem of $\mathscr{S}\backslash\{F_2\}$. In other words: if $\mathscr{S}\backslash\{F_1\}$ is not connected, then we can replace (x_1, F_1) by some (x_2, F_2) in such a manner that the maximum number of members of a connected subsystem of $\mathscr{S}\backslash\{F_2\}$ is larger than the maximum number of members of a connected subsystem of $\mathscr{S}\backslash\{F_1\}$. Continuing this procedure eventually leads to a pair (x_0, F_0) with the property that $\mathscr{S}\backslash\{F_0\}$ is connected.

Now, let (x_1, F_1) and $\mathscr{S}_1$ be as explained above. We first prove that $\mathscr{S}_1 \cup \{F_1\}$ is connected. Let y be any vertex of $\mathscr{S}_1$; note that $y \neq x_1$ since F_1 is the only member of $\mathscr{S}$ containing x_1 and $F_1 \notin \mathscr{S}_1$. By the connectedness of $\mathscr{S}$ there is a path in $\mathscr{G}(\mathscr{S})$ joining y and x_1. Traversing this path from y to x_1, let F be a member of $\mathscr{S}$ containing the first edge of the path not in $\mathscr{S}_1$. (Since x_1 is not in $\mathscr{S}_1$, such an edge certainly exists.) Then clearly $\mathscr{S}_1 \cup \{F\}$ is connected. By the maximality property of $\mathscr{S}_1$ we must have $F = F_1$, whence $\mathscr{S}_1 \cup \{F_1\}$ is connected, as desired. Let $\mathscr{S}'_1 := \mathscr{S}\backslash(\mathscr{S}_1 \cup \{F_1\})$. Then $\mathscr{S}'_1$ is non-empty, possibly disconnected. By Lemma 19.2, $\mathscr{S}'_1$ has external vertices. Not every external vertex of $\mathscr{S}'_1$ can be in F_1. For then every path joining a vertex of $\mathscr{S}'_1$ and a vertex of P not in $\mathscr{S}'_1$ would have to pass through a vertex of F_1, whence the subgraph of $\mathscr{G}(P)$ spanned by ext $P\backslash$ext F_1 would be disconnected, contradicting Theorem 15.5. Let x_2 be an external vertex of $\mathscr{S}'_1$ not in F_1, and let F_2 be the unique member of $\mathscr{S}'_1$ containing x_2. Then actually x_2 is external in $\mathscr{S}$. For if not, then x_2 would have to be a vertex of some member F of $\mathscr{S}_1$, since F_2 is the only member of $\mathscr{S}'_1$ containing x_2, and x_2 is not in F_1; but then $\mathscr{S}_1 \cup \{F_2\}$ would be connected, contradicting the maximality property of $\mathscr{S}_1$. Hence, x_2 is external in $\mathscr{S}$, the facet F_2 is the unique member of $\mathscr{S}$ containing x_2, and $\mathscr{S}_1 \cup \{F_1\}$ is a connected subsystem of $\mathscr{S}\backslash\{F_2\}$, as desired. □

Lemma 19.4. *Let $\mathscr{S}$ be a connected facet system in a simple d-polytope P. Assume that at least one vertex of P is not in $\mathscr{S}$, and that $\mathscr{S}$ has at least two members. Let (x_0, F_0) be as in Lemma 19.3. Then at least $d - 1$ vertices of P are internal in $\mathscr{S}$ but external in $\mathscr{S}\backslash\{F_0\}$.*

Proof. By the connectedness of $\mathscr{S}$, there is a member F of $\mathscr{S}$ with $F \neq F_0$ and $F \cap F_0 \neq \emptyset$. Then by Theorem 12.14, the face $F \cap F_0$ has dimension $d - 2$, whence F and F_0 have at least $d - 1$ vertices in common. Being vertices of two members of $\mathscr{S}$, such $d - 1$ vertices are all internal in $\mathscr{S}$. So, if they are all external in $\mathscr{S}\backslash\{F_0\}$, we have the desired conclusion. If they are not all external in $\mathscr{S}\backslash\{F_0\}$, one of the vertices, say y, is internal in $\mathscr{S}\backslash\{F_0\}$. In particular, $y \neq x_0$. Then by Theorem 15.7 there are $d - 1$ independent paths in $\mathscr{G}(\mathscr{S})$ joining x_0 and y. Traversing the ith path from x_0 to y, let x_i be the first vertex which is in $\mathscr{S}\backslash\{F_0\}$. Then the preceding edge $[u_i, x_i]$ is not in $\mathscr{S}\backslash\{F_0\}$, and therefore x_i is external in $\mathscr{S}\backslash\{F_0\}$. In particular, $x_i \neq x_0$ and $x_i \neq y$. Moreover, since $[u_i, x_i]$ is not in $\mathscr{S}\backslash\{F_0\}$, it must be in F_0, whence x_i is a vertex of F_0. Since x_i is also a vertex of $\mathscr{S}\backslash\{F_0\}$, we see that x_i belongs to at least two members of $\mathscr{S}$, showing that x_i is internal in $\mathscr{S}$. In conclusion, the $d - 1$ vertices $x_1, \ldots, x_{d-1}$ are internal in $\mathscr{S}$ but external in $\mathscr{S}\backslash\{F_0\}$. □

Lemma 19.5. *Let $\mathscr{S}$ be a facet system in a simple d-polytope P such that at least one vertex of P is not in $\mathscr{S}$. Then there are at least d facets $G_1, \ldots, G_d$ of P such that $G_1, \ldots, G_d$ are not in $\mathscr{S}$ but each contains a $(d - 2)$-face which is in $\mathscr{S}$.*

Proof. Let x be a vertex of P not in $\mathscr{S}$. Let Q be a dual of P in $\mathbb{R}^d$, and let ψ be an anti-isomorphism from $(\mathscr{F}(P), \subset)$ onto $(\mathscr{F}(Q), \subset)$. Writing

$$\mathscr{S} = \{F_1, \ldots, F_m\},$$

x is not a vertex of any of the F_i's, whence the facet $\psi(\{x\})$ of Q does not contain any of the vertices $\psi(F_i)$ of Q, cf. Theorem 9.8. Let z be a point of $\mathbb{R}^d$ outside Q but "close" to $\psi(\{x\})$ such that every vertex of Q is also a vertex of $Q' := \operatorname{conv}(Q \cup \{z\})$; then the vertices of Q' are the vertices of Q plus the vertex z and the edges of Q' are the edges of Q plus the edges $[z, u]$, where $u \in \operatorname{ext} \psi(\{x\})$. (Supposing that $o \in \operatorname{int} P$, one may take Q' to be the polar of a polytope obtained by truncating the vertex x of P, cf. Section 11.) By Theorem 15.6 there are d independent paths in $\mathscr{G}(Q')$ joining the vertices z and $\psi(F_1)$. Traversing the ith path from z to $\psi(F_1)$, let y_i be the vertex preceding the first of any of the vertices $\psi(F_1), \ldots, \psi(F_m)$ on the path. Then by duality,

$\psi^{-1}(\{y_1\}), \ldots, \psi^{-1}(\{y_d\})$ are d facets of P not in $\mathscr{S}$, each having a $(d - 2)$-face in common with some member of $\mathscr{S}$. □

We are now in position to prove the Lower Bound Theorem:

Proof (Theorem 19.1). We divide the proof into four parts. In Part A we prove the inequality for $j = 0$, and in Part B we prove the inequality for $j = d - 2$; here the lemmas above are used. In Part C we cover the remaining values of j; the proof is by induction. Finally, in Part D we exhibit polytopes for which we have equality.

A. We choose a vertex x of P and let

$$\mathscr{S} := \{F \in \mathscr{F}_{d-1}(P) \mid x \notin F\}.$$

Then $\mathscr{G}(\mathscr{S})$ is the subgraph of $\mathscr{G}(P)$ spanned by $\operatorname{ext} P\backslash\{x\}$, whence, by Theorem 15.5, $\mathscr{S}$ is a connected facet system. The number of members of $\mathscr{S}$ is $p - d$.

Only one vertex of P is not in $\mathscr{S}$, namely, the vertex x. The d vertices of P adjacent to x are external vertices of $\mathscr{S}$, and they are the only external vertices of $\mathscr{S}$. Hence, the number of internal vertices of $\mathscr{S}$ is $f_0(P) - (d + 1)$.

If $p = d + 1$, then P is a d-simplex and the inequality holds with equality. If $p \geq d + 2$, we remove facets from $\mathscr{S}$ one by one by successive applications of Lemma 19.3. At each removal, at least $d - 1$ vertices change their status from internal to external by Lemma 19.4. After $p - d - 1$ removals, we end up with a one-membered facet system. The total number of vertices which during the removal process have changed their status is therefore at least

$$(p - d - 1)(d - 1).$$

Since the number of internal vertices equals $f_0(P) - (d + 1)$, it follows that

$$f_0(P) - (d + 1) \geq (p - d - 1)(d - 1),$$

whence

$$f_0(P) \geq (d - 1)p - (d + 1)(d - 2),$$

as desired.

B. This part is divided into two steps. We first prove that if there is a constant K depending on d only such that

$$f_{d-2}(P) \geq d f_{d-1}(P) - K \tag{1}$$

for all simple d-polytopes P, then the desired inequality

$$f_{d-2}(P) \geq d f_{d-1}(P) - (d^2 + d)/2 \tag{2}$$

must hold. Then, in the second step, we show that (1) holds with $K = d^2 + d$.

Suppose that the inequality (2) does not hold in general. Then there is a simple d-polytope P in $\mathbb{R}^d$ such that

$$f_{d-2}(P) = df_{d-1}(P) - (d^2 + d)/2 - r$$

for some $r > 0$. Let Q be a dual of P in $\mathbb{R}^d$. Then Q is a simplicial d-polytope with

$$f_1(Q) = df_0(Q) - (d^2 + d)/2 - r.$$

By Theorem 11.10 we may assume that there is a facet F of Q such that the orthogonal projection of $\mathbb{R}^d$ onto the hyperplane aff F maps $Q\setminus F$ into ri F. Let Q' denote the polytope obtained by reflecting Q in aff F. Then $Q_1 := Q \cup Q'$ is again a d-polytope by the property of F. It is clear that Q_1 is simplicial. Since F has d vertices, we have

$$f_0(Q_1) = 2f_0(Q) - d,$$

and since F has

$$\binom{d}{2} = (d^2 - d)/2$$

edges, we have

$$f_1(Q_1) = 2f_1(Q) - (d^2 - d)/2.$$

We then get

$$\begin{aligned} f_1(Q_1) &= 2(df_0(Q) - (d^2 + d)/2 - r) - (d^2 - d)/2 \\ &= df_0(Q_1) - (d^2 + d)/2 - 2r. \end{aligned}$$

Let P_1 be a dual of Q_1. Then P_1 is a simple d-polytope with

$$f_{d-2}(P_1) = df_{d-1}(P_1) - (d^2 + d)/2 - 2r$$

This shows that P_1 fails to satisfy (2) by at least $2r$ faces of dimension $d - 2$. Continuing this construction we conclude that no inequality of the form (1) can hold for all simple d-polytopes. This completes the first step.

To carry out the second step, let P be any simple d-polytope, and let $p := f_{d-1}(P)$. Let x and $\mathscr{S}$ be as in Part A. If $p = d + 1$, then P is a d-simplex, whence

$$\begin{aligned} f_{d-2}(P) &= \binom{d+1}{d-1} \\ &= d(d+1) - (d^2 + d)/2 \\ &= dp - (d^2 + d)/2 \\ &> dp - (d^2 + d), \end{aligned}$$

as desired. For $p \geq d + 2$, we shall remove the facets in $\mathscr{S}$ one by one by successive applications of Lemma 19.3 as we did in Part A. Let F_i denote the

ith member of $\mathscr{S}$ to be removed, let x_i denote a corresponding external vertex of $\mathscr{S}\setminus\{F_1, \ldots, F_{i-1}\}$ contained in F_i, and let

$$\mathscr{S}_i := \{F_i \cap F_j | F_i \cap F_j \neq \emptyset, j = i + 1, \ldots, p\}, \qquad i = 1, \ldots, p - d - 1.$$

Then $\mathscr{S}_i$ is a facet system in F_i, cf. Theorem 12.14.

Now, let us say that a $(d - 2)$-face G of F_i is of type 1 in F_i if G is not in $\mathscr{S}_i$ but some $(d - 3)$-face of G is in $\mathscr{S}_i$. Lemma 19.5 can be applied to the facet system $\mathscr{S}_i$ in F_i, for x_i is a vertex of F_i not in $\mathscr{S}_i$. As a result we get $d - 1$ $(d - 2)$-faces of F_i of type 1. Note that a $(d - 2)$-face of type 1 in F_i is not a face of any F_j with $j > i$.

For $i = 1, \ldots, p - d - 1$, let

$$q_i := \max\{j | i < j, F_i \cap F_j \neq \emptyset\}.$$

Then $G_i := F_i \cap F_{q_i}$ is a $(d - 2)$-face of F_i which we shall call a $(d - 2)$-face of type 2 in F_i. Note that F_i and F_{q_i} are the only facets of P containing G_i, cf. Theorem 12.14, that G_i is not at the same time of type 1 in F_i, and that G_i is neither of type 1 nor type 2 in F_{q_i}.

The discussion above now shows that for $i = 1, \ldots, p - d - 1$, the number of $(d - 2)$-faces contributed by F_i is at least d, namely, $d - 1$ of type 1 and one of type 2. Therefore, the total number of $(d - 2)$-faces of P is at least

$$(p - d - 1)d = dp - (d^2 + d),$$

as desired.

C. Using induction on d we shall prove that the inequality holds for the remaining values of j, namely, $j = 1, \ldots, d - 3$. We first note that for $d = 3$ there are no such remaining values; this ensures the start of the induction. So, let $d \geq 4$ and assume that the inequality holds for dimension $d - 1$ and $j = 1, \ldots, (d - 1) - 3$. Let P be a simple d-polytope with p facets, and let j have any of the values $1, \ldots, d - 3$. By a *j-incidence* we shall mean a pair (F, G) where F is a facet of P and G is a j-face of F. (This notion of incidence differs from the one used in the proof of the Upper Bound Theorem.) It is clear that the number of j-incidences equals

$$\sum_{F \in \mathscr{F}_{d-1}(P)} f_j(F).$$

Moreover, since each j-face of P is contained in precisely $d - j$ facets, the number of j-incidences also equals $(d - j)f_j(P)$. Hence,

$$(d - j)f_j(P) = \sum_{F \in \mathscr{F}_{d-1}(P)} f_j(F). \tag{3}$$

We next note that for any facet F of P we have

$$f_j(F) \geq \binom{d-1}{j+1} f_{d-2}(F) - \binom{d}{j+1}(d - 2 - j); \tag{4}$$

in fact, for $j = 1, \ldots, d - 4$ this follows from the induction hypothesis applied to F, and for $j = d - 3$ it follows from the result of Part B applied to F. Combining (3) and (4) we obtain

$$(d-j)f_j(P) \geq \sum_{F \in \mathscr{F}_{d-1}(P)} \left(\binom{d-1}{j+1} f_{d-2}(F) - \binom{d}{j+1}(d-2-j)\right)$$
$$= \binom{d-1}{j+1} \sum_{F \in \mathscr{F}_{d-1}(P)} f_{d-2}(F) - \binom{d}{j+1}(d-2-j) \sum_{F \in \mathscr{F}_{d-1}(P)} 1$$
$$= \binom{d-1}{j+1} \sum_{F \in \mathscr{F}_{d-1}(P)} f_{d-2}(F) - \binom{d}{j+1}(d-2-j)p.$$

Here

$$\sum_{F \in \mathscr{F}_{d-1}(P)} f_{d-2}(F) = 2f_{d-2}(P)$$

since each $(d-2)$-face of P is contained in precisely two facets. Hence,

$$(d-j)f_j(P) \geq \binom{d-1}{j+1} 2f_{d-2}(P) - \binom{d}{j+1}(d-2-j)p.$$

We next apply the result of Part B to P, obtaining

$$(d-j)f_j(P) \geq \binom{d-1}{j+1} 2\left(dp - \binom{d+1}{d-1}\right) - \binom{d}{j+1}(d-2-j)p.$$

An easy calculation shows that the right-hand side of this inequality may be rewritten as

$$(d-j)\left(\binom{d}{j+1}p - \binom{d+1}{j+1}(d-1-j)\right).$$

Cancelling the factor $d - j$, we obtain the desired inequality.

D. It is easy to see that we have equality for all j when P is a d-simplex. Truncation of one vertex of a simple d-polytope P with p facets produces a simple d-polytope P' with $p + 1$ facets, with

$$\binom{d}{j+1}$$

more j-faces than P for $1 \leq j \leq d - 2$, and with $d - 1$ more vertices than P, cf. Theorem 12.18. It is easy to see that if we have equality for P, then we also have equality for P'. Hence, the desired polytopes may be obtained from a d-simplex by repeated truncation of vertices. This completes the proof of Theorem 19.1. □

It would be desirable to have a more direct proof of the Lower Bound Inequalities than the one given in Parts A, B and C above. As a beginning, one

could think of a direct proof of the inequality for $j = d - 2$, replacing the two-step proof of Part B. In the second step we proved that (1) holds with $K = d^2 + d$. Compared to the desired inequality, the deficit amounts to $(d^2 + d)/2$. However, when counting the $(d-2)$-faces we did not count those containing x; the number of such $(d-2)$-faces equals

$$\binom{d}{d-2} = (d^2 - d)/2.$$

This improvement does not yield the desired inequality, but it reduces the deficit to d.

In Part D of the proof of Theorem 19.1, we showed that we have equality for the *truncation polytopes*, i.e. the polytopes obtained from simplices by successive truncations of vertices. For $d \geq 4$ it is known that if $f_j(P) = \varphi_j(d, p)$ for just one value of j, then P must be a truncation polytope. For $d = 3$ the situation is different. As we know, all simple 3-polytopes yield equality. On the other hand, there are simple 3-polytopes which are not truncation polytopes, for example, the parallellotopes.

In Section 18 it was indicated that the upper bound $\Phi_j(d, p)$ is also valid for non-simple polytopes. In contrast to this, little seems to be known about lower bounds for non-simple polytopes.

In its dual form, the Lower Bound Theorem may be stated as follows:

Corollary 19.6. *For any simplicial d-polytope P with p vertices we have*

$$f_j(P) \geq \varphi_{d-1-j}(d, p), \qquad j = 1, \ldots, d-1.$$

Moreover, there are simplicial d-polytopes P with p vertices such that

$$f_j(P) = \varphi_{d-1-j}(d, p), \qquad j = 1, \ldots, d-1.$$

Equality in Corollary 19.7 is attained by the duals of the truncation polytopes, and, for $d \geq 4$, only by these. They are the polytopes obtained from simplices by successive addition of pyramids over facets; they are called *stacked polytopes*.

It is interesting to note that the Lower Bound Inequalities are closely related to inequalities between the numbers $g_i(P)$ introduced in Section 18. For details, see Section 20.

§20. McMullen's Conditions

At the beginning of Section 16 it was indicated that it is not known how to characterize the f-vectors of d-polytopes among all d-tuples of positive integers. However, the more restricted problem of characterizing the f-vectors of simple (or simplicial) d-polytopes has recently been solved. It was