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The Polytope Algebra

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Let $\mathbb{F}$ be an ordered field, and let $\mathscr{P}$ denote the family of all convex polytopes in the d-dimensional vector space V over $\mathbb{F}$. The universal abelian group Π corresponding to the translation invariant valuations on $\mathscr{P}$ has generators $[P]$ for $P \in \mathscr{P}$ (with $[\emptyset]=0$), satisfying the relations (V) $[P \cup Q]+[P \cap Q]=[P]+[Q]$ whenever $P, Q, P \cup Q \in \mathscr{P}$, and (T) $[P+t]=[P]$ for $P \in \mathscr{P}$ and $t \in V$. With multiplication induced by (M) $[P]\cdot[Q]=[P+Q]$, Π is almost a graded commutative algebra over $\mathbb{F}$, in that $\Pi=\bigoplus_{r=0}^{d} \Xi_r$, with $\Xi_0 \cong \mathbb{Z}$, Ξ_r a vector space over $\mathbb{F}$ ($r \geqslant 1$), and $\Xi_r \cdot \Xi_s = \Xi_{r+s}$ ($r, s \geqslant 0$, $\Xi_r=\{0\}$ for $r>d$). The dilatation (D) $\Delta(\lambda)[P]=[\lambda P]$ for $P \in \mathscr{P}$ and $\lambda \in \mathbb{F}$ is such that $\Delta(\lambda)x=\lambda^r x$ for $x \in \Xi_r$ and $\lambda \geqslant 0$. Negative dilatations arise from the Euler map (E) $[P] \mapsto [P]^* := \sum_F (-1)^{\dim F}[F]$ (the sum extending over all faces F of P), since $\Delta(\lambda)x=\lambda^r x^*$ for $x \in \Xi_r$ and $\lambda<0$. Separating group homomorphisms for Π are the frame functionals, which give the volumes of the faces of polytopes determined by successive support hyperplanes in sequences of directions. Two isomorphisms on Π are described: one related to cones of outer normal vectors, and the other to the polytope groups, obtained from Π by discarding polytopes of dimension less than d. Various applications of the polytope algebra are given, including a theory of mixed polytopes, which has implications for mixed valuations.

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1. INTRODUCTION

As we should always remember, the very word "geometry" suggests comparison of measurements such as area and perimeter of different figures. An old question, mentioned by Gauss and crystallized by Hilbert in his Third Problem, is whether a satisfactory theory of volume of polytopes can be formulated in terms of equidissectability (or equicomplementability). It has been known from at least the time of Archimedes that the problem could be dealt with by the "method of exhaustion." (Here, as elsewhere, we shall not give the original historical references, but instead refer the reader to the works we do cite, and in particular to the survey article [9].) In the precise terms in which Hilbert phrased it, Dehn had already found the required counterexample before the problem had been published. Nevertheless, the problem itself provoked investigations into equidissectability under various groups of motions, which culminated in the complete solution of the translation case by Jessen and Thorup [4] and, independently, by Sah [12]. (When the full group of isometries is allowed, the problem remains open in five or more dimensions.)

Volume, and functions such as surface area and the Euler characteristic, are examples of valuations, and their investigation provides another strand to the story. Indeed, the close connexion between valuations and dissections was already used by Dehn to provide his counterexample, although the formal development of the theory undoubtedly owes most to Hadwiger [3]. (Hadwiger, incidentally, showed that, in a somewhat different sense from that of Hilbert and Dehn—in essence by imposing the weakest form of the assumption made by Archimedes—valuations and dissectability can lead to a satisfactory treatment of volume.)

Jessen and Thorup, and Sah, built on Hadwiger's work by considering the universal group corresponding to translation invariant simple valuations (*simple* here refers to those valuations which vanish on polytopes of less than the full dimension; we shall give precise definitions of the terms we use in Section 2 below). They deal with polytopes in a finite dimensional affine space over an arbitrary ordered field, and show (among other things) that the corresponding polytope group is a vector space over that field. A crucial feature of their treatments is that they also describe a family of homomorphisms (into the base field) which separates the group.

In this paper, we shall describe the corresponding universal group for the translation invariant valuations which are not necessarily simple; in other words, we no longer work strictly with dissections, because we do not discard lower dimensional polytopes. The name *polytope algebra* which we give this group indicates that it has a richer structure than that of the polytope group of the previous paragraph; indeed, it fails to be a genuine graded commutative algebra over the base field in just one trivial respect. The grading arises from scaling, or dilatation, by non-negative elements of the field; negative dilatations involve Euler-type relations.

We shall construct two group isomorphisms between the polytope algebra and other groups, one strongly reminiscent of the intrinsic volumes (or quermassintegrals), and the other related to the polytope group. We shall also discuss other groups connected with the polytope algebra, and develop a theory of mixed polytopes, which generalize mixed valuations.

For convenience, we collect the statements of the basic definitions and the five main theorems in Section 2. The numbering of these theorems corresponds to an orderly description of the structure of the polytope algebra, and bears little relationship to the order in which they are proved.

Some of the results are just universalized versions of theorems on valuations which have been proved elsewhere, and so little purpose would be served by reproducing their proofs with obvious changes of language. But details of most of the proofs of the main theorems are given, even though in a number of respects they strongly resemble the corresponding theory of the polytope group. In part, this is because some of the differences are a little subtle, and in pointing out how the earlier proofs can be modified we find that not much can be omitted. Also, however, while largely following [4], we have chosen in some places to follow [12]. Another distinguishing feature is the presence of a genuine multiplication. This permits a different line of attack, and also allows us to introduce at an early stage the useful concept of the logarithm of a polytope.

An early draft of this paper was written in 1984/1985; in that, the base field was just the real field $\mathbb{R}$, multiplication only appeared as an afterthought, and the rest of Theorems 1 and 2 was established by means of an inductive proof of Theorem 4. The present approach has enabled us to mimic much of the corresponding parts of [4, 12], and so construct a parallel theory from which most of the earlier results can be deduced.

2. Basic Definitions and Main Theorems

As we said above, in this section we shall state the basic definitions and main theorems.

Let $\mathbb{F}$ be an ordered, but not necessarily archimedean, field, and let V be

a d-dimensional vector space over $\mathbb{F}$, which is, of course, isomorphic to the coordinate vector space $\mathbb{F}^d$. In many ways, though, it is the affine structure of V which is of interest. The topology of V is that induced by the order topology of $\mathbb{F}$.

Though it could be avoided, we shall find it convenient to endow V with a (positive definite) inner product $\langle \cdot, \cdot \rangle$, and orthogonality will always refer to this. In many cases, the orthogonality is only used to set up an isomorphism between V and its dual space. However, since the Gram–Schmidt process will turn an arbitrary basis of a (linear) subspace L of V into an orthogonal basis, orthogonal projection onto L can be defined.

We shall mostly deal with convex subsets of V, where, as usual, $C \subseteq V$ is *convex* if $(1-\lambda)v + \lambda w \in C$ whenever $v, w \in C$ and $0 \leqslant \lambda \leqslant 1$ (with $\lambda \in \mathbb{F}$, of course, but this will be a general assumption about scalars unless specified otherwise). This purely algebraic definition ensures that all the standard results about convex sets, which are usually established in $\mathbb{R}^d$, carry over to convex sets in V.

Two families of convex sets are of importance here. A *polytope* is the convex hull conv S of a finite set S in V. The empty set $\varnothing$ is a particular example of a polytope. The family of all polytopes in V is denoted $\mathscr{P} = \mathscr{P}(V)$. The *dimension* dim P of a polytope P is the (algebraic) dimension of its affine hull aff P; a k-dimensional polytope is called briefly a *k-polytope*. (Here, and elsewhere when it is relevant, we follow the notation and terminology of [2].)

A (polyhedral) *cone* is the positive hull pos S of a finite subset S of V, so that the origin o of V is always an apex of a cone. The family of cones in V is denoted $\mathscr{C} = \mathscr{C}(V)$.

Observe that a polytope is just a bounded intersection of finitely many closed half-spaces, while a cone is an intersection of finitely many closed half-spaces whose bounding hyperplanes contain o.

Let $\mathscr{F} = \mathscr{P}$ or $\mathscr{C}$. A function ϕ on $\mathscr{F}$, taking values in some abelian group, is called a *valuation* if $\phi(P \cup Q) + \phi(P \cap Q) = \phi(P) + \phi(Q)$ whenever $P, Q \in \mathscr{F}$ are such that $P \cup Q \in \mathscr{F}$ also (note that $P \cap Q \in \mathscr{F}$ always). Further, ϕ is said to be *translation invariant* if $\phi(P+t) = \phi(P)$ for each $P \in \mathscr{F}$ and translation vector $t \in V$ (this definition has no force if $\mathscr{F} = \mathscr{C}$, but for convenience will be allowed to stand in definitions or results which otherwise apply to both classes, as immediately below). Here, the *Minkowski* or *vector sum* of two subsets S, T of V is defined by

$$S + T = \{v + w \mid v \in S,\ w \in T\},$$

and $S + t := S + \{t\}$. By convention, $\phi(\varnothing) = 0$ for every valuation ϕ.

If L is a (linear) subspace of V, we write

$$\mathscr{F}(L)=\{P\in\mathscr{F}\mid P\subseteq L+t \text{ for some } t\in V\}.$$

A valuation ϕ on $\mathscr{F}(L)$ is called *L-simple* if $\phi(P)=0$ for all $P\in\mathscr{F}(L)$ with $\dim P<\dim L$.

The *polytope algebra* $\Pi=\Pi(V)$ is (initially) the abelian group with a generator $[P]$ for each $P\in\mathscr{P}$ (and $[\varnothing]=0$); these generators satisfy the relations:

(V) $[P\cup Q]+[P\cap Q]=[P]+[Q]$, whenever $P, Q\in\mathscr{P}$ are such that $P\cup Q\in\mathscr{P}$ also;

(T) $[P+t]=[P]$, for each $P\in\mathscr{P}$ and $t\in V$.

We shall refer to $[P]$ as the *class* of P in Π.

We shall make the obvious connexion between the definition of a translation invariant valuation on $\mathscr{P}$ and the relations (V) and (T) explicit in Lemma 1 (Section 3 below).

We immediately turn Π into a ring. The *multiplication* is defined on the generators of Π by:

(M) $[P]\cdot[Q]=[P+Q]$, for all $P, Q\in\mathscr{P}$,

with the Minkowski sum $P+Q$ as above. Lemma 7 (Section 4) will show that (M) indeed induces a multiplication on Π.

For $\lambda\in\mathbb{F}$, the *dilatation* $\Delta(\lambda)$ is defined on the generators of Π by:

(D) $\Delta(\lambda)[P]=[\lambda P]$, for $P\in\mathscr{P}$,

where for S a subset of V

$$\lambda S=\{\lambda v\mid v\in S\}$$

is the *scalar multiple* or *dilatate* of S by λ. In Section 5 (Corollary 2 to Theorem 6), we shall see that $\Delta(\lambda)$ is a ring endomorphism of Π.

We can now state the main structure theorems.

THEOREM 1. *The polytope algebra Π is almost a graded commutative algebra over $\mathbb{F}$, in the following sense:*

(a) *as an abelian group, Π admits a direct sum decomposition*

$$\Pi=\bigoplus_{r=0}^{d}\Xi_r;$$

(b) *under multiplication,*

$$\Xi_r\cdot\Xi_s=\Xi_{r+s},$$

for $r, s=0, \ldots, d$ ($\Xi_r=\{0\}$ for $r>d$);

(c) *$\Xi_0\cong\mathbb{Z}$, and for $r=1, \ldots, d$, Ξ_r is a vector space over $\mathbb{F}$ (with $\Xi_d\cong\mathbb{F}$);*

(d) *if $x, y\in Z_1:=\bigoplus_{r=1}^{d}\Xi_r$, and $\lambda\in\mathbb{F}$, then $(\lambda x)y=x(\lambda y)$ $(=\lambda(xy))$;*

(e) *the dilatations $\Delta(\lambda)$ are algebra endomorphisms of Π, and for $r=0, \ldots, d$, if $x\in\Xi_r$ and $\lambda\geqslant 0$, then*

$$\Delta(\lambda)x=\lambda^r x,$$

where $\lambda^0=1$.

The *Euler map* * is defined on the generators of Π by:

(E) $[P]^*=\sum_F(-1)^{\dim F}[F]$, for $P\in\mathscr{P}$, where the sum (here and elsewhere) extends over all faces F of P.

THEOREM 2. *The Euler map is an involutory automorphism of Π. Moreover, for $r=0, \ldots, d$, if $x\in\Xi_r$ and $\lambda<0$, then*

$$\Delta(\lambda)x=\lambda^r x^*.$$

We next describe the separating group homomorphisms on Π. If u is a non-zero vector in V and $P\in\mathscr{P}$, then the *face of P in direction u* is defined to be

$$P_u=\{v\in P\mid\langle v, u\rangle=h(P, u)\},$$

where

$$h(P, u)=\max\{\langle w, u\rangle\mid w\in P\}$$

is the *support functional* of P in direction u. Thus P_u is the intersection of P with its support hyperplane with outer normal u. If $U=(u_1, \ldots, u_k)$ is a *k-frame*, that is, an ordered orthogonal set of k vectors, we define recursively

$$P_U=(P_{(u_1,\ldots,u_{k-1})})_{u_k},$$

starting with $P_\varnothing=P$ (we allow $\varnothing$ as a frame).

We shall identify the highest grade term Ξ_d in Theorem 1 with volume (see Section 7). More generally, every subspace L of V admits a (within scaling) unique volume functional $\mathrm{vol}_L:\mathscr{P}(L)\to\mathbb{F}$. If U is a $(d-r)$-frame, we write $\mathrm{vol}_U:=\mathrm{vol}_L$ if

$$L=U^\perp:=\{v\in V\mid\langle v, u\rangle=0 \text{ for each } u\in U\}$$

is the orthogonal complement of U in V, and we call the mapping $f_U: \mathscr{P} \to \mathbb{F}$ defined by

$$f_U(P) = \mathrm{vol}_U P_U$$

a *frame functional of type r*. Frame functionals induce homomorphisms on Π (see Section 5, Theorem 7), and we have:

THEOREM 3. *The frame functionals separate Π; that is, if $x \in \Pi$ is such that $f_U(x) = 0$ for every frame U, then $x = 0$.*

Let $\mathscr{F} = \mathscr{P}$ or $\mathscr{C}$ as before, and let L be a subspace of V. The abelian group with a generator $\langle P \rangle$ for each $P \in \mathscr{F}(L)$, satisfying the relations (V), (T) (for $\mathscr{F} = \mathscr{P}$) and

(S) $\langle P \rangle = 0$, for $P \in \mathscr{F}(L)$ with $\dim P < \dim L$,

is the *polytope group* $\hat{\Pi}(L)$ or the *cone group* $\hat{\Sigma}(L)$, respectively. The *full polytope group* $\hat{\Pi}$ and the *full cone group* $\hat{\Sigma}$ are defined by

$$\hat{\Pi} = \bigoplus_L \hat{\Pi}(L), \qquad \hat{\Sigma} = \bigoplus_L \hat{\Sigma}(L),$$

the direct sums in each case extending over all subspaces L of V, including $\{o\}$ and V itself.

The first isomorphism theorem for Π is

THEOREM 4. $\Pi \cong \hat{\Pi}$.

For the second, we begin by defining the *outer* (or *normal*) *cone* $N(F, P)$ to a polytope or cone P at its non-empty face F by

$$N(F, P) = \{u \in V \mid \langle v, u \rangle = h(P, u) \text{ for every } v \in F\}.$$

That is, $N(F, P)$ is the set of outer normal vectors to support hyperplanes of P which contain F (allowing o as such a vector also). The subspace L of V parallel to aff F, written $L \| F$, is the orthogonal complement of $N(F, P)$, and we write vol $F := \mathrm{vol}_L F$. We denote by $n(F, P) := \langle N(F, P) \rangle$ the *intrinsic* class of $N(F, P)$, meaning its class in $\hat{\Sigma}(\mathrm{lin}\, N(F, P))$. The mapping $\sigma: \mathscr{P} \to \mathbb{F} \otimes \hat{\Sigma}$ defined by

$$\sigma(P) := \sum_F \mathrm{vol}\, F \otimes n(F, P)$$

induces a homomorphism on Π (see Section 12, Lemma 37), and we have

THEOREM 5. *The mapping $\sigma: \Pi \to \mathbb{F} \otimes \hat{\Sigma}$ is injective.*

3. Preliminary Remarks

Before we embark on the main part of the proofs of the theorems, we make some general remarks about valuations and their extensions, and about particular classes of polytopes.

We first make explicit the relationship between valuations and the polytope algebra. A fact which we shall often use without much comment is

LEMMA 1. *Let $\mathscr{G}$ be an abelian group. A mapping $\phi: \mathscr{P} \to \mathscr{G}$ is a translation invariant valuation if and only if ϕ induces a (group) homomorphism from Π to $\mathscr{G}$.*

We shall invariably denote this homomorphism by the same symbol, and not distinguish between it and the translation invariant valuation to which it corresponds; that is, we write $\phi([P]) = \phi(P)$. Lemma 1 enables us to lift known results about translation invariant valuations to Π; observe, in particular, that the mapping $P \mapsto [P]$ is a translation invariant valuation.

Note that there is an exactly analogous relationship between L-simple translation invariant valuations on $\mathscr{P}(L)$ and homomorphisms on the polytope group $\hat{\Pi}(L)$, and similarly for $\mathscr{C}(L)$ and $\hat{\Sigma}(L)$.

A useful variant of the idea of valuation is the following. We call a mapping ϕ on $\mathscr{P}$ (into some abelian group) a *weak valuation* if $\phi(P) + \phi(P \cap H) = \phi(P \cap H^-) + \phi(P \cap H^+)$ whenever $P \in \mathscr{P}$ and H is a hyperplane in V which bounds the two closed half-spaces H^- and H^+. It was shown by Sallee [14] that

LEMMA 2. *A mapping on $\mathscr{P}$ is a valuation if and only if it is a weak valuation.*

This lemma implies that we can replace the condition (V) in the definition of Π by

(W) $[P] + [P \cap H] = [P \cap H^-] + [P \cap H^+]$, for $P \in \mathscr{P}$ and H a hyperplane bounding the closed half-spaces H^- and H^+.

A modification of an approach due to Groemer [1] yields many results concerning extensions of valuations, or suitable restrictions of their domain of definition. The *characteristic function* $S^\dagger$ of a subset S of V is defined (in the usual way) by

$$S^\dagger(v) = \begin{cases} 1, & \text{if } v \in S, \\ 0, & \text{if } v \notin S. \end{cases}$$

The subgroup of functions on V taking values in $\mathbb{Z}$ which is generated by the functions $P^{\dagger}$ with $P \in \mathscr{P}$ is denoted by $X(\mathscr{P})$. The crucial observation of Groemer [1] is:

LEMMA 3. *A mapping on $\mathscr{P}$ (into some abelian group) is a valuation if and only if it induces a homomorphism on $X(\mathscr{P})$.*

Since a homomorphism on $X(\mathscr{P})$ is defined uniquely on any charac-
teristic function (of some subset of V) which happens to lie in $X(\mathscr{P})$, we
deduce certain important consequences. As in [9], we denote by $U(\mathscr{P})$ the
family of finite unions of polytopes in $\mathscr{P}$; further, we write

$$\bar{U}(\mathscr{P}) = \{A \setminus B \mid A, B \in U(\mathscr{P})\}.$$

LEMMA 4. *A valuation on $\mathscr{P}$ admits a unique extension to a valuation on $\bar{U}(\mathscr{P})$.*

With $\bar{U}(\mathscr{P})$ replaced by $U(\mathscr{P})$, this result is due to Volland [17]. It is of interest to sketch a proof of this important lemma. First observe that, if $A, B \subseteq V$, then

$$(A \cap B)^{\dagger} = A^{\dagger} B^{\dagger}.$$

Since the relationship for complements is

$$(V \setminus S)^{\dagger} = 1 - S^{\dagger},$$

or, more generally,

$$(A \setminus B)^{\dagger} = A^{\dagger}(1 - B^{\dagger}) = A^{\dagger} - A^{\dagger} B^{\dagger},$$

that for unions is

$$1 - (A_1 \cup \cdots \cup A_n)^{\dagger} = (1 - A_1^{\dagger}) \cdots (1 - A_n^{\dagger}).$$

The proof of Lemma 4 is now straightforward. The formula for the charac-
teristic function of a general element of $U(\mathscr{P})$ follows at once from this last
expression for the union (note that $1 = V^{\dagger}$, which, of course, is not in
$X(\mathscr{P})$, occurs on both sides of the expression). The expansion of this
formula gives the familiar *inclusion–exclusion principle* for valuations (see
[9, (1.2)]).

For our purposes, we must note two consequences of Lemma 4. As men-
tioned above, a polytope is a bounded intersection of finitely many closed
half-spaces. On occasions, though, it is more convenient to work with
bounded intersections of finitely many half-spaces, which are either closed
or open; we call these *partly open* polytopes, and denote the family of them
by $\mathscr{P}_{po}$. Recalling that a *decomposition* of a set is an expression of that set
as a disjoint union of subsets, our first consequence of Lemma 4 is:

COROLLARY. *A valuation ϕ on $\mathscr{P}$ admits a unique extension to $\mathscr{P}_{po}$. Moreover, if $Q_1, \ldots, Q_k \in \mathscr{P}_{po}$ decompose $Q \in \mathscr{P}_{po}$, then*

$$\phi(Q) = \sum_{j=1}^{k} \phi(Q_j).$$

We shall discuss the even more special case of relatively open polytopes
in Section 19.

A *simplex* is the convex hull $\operatorname{conv}\{v_0, \ldots, v_k\}$ of an affinely independent
set $\{v_0, \ldots, v_k\}$ in V; more specifically, this is a *k-simplex*, since it has
dimension k. A result admitting many proofs (see, for example, [9, Sect. 6],
which uses [18]; for a nice proof, see [16]) is:

LEMMA 5. *If $P \in \mathscr{P}$, then there is a simplicial complex in V whose underlying point-set is P.*

Combining this with Lemma 4, we have:

COROLLARY. *The group Π is generated by the classes of the simplices in $\mathscr{P}$.*

4. MULTIPLICATION

An important role in our treatment is played by the multiplication on Π
induced by Minkowski addition. In [4, 12] a product structure is also
introduced, but it only gives a product mapping from $\hat{\Pi}(L) \otimes \hat{\Pi}(M)$ to
$\hat{\Pi}(L+M)$, when L and M are supplementary (linear) subspaces of V. (The
product discussed in [1], however, does correspond to ours.) Initially, we
shall use our multiplication in a very similar way, but we shall soon see
examples of its greater power and generality.

Of course, we must first establish that our definition does lead to a mul-
tiplication on Π; we do that here.

LEMMA 6. *With addition satisfying* (V) *and* (T), *and multiplication defined by* (M) *and extended by linearity, Π is a commutative ring with unity.*

All the properties of a commutative ring with unity are easily verified,
except those which we now discuss. We first observe that (M) is compatible
with the translation invariance (T). Next, note that $\varnothing + P = \varnothing$ for every
$P \in \mathscr{P}$, from which we conclude that $0 \cdot [P] = [\varnothing] \cdot [P] = [\varnothing + P] =$
$[\varnothing] = 0$, and hence $0 \cdot x = 0$ for every $x \in \Pi$. (By the way, this is wha
would oblige us to adopt the convention $[\varnothing] = 0$, if it were not otherwis

obvious.) Then we define $1 := [o]$ to be the class of a point (we write $[t]$ for $[\{t\}]$ if $t \in V$; from (T), $[t] = [o]$ for each $t \in V$), which gives the unity of Π.

The only real problem is caused by the extension of multiplication to Π by linearity, so that the distributive law $x(y+z) = xy + xz$ holds for all $x, y, z \in \Pi$. In other words, we must check that (M) is compatible with the valuation property (V). Now, if $P, Q_1, Q_2 \in \mathscr{P}$, then

$$P + (Q_1 \cup Q_2) = (P + Q_1) \cup (P + Q_2),$$

while if $Q_1 \cup Q_2 \in \mathscr{P}$ also, then, as shown in [3, 1.2.2],

$$P + (Q_1 \cap Q_2) = (P + Q_1) \cap (P + Q_2).$$

In this latter case,

$$\begin{aligned}[P] \cdot [Q_1 \cup Q_2] + [P] \cdot [Q_1 \cap Q_2] &= [P + (Q_1 \cup Q_2)] + [P + (Q_1 \cap Q_2)] \\ &= [(P + Q_1) \cup (P + Q_2)] + [(P + Q_1) \cap (P + Q_2)] \\ &= [P + Q_1] + [P + Q_2] \\ &= [P] \cdot [Q_1] + [P] \cdot [Q_2],\end{aligned}$$

as required. This completes the proof of the lemma.

In view of Lemma 6, the multiplication on Π extends to classes of elements of $\bar{U}(\mathscr{P})$, and, in particular, to classes of partly open polytopes. However, while in general this extension does not correspond in a natural way to the geometric Minkowski sum, there is one important exception.

LEMMA 7. *Let L and M be supplementary subspaces of V, let $A, B \in \bar{U}(\mathscr{P})$ be such that $A \subseteq L$ and $B \subseteq M$, and let a, b be their classes in Π. Then the class of $A + B$ is ab.*

The important observation here is that, if $A \subseteq L$ and $B, C \subseteq M$ satisfy $B \cap C = \varnothing$, then $(A + B) \cap (A + C) = \varnothing$ (this is clearly not generally true for arbitrary subsets A, B, C of V). The proof of Lemma 4 will now easily show that the extension from $\mathscr{P}$ to $\bar{U}(\mathscr{P})$ and Minkowski addition are compatible in this special case, and Lemma 7 then follows.

We end this section with a remark. In view of the existence of multiplication, the condition (T) for translation invariance can be expressed as

$$[P]([t] - 1) = 0$$

for all $P \in \mathscr{P}$ and $t \in V$. It follows that we can replace (T) by

(T′) $[t] = [o]$, for every $t \in V$.

5. HOMOMORPHISMS AND ENDOMORPHISMS

We recall that, if A, B are two algebras over the same field $\mathbb{F}$, then a mapping $\Psi: A \to B$ is called a *homomorphism* if it satisfies

(A1) $\Psi(x + y) = \Psi x + \Psi y$,

(A2) $\Psi(xy) = (\Psi x)(\Psi y)$,

(A3) $\Psi(\lambda x) = \lambda(\Psi x)$,

whenever $x, y \in A$ and $\lambda \in \mathbb{F}$. In our case, we shall have $A = \Pi(V)$ and $B = \Pi(W)$ for two finite dimensional vector spaces V, W over $\mathbb{F}$, and then (A3) only applies for $x \in Z_1(V)$. Further, then, if Ψ only satisfies (A1), it is a *group* homomorphism, while if it satisfies (A1) and (A2), it is a *ring* homomorphism. If $A = B$ (or $V = W$), we refer to Ψ as an *endomorphism*, and an invertible endomorphism is an *automorphism*.

Two kinds of endomorphism of Π are of particular importance. Since we have yet to introduce the full algebra structure of Π, in the following two theorems we only prove the ring endomorphism (or homomorphism) properties; the remainder of the proofs will be postponed to the end of Section 11.

THEOREM 6. *Let V, W be vector spaces over $\mathbb{F}$, and let $\Phi: V \to W$ be an affine mapping. Then Φ induces a homomorphism from $\Pi(V)$ to $\Pi(W)$, which is also denoted Φ, by $\Phi[P] = [\Phi P]$ for $P \in \mathscr{P}$. Moreover, Φ commutes with the dilatations.*

Since an affine mapping is just a linear mapping followed by a translation, in view of (T) we can suppose Φ to be linear. In addition, since $\Phi(P + t) = \Phi P + \Phi t$ for $P \in \mathscr{P}$ and $t \in V$, the action of Φ is compatible with (T). For compatibility with (V), if $P, Q \in \mathscr{P}$, then trivially $\Phi(P \cup Q) = \Phi P \cup \Phi Q$, while if $P \cup Q \in \mathscr{P}$ also, then $\Phi(P \cap Q) = \Phi P \cap \Phi Q$ (consider the intersection of P and Q with $\Phi^{-1}w$, for $w \in W$). Thus Φ preserves (V), and so extends by linearity to Π. Finally, if $P, Q \in \mathscr{P}$, then $\Phi(P + Q) = \Phi P + \Phi Q$, and hence Φ respects (M) also, and thus preserves products, by the way (M) extends to Π.

For the last part, since $\Phi(\lambda P) = \lambda(\Phi P)$ for $P \in \mathscr{P}$ and $\lambda \in \mathbb{F}$, Φ commutes (in the obvious sense) with dilatations.

For our purposes, two consequences of Theorem 6 are usually more important.

COROLLARY 1. *An affine mapping $\Phi: V \to V$ induces an endomorphism $\Phi: \Pi(V) \to \Pi(V)$, which commutes with the dilatations.*

COROLLARY 2. *The dilatations $\Delta(\lambda)$ induce endomorphisms of Π.*

The other kind of endomorphism arises in quite a different way.

THEOREM 7. *Let U be a frame in V. Then the mapping $P \mapsto P_U$ on $\mathscr{P}$ induces an endomorphism $x \mapsto x_U$ of Π, defined on the generators by $[P]_U := [P_U]$, which commutes with non-negative dilatations.*

Let us remark here that the mapping $P \mapsto P_U$ only depends on the directions of the vectors in U, so that, if $U = (u_1, ..., u_k)$, then we can replace U by $(\mu_1 u_1, ..., \mu_k u_k)$ with $\mu_i > 0$ $(i = 1, ..., k)$, to obtain the same mapping.

It is clear that we need only prove Theorem 7 for the special case $P \mapsto P_u$, with $u \neq o$ a single vector. The translation invariance (T) is trivial, since $(P+t)_u = P_u + t$. For (V), let $P, Q \in \mathscr{P}$ be such that $P \cup Q \in \mathscr{P}$ also. There are two possibilities. If the support hyperplane H to $P \cup Q$ with outer normal u meets both P and Q, then

$$(P \cup Q)_u = P_u \cup Q_u, \ (P \cap Q)_u = P_u \cap Q_u.$$

If, say, H meets P alone, then

$$(P \cup Q)_u = P_u, \qquad (P \cap Q)_u = Q_u.$$

In either case, (V) is preserved. Further (see [2, 15.1.1]),

$$(P+Q)_u = P_u + Q_u, \qquad (\lambda P)_u = \lambda P_u,$$

for $P, Q \in \mathscr{P}$ and $\lambda \geq 0$. Arguments exactly analogous to those used to prove Theorem 6 now show that $[P] \mapsto [P]_u$ induces a ring endomorphism of Π which commutes with non-negative dilatations. Thus we have Theorem 7 (again, except for the algebra property).

Observe that we cannot allow negative dilatations in Theorem 7. Indeed, we have

$$(\Delta(-1)x)_u = \Delta(-1)(x_{-u}).$$

6. THE RATIONAL STRUCTURE

In this section, we begin the proof of Theorem 1 by establishing a weaker version, with our given field $\mathbb{F}$ replaced by the rational field $\mathbb{Q}$ in various places.

It is clear from the statement of Theorem 1 that the subgroup (actually subring) Ξ_0 of Π generated by the class 1 of a point plays a somewhat anomalous role. We could get around the problem by replacing $\Xi_0 \cong \mathbb{Z}$ (the integers) by the tensor product $\mathbb{F} \otimes \Xi_0 \cong \mathbb{F}$ (tensor products are always over $\mathbb{Z}$). Although we should then obtain a genuine algebra over $\mathbb{F}$, the geometric meaning of Ξ_0 would be blurred. So, we shall pursue an alternative course, and begin by hiving off Ξ_0.

As our notation 1 for $[o]$ suggests, we shall identify Ξ_0 with $\mathbb{Z}$ by writing

$$n = \begin{cases} 1 + \cdots + 1 & (n \text{ times}),\ n \geq 0, \\ -(1 + \cdots + 1) & (-n \text{ times}),\ n < 0, \end{cases}$$

where, in these expressions, $1 = [o]$.

Let Z_1 denote the subgroup of Π generated by all elements of the form $[P] - 1$, with $P \in \mathscr{P} \backslash \{\varnothing\}$.

LEMMA 8. *As an abelian group, Π has a direct sum decomposition*

$$\Pi = \Xi_0 \oplus Z_1.$$

The projection from Π onto Ξ_0 is the dilatation $\Delta(0)$. Further, Z_1 is an ideal in Π, and $z \in Z_1$ if and only if $\Delta(0)z = 0$.

A general element of Π can be expressed as a sum

$$x = \sum_{j=1}^{k} \varepsilon_j [P_j],$$

where $\varepsilon_j = \pm 1$ and $P_j \in \mathscr{P} \backslash \{\varnothing\}$ $(j = 1, ..., k)$. Writing this as

$$x = \sum_{j=1}^{k} \varepsilon_j + \sum_{j=1}^{k} \varepsilon_j([P_j] - 1)$$

expresses x as a member of $\Xi_0 + Z_1$. Further, $x \in Z_1$ if and only if $\sum_{j=1}^{k} \varepsilon_j = 0$, and so the sum is direct.

It is almost obvious that $\Delta(0)[P] = 1$ for every $P \in \mathscr{P} \backslash \{\varnothing\}$. To confirm this, we argue as follows. Since every two k-simplices are affinely equivalent, we see from Theorem 6 that the value of $\Delta(0)[T^k]$ for a k-simplex T^k depends only on the dimension k. But for $k \geq 1$, a k-simplex T^k can be split into two k-simplices by a hyperplane H which separates two vertices of T^k and contains the remaining $k-1$. Since $T^{k-1} := H \cap T^k$ is a $(k-1)$-simplex, the weak valuation property (W) shows that $\Delta(0)[T^k] = \Delta(0)[T^{k-1}]$. We conclude that $\Delta(0)[T] = \Delta(0)1 = 1$ for every non-empty simplex T. Then the mapping $[P] \mapsto 1$, which clearly induces an endomorphism of Π, coincides with $[P] \to \Delta(0)[P]$ on the generators of Π (see Lemma 5), and so is given by $\Delta(0)$.

The characterization of Z_1 follows immediately. Finally, if $z \in Z_1$ and $x \in \Pi$, then $\Delta(0)(xz) = \Delta(0)x \cdot \Delta(0)z = 0$, so that $xz \in Z_1$, and hence Z_1 is an ideal (this can be seen in several other ways as well). This completes the proof of the lemma.

Care does need to be taken over the behavior of $\Delta(0)$. Just because $0 \cdot S = \{o\}$ for every non-empty subset S of V, it does not mean that

$\Delta(0)[S]=1$ for every non-empty S in $\bar{U}(\mathscr{P})$ (the proof of Lemma 8 makes this very clear). Various other ways of seeing that $\Delta(0)[P]=1$ for $P\in\mathscr{P}\backslash\{\emptyset\}$ will become clear below (in Lemmas 10 and 11, for instance); we may observe that $\Delta(0)[P]=\Delta(0)[2P]=\Delta(0)[P]^2=(\Delta(0)[P])^2$ already implies that $\Delta(0)[P]=0$ or 1.

A pivotal role in our treatment is played by the analogues of the canonical simplex dissections of [3]. The presence of these analogues enables us to mimic many of the proofs of [4] or [12], after a suitable change of language.

Suppose that $a_0, a_1, ..., a_k \in V$ are such that $\{a_1, ..., a_k\}$ is linearly independent. We write

$$T(a_1, ..., a_k)=\operatorname{conv}\{a_0, a_0+a_1, ..., a_0+\cdots+a_k\},$$

which is a k-simplex, and define

$$s(a_1, ..., a_k)=[T(a_1, ..., a_k)]-[T(a_1, ..., a_{k-1})],$$

with $s(\emptyset)=1$. This is the class of a partly open simplex (lacking one facet), and plays the role of $[a_1, ..., a_k]$ in [4] or $/a_1/\cdots/a_k/$ in [12]. Of course, condition (T) ensures that $s(a_1, ..., a_k)$ does not depend on a_0, which justifies our not mentioning it. Indeed, it is usually convenient to assume that $a_0=o$.

An obvious first remark is:

LEMMA 9. *The various classes $s(a_1, ..., a_k)$ (with $\{a_1, ..., a_k\}\subseteq V$ linearly independent) generate Π; the classes with $k\geq 1$ generate Z_1.*

By the corollary to Lemma 5, the classes of the simplices generate Π. But, from the definition,

$$[T(a_1, ..., a_k)]=\sum_{j=0}^{k} s(a_1, ..., a_j),$$

and this and Lemma 8 yield the lemma.

The first canonical simplex dissection is

LEMMA 10. *For $\lambda, \mu\geq 0$,*

$$\Delta(\lambda+\mu)s(a_1, ..., a_k)=\sum_{j=0}^{k}(\Delta(\lambda)s(a_1, ..., a_j))(\Delta(\mu)s(a_{j+1}, ..., a_k)).$$

The discussion of Section 3 helps us to visualize what is happening here.

The jth term of the sum is just the class of the partly open polytope

$$\left\{\sum_{i=1}^{k}\xi_i a_i \mid \lambda+\mu\geq\xi_1\geq\cdots\geq\xi_j>\lambda\geq\xi_{j+1}\geq\cdots\geq\xi_k>0\right\},$$

and the disjoint union of these is the original partly open simplex

$$\left\{\sum_{i=1}^{k}\xi_i a_i \mid \lambda+\mu\geq\xi_1\geq\cdots\geq\xi_k>0\right\},$$

whose class is $\Delta(\lambda+\mu)s(a_1, ..., a_k)$. Lemmas 4 and 7 then apply.

Lemma 10 and an induction argument yield the analogue of the second canonical simplex dissection.

LEMMA 11. *For $k\geq 1$ and integer $n\geq 0$,*

$$\Delta(n)s(a_1, ..., a_k)=\sum_{r=1}^{k}\binom{n}{r}z_r,$$

where

$$z_r=\sum_{0=j(0)<j(1)<\cdots<j(r)=k}\ \prod_{i=1}^{r} s(a_{j(i-1)+1}, ..., a_{j(i)})$$

is independent of n.

An alternative proof applies the corollary to Lemma 4 to the decomposition of the partly open simplex

$$\left\{\sum_{i=1}^{k}\xi_i a_i \mid n\geq\xi_1\geq\cdots\geq\xi_k>0\right\}$$

by the half-open strips

$$\left\{\sum_{i=1}^{k}\xi_i a_i \mid m-1<\xi_j\leq m\right\}$$

for $j=1, ..., k$ and $m=1, ..., n$.

As a consequence of Lemma 11, we have

LEMMA 12. *Let $x\in\Pi$. Then there are unique $y_0\in\Xi_0$ and $y_1, ..., y_d\in Z_1$, such that, for all integers $n\geq 0$,*

$$\Delta(n)x=\sum_{r=0}^{d}\binom{n}{r}y_r.$$

The existence of the expression follows from Lemma 11, and the fact that, by Lemma 9, the classes $s(a_1, ..., a_k)$ generate Π. For uniqueness, we note that the y_r can be calculated from various dilatates of x. Indeed, for any integer $n \geqslant 0$, the $(n+1)\times(n+1)$ matrix with (i,j)-entry $\binom{i}{j}$, for $i=0, ..., n$ and $j=0, ..., n$, is invertible over $\mathbb{Z}$, since it is triangular with diagonal entries 1. The inverse matrix is easily calculated, and we then obtain

$$y_r = \sum_{n=0}^{r} (-1)^{r-n}\binom{r}{n} \Delta(n)x,$$

which is the required expression for y_r. This proves the lemma.

We can put $y_r = 0$ for $r > d$ in the expression of Lemma 12, and deduce

COROLLARY. *For $r > d$,*

$$\sum_{n=0}^{r} (-1)^{r-n}\binom{r}{n} \Delta(n) = 0.$$

Now let $P \in \mathscr{P}\backslash\{\varnothing\}$. If we compare the expression

$$\Delta(n)[P] = [nP] = [P]^n = (1 + ([P]-1))^n = 1 + \sum_{r=1}^{n}\binom{n}{r}([P]-1)^r$$

with Lemma 12 and its proof (compare the corollary), we deduce

LEMMA 13. *If $P \in \mathscr{P}\backslash\{\varnothing\}$, then $([P]-1)^r = 0$ for $r > d$.*

Let Z_r be the subgroup of Z_1 generated by all elements of the form $([P]-1)^j$, with $P \in \mathscr{P}\backslash\{\varnothing\}$ and $j \geqslant r$. Writing $Z_0 = \Pi$, from the definition we have the filtration

$$Z_0 \supseteq Z_1 \supseteq \cdots \supseteq Z_d \supseteq Z_{d+1} = \{0\}.$$

Because $\Delta(\lambda)([P]-1)^j = ([\lambda P]-1)^j$, we conclude

LEMMA 14. *If $\lambda \in \mathbb{F}$, then $\Delta(\lambda) Z_r \subseteq Z_r$.*

If we rewrite the expression above as

$$\Delta(n)([P]-1) = \sum_{k=1}^{d}\binom{n}{k}([P]-1)^k,$$

take jth powers of both sides, and again use the fact that $\Delta(n)$ is a ring endomorphism, we obtain

LEMMA 15. *If $x \in Z_r$, then $\Delta(n)x - n^r x \in Z_{r+1}$.*

This holds for the generators $([P]-1)^j$ $(j \geqslant r)$ of Z_r, and so it holds generally.

We are now in a position to show that Z_1 is a vector space over $\mathbb{Q}$. Since Z_1 is an abelian group, it suffices to prove that Z_1 is uniquely divisible, meaning that, given any $x \in Z_1$ and any integer $m \geqslant 2$, there exists a unique $y \in Z_1$, such that $x = my$.

LEMMA 16. *Z_1 is torsion free.*

Let $x \in Z_1$ be a torsion element, say $nx = 0$ with $n \geqslant 2$ an integer. We show by induction that $x \in Z_r$ for all r. Indeed, if $x \in Z_r$, then

$$\Delta(n)x = \Delta(n)x - n^{r-1}\cdot nx \in Z_{r+1},$$

by Lemma 15. Thus $x \in \Delta(n^{-1}) Z_{r+1} = Z_{r+1}$, by Lemma 14, and since $Z_{d+1} = \{0\}$, the lemma follows.

LEMMA 17. *Z_1 is divisible.*

Let $x \in Z_1$ and $m \geqslant 2$ an integer. If $x \in Z_d$, then by Lemmas 14 and 15,

$$x = \Delta(m)\Delta(m^{-1})x = m\cdot m^{d-1}\Delta(m^{-1})x,$$

so that $m^{-1}x$ exists (and is unique by Lemma 16). We now use backward induction on r. If $x \in Z_r$, then

$$y = x - m\cdot m^{r-1}\Delta(m^{-1})x \in Z_{r+1},$$

so that $m^{-1}y \in Z_{r+1}$ exists, and thus

$$m^{-1}x = m^{r-1}\Delta(m^{-1})x + m^{-1}y \in Z_r$$

exists also. The lemma follows at once.

At this stage, we could now follow [4] or [12] in expanding the binomial coefficients $\binom{n}{r}$ in Lemma 12 as polynomials in n with rational coefficients, and collecting together the terms in n^r for each $r = 0, ..., d$. However, an alternative approach using the rational algebra structure is quicker and yields more information.

From Lemma 8, Z_1 is closed under multiplication; further, if $x, y \in Z_1$ and $\lambda = m/n \in \mathbb{Q}$, then $(\lambda x)y = \lambda(xy)$, since both sides are the unique solution z to the equation $nz = (mx)y = m(xy)$. Since Z_1 is generated by the nilpotent elements $[P]-1$ (with $P \in \mathscr{P}\backslash\{\varnothing\}$), every element of Z_1 is nilpotent; that is, Z_1 is a nil ideal of Π. It follows that we can define the *logarithm* and *exponential* mappings in the usual way by

$$\log(1+z) = \sum_{k\geqslant 1} \frac{(-1)^{k-1}}{k} z^k,$$

$$\exp z = \sum_{k\geqslant 0} \frac{1}{k!} z^k$$

(with $z^0 = 1$), for every $z \in Z_1$. The ordinary properties of log and exp carry over, namely

LEMMA 18. *The mappings* log *and* exp *are inverse mappings, and satisfy*

(a) $\log(x_1 x_2) = \log x_1 + \log x_2$, *when* $\Delta(0)x_1 = \Delta(0)x_2 = 1$;

(b) $\exp(z_1 + z_2) = \exp z_1 \cdot \exp z_2$, *when* $z_1, z_2 \in Z_1$.

In particular, $\log[P]$ is defined for every $P \in \mathscr{P}\backslash\{\varnothing\}$; for brevity (but see also Section 8 below), we write $\log P := \log[P]$. Putting $z = [P] - 1$, we recognize $\log P$ as the coefficient of n in the expansion of $[nP] = [P]^n$ given by Lemma 12.

Indeed, since $\log[nP] = \log[P]^n = n \log[P]$, and $\Delta(\lambda) \log P = \log(\lambda P)$ for $\lambda \in \mathbb{Q}$, we deduce at once

LEMMA 19. *For* $P \in \mathscr{P}\backslash\{\varnothing\}$ *and rational* $\lambda \geqslant 0$, $\Delta(\lambda) \log P = \lambda \log P$.

We now invert this relation. If $P \in \mathscr{P}\backslash\{\varnothing\}$, let $p = \log P \in Z_1$, and suppose that $\lambda \geqslant 0$ is rational. Since $\Delta(\lambda)$ is a ring endomorphism of Π, we have

$$[\lambda P] = \Delta(\lambda)[P] = \Delta(\lambda) \exp p = \exp(\Delta(\lambda)\, p)$$

$$= \exp(\lambda p) = \sum_{r=0}^{d} \lambda^r \cdot \frac{1}{r!} p^r.$$

The sum terminates at $r = d$, because the expression for $p = \log P$ and $([P]-1)^{d+1} = 0$ imply $p^{d+1} = 0$ also.

For $r = 1, \ldots, d$, we define the rth *weight space* Ξ_r to be the subgroup of Π generated by all the elements p^r (or $(1/r!)\, p^r$), with $p = \log P$ for some $P \in \mathscr{P}\backslash\{\varnothing\}$. Then

LEMMA 20. $\Pi = \bigoplus_{r=0}^{d} \Xi_r$. *Moreover*, $x \in \Xi_r$ *if and only if, for any single rational* $\lambda > 0$ *with* $\lambda \neq 1$, $\Delta(\lambda)x = \lambda^r x$.

From the definition, if $x \in \Xi_r$ and $\lambda \geqslant 0$ is rational, we have $\Delta(\lambda)x = \lambda^r x$.

Again from the definition, Π is the sum of the Ξ_r. If $x_r \in \Xi_r$ $(r = 0, \ldots, d)$ are such that $\sum_{r=0}^{d} x_r = 0$, then

$$0 = \Delta(\lambda) \sum_{r=0}^{d} x_r = \sum_{r=0}^{d} \lambda^r x_r$$

for every rational number $\lambda \geqslant 0$, and so, since $\Pi = \Xi_0 \oplus Z_1$ is the sum of a copy of $\mathbb{Z}$ and a rational vector space, $x_r = 0$ for each r; that is, the sum is direct.

If $x \in \Pi$ and $\lambda = m/n > 0$ $(m \neq n)$ are such that $\Delta(\lambda)x = \lambda^r x$, then express x as $x = \sum_{k=0}^{d} x_k$, with $x_k \in \Xi_k$ $(k = 0, \ldots, d)$. Applying $\Delta(\lambda)$, we have

$$\lambda^r x = \Delta(\lambda)x = \sum_{k=0}^{d} \lambda^k x_k.$$

Multiply the first expression by m^r and the second by n^r, and subtract one from the other, to obtain

$$\sum_{k\neq r} (m^r - n^r \lambda^k)\, x_k = 0.$$

But $m^r - n^r \lambda^k \neq 0$ for $k \neq r$, so that $x_k = 0$ for $k \neq r$, and hence $x = x_r \in \Xi_r$, as claimed. This proves the lemma.

In fact, in the notation introduced above, we can easily see that $Z_s = \bigoplus_{r=s}^{d} \Xi_r$ for each $s = 0, \ldots, d$. If $x \in \Pi$, let $x = \sum_{r=0}^{d} x_r$ with $x_r \in \Xi_r$ $(r = 0, \ldots, d)$; we call x_r the *r-component* of x.

If $r, s = 0, \ldots, d$, and $x \in \Xi_r$, $y \in \Xi_s$, then taking $\lambda = 2$ (say) in Lemma 20, we see that

$$\Delta(2)(xy) = (\Delta(2)x)(\Delta(2)\, y) = 2^r x \cdot 2^s y = 2^{r+s} xy,$$

so that $xy \in \Xi_{r+s}$. Since Ξ_{r+s} is generated by the elements p^{r+s}, with $p \in \mathscr{P}\backslash\{\varnothing\}$, and $p^r \in \Xi_r$, $p^s \in \Xi_s$, it follows that $\Xi_r \cdot \Xi_s = \Xi_{r+s}$.

Thus we have now established all of Theorem 1, with the scalars or dilatations restricted to rationals, except for the characterization of Ξ_d, which will be considered in Section 7.

We end this section by remarking on some implications of these results for valuations. We say that a valuation ϕ on $\mathscr{P}$ is *homogeneous of degree* r if $\phi(nP) = n^r \phi(P)$ for all $P \in \mathscr{P}$ and all integers $n \geqslant 0$. Then we have (compare Section 6):

THEOREM 8. *Let* ϕ *be a translation invariant valuation on* $\mathscr{P}$. *Then* ϕ *admits a unique decomposition* $\phi = \sum_{r=0}^{d} \phi_r$, *where* ϕ_r *is a translation invariant valuation on* $\mathscr{P}$ *which is homogeneous of degree* r.

The proof is immediate; we just define ϕ_r to be the restriction of ϕ to Ξ_r, so that $\phi_r(P)=\phi([P]_r)$, where $[P]$ is the r-component of $[P]$ for $P\in\mathscr{P}$. (The usual conventions of Lemma 1 apply.) Then for integer $n\geqslant 0$,

$$\phi_r(nP)=\phi([nP]_r)=\phi(n^r[P]_r)=n^r\phi([P]_r)=n^r\phi_r(P),$$

as claimed.

Note, in fact, that we actually have $\phi_r(\lambda P)=\lambda^r\phi_r(P)$ for all rational $\lambda\geqslant 0$, with the implication that the image of Π under ϕ_r is a divisible subgroup of the target group for $r\geqslant 1$.

The uniqueness part of Theorem 8 has a useful consequence.

COROLLARY. *Let ϕ be a translation invariant valuation on $\mathscr{P}$ which is homogeneous of degree r. If $s\neq r$, then ϕ vanishes on Ξ_s.*

We shall particularly want to apply this corollary to the frame functionals. As is clear, and will be made even clearer after the discussion of volume in Section 7, a frame functional of type r is homogeneous of degree r.

7. VOLUME

In this section, we shall verify the isomorphism $\Xi_d\cong\mathbb{F}$ of Theorem 1(c). The isomorphism is given by volume; this important notion turns up as well as in the main Theorems 3 and 5.

LEMMA 21. *As an abelian group, $\Xi_d\cong\mathbb{F}$.*

The definition of Ξ_d as the set of d-components of elements of Π, the fact that these d-components are the coefficients of n^d in the polynomial expansions of the $\Delta(n)x$ for $x\in\Pi$, and the second canonical simplex dissection Lemma 11, show that the only generators $s(a_1,\ldots,a_k)$ of Π which can contribute to Ξ_d are those for which $k=d$. The corresponding d-component is

$$\frac{1}{d!}s(a_1)\cdots s(a_d).$$

Now $s(a_1)\cdots s(a_d)$ is the class of the half-open parallelotope

$$\left\{\sum_{i=1}^{d}\xi_i a_i \,|\, 0<\xi_i\leqslant 1\ (1,\ldots,d)\right\},$$

The order of the terms $s(a_i)$ is immaterial, and we can clearly replace any a_i by $-a_i$, since $s(a_i)=s(-a_i)$ from the translation invariance (T). Finally, if $i\neq j$ and $\lambda\in\mathbb{F}$, we can replace a_i by $a_i+\lambda a_j$. To see this, note that the previous remark shows that we can assume that $\lambda>0$. If q is the class of

$$\{\xi_i a_i+\xi_j a_j \,|\, 0<\xi_i\leqslant 1,\ 0<\xi_j\leqslant 1+\lambda\xi_i\},$$

the decompositions of the latter by the two open half-planes

$$\{\xi_i a_i+\xi_j a_j \,|\, \xi_j>1\},$$
$$\{\xi_i a_i+\xi_j a_j \,|\, \xi_j>\lambda\xi_i\}$$

yield the equations

$$\begin{aligned} q&=s(a_i)s(a_j)+s(a_i,\lambda a_j)\\ &=s(a_i,\lambda a_j)+s(a_i+\lambda a_j)s(a_j). \end{aligned}$$

whence $s(a_i)s(a_j)=s(a_i+\lambda a_j)s(a_j)$.

If a fixed basis $\{e_1,\ldots,e_d\}$ of V is now chosen, then the theory of elementary row operations on matrices shows that the above operations suffice to transform $s(a_1)\cdots s(a_d)$ into $s(\mu e_1)\cdots s(e_d)$, where $\mu=|\det(a_1,\ldots,a_d)|$, the determinant being relative to the given basis. Since $s((\mu+\nu)e_1)=s(\mu e_1)+s(\nu e_1)$ for $\mu,\nu\geqslant 0$, we conclude immediately that the correspondence

$$s(a_1)\cdots s(a_d)\mapsto|\det(a_1,\ldots,a_d)|$$

induces an isomorphism between the abelian groups Ξ_d and $\mathbb{F}$.

This isomorphism on Ξ_d, the homomorphism it induces on Π, and the corresponding translation invariant valuation on P are all called *volume*, which is denoted vol.

There is clearly a scaling factor involved in the definition of volume, arising from the choice of basis of V. However, apart from this, volume is unique. The characterization of volume by Hadwiger [3, Sect. 2.1.3], is only available if $\mathbb{F}$ is archimedean, but we can modify it as follows.

LEMMA 22. *Let $\phi:\mathscr{P}\to\mathbb{F}$ be a translation invariant valuation, which is homogeneous of degree d, and is such that $\phi(P)\geqslant 0$ for all $P\in\mathscr{P}$. Then ϕ is a non-negative multiple of volume.*

If L is a linear subspace of V, of dimension $k\geqslant 1$, then Theorem 6 shows that the subring $\Pi(L)$ of Π is isomorphic to $\Pi(\mathbb{F}^k)$, with $\mathbb{F}^k$ the usual coordinate vector space. Thus $\Xi_k(L)\cong\mathbb{F}$ also, and the isomorphism yields a volume vol_L on $\Pi(L)$ or $\mathscr{P}(L)$, which is an L-simple translation invariant valuation, homogeneous of degree k. If $L=\{o\}$ is the trivial subspace, we scale naturally by defining $\mathrm{vol}_{\{o\}}1=1$.

We sometimes wish to have a volume vol_L for each subspace L of V. To avoid needing to appeal to the axiom of choice, to specify a particular scaling of vol_L for each L, we can proceed as follows. Let Q be a fixed polytope in V with $o \in \mathrm{int}\, Q$, for example, $Q = \mathrm{conv}\{e_0, e_1, ..., e_d\}$, where $\{e_1, ..., e_d\}$ is any basis of V, and $e_0 = -(e_1 + \cdots + e_d)$. Then $Q \cap L$ is a polytope of dimension $\dim L$ for every subspace L of V, and so we can choose the scaling so that $\mathrm{vol}_L(Q \cap L) = 1$ for every L. We call this the *scaling induced by* Q.

8. The First Weight Space

While it is not necessary at this stage, it is helpful to give an alternative description of the first weight space Ξ_1. By definition, Ξ_1 is generated by the elements $\log P$, with $P \in \mathscr{P}\backslash\{\varnothing\}$. Since $\log P$ is just the 1-component of P, we deduce

Lemma 23. *The mapping* log *induces a translation invariant valuation on* $\mathscr{P}$.

However, we shall not define $\log \varnothing$, allowing the conflict between writing $\log \varnothing = 0$ on the basis of Lemma 23 and the "natural" definition $\log \varnothing = \log 0 = -\infty$ (whatever this might mean!) to remain unresolved. We note that the property $\log(P+Q) = \log P + \log Q$ (obtained by setting $x_1 = [P]$, $x_2 = [Q]$ in Lemma 18) and the valuation property (V) ensure that, if $P, Q \in \mathscr{P}\backslash\{\varnothing\}$ are such that $P \cup Q \in \mathscr{P}$ also, then

$$\log((P \cup Q) + (P \cap Q)) = \log(P+Q).$$

In fact, this is also a consequence of a result of Sallee [13]:

Lemma 24. *Let* $P, Q \in \mathscr{P}$ *satisfy* $P \cup Q \in \mathscr{P}$ *also. Then*

$$(P \cup Q) + (P \cap Q) = P + Q.$$

Compare also with Section 15 below, whose results do not depend on those of this section.

We next have (compare [2, Sect. 15.1]):

Lemma 25. *Let* $P, Q_1, Q_2 \in \mathscr{P}\backslash\{\varnothing\}$ *be such that* $P + Q_1 = P + Q_2$. *Then* $Q_1 = Q_2$.

In fact, we observe that

$$Q_i = \{v \in V \mid P + v \subseteq P + Q_i\}.$$

Now let $\mathscr{P}_T$ denote the equivalence classes of pairs (P, Q), with $P, Q \in \mathscr{P}\backslash\{\varnothing\}$, under the relation

$$(P, Q) \sim (P', Q') \Leftrightarrow P + Q' = P' + Q + t$$

for some translation vector $t \in V$. Then

Lemma 26. $\mathscr{P}_T$ *is an abelian group, under the addition*

$$(P, Q) + (P', Q') = (P + P', Q + Q').$$

The cancellation law of Lemma 25 ensures that $\sim$ is an equivalence relation. The identity in $\mathscr{P}_T$ is $(\{o\}, \{o\})$, and the inverse is given by $-(P, Q) = (Q, P)$.

We now have the following isomorphism theorem.

Theorem 9. *The mapping* $\log\colon \mathscr{P}\backslash\{\varnothing\} \to \Xi_1$ *induces an isomorphism between* $\mathscr{P}_T$ *and* Ξ_1.

We extend log to $\mathscr{P}_T$ by defining

$$\log(P, Q) = \log P - \log Q,$$

for $(P, Q) \in \mathscr{P}_T$. The extension is well defined, because if $(P, Q) \sim (P', Q')$, say $P + Q' = P' + Q + t$ with $t \in V$, then

$$\begin{aligned} \log P + \log Q' &= \log(P + Q') \\ &= \log(P' + Q) \\ &= \log P' + \log Q, \end{aligned}$$

so that $\log P - \log Q = \log P' - \log Q'$, as required.

On the other hand, in view of Lemma 24 and the definition of addition in $\mathscr{P}_T$, the mapping $\phi\colon \mathscr{P}\backslash\{\varnothing\} \to \mathscr{P}_T$ defined by $\phi(P) = (P, \{o\})$ is a translation invariant valuation, and so induces a homomorphism $\phi\colon \Pi \to \mathscr{P}_T$. But

$$\phi(nP) = (nP, \{o\}) = n(P, \{o\}) = n\phi(P)$$

for integral $n \geq 0$, so that ϕ is homogeneous of degree 1, and hence, by the corollary to Theorem 8, acts effectively on Ξ_1. Therefore, on the generators $[P]$ $(P \in \mathscr{P}\backslash\{\varnothing\})$ of Π, we have

$$(P, \{o\}) = \phi([P]) = \phi(\log P).$$

It follows that log and ϕ are inverse homomorphisms, and this is the theorem.

9. The Algebra Structure I

We now embark on the process of extending the range of the scalars occurring in Theorem 1 from $\mathbb{Q}$ to $\mathbb{F}$. This will be done over the next three sections; Section 10 will contain the proof of the separation Theorem 3.

Our first step is straightforward.

Lemma 27. *Ξ_1 is a vector space over $\mathbb{F}$.*

We present two proofs. The first employs Theorem 9. There is a natural scalar multiplication on the group $\mathscr{P}_T$, namely

$$\lambda(P, Q) = \begin{cases} (\lambda P, \lambda Q), & \text{if } \lambda \geqslant 0, \\ (-\lambda Q, -\lambda P), & \text{if } \lambda < 0. \end{cases}$$

With the given (vector) addition on $\mathscr{P}_T$, the axioms of a vector space are easily checked. In fact, the only problem is caused by scalar multiplication by $\lambda + \mu$ when $\lambda\mu < 0$. If $\lambda + \mu \geqslant 0$ (the other case is similar), with, say, $\lambda > 0$, $\mu < 0$, then

$$\begin{aligned} \lambda(P, Q) + \mu(P, Q) &= (\lambda P, \lambda Q) + (-\mu Q, -\mu P) \\ &= (\lambda P - \mu Q, \lambda Q - \mu P) \\ &= ((\lambda+\mu)P - \mu(P+Q), (\lambda+\mu)Q - \mu(P+Q)) \\ &= ((\lambda+\mu)P, (\lambda+\mu)Q) \\ &= (\lambda+\mu)(P, Q), \end{aligned}$$

where we have cancelled the terms $-\mu(P+Q)$ using the definition of $\mathscr{P}_T$ (note that $-\mu > 0$). We now appeal to Theorem 9.

Alternatively, we can start from the first canonical simplex dissection Lemma 10. In that, all the terms for $j = 1, \ldots, k-1$ ($k \geqslant 1$; that is, excepting the first and last) lie in Z_2, since each class $s(b_1, \ldots, b_j)$ ($j \geqslant 1$) of a partly open simplex lies in $Z_1 = \bigoplus_{r=1}^{d} \Xi_r$. Writing $s_1 = s_1(a_1, \ldots, a_k)$ for the 1-component of $s(a_1, \ldots, a_k)$, we therefore deduce that, for $\lambda, \mu \geqslant 0$,

$$\Delta(\lambda+\mu)s_1 = \Delta(\lambda)s_1 + \Delta(\mu)s_1.$$

Clearly also,

$$\Delta(\lambda\mu)s_1 = \Delta(\lambda)\,\Delta(\mu)s_1$$

for all $\lambda, \mu \in \mathbb{F}$. Since the classes s_1 generate Ξ_1, we conclude that the same relations hold, with a general $x \in \Xi_1$ replacing s_1.

The scalar multiplication on Ξ_1 is now defined by

$$\lambda x = \begin{cases} \Delta(\lambda)x, & \text{if } \lambda \geqslant 0, \\ -\Delta(-\lambda)x, & \text{if } \lambda < 0, \end{cases}$$

for $x \in \Xi_1$ and $\lambda \in \mathbb{F}$. Again, all the axioms of a vector space over $\mathbb{F}$ are easily verified, with scalar multiplication by $\lambda + \mu$ with $\lambda\mu < 0$ causing the only problem. We have to approach this indirectly. This time, let us take $\lambda + \mu \leqslant 0$, with $\lambda < 0$, $\mu > 0$. Then for all $x \in \Xi_1$,

$$\begin{aligned} \lambda x &= -\Delta(-\lambda)x \\ &= -\Delta(-(\lambda+\mu)+\mu)x \\ &= -\Delta(-(\lambda+\mu))x - \Delta(\mu)x \\ &= (\lambda+\mu)x - \mu x, \end{aligned}$$

or $(\lambda+\mu)x = \lambda x + \mu x$, as we require.

We may observe that the isomorphism of Theorem 9 is compatible with the definition of scalar multiplication in $\mathscr{P}_T$ and Ξ_1, and so becomes one of vector spaces over $\mathbb{F}$.

As we shall remark in Section 11, it is the case (d) of Theorem 1 with $x, y \in \Xi_1$ which enables us to impose the full vector space structure on Z_1 (or on each Ξ_r with $r \geqslant 2$). To prove this case, we shall need to adapt the geometric construction of Thorup in [4]. A somewhat paradoxical situation arises. The argument of [4] directly applied would only prove $(\lambda x)y = x(\lambda y)$ for $\lambda \in \mathbb{F}$, in case $x \in \Xi_1(L)$ and $y \in \Xi_1(M)$ for some supplementary subspaces L and M of V, which is insufficient (but see Section 10 below). We shall get around this problem, as we have hinted earlier, by proving the separation Theorem 3 before we have completed the proof of Theorem 1. Curiously, we shall then find that we need an even less general case of (d) than that just mentioned; it is enough to take L and M a hyperplane and complementary line.

So, let H be a hyperplane in V (passing through the origin o), and let E be a line segment in a line complementary to H. We write

$$e = \log E = [E] - 1.$$

Lemma 28. *If $x \in \Xi_1(H)$ and $\lambda > 0$, then $(\lambda e)x = e(\lambda x)$.*

The idea of the proof of this lemma is to establish it first for (the 1-components of classes of) certain special polytopes x, and then to show that these x generate the simplex classes in $\Xi_1(H)$.

We can appeal to induction on k, and so remark that we need only prove the lemma for the $x = s_1(a_1, \ldots, a_k)$ in $\Xi_1(H)$ with $k = d-1$. (The case

$k=1$ is particularly easy, because this is a consequence of the discussion of area ($=$ volume) when $d=2$; see Section 7.)

The construction which we generalize from [4] is perhaps clarified by a little extra notation. Let $\{b_1, ..., b_d\}$ be a fixed basis of V, and for $j=1, ..., d-1$ and v a positive rational number, let $\Omega_j(v)$ be the endomorphism of Π induced by the linear mapping

$$b_i \mapsto \begin{cases} b_i, & \text{if } i \leqslant j, \\ vb_i, & \text{if } i > j. \end{cases}$$

Further, define

$$\Phi_j(v) = I - v^{-1}\Omega_j(v),$$
$$\Psi_j(v) = I - \Omega_j(v),$$

where I is the identity endomorphism. These $\Phi_j(v)$ and $\Psi_j(v)$, for different values of j and v, are mutually commuting group endomorphisms of Π.

For $k=1, ..., d-1$, let $L=\text{lin}\{b_1, ..., b_k\}$, $M=\text{lin}\{b_{k+1}, ..., b_d\}$, and $y \in \Xi_1(L)$, $z \in \Xi_1(M)$. Then we can easily see that

$$\Phi_j(v)(yz) = \begin{cases} (\Psi_j(v)y)z, & \text{if } j<k, \\ 0, & \text{if } j=k, \\ y(\Phi_j(v)z), & \text{if } j>k. \end{cases}$$

If $y, z \in \Xi_1$, we write

$$y * z = (\lambda y)z - y(\lambda z),$$

so that we must show that $x * e = 0$. Choosing, as we may, $\{b_1, ..., b_{d-1}\}$ as a basis of H and $E=\text{conv}\{o, b_d\}$, and applying Lemma 10 to the 2-component of

$$\Delta(\lambda+\mu)s(b_1, ..., b_d) - \Delta(\mu+\lambda)s(b_1, ..., b_d) = 0$$

with $\mu=1$, we see that

$$\sum_{k=1}^{d-1} s_1(b_1, ..., b_k) * s_1(b_{k+1}, ..., b_d) = 0.$$

Letting $v_1, ..., v_{d-2}$ be any positive rationals, applying $\Phi_1(v_1), ..., \Phi_{d-2}(v_{d-2})$ to this relation, and using the remarks above, we deduce that

$$(\Psi_1(v_1) \cdots \Psi_{d-2}(v_{d-2})s_1(b_1, ..., b_{d-1})) * e = 0,$$

since $e = s_1(b_d)$. Thus Lemma 28 holds for the special classes of the form

$$x = \Psi_1(v_1) \cdots \Psi_{d-2}(v_{d-2})s_1(b_1, ..., b_{d-1}).$$

In fact, we shall only need to consider the cases where $v_i = n_i^{-1}$, with n_i a positive integer for $i=1, ..., d-2$.

We must now show how to recover a general class $s_1(a_1, ..., a_{d-1})$ (with $\{a_1, ..., a_{d-1}\} \subseteq H$ linearly independent) from these special classes. Once again, we generalize the ideas of [4]. If L is a subspace of V, $Q \subseteq L$ a partly open polytope, $v \notin L$ a point, and $n \geqslant 2$ an integer, then

$$\{(1-\mu)v + \mu w \mid w \in Q, 1/n < \mu \leqslant 1\}$$

is called a *stump with base* Q, or *over* Q. A *k-fold stump* is a stump over a $(k-1)$-fold stump. A stump over a point is a half-open line segment; then x (as above) is the 1-component of the class of a pyramid (with missing apex), whose base is a $(d-1)$-fold stump over a point.

To avoid constant repetition, let us take the phrase "the 1-component of the class of" as read in what follows. Moreover, a simplex will always lack a facet (so that its class is an $s(a_1, ..., a_k)$). The construction of a simplex from stumps proceeds by induction, in the following way: if we have all stumps over $(k-1)$-simplices, then we have all (stumps over) k-simplices. We thus work backwards, "unstumping" the last stumped base.

The argument of [4] is easily modified, if we replace the dissections which occur there by decompositions into partly open polytopes. The class $s(c_1, ..., c_k)$ of a k-simplex is represented by

$$S = \left\{ \sum_{i=1}^{k} \xi_i c_i \mid 1 \geqslant \xi_1 \geqslant \xi_2 \geqslant \cdots \geqslant \xi_k > 0 \right\}.$$

We now define, for $m=0, ..., k$,

$$S_m^1 = \left\{ \sum_{i=1}^{k} \xi_i c_i \mid \xi_1 \geqslant \xi_2 \geqslant \cdots \geqslant \xi_k > 0, m < \xi_1 \leqslant m+1 \right\},$$

and, for $n=2, ..., k$,

$$S_m^n = \left\{ \sum_{i=1}^{k} \xi_i c_i \in S_{m+1}^{n-1} \mid \xi_{n-1} < \xi_n - 1 \right\}.$$

Effectively, we have here $m+n \leqslant k+1$. We further write

$$t_n = \sum_{j=1}^{n} c_j,$$

for $n=1, \ldots, k$. We can easily check the decompositions

$$S^n_{m+1} = (S^n_m + t_n) \cup S^{n+1}_m$$

for $n<k$, while

$$S^k_{m+1} = S^k_m + t_k.$$

Now each union $S^1_1 \cup \cdots \cup S^1_m$ is a stump over a $(k-1)$-simplex, and so we can construct each individual S^1_m $(m \geqslant 1)$ from stumps. We then obtain successively all the S^n_m with $m \geqslant 1$. But for $n=k$, we therefore have S^k_0, and reversing all the steps with $m=0$, we eventually obtain $S^1_0 = S$. This, and the induction argument outlined above, complete the proof of Lemma 28.

In stating the following consequence of Lemma 28, we make the inductive assumption that Theorem 1 has been established for $\Pi(H)$. Implicit also is a forward reference to Section 11, for the details of extending the vector space structure to Ξ_r for $r \geqslant 2$.

COROLLARY. *With e, H as above, if $y \in Z_1(H)$ and $\lambda \in \mathbb{F}$, then $(\lambda e)y = e(\lambda y)$.*

10. SEPARATION

We now depart more radically from the pattern of proof of [4]. In order to complete the proof of the structure Theorem 1, we shall first prove the separation Theorem 3. However, the method of proof of Theorem 3 still follows quite closely the corresponding part of [4].

Let H be a hyperplane and L the orthogonal line in V, both containing the origin o, let E be a line segment in L, and let $e = \log E$. We denote by Λ the subgroup of Π generated by all elements of the form $(\lambda e)y$, with $\lambda \in \mathbb{F}$ and $y \in \Pi(H)$. The first step in proving Theorem 3 is to show that, if $x \in \Pi$ is such that $f_U(x)=0$ for all frame functionals f_U, then $x \in \Lambda$.

Let u be any non-zero vector in V, and let H_u be the (linear) hyperplane orthogonal to u. The mapping $x \mapsto x_u$ is a ring endomorphism of Π (Theorem 7). Using frames $U=(u_1, \ldots, u_k)$ with $u_1 = u$, the inductive assumption that Theorem 3 holds in $\Pi(H_u)$ shows that, if $f_U(x)=0$ for all frames U, then $x_u = 0$.

The quotient map $\Pi \to \Pi/\Lambda$ has the following description. Suppose that $L = \operatorname{lin}\{b\}$, and let H^+ be that half-space bounded by H which contains b. If $u \notin H$ and $Q \in \mathscr{P}(H_u)$, then suppose Q translated so that $Q \subseteq H^+$, let Q_0 be the image of Q under orthogonal projection on to H, and write $\bar{Q} = \operatorname{conv}(Q \cup Q_0)$. Then $[\bar{Q}]$ is determined by Q up to an element of Λ, and so the class $[\bar{Q}]_\Lambda$ of $\bar{Q}$ in Π/Λ depends only upon $[Q]$, and determines a homomorphism $y \mapsto y_\Lambda$ of $\Pi(H_u)$ into Π/Λ.

Now let $P \in \mathscr{P}$. From P, we obtain two elements of $U(\mathscr{P})$, namely, its *upper* and *lower boundaries* P_+ and P_-, defined by

$$P_+ = \{v \in P \mid v + \mu b \notin P \text{ for all } \mu > 0\},$$
$$P_- = \{v \in P \mid v + \mu b \notin P \text{ for all } \mu < 0\}.$$

Using Lemma 4 (or the inclusion–exclusion principle), we see that the three classes $[\bar{P}_+]$, $[\bar{P}_-]$, and $[P_-]$ are all well defined (assuming P translated so that $P \subseteq H^+$), and

$$[P] = [\bar{P}_+] - [\bar{P}_-] + [P_-].$$

We now factor out by Λ. We decompose $x \in \Pi$ into three terms $\bar{x}_+, \bar{x}_-$, and x_-, corresponding to the decomposition of $[P]$ above, so that $x = \bar{x}_+ - \bar{x}_- + x_-$. If $x_u = 0$ for each $u \notin H$, then $x_- = 0$ anyway. Modulo Λ, we must also have $\bar{x}_+ = 0 = \bar{x}_-$, so that $x_\Lambda = 0$, or $x \in \Lambda$, as required. This completes the first step.

We can thus express x in the form

$$x = \lambda_0 e + \sum_{j=1}^{m} (\lambda_j e) y_j,$$

where $\lambda_0, \ldots, \lambda_m \in \mathbb{F}$ and $y_1, \ldots, y_m \in Z_1(H)$. The corollary to Lemma 26 shows that we can write this in the form $x = \lambda_0 e + ey$, where $y = \sum_{j=1}^m \lambda_j y_j \in Z_1(H)$.

We now apply the frame functionals f_U, with $U \subseteq H$. From $f_U(x)=0$ for any single such f_U of type 1 follows $\lambda_0 = 0$. Now let f_U be such a functional of type $r \geqslant 2$. We can always rescale volumes, if necessary, so that, for each subspace M of H,

$$\operatorname{vol}_{L+M}(E+Q) = \operatorname{vol}_M Q$$

for $Q \in \mathscr{P}(M)$. The frame U also gives rise to a frame functional f'_U on $\Pi(H)$, this time of type $r-1$, and our choice of scaling shows that

$$f_U(ey) = f'_U(y)$$

for each $y \in \Pi(H)$. But now, if $x = ey$ is such that $f_U(x)=0$ for each such frame $U \subseteq H$, we have $f'_U(y)=0$, and again the inductive assumption that Theorem 3 holds in $\Pi(H)$ yields $y=0$, and hence $x=0$, as claimed. This completes the proof of Theorem 3.

In view of the fact that, by the corollary to Theorem 8 (and the following remark), frame functionals of type r vanish identically on Ξ_s unless $r=s$, we deduce

COROLLARY. *For each $r=0, \ldots, d$, the frame functionals of type r separate Ξ_r.*

The separating frame functionals f_U are not, in fact, independent. A *syzygy* is a non-trivial linear relationship $\sum_U \alpha_U f_U = 0$ between them. We do not insist on such syzygies involving only finitely many terms; indeed, in all but one trivial case, we shall see that they generally do not.

We can obviously confine our attention to syzygies between frame functionals of the same type. If U is a d-frame, then $f_U(x) = \Delta(0)x$ is actually independent of U, and hence

LEMMA 29. *For every two d-frames U, U', $f_U = f_{U'}$.*

Since $f_\varnothing = \mathrm{vol}$ is the only frame functional of type d, we henceforth consider frame functionals of some type r, with $1 \leqslant r \leqslant d-1$. We know of two further kinds of syzygy, which correspond naturally to syzygies between the Hadwiger functionals h_U (see [12, Chap. 5] and Section 17 below).

The next kind derives from the analogue of Minkowski's theorem relating the areas and normal vectors of facets of polytopes (see [2, Sect. 15.3]). Let L, M be two subspaces of V of the same dimension, with corresponding volumes vol_L, vol_M, and let Φ_L denote orthogonal projection on to L. By the essential uniqueness of volume (Lemma 22), there is a non-negative scalar $\theta(L, M)$, such that

$$\mathrm{vol}_L(\Phi_L P) = \theta(L, M)\,\mathrm{vol}_M P$$

for each $P \in \mathscr{P}(M)$. If U is a fixed frame, and v, w are vectors orthogonal to U, write $L_v = (U, v)^\perp$ and

$$\tau(U, v, w) = \mathrm{sign}\langle v, w\rangle\,\theta(L_v, L_w).$$

By considering the areas of the projections of the facets of a polytope in $\mathscr{P}(U^\perp)$ on to L_v, we obtain

LEMMA 30. *For each frame U and fixed $v \in U$*

$$\sum_{w \in U} \tau(U, v, w)\, f_{(U, w)} = 0.$$

We observe that the sum in Lemma 30 is infinite (if we exclude $(d-1)$-frames U, according to our remarks above). Now general infinite linear combinations of frame functionals are not permitted, in contrast to the situation for Hadwiger functionals h_U; the latter vanish on polytopes of less than full dimension anyway, and if $P \in \mathscr{P}$ is d-dimensional, then $h_U(P) \neq 0$ for only finitely many frames U. However, if U is a fixed frame as in Lemma 30, then again for a given polytope P, we have $f_{(U,w)}(P) \neq 0$ for only finitely many $w \in U$.

Similar considerations must be borne in mind in constructing the third kind of syzygy. If $P \in \mathscr{P}$, then for only finitely many frames (v, w) spanning a fixed plane L is it true that $P_{(v,w)} \neq P_{(v,-w)}$ (consider the orthogonal projection $\Phi_L P$, which is a polygon, line segment, or point). Moreover, if we choose a fixed orientation (v, w) in L, and rotate v according to this orientation, then for two successive values v_1, v_2 of v for which inequality does prevail, we have $P_{(v_1, w_1)} = P_{(v_2, -w_2)}$. Applying this to polytopes $P_{U'}$ with $L \subseteq (U')^\perp$, and looking at faces in direction a further frame $U'' \subseteq (U', L)^\perp$, we obtain

LEMMA 31. *Let (U', U'') be a frame in V, and let L be a plane in V with $L \subseteq (U', U'')^\perp$. Then*

$$\sum_{(v,w) \subseteq L} \left(f_{(U', v, w, U'')} - f_{(U', v, -w, U'')}\right) = 0,$$

where the sum extends over all frames (v, w) in L of a given orientation.

Note, by the way, that the summation above is really only over $v \in L$, since the orientation and $\langle v, w\rangle = 0$ determine w (and, as usual in talking about frames, only the directions of the vectors are significant).

We refer to the syzygies of Lemmas 29, 30, and 31 as *syzygies of the first, second, and third kind*, respectively. We wish to propose:

Conjecture 1. *Every syzygy between frame functionals is a consequence of syzygies of these three kinds.*

The syzygies of the first kind need no further comment. For the rest, we have:

THEOREM 10. *The only syzygies between frame functionals f_U of type $r \leqslant d-1$, where $U = U(v)$ depends on a single vector v, are those of the second and third kind.*

We sketch the proof. If U depends just on v, it is of the form $U = (U_1, U_2(v), U_3)$, with U_1, U_3 fixed frames, and $U_2(v)$ varying over frames in some fixed subspace $L \subseteq U_1^\perp \cap U_3^\perp$. We clearly lose no generality if we take $U_1 = \varnothing$ (alternatively, we work in $\Pi(U_1^\perp)$). We consider separately the cases $\dim L = 1, 2$ or $\dim L \geqslant 3$. For $\dim L = 1$ and $d \geqslant 2$, for suitable $P \in \mathscr{P}$ there is no relationship between P_v and P_{-v}, which thus excludes this case. For $\dim L = 2$, we necessarily have the relationships $P_{(v_1, w_1)} = P_{(v_2, -w_2)}$ (with $w_i \in v_i^\perp$ for $i = 1, 2$) as above, when the general equation $P_{(v,w)} = P_{(v,-w)}$ fails (see also Section 13 below), but again suitable choices of P show that we have no others; this yields the third kind of syzygy. Finally, if $\dim L \geqslant 3$, suitable simplices yield Minkowski's theorem (that is, Lemma 30), but deny other relationships. The theorem follows.

11. The Algebra Structure II

We are now in a position to complete the proof of Theorem 1. We first need a special case of Theorem 1(d).

LEMMA 32. *If* $d=2$, *and* $x, y \in \Xi_1$, $\lambda \in \mathbb{F}$, *then* $(\lambda x)y = x(\lambda y)$.

Let P, Q be two fixed polygons or line segments in the plane, and let $\lambda, \mu \geqslant 0$ be variable scalars. An application of the lifting theorem of Walkup and Wets [18] shows that $\lambda P + \mu Q$ admits a dissection (up to translation) into λP, μQ and sets of the form $\lambda E + \mu F$, where E is an edge of P and F an edge of Q. Considering the 2-component of

$$[\lambda P + \mu Q] = (1 + \lambda p + \tfrac{1}{2}\Delta(\lambda)\, p^2)(1 + \mu q + \tfrac{1}{2}\Delta(\mu)\, q^2),$$

where $p = \log P$, $q = \log Q$, using Lemma 4 (or the inclusion–exclusion principle), and noting that the 2-components (areas) of points and line segments vanish, we deduce that

$$(\lambda p)(\mu q) = \sum_{E,F} (\lambda e)(\mu f),$$

where E, F are as above, and $e = \log E$, $f = \log F$. But the analysis in Section 7 shows that the area term $(\lambda e)(\mu f)$ depends only on $(e, f$ and) the product $\lambda\mu$. The same is therefore true of $(\lambda p)(\mu q)$.

In particular, $(\lambda p)q = p(\lambda q)$ for all $\lambda \geqslant 0$. But the definition of scalar multiplication in Ξ_1 shows that we need only consider this case (this remark holds good below as well). Thus the lemma holds for the generators of Ξ_1, and so it holds for all $x, y \in \Xi_1$, which proves the lemma.

Theorem 1(d) for general dimension d and $x, y \in \Xi_1$ now follows. As above, it is enough to prove that $(\lambda p)q = p(\lambda q)$, whenever $p = \log P$, $q = \log Q$ for some $P, Q \in \mathscr{P}\backslash\{\varnothing\}$ and $\lambda \geqslant 0$. In turn, the corollary to the separation Theorem 3 shows that we need only show that $((\lambda p)q - p(\lambda q))_U = 0$ for each $(d-2)$-frame U. But recalling that the mapping $x \mapsto x_U$ is a ring endomorphism of Π which commutes with non-negative dilatations (by those parts of Theorem 7 which we have proved so far), and that, by definition, $\lambda p = \Delta(\lambda)p$ since $p \in \Xi_1$ (and similarly for q), we have

$$\begin{aligned}((\lambda p)q)_U &= (\lambda p)_U q_U = (\lambda p_U) q_U \\ &= p_U(\lambda q_U) = p_U(\lambda q)_U = (p(\lambda q))_U.\end{aligned}$$

Thus $(\lambda p)q = p(\lambda q)$, and consequently we have this more general case of Theorem 1(d).

All that remains of Theorem 1 is the rest of (d), and the extensions of the scalar multiplication of (c) and the dilatation of (e) from $\mathbb{Q}$ to $\mathbb{F}$. We do all these together.

A typical generator of Ξ_r $(r \geqslant 2)$ is of the form $x_1 \cdots x_r$, where $x_i \in \Xi_1$ for $i = 1, ..., r$. (In fact, we could take it to be of the form p^r, where $p = \log P$ for some $P \in \mathscr{P}\backslash\{\varnothing\}$, but this is needlessly specialized.) We define the scalar multiplication by

$$\lambda(x_1 \cdots x_r) = (\lambda x_1) x_2 \cdots x_r$$

for $\lambda \in \mathbb{F}$. Induction on r, starting with the case $r=2$ (that part of Theorem 1(d) proved above), shows that it is irrelevant to which factor x_i the scalar λ is attached. This remark, applied to the generators of Z_1, also establishes (d) in full generality.

Now, all the properties of a vector space over $\mathbb{F}$ are easily verified, except for the distributivity property

$$\lambda(y+z) = \lambda y + \lambda z,$$

for $\lambda \in \mathbb{F}$ and $y, z \in \Xi_r$, in other words, that scalar multiplication by λ is a group endomorphism of Ξ_r. But for our generator $x_1 \cdots x_r$ and $\lambda \geqslant 0$, we have

$$\begin{aligned}\Delta(\lambda)(x_1 \cdots x_r) &= (\Delta(\lambda)x_1) \cdots (\Delta(\lambda)x_r) \\ &= (\lambda x_1) \cdots (\lambda x_r) \\ &= \lambda^r(x_1 \cdots x_r).\end{aligned}$$

This, then, is Theorem 1(e).

Finally, we can write $\lambda \geqslant 0$ as a rational linear combination

$$\lambda = \sum_{k=0}^{r} \alpha_k(\lambda + k)^r$$

for some $\alpha_0, ..., \alpha_r \in \mathbb{Q}$ (valid for all λ). Since each $\Delta(\lambda+k)$ is a group endomorphism of the rational vector space Ξ_r, so is $\sum_{k=0}^{r} \alpha_k \Delta(\lambda+k)$. But

$$\left(\sum_{k=0}^{r} \alpha_k \Delta(\lambda+k)\right)(x_1 \cdots x_r) = \left(\sum_{k=0}^{r} \alpha_k(\lambda+k)^r\right)(x_1 \cdots x_r) = \lambda(x_1 \cdots x_r),$$

and this yields Theorem 1(c), and completes the proof.

Note that we must now have $\Xi_d \cong \mathbb{F}$ as a vector space over $\mathbb{F}$, in a natural way that was perhaps already apparent in Section 7.

Let us make one final remark about the corollary to Lemma 28. There, we can now assume that $y \in \Xi_r(H)$ for some $r \geqslant 1$, say a basic $y = y_1 \cdots y_r$, with $y_i \in \Xi_1(H)$ for $i = 1, ..., r$. Then for $\lambda \geqslant 0$ (as usual sufficient), we have

$$(\lambda e)y = (\lambda e)y_1 ... y_r = e(\lambda y_1)y_2 ... y_r = e(\lambda y),$$

by the definition of scalar multiplication in $\Xi_r(H)$.

We left the proofs of Theorems 6 and 7 incomplete, in that we could not, in Section 5, prove that the two kinds of homomorphism occurring there were full algebra homomorphisms. As an obvious first remark:

LEMMA 33. *The homomorphisms of Theorems* 6 *and* 7 *act as group homomorphisms on each weight space* Ξ_r.

This follows directly from Lemma 20 (with, say, $\lambda = 2$), and the fact that these homomorphisms commute with non-negative dilatations.

The full algebra properties

$$\Phi(\lambda x) = \lambda \Phi x; \qquad (\lambda x)_U = \lambda x_U$$

now follow at once, if we bear in mind the definition of scalar multiplication in Ξ_r for $r \geqslant 1$, and the proof of the last parts of Theorem 1 just above.

We conclude the discussion of Theorem 1 with an observation.

THEOREM 11. *Let* $0 \leqslant r \leqslant d-1$, *and let* $x \in \Xi_r$ *with* $x \neq 0$. *Then there is a* $y \in \Xi_1$, *such that* $xy \neq 0$.

The case $r = 0$ is trivial, since any $y \in \Xi_1$ with $y \neq 0$ will do (bear in mind Lemma 16); we may thus suppose that $r \geqslant 1$, and hence that $d \geqslant 2$. We first consider the case $r = d-1$. For each direction u, x_u is a pure $(d-1)$-volume term, and since $x \neq 0$, there is at least one, and at most finitely many directions u_i, such that $\alpha_i := x_{u_i} \neq 0$ (we take $\alpha_i \in \mathbb{F}$ here). There are

constants $\kappa_i > 0$, such that, if $P \in \mathscr{P}$ has support function $h(P, u) = \max\{\langle v, u\rangle \mid v \in P\}$, as in Section 2, then for $p = \log P$, we have

$$px = \sum_i \kappa_i h(P, u_i)\,\alpha_i$$

(this is the usual calculation of mixed volume, with the constant κ_i depending on the normalization of the $(d-1)$-volume α_i; see also Section 15 below).

We now pick any $a, b \in V$ with $a \neq b$, such that $\langle a, u\rangle = \langle b, u\rangle$ for exactly one $u = u_i$ (among those u_i above). For $\lambda > 0$, write $Y_\lambda = \text{conv}\{o, \lambda a, b\}$, and let $y_\lambda = \log Y_\lambda$. By direct calculation, for λ sufficiently near 1, we have

$$xy_\lambda - xy_1 = \psi(\lambda) + \begin{cases} 0, & \text{if } \lambda \leqslant 1, \\ \kappa(\lambda-1)\langle a, u\rangle \alpha, & \text{if } \lambda \geqslant 1, \end{cases}$$

where $\psi(\lambda)$ is some linear function, and $\kappa = \kappa_i$, $\alpha = \alpha_i$. Thus xy_λ cannot be constant, and so for some $y = y_\lambda$, we have $xy \neq 0$.

Now let $r < d-1$. If $x \in \Xi_r$ with $x \neq 0$, then from the separation Theorem 3, we can find a vector $u \neq o$ such that $x_u \neq 0$. Since $x_u \in \Xi_r(H_u)$, with H_u as usual the hyperplane orthogonal to u, by induction on d we can find a $y \in \Xi_1(H_u)$ such that $x_u y \neq 0$. But we have $(xy)_u = x_u y_u = x_u y$, and consequently $xy \neq 0$, as we wished to show.

12. THE CONE GROUP

The definitions of the cone groups $\hat{\Sigma}(L)$ and $\hat{\Sigma}$ were given in Section 2, but for convenience we repeat them here.

Let L be a subspace of V, and let $\mathscr{C}(L)$ denote the family of all cones (that is, polyhedral cones with apex o) in L. The *cone group* $\hat{\Sigma}(L)$ is the abelian group with generators $\langle K\rangle$ for $K \in \mathscr{C}(L)$, which satisfy the relations

(V) $\langle K_1 \cup K_2\rangle + \langle K_1 \cap K_2\rangle = \langle K_1\rangle + \langle K_2\rangle$, whenever $K_1, K_2 \in \mathscr{C}(L)$ are such that $K_1 \cup K_2 \in \mathscr{C}(L)$ also;

(S) $\langle K\rangle = 0$, if $K \in \mathscr{C}(L)$ satisfies $\dim K < \dim L$.

The *full cone group* $\hat{\Sigma}$ is defined to be $\hat{\Sigma} = \bigoplus_L \hat{\Sigma}(L)$, the direct sum extending over all subspaces L of V, including $\{o\}$ and V itself. We also write $\hat{\Sigma}^k = \bigoplus_{\dim L = k} \hat{\Sigma}(L)$, so that $\hat{\Sigma} = \bigoplus_{k=0}^{d} \hat{\Sigma}^k$.

Two cones are associated with a polyhedral set P (in our case, a member of $\mathscr{P}$ or $\mathscr{C}$) and a non-empty face F of P. The first is the *inner* (or *angle*) *cone*

$$A(F, P) = \text{pos}(P - F),$$

which is, after translation of its apex to o, the cone generated by P from any relatively interior point of F. The other is the *outer* (or *normal*) *cone* $N(F, P)$, which is (as defined in Section 2) the cone of outer normal vectors to support hyperplanes of P which contain F.

If the polar cone K^o of a cone K is defined (in the usual way) by

$$K^o = \{u \in V \mid \langle u, v\rangle \leqslant 0 \text{ for all } v \in K\},$$

so that $K^{oo} = K$ again, then we have:

LEMMA 34. *If P is a polyhedral set and F a non-empty face of P, then*

$$N(F, P) = A(F, P)^o; \qquad A(F, P) = N(F, P)^o.$$

In all that follows, the class of a cone $K \in \mathscr{C}$ is taken intrinsically; in other words, $\langle K\rangle$ is the class of K in $\hat{\Sigma}(L)$, where $L = \operatorname{lin} K$ is the smallest subspace containing K. Thus, $\langle K\rangle \neq 0$.

The classes of $A(F, P)$ and $N(F, P)$ in $\hat{\Sigma}$ are denoted $a(F, P)$ and $n(F, P)$, respectively. In [8], we proved the following result (with $\mathbb{F} = \mathbb{R}$, but the proof carries over directly):

LEMMA 35. (a) *Let $K \in \mathscr{C}$. Then*

$$\sum_F (-1)^{\dim F} a(F, K) = (-1)^{\dim K} \langle -K\rangle.$$

(b) *Let $P \in \mathscr{P}$ with* $\dim P > 0$. *Then*

$$\sum_F (-1)^{\dim F} a(F, P) = 0.$$

Such sums always extend over all non-empty faces F (of K or P). These relations are abstract versions of theorems of Sommerville and Brianchon and Gram, respectively. Note the occurrence of the opposite cone $-K$ on the right side of the first relation.

One case of Lemma 35(a) is of particular importance (see Section 14). Bearing in mind Lemma 34, we easily see that, if G is a face of P which contains F, then the inner cone of $N(F, P)$ at its face $N(G, P)$ is just $N(F, G)$. We therefore deduce

LEMMA 36. *Let P be a polyhedral set in V, and let F be a non-empty face of P. Then*

$$\sum_{F \subseteq G \subseteq P} (-1)^{\dim G} n(F, G) = (-1)^{\dim F} n(-F, -P).$$

We next repeat more results from [8] (which were actually proved in a concrete form in [5] with $\mathbb{F} = \mathbb{R}$, but again the essential geometry translates). As before, if Q is a polyhedral set and L a subspace of V, we write $Q \parallel L$ to mean that $\operatorname{aff} Q$ is a translate of L.

LEMMA 37. *Let $\mathscr{G}$ be an abelian group, and for each subspace L of V, let $\psi_L : \mathscr{F}(L) \to \mathscr{G}$ be an L-simple translation invariant valuation, where $\mathscr{F} = \mathscr{P}$ or $\mathscr{C}$. If $\psi : \mathscr{F} \to \mathscr{G}$ is defined by $\psi(P) = \psi_L(P)$ if $P \parallel L$, then the mapping $\phi : \mathscr{F} \to \mathscr{G} \otimes \hat{\Sigma}$ given by*

$$\phi(P) = \sum_F \psi(F) \otimes n(F, P)$$

is a translation invariant valuation.

Of course, here and in the next lemma, translation invariance is irrelevant if $\mathscr{F} = \mathscr{C}$.

LEMMA 38. *Let $\mathscr{G}$ be an abelian group, and let $\phi : \mathscr{F} \to \mathscr{G}$ be a translation invariant valuation, where $\mathscr{F} = \mathscr{P}$ or $\mathscr{C}$. Then for each subspace L of V, the mapping $\psi_L : \mathscr{F}(L) \to \mathscr{G} \otimes \hat{\Sigma}$ defined by*

$$\psi_L(P) = \begin{cases} \sum_F \phi(F) \otimes (-1)^{\dim P - \dim F} a(F, P), & \text{if } P \parallel L, \\ 0, & \text{otherwise}, \end{cases}$$

is an L-simple translation invariant valuation.

We shall use more concrete versions of these lemmas in Sections 16 and 17.

13. The Second Isomorphism Theorem

In this section, we shall establish the second isomorphism Theorem 5. For convenience, and bearing in mind Theorem 8, we shall restate the result in a rather stronger form. We suppose, as in Section 7, that we have picked a volume vol_L for each subspace L of V, and that, as usual, $\operatorname{vol} P = \operatorname{vol}_L P$, where $P \parallel L$.

LEMMA 39. *For each $r = 0, \ldots, d$, the mapping $\sigma_r : \mathscr{P} \to \mathbb{F} \otimes \hat{\Sigma}$ defined by*

$$\sigma_r(P) = \sum_{F^r} \operatorname{vol} F^r \otimes n(F^r, P),$$

where the sum extends over the r-faces F^r of P, induces an injective (vector space) homomorphism from Ξ_r into $\mathbb{F} \otimes \hat{\Sigma}^{d-r}$.

$\mathbb{F}\otimes\hat{\Sigma}$ inherits its structure as a vector space over $\mathbb{F}$ from its first component. Of course, $\Xi_0\cong\mathbb{Z}$, and so lacks a vector space structure. It is convenient to treat this case first. Each vertex F^0 of P is a point, with volume vol $F^0=1$. The outer cones $N(F^0, P)$ to P at these vertices form a dissection of V, and hence

$$\sigma_0(P)=1\otimes\langle V\rangle$$

for each $P\in\mathscr{P}\backslash\{\varnothing\}$. The isomorphism $k\mapsto k\otimes\langle V\rangle$ shows that σ_0 is an injection.

In general, by Lemma 37, σ_r is a translation invariant valuation, and since σ_r is homogeneous of degree r (since each volume occurring is also), we see that σ_r maps Ξ_r into $\mathbb{F}\otimes\hat{\Sigma}^{d-r}$. The case $r=d$ is also easy, since $\hat{\Sigma}^0=\mathbb{Z}$ is generated by the class of the subspace $\{o\}$, so that every element of $\mathbb{F}\otimes\hat{\Sigma}^0$ is uniquely representable in the form $\lambda\otimes\langle\{o\}\rangle$, for some $\lambda\in\mathbb{F}$.

So, now let us suppose that $1\leqslant r\leqslant d-1$. We shall show how $\sigma_r(P)$ determines $f_U(P)$ for each frame functional f_U of type r, and the separating property of these f_U will show that σ_r is injective. If $U=(u_1, ..., u_{d-r})$ is a $(d-r)$-frame, and if we write $F_0=P$ and $F_j=P_{(u_1,...,u_j)}$ for $j=1, ..., d-r$, then $f_U(P)=\mathrm{vol}_L F_{d-r}$, where $L=U^{\perp}$ is the r-dimensional subspace orthogonal to U. Now the condition $F_j=(F_{j-1})_{u_j}$ says that

$$\begin{aligned} u_j&\in\mathrm{relint}\, N(F_j, F_{j-1})\\ &=\mathrm{relint}(N(F_j, P)-N(F_{j-1}, P)), \end{aligned}$$

the latter relation holding since $N(F, G)=N(F, P)-N(G, P)$ is the inner cone to $N(F, P)$ at its face $N(G, P)$, as mentioned in Section 12 above. Conversely, if these conditions hold for some chain

$$P=F_0\supseteq F_1\supseteq\cdots\supseteq F_{d-r}=F$$

of faces of P, then $F=P_U$.

This motivates the following definition. We say that the $(d-r)$-dimensional cone K is *adapted* to the $(d-r)$-frame $U=(u_1, ..., u_{d-r})$ if $K\subseteq\mathrm{lin}\, U$, and K has a chain of faces $\{o\}=K_0\subseteq K_1\subseteq\cdots\subseteq K_{d-r}$, with $u_j\in\mathrm{relint}(K_j-K_{j-1})$ for $j=1, ..., d-r$. Suppose now that $x\in\Xi_r$ is such that

$$\sigma_r(x)=\sum_K\mu_K\otimes\langle K\rangle,$$

where the sum extends over a finite set of $(d-r)$-dimensional cones K. Then the above discussion shows that, for each $(d-r)$-frame U,

$$f_U(x)=\sum\{\mu_K\mid K \text{ is adapted to } U\}.$$

In particular, if $\sigma_r(x)=0$, then $f_U(x)=0$ for every frame functional f_U of type r, and hence, by the corollary to Theorem 3, $x=0$. That is, σ_r is injective, as claimed.

The $\sigma_r(P)$ are, in some sense, the abstract analogues of the intrinsic r-volumes $V_r(P)$ (or $(d-r)$th quermassintegrals $W_{d-r}(P)$; see [5] or [9, Sect. 3]). Theorem 5 itself can also be regarded as an abstract version of the main Theorem 2 of [7], since the identity map from Π into itself is obviously dilatation continuous in the sense of that paper.

We conclude this section with a few remarks about the image of the mapping σ. We observe first that the range of definition of the frame functionals can be extended to $\mathbb{F}\otimes\hat{\Sigma}$, as the concept of "adapted" shows. Indeed, if we denote by G the vector space over $\mathbb{F}$ generated by the frame functionals (of course, only finite linear combinations are allowed here), then $\mathbb{F}\otimes\hat{\Sigma}$ and G are easily seen to be dual vector spaces.

The image space im σ and the syzygies between the frame functionals are clearly closely related. On im σ, the syzygies of the first kind are trivial, but on $\mathbb{F}\otimes\hat{\Sigma}$ they are not. Neither, naturally, are the syzygies of the second kind trivial. However, it is not hard to see that the syzygies of the third kind also hold on the whole of $\mathbb{F}\otimes\hat{\Sigma}$. It is therefore natural to pose:

Conjecture 2. (a) im $\sigma_0=\mathbb{Z}\otimes\langle V\rangle$, *and is determined by the syzygies of the first kind, and the conditions* $f_U(x)\in\mathbb{Z}$.

(b) *For* $r=1, ..., d-1$, im σ_r *is determined solely by the syzygies of the second kind.*

(c) im $\sigma_d=\mathbb{F}\otimes\langle\{o\}\rangle$ $(\cong\mathbb{F})$.

In fact, (a) and (c) are true, as we know.

14. The Euler Map and Negative Dilatations

We recall from Section 2 that the *Euler map* $*: \Pi\to\Pi$ is defined on the generators $[P]$ of Π $(P\in\mathscr{P})$ by:

(E) $[P]^*=\sum_F(-1)^{\dim F}[F]$.

The first stage in proving Theorem 2 is a universalized form of a result of Sallee [14]:

Lemma 40. *The Euler map induces a group endomorphism of* Π, *and, indeed, of each weight space* Ξ_r.

The first assertion is proved using Lemma 2; we shall not reproduce the details. The second then follows easily from Lemma 20.

If we apply the Euler map to $1=[o]$, we obtain $1^*=1$. But, by

Lemma 8, the 0-component of $[P]$ is also 1, for every $P \in \mathscr{P}\backslash\{\varnothing\}$. Lemma 40 thus yields:

LEMMA 41. *Each $P \in \mathscr{P}\backslash\{\varnothing\}$ satisfies the Euler relation*

$$\sum_F (-1)^{\dim F} = 1.$$

It should be noted that this is not the most straightforward proof of Euler's theorem.

The connexion between Euler-type relations and negative dilatations was first observed by Sallee [14]; subsequently, many Euler-type relations were discovered (see [6, Sect. 6] or [9, Sect. 12] for details), and the general relationship was elucidated in [6].

If $x \in \Pi$, we write for brevity in what follows $\bar{x} = \Delta(-1)x$. The mapping $x \mapsto \bar{x}$ is obviously involutory, and, by the corollary to Theorem 6, is an algebra automorphism of Π. The core of Theorem 2 is contained in:

LEMMA 42. *Let $r = 0, \ldots, d$. If $x \in \Xi_r$, then $\bar{x} = (-1)^r x^*$.*

It is enough to verify this for the r-component of a generator $[P]$ of Π. We employ the injection $\sigma_r \colon \Xi_r \to \mathbb{F} \otimes \hat{\Sigma}^{d-r}$ (see Lemma 39). Then, with F^r in the sums below running over the r-faces of P, and using Lemma 36, we have

$$\begin{aligned}
\sigma_r([P]_r^*) &= \sum_{G \subseteq P} (-1)^{\dim G} \sigma_r([G]_r) \\
&= \sum_{G \subseteq P} (-1)^{\dim G} \left\{ \sum_{F^r \subseteq G} \operatorname{vol} F^r \otimes n(F^r, G) \right\} \\
&= \sum_{F^r \subseteq P} \operatorname{vol} F^r \otimes \left\{ \sum_{F^r \subseteq G \subseteq P} (-1)^{\dim G} n(F^r, G) \right\} \\
&= \sum_{F^r \subseteq P} \operatorname{vol}(-F^r) \otimes \{(-1)^r n(-F^r, -P)\} \\
&= \sigma_r((-1)^r [-P]_r).
\end{aligned}$$

Thus $[-P]_r = (-1)^r [P]_r^*$, as we wished to show.

More generally, for $\lambda < 0$, write $\lambda = (-\lambda)(-1)$; if $x \in \Xi_r$, there then follows, using Lemma 42 and Theorem 1(e), that $\Delta(\lambda)x = \lambda^r x^*$, as required.

The rest of Theorem 2 follows easily as well. Since the mapping $x \mapsto (-1)^r x$ for $x \in \Xi_r$ $(r = 0, \ldots, d)$ trivially induces an involutory algebra automorphism of Π, Lemma 42 and the remark before it show that $x \mapsto x^*$ is also an involutory algebra automorphism.

The relationships of Lemma 42 can be stated in a more picturesque way. The invertible elements of Π are clearly just those of the form $\pm(1 + z)$, with $z \in Z_1$. In particular, if $P \in \mathscr{P}\backslash\{\varnothing\}$, then $[P]$ is invertible. Now, if $p = \log P$, then obviously $[P]^{-1} = \exp(-p)$. But Lemma 42 for $r = 1$ implies that $-p = \bar{p}^*$, in the notation used there. Exponentiating, and using the fact that $x \mapsto \bar{x}$ and $x \mapsto x^*$ are algebra automorphisms, we deduce:

THEOREM 12. *Let $P \in \mathscr{P}\backslash\{\varnothing\}$. Then $[P]^{-1} = [-P]^*$.*

Theorems 2 and 6 also immediately yield:

THEOREM 13. *The homomorphism $\Phi \colon \Pi(V) \to \Pi(W)$ induced by an affine mapping $\Phi \colon V \to W$ commutes with the Euler map.*

From a geometric point of view this is curious, since if rank $\Phi < \dim V$, then for $P \in \mathscr{P}$, the facial structures of P and ΦP are not particularly closely related.

We sometimes write the 0-component $\Delta(0)x$ of $x \in \Pi$ as $\chi(x) = \chi(x)1$, and call $\chi(x)$ the *Euler characteristic* of x. Then χ can be characterized in the following way:

THEOREM 14. *Let R be any ring without nilpotent elements (for example, an integral domain), and let $\phi \colon \Pi \to R$ be a non-trivial ring homomorphism. Then there is an idempotent $i \in R$, such that $\phi(x) = \chi(x) i$ for all $\chi \in \Pi$.*

If $x \in Z_1$, then $x^{d+1} = 0$, and hence $0 = \phi(x^{d+1}) = \phi(x)^{d+1}$, so that $\phi(x) = 0$ also. Thus $i = \phi(1) \neq 0$, and $i^2 = \phi(1)^2 = \phi(1^2) = \phi(1) = i$, so that i is an idempotent. There follows at once $\phi(x) = \chi(x) i$, as claimed.

15. MIXED POLYTOPES

If $\mathscr{X}$ is a rational vector space, and $\phi \colon \mathscr{P} \to \mathscr{X}$ a translation invariant valuation, then it is known (see [6] and the Appendix to [9]) that, for $P_1, \ldots, P_k \in \mathscr{P}$ and rationals $\lambda_1, \ldots, \lambda_k \geq 0$, there is a polynomial expansion

$$\begin{aligned}
&\phi(\lambda_1 P_1 + \cdots + \lambda_k P_k) \\
&\quad = \sum_{r_i \geq 0} \binom{r_1 + \cdots + r_k}{r_1 \; \cdots \; r_k} \lambda_1^{r_1} \cdots \lambda_k^{r_k} \phi(P_1, r_1; \ldots; P_k, r_k),
\end{aligned}$$

where

$$\binom{r_1 + \cdots + r_k}{r_1 \; \cdots \; r_k} = \frac{(r_1 + \cdots + r_k)!}{r_1! \cdots r_k!}$$

is the multinomial coefficient. The coefficient $\phi(P_1, r_1; \ldots; P_k, r_k)$ is a translation invariant valuation which is homogeneous in P_i of degree r_i $(i = 1, \ldots, k)$; it is called a *mixed valuation*. If $P_1 = \cdots = P_k = P$, say, and $r = r_1 + \cdots + r_k$, then, in the notation of Theorem 8, $\phi(P_1, r_1; \ldots; P_k, r_k) = \phi_r(P)$ is the rth homogeneous component of $\phi(P)$.

We shall shortly see that this result is a consequence of our general theory. One approach to it has been to develop a corresponding theory of mixed polytopes; this was attempted by Meier [10], though his argument appears at one point to be flawed. In [9], an alternative approach was outlined, though there only within the context of the polytope group $\hat{\Pi}(V)$.

However, working with the polytope algebra makes it clear what we must do. The general mixed valuation is of the form $\phi(P_1, \ldots, P_r)$, where we suppress the mention of $r_i = 1$, since

$$\phi(P_1, r_1; \ldots; P_k, r_k) = \phi(P_1, \ldots, P_1, \ldots, P_k, \ldots, P_k),$$

where P_i is repeated r_i times (and omitted if $r_i = 0$). If $2 \leqslant r \leqslant d$, let $P_1, \ldots, P_r \in \mathscr{P}\backslash\{\varnothing\}$ (but not necessarily distinct), and let $p_i = \log P_i$ $(i = 1, \ldots, r)$. Then the *mixed polytope* (*class*) of $P_1, \ldots, P_r$ is defined to be

$$m(P_1, \ldots, P_r) = \frac{1}{r!} p_1 \cdots p_r.$$

In particular, if $P_1 = \cdots = P_r = P$, then $m(P, \ldots, P) = [P]_r$ is the r-component of P.

Expanding $[\lambda_1 P_1 + \cdots + \lambda_k P_k] = \exp(\lambda_1 p_1 + \cdots + \lambda_k p_k)$ as a polynomial in the rational numbers $\lambda_i \geqslant 0$, and applying the valuation $\phi: \mathscr{P} \to \mathscr{X}$, then yields the result above. Of course, the general mixed valuation is $\phi(P_1, \ldots, P_r) = \phi(m(P_1, \ldots, P_r))$.

This approach to mixed valuations (and, in particular, in case $r = d$ to mixed volumes) clarifies a number of previously known results. We give a few examples.

The first concerns an observation made originally about mixed volumes by Groemer [1]; there it was stated for convex bodies in case $\mathbb{F} = \mathbb{R}$, but its essence is algebraic. A neater proof was given in [9], and what follows is an abstract version of this.

THEOREM 15. *Let $P, Q \in \mathscr{P}\backslash\{\varnothing\}$ be such that $X = P \cup Q \in \mathscr{P}$ also, let $Y = P \cap Q$, and write $p = \log P$, and so on. Then $pq = xy$.*

Equating r-components of the valuation property equation (V) and multiplying by $r!$ yields

$$p^r + q^r = x^r + y^r \qquad (r = 0, \ldots, d).$$

Hence

$$pq = \tfrac{1}{2}((p+q)^2 - (p^2 + q^2)) = \tfrac{1}{2}((x+y)^2 - (x^2 + y^2)) = xy,$$

as we wished to show.

Observe also that the equations $p + q = x + y$ and $pq = xy$ imply that $p^r + q^r = x^r + y^r$ for each $r = 0, \ldots, d$. Exponentiating $p + q = x + y$ also yields $[P + Q] = [X + Y]$, which is a weaker version of Lemma 24.

Another example involves summands of polytopes (see [2, Chapt. 15]). Let $P, Q \in \mathscr{P}\backslash\{\varnothing\}$ be such that there exists a rational $\bar{\lambda} > 0$, with the property that, for each (rational) λ satisfying $0 \leqslant \lambda \leqslant \bar{\lambda}$, there is a $P_\lambda \in \mathscr{P}$ with $P = P_\lambda + \lambda Q$ (it is enough, in fact, to take $\lambda = \bar{\lambda}$ here). Writing $p = \log P$, and so on, we have $p = p_\lambda + \lambda q$, or $p_\lambda = p - \lambda q$. The r-component of $[P_\lambda]$ is thus

$$\frac{1}{r!}(p - \lambda q)^r = \frac{1}{r!} \sum_{s=0}^{r} (-1)^s \binom{r}{s} \lambda^s p^{r-s} q^s.$$

Now let $\mathscr{X}$ be a rational vector space, as before. We conclude from our discussion the following

THEOREM 16. *Let $\phi: \mathscr{P} \to \mathscr{X}$ be a translation invariant valuation which is homogeneous of degree r. With the above notation, for rational λ with $0 \leqslant \lambda \leqslant \bar{\lambda}$, $\phi(P_\lambda) = \sum_{s=0}^{r} (-\lambda)^s \binom{r}{s} \phi(P, r-s; Q, s)$.*

The traditional proof of this involves expressing P_λ as $P_\lambda = P_{\bar{\lambda}} + (\bar{\lambda} - \lambda)Q$, expanding $\phi(P_\lambda)$ as a polynomial in $\bar{\lambda} - \lambda$, and comparing coefficients with those of $\phi(P + \mu Q)$ for $\mu \geqslant 0$ and $\lambda \leqslant 0$.

These two results admitted proofs within existing valuation theory. The last, in contrast, uses the multiplicative structure in an essential way.

THEOREM 17. *Let ϕ be a translation invariant valuation on $\mathscr{P}$ which is homogeneous of degree r. Then for fixed $P_1, \ldots, P_r \in \mathscr{P}\backslash\{\varnothing\}$ and variable $\lambda_1, \ldots, \lambda_r \geqslant 0$, the value of the mixed valuation $\phi(\lambda_1 P_1, \ldots, \lambda_r P_r)$ depends only on the product $\lambda_1 \cdots \lambda_r$.*

The reason is simple: the corresponding mixed polytope is

$$m(\lambda_1 P_1, \ldots, \lambda_r P_r) = \frac{1}{r!} (\lambda_1 p_1) \cdots (\lambda_r p_r) = (\lambda_1 \cdots \lambda_r) \cdot \frac{1}{r!} p_1 \cdots p_r,$$

where $p_i = \log P_i$ $(i = 1, \ldots, r)$, and the theorem follows at once.

16. Inner and Outer Angles

In preparation for discussing the isomorphism between Π and the full polytope group $\hat{\Pi}$, we must return to the topic of cones. A homomorphism ω on the full cone group $\hat{\Sigma}$ is identified with a family of L-simple valuations ω_L on $\mathscr{C}(L)$, one for each subspace L of V (including $\{o\}$ and V itself). We call ω an *angle* (*functional*) if ω takes values in $\mathbb{F}$, with $\omega(L)$ $(=\omega_L(L)) = 1$ for each subspace L.

As has been pointed out by Betke (private communication):

Lemma 43. *Angle functionals on $\hat{\Sigma}$ exist.*

We refer back to Section 7, where we chose a volume vol_L in each subspace L, whose scaling was induced by a polytope Q with $o \in \mathrm{int}\, Q$. We now define the angle ω_L on $\mathscr{C}(L)$ by

$$\omega_L(K) = \mathrm{vol}_L(K \cap Q)$$

for $K \in \mathscr{C}(L)$. This clearly gives an L-simple valuation, with $\omega_L(L) = 1$, since the scaling of vol_L is induced by Q.

It should be noted, however, that angles do not necessarily arise in this way. As a variant on this construction, any polytope Q_L in L with $\dim Q_L = \dim L$ will give rise to an angle on $\mathscr{C}(L)$ as above, even if $o \notin \mathrm{relint}\, Q_L$. Our choice of Q in Section 7 shows that angles need not be centrally symmetric. There is no reason for them to be non-negative either; for example, pick Q_1, Q_2 in L which are strictly separated by a hyperplane through o, whose positive volumes satisfy $\mathrm{vol}_L\, Q_1 \neq \mathrm{vol}_L\, Q_2$, and define

$$\omega_L(K) = (\mathrm{vol}_L(K \cap Q_1) - \mathrm{vol}_L(K \cap Q_2))/(\mathrm{vol}_L\, Q_1 - \mathrm{vol}_L\, Q_2).$$

Denoting by $a(F, G)$ and $n(F, G)$ the classes of the inner and outer cones to a polyhedral set G at its face F, we define *inner* and *outer angles* to G at F by

$$\alpha(F, G) = \omega(a(F, G)),$$
$$\nu(F, G) = \omega(n(F, G)),$$

where ω is some angle functional, not necessarily the same at each occurrence. We call inner and outer angles α and ν *inverse* if

$$\sum_J (-1)^{\dim J - \dim F} \alpha(F, J) \nu(J, G) = \delta(F, G),$$

where

$$\delta(F, G) = \begin{cases} 1, & \text{if } F = G, \\ 0, & \text{if } F \neq G. \end{cases}$$

It is convenient here to adopt the language of the incidence algebra of functions on the faces of polyhedral sets (see [11]). The incidence algebra consists of functions κ on ordered pairs (F, G) of faces, taking values in some ring (in our case, $\mathbb{F}$). These are such that $\kappa(F, G) = 0$ unless F is a face of G. Addition and multiplication of such functions are defined by

$$(\kappa + \lambda)(F, G) = \kappa(F, G) + \lambda(F, G),$$
$$(\kappa\lambda)(F, G) = \sum_J \kappa(F, J)\lambda(J, G).$$

The values $\kappa(F, G)$ can be thought of as entries in a triangular matrix, and the defining condition then implies:

Lemma 44. *If α and ν are inverse inner and outer angles, then*

$$\sum_J (-1)^{\dim G - \dim J} \nu(F, J)\, \alpha(J, G) = \delta(F, G).$$

An obvious result to which we shall often wish to appeal when we pass from Lemmas 37 and 38, involving inner and outer cone classes, to their concrete versions involving inner and outer angles, is:

Lemma 45. *Let $\mathscr{X}$ be a vector space over $\mathbb{F}$, and let ω be an angle on $\hat{\Sigma}$. Then the mapping $\pi: \mathscr{X} \otimes \hat{\Sigma} \to \mathscr{X}$ defined by $\pi(x \otimes c) = \omega(c)x$ $(x \in \mathscr{X},\ c \in \hat{\Sigma})$ is a homomorphism.*

The crucial result of this section is:

Lemma 46. *If ν is an outer angle, then there exists an inverse inner angle α, and conversely.*

The inverse α of ν certainly exists in the incidence algebra, since ν corresponds to a triangular matrix with diagonal entries $\nu(F, F) = 1$. However, this does not immediately ensure that α is an inner angle.

We therefore proceed as follows. We first construct an auxiliary inner angle $\bar{\alpha}$, which will be such that $\bar{\alpha}(F, G) = \alpha(-F, -G)$, and to do this, we need to find a corresponding angle functional ω. We do this by induction on the dimension of the subspace L of V, beginning with $\omega(\{o\}) = 1$.

So, suppose that we have constructed ω (and the corresponding inner angle $\bar{\alpha}$) in such a way that, whenever K is a cone with $\dim K < \dim L$, then

$$\sum_F \omega(F)\nu(F, K) = 1$$

(here, $\omega(F)=\bar{\alpha}(A, F)$, where A is the face of apices of K). We now define ω_L on $\mathscr{C}(L)$ by

$$\omega_L(K)=1-\sum_{\dim F<\dim L}\omega(F)\nu(F, K).$$

The mapping $K\mapsto 1$ is certainly a valuation on $\mathscr{C}$ (though not simple; bear in mind that all cones are non-empty and convex), and so is the mapping

$$K\mapsto \sum_{\dim F<\dim L}\omega(F)\nu(F, K),$$

by Lemmas 37 and 45, since the condition $\dim F<\dim L$ ensures that $\omega(F)$ is already defined. Thus ω_L is a valuation on $\mathscr{C}(L)$; it is simple by the inductive assumption made above, and $\omega_L(L)=1$ since L is the only face of itself.

We next set $\alpha(F, G)=\bar{\alpha}(-F, -G)$ $(=\omega(-A(F, G)))$. From the Euler relation for cones (see [5] or [8]), and Lemma 35(a) (with α replacing a), we deduce

$$\begin{aligned}\delta(F, G)&=\sum_{F\subseteq K\subseteq G}(-1)^{\dim K-\dim F}\\ &=\sum_{F\subseteq K\subseteq G}(-1)^{\dim K-\dim F}\left\{\sum_J\alpha(-K, -J)\nu(J, G)\right\}\\ &=\sum_J\left\{\sum_{F\subseteq K\subseteq J}(-1)^{\dim K-\dim F}\alpha(-K, -J)\right\}\nu(J, G)\\ &=\sum_J(-1)^{\dim J-\dim F}\alpha(F, J)\nu(J, G),\end{aligned}$$

as required.

The proof with α and ν interchanged is similar, or can be deduced from the first case by polarity.

17. The Polytope Groups

The polytope group $\hat{\Pi}(L)$ is derived from the subalgebra $\Pi(L)$ of Π by imposing the extra conditions (S) which correspond to simple valuations; in other words, as a group, $\hat{\Pi}(L)$ is a quotient of $\Pi(L)$. Before we prove the first isomorphism Theorem 4, we shall derive the structure theorem for $\hat{\Pi}(L)$ of [4] or [12] from that of Π in Theorem 1.

We begin by recalling that, up to isomorphism, $\hat{\Pi}(L)$ only depends on $\dim L$, because of Theorem 6. So, we need only consider $\hat{\Pi}^d=\hat{\Pi}(V)$ itself. We have:

Theorem 18. (a) $\hat{\Pi}^0\cong\mathbb{Z}$.

(b) *For $d\geqslant 1$, $\hat{\Pi}^d$ admits a direct sum decomposition*

$$\hat{\Pi}^d=\bigoplus_{r=1}^{d}\hat{\Xi}_r^d$$

into vector spaces $\hat{\Xi}_r^d$ over $\mathbb{F}$ $(r=1, \dots, d)$. Moreover, dilatations act on $\hat{\Pi}^d$ by

$$\Delta(\lambda)x=\begin{cases}\lambda^r x, & \lambda\geqslant 0,\\ (-1)^d\lambda^r x, & \lambda<0,\end{cases}$$

for $x\in\hat{\Xi}_r^d$.

Part (a) is obvious, since $\hat{\Pi}^0$ is generated by the class 1 of a point $(\{o\})$. In fact, we can (and shall) identify $\hat{\Pi}^0$ with $\Pi(\{o\})$ in the natural way.

So, now suppose that $d\geqslant 1$. Let Π^S be the additive subgroup of Π generated by the polytope classes $[P]$ with $P\in\mathscr{P}$ and $\dim P<d$. Then the dilatations clearly act as group endomorphisms of Π^S, and so Π^S also admits a direct sum decomposition

$$\Pi^S=\bigoplus_{r=0}^{d}\Xi_r^S,$$

where $\Xi_0^S=\Xi_0\cong\mathbb{Z}$, and Ξ_r^S is a vector subspace of Ξ_r for $r=1, \dots, d$. In fact, $\Xi_d^S=\{0\}$, since volume vanishes on Ξ^S. Taking quotients yields the direct sum decomposition for $\hat{\Pi}^d$; note that $\hat{\Xi}_d^d\cong\Xi_d\cong\mathbb{F}$ again gives us volume.

The action of the dilatation $\Delta(\lambda)$ on $\hat{\Xi}_r^d$ for $\lambda\geqslant 0$ is directly inherited from its action on Ξ_r. For $\lambda<0$, the action involves the Euler map. But in $\hat{\Pi}^d$, we have $\langle F\rangle=0$ if F is a face of P with $\dim F<d$. In other words, $\langle P\rangle^*=(-1)^d\langle P\rangle$ for all $P\in\mathscr{P}$, and so if $x\in\hat{\Xi}_r^d$ and $\lambda<0$, then

$$\Delta(\lambda)x=\lambda^r x^*=(-1)^d\lambda^r x,$$

as claimed. This proves the theorem.

Let us remark that, although the assumption $\langle -P\rangle=(-1)^d\langle P\rangle$ was made in [4], it can be seen here to be unnecessary (contrast [12, Proposition 2.5.5]).

While we are considering polytope groups, we shall establish a representation theorem analogous to Theorem 5. We must first quote the separation theorem for $\hat{\Pi}^d$.

If $U=(u_1, ..., u_k)$ is a k-frame, and $E=(\varepsilon_1, ..., \varepsilon_k)$ with $\varepsilon_i=\pm 1$ $(i=1, ..., k)$, we write $EU=(\varepsilon_1 u_1, ..., \varepsilon_k u_k)$ and $\operatorname{sgn} E=\varepsilon_1 ... \varepsilon_k$. Then a *Hadwiger functional of type r* is a mapping of the form

$$h_U=\sum_E \operatorname{sgn} E \cdot f_{EU},$$

where U is a $(d-r)$-frame and f_U is the corresponding frame functional. The case $U=\emptyset$ just gives volume.

The Hadwiger functionals are simple translation invariant valuations, and so induce homomorphisms on $\hat{\Pi}^d$ (we shall say more about this below). In fact, we have (see [4] or [12]):

LEMMA 47. *The Hadwiger functionals separate* $\hat{\Pi}^d$.

It is convenient, and not too confusing, to identify a homomorphism on $\hat{\Pi}^d$ with the corresponding homomorphism on Π which vanishes on Π^S (so that we suppress the quotient map from Π to $\Pi/\Pi^S \cong \hat{\Pi}^d$). Hence, in particular, the Hadwiger functionals are regarded indiscriminately as homomorphisms on Π or on $\hat{\Pi}^d$.

Let $\hat{\Gamma}$ denote the subgroup of $\hat{\Sigma}$ generated by the classes of cones in $\mathscr{C}$ which contain a line, and so have faces of apices of positive dimension. The important step in our discussion is:

LEMMA 48. *Let* $\sigma: \Pi \to \mathbb{F} \otimes \hat{\Sigma}$ *be the injection of Theorem* 5, *and let* $x \in \Pi$. *Then* $x \in \Pi^S$ *if and only if* $\sigma(x) \in \mathbb{F} \otimes \hat{\Gamma}$.

Since the face of apices of the outer cone $N(F, P)$ has dimension $d-\dim P$, we see that $n(F, P) \in \hat{\Gamma}$ whenever $\dim P<d$, and so $x \in \Pi^S$ implies $\sigma(x) \in \mathbb{F} \otimes \hat{\Gamma}$.

For the converse, we consider in more detail the effect of a Hadwiger functional. The volume term in Π corresponds to the subgroup $\mathbb{F} \otimes \hat{\Sigma}^0$, so that $\sigma(x) \in \mathbb{F} \otimes \hat{\Gamma}$ implies that $\operatorname{vol} x=0$. So, let h_U be a Hadwiger functional of type $r<d$, and consider $h_U(P)$ for $P \in \mathscr{P} \backslash \{\emptyset\}$. Now $h_U(P)=0$ anyway unless $\dim P_{EU}=r$ for some $E=(\varepsilon_1, ..., \varepsilon_r)$. On the other hand, if (for simplicity) $\dim P_U=r$ but $\dim P<d$, then the decreasing sequence

$$F_j=P_{(u_1, ..., u_j)} \qquad (j=0, ..., d-r)$$

of faces of P is such that, for some minimal j, $F_{j-1}=F_j$. With $E=(\varepsilon_1, ..., \varepsilon_{d-r})$ such that $\varepsilon_j=-1$ and $\varepsilon_i=1$ for $i \neq j$, the terms $f_U(P)$ and $f_{EU}(P)$ of $h_U(P)$ cancel. We conclude that $h_U(P)=0$ if $\dim P<d$. But conversely, referring back to the proof of Theorem 5, we can see that if U is adapted to a cone K whose class lies in $\hat{\Gamma}$, then so is EU for some such E of the kind just mentioned, with j minimal such that $K_j=K_{j-1}$ (in the notation of that theorem). We conclude that, if $\sigma(x) \in \mathbb{F} \otimes \hat{\Gamma}$, then $h_U(x)=0$ for every Hadwiger functional h_U, and so, by Lemma 47, $x \in \Pi^S$. This proves the lemma.

If we write $\bar{c}$ for the image of c under the quotient map from $\hat{\Sigma}$ to $\hat{\Sigma}/\hat{\Gamma}$, then there immediately follows from Lemma 48 the promised isomorphism theorem for $\hat{\Pi}^d$.

THEOREM 19. *The map* $\bar{\sigma}: \mathscr{P} \to \mathbb{F} \otimes (\hat{\Sigma}/\hat{\Gamma})$, *given by*

$$\bar{\sigma}(P)=\sum_F \operatorname{vol} F \otimes \overline{n(F, P)},$$

induces an injective homomorphism on $\hat{\Pi}^d$.

First, $\bar{\sigma}$ induces a homomorphism on Π, using Theorem 5 and the fact that $\mu \otimes c \to \mu \otimes \bar{c}$ is a homomorphism from $\mathbb{F} \otimes \hat{\Sigma}$ onto $\mathbb{F} \otimes (\hat{\Sigma}/\hat{\Gamma})$. Second, Lemma 48 shows that, if $x \in \Pi$, then $\bar{\sigma}(x)=0$ if and only if $x \in \Pi^S$. Thus $\bar{\sigma}$ indeed induces an injective homomorphism on $\hat{\Pi}^d$, as claimed.

We end this section with a remark. In [12] (see p. 40), Sah uses logarithms and exponentials as an accounting device to investigate the relationship between $\hat{\Xi}_1^d$ and $\hat{\Pi}^d$. We can now see these as the shadows of the genuine log and exp, under the projection from Π on to $\hat{\Pi}^d$.

18. THE FIRST ISOMORPHISM THEOREM

As in Section 2, the full polytope group is defined to be $\hat{\Pi}=\bigoplus_L \hat{\Pi}(L)$. To prove the isomorphism $\Pi \cong \hat{\Pi}$ of Theorem 4, we employ any pair of inverse inner and outer angles α and ν of Lemma 46. We construct homomorphisms $\phi: \Pi \to \hat{\Pi}$ and $\psi: \hat{\Pi} \to \Pi$ as follows.

First, we define the mapping $\phi: \mathscr{P} \to \hat{\Pi}$ by

$$\phi(P)=\sum_F \nu(F, P)\langle F\rangle,$$

where $\langle F\rangle$ is now the intrinsic class of F (in $\hat{\Pi}(L)$, with L such that $F \| L$). It might appear that we run into trouble with the vertices F^0 of P, since $\hat{\Xi}_0^d \cong \mathbb{Z}$ is not a vector space over $\mathbb{F}$, but note that $\sum_{F^0} \nu(F^0, P)=1$, because the outer cones to the vertices of P dissect V, and ν is an outer angle. By Lemmas 37 and 45, ϕ is a translation invariant valuation on $\mathscr{P}$, and so induces a homomorphism $\phi: \Pi \to \hat{\Pi}$.

Next, for each subspace L of V, we define the mapping $\psi_L: \mathscr{P}(L) \to \Pi$ by

$$\psi_L(P)= \begin{cases}\sum_F(-1)^{\dim P-\dim F} \alpha(F, P)[F], & \text{if } P \| L, \\ 0, & \text{otherwise.}\end{cases}$$

Again, we might appear to have problems with the 0-components of the classes $[F]$, but we observe that Lemmas 35(b) and 45 ensure that the corresponding contribution is 1 if $\dim P = 0$, and 0 otherwise. Then Lemmas 38 and 45 show that ψ_L is an L-simple valuation for each L, and so these ψ_L induce a homomorphism $\psi: \hat{\Pi} \to \Pi$.

The definition of inverse inner and outer angles and Lemma 44 easily show that ϕ and ψ are inverse homomorphisms. Thus, $\Pi \cong \hat{\Pi}$, which is Theorem 4.

This proof closely parallels the proof in [6] of the relationship between general and simple valuations. However, there it had to be assumed that the valuations were real-valued (in the case considered, $\mathbb{F} = \mathbb{R}$); this treatment removes that special assumption. Note that the isomorphism constructed above is obviously compatible with dilatations.

19. Relatively Open Polytopes

In [15], Schneider has shown how to obtain a theory of translation equidecomposability of unions of polytopes in $\mathbb{R}^d$, based on relatively open polytopes. The analogous theory is valid over any archimedean field $\mathbb{F}$, although Schneider's argument will still need real-valued functionals. However, for non-archimedean fields, standard examples show that here we must allow complementation. We shall briefly outline Schneider's theory, and provide a simpler separation theorem.

With $\mathscr{P}$ having its usual meaning, let $\tilde{\Pi}$ be the abelian group, with a generator $[\![P]\!]$ for each $P \in \mathscr{P}$ (and $[\![\varnothing]\!] = 0$), and with relations

($\tilde{\text{V}}$) $[\![P]\!] = [\![P \cap H^+]\!] + [\![P \cap H^-]\!] + [\![P \cap H]\!]$, whenever $P \in \mathscr{P}$ and H is a hyperplane bounding the closed half-spaces H^+ and H^-, which cuts P properly (so that $P \not\subseteq H^+$ and $P \not\subseteq H^-$),

and the translation invariance property (T).

The intuitive picture is that $[\![P]\!]$ is the class of relint P, the relative interior of P. Thus, in fact, ($\tilde{\text{V}}$) is really the analogue of the weak valuation property (W).

The basis of our discussion is a remark made in [9] in the context of Euler-type relations for valuations. Recall that in Section 3 we defined the family $\mathscr{P}_{\text{po}}$ of partly open polytopes, and observed that valuations on $\mathscr{P}$ extend to $\mathscr{P}_{\text{po}}$. Then we have:

Lemma 49. *For $P \in \mathscr{P}$, $[\text{relint } P] = (-1)^{\dim P} [P]^*$.*

Since P is the disjoint union of the relative interiors of its faces, we have

$$[P] = \sum_F [\text{relint } F].$$

Möbius inversion (see [11]) then leads to

$$[\text{relint } P] = \sum_F (-1)^{\dim P - \dim F} [F] = (-1)^{\dim P} [P]^*,$$

since the Möbius function on the lattice of faces of a polytope (or of a cone) is $\mu(F, G) = (-1)^{\dim G - \dim F}$. This is the lemma.

Now, the condition ($\tilde{\text{V}}$) (and our intuitive picture) gives an isomorphism between $\tilde{\Pi}$ and Π, namely $[\![P]\!] \leftrightarrow (-1)^{\dim P} [P]^*$. In view of Theorem 2, a less natural, but more convenient formulation is:

Theorem 20. *$\tilde{\Pi}$ and Π are isomorphic, under the correspondence $[\![P]\!] \leftrightarrow (-1)^{\dim P} [P]$ between their generators.*

In hindsight, we can also see this by comparing ($\tilde{\text{V}}$) and (W).

The separation criterion is now easily obtained. The *modified frame functional* $\tilde{f}_U$ is defined by $\tilde{f}_U(P) = (-1)^{\dim P} f_U(P)$. These induce homomorphisms on $\tilde{\Pi}$, and from Theorem 3 we deduce:

Theorem 21. *The modified frame functionals separate $\tilde{\Pi}$.*

We refer to [15] for the details of the equidecomposability over an archimedean field.

20. Invariance with Respect to Other Groups

Let G be any group of affinities of V which contains the group T of translations ($T \cong V$, as abelian groups). We can define a new group $\Pi_G = \Pi(V; G)$ by taking, as before, a generator $[P]_G$ for each $P \in \mathscr{P}$ ($[\varnothing]_G = 0$), with these generators satisfying the relations (V) and

(G) $[\Phi P]_G = [P]_G$ whenever $P \in \mathscr{P}$ and $\Phi \in G$.

Thus $\Pi = \Pi_T$.

If $G \neq T$, we now only have an abelian group structure, since Minkowski addition will not be compatible with the group operations in G. However, as an abelian group:

Theorem 22. *Π_G is a quotient group of Π, and admits a direct sum decomposition*

$$\Pi_G = \bigoplus_{r=0}^{d} \Xi_r,$$

such that $\Xi_0 \cong \mathbb{Z}$, *and for* $r = 1, \dots, d$, Ξ_r *is a vector space over* $\mathbb{F}$. *Moreover, the dilatations act on* Ξ_r *by*

$$\Delta(\lambda)x = \begin{cases} \lambda^r x, & \text{for } \lambda \geqslant 0, \\ \lambda^r x^*, & \text{for } \lambda < 0, \end{cases}$$

if $x \in \Xi_r$, *where* $x \mapsto x^*$ *is the Euler map.*

We obtain the direct sum decomposition by virtue of Theorem 6, whose Corollary 1 says that endomorphisms of Π induced by affinities commute with dilatations.

For most groups G, we can at present say no more than this about Π_G. However, there are two special cases.

THEOREM 23. *Let G contain a dilatation by some $\lambda \neq \pm 1$. Then $\Pi_G \cong \mathbb{Z}$.*

If $\lambda < 0$, we replace λ by λ^2; thus we can assume that $\lambda > 0$. The action of the dilatations implies that, if $x \in \Xi_r$ with $r > 0$, then $\lambda^r x = \Delta(\lambda)x = x$, and so, since $\lambda \neq 1$, we have $x = 0$. Thus $\Xi_r = \{0\}$ for $r > 0$, and the theorem follows.

Let A denote the group of all affinities of V, and EA the subgroup of *equiaffinities*, that is, the mappings of the form $v \mapsto \Phi v + t$, where Φ is a linear mapping with $\det \Phi = \pm 1$. First, as a consequence of Theorem 23, we have:

COROLLARY. $\Pi_A \cong \mathbb{Z}$.

Then we have:

THEOREM 24. *For $d \geqslant 1$, $\Pi_{EA} \cong \mathbb{Z} \oplus \mathbb{F}$.*

On each proper subspace L of V, EA induces a dilatation by some $\lambda > 1$, and we conclude at once that the subgroup Π^S_{EA} of Π_{EA} generated by the polytopes of dimension lower than d is isomorphic to $\mathbb{Z}$, generated by 1. Since two d-simplices are EA-equivalent if and only if they have the same volume, we see that the corresponding polytope group $\hat{\Pi}^d_{EA}$ is such that $\Pi_{EA}/\Pi^S_{EA} \cong \hat{\Pi}^d_{EA} \cong \mathbb{F}$. Thus the only terms of the direct sum decomposition of Π_{EA} which survive are $\Xi_0 \cong \mathbb{Z}$ and $\Xi_d \cong \mathbb{F}$, and the theorem follows at once.

If G contains a linear mapping Φ with $\det \Phi \neq \pm 1$, then certainly $\Xi_d = \{0\}$, since Ξ_d possesses the automorphism $x \mapsto |\det \Phi| x$. However, this does not necessarily mean that $\Xi_r = \{0\}$ for each $r = 1, \dots, d$. For example, if G consists of all mappings of the form

$$(\alpha_1, \alpha_2, \dots, \alpha_d) \mapsto (\lambda \alpha_1, \alpha_2, \dots, \alpha_d) + t,$$

with $\lambda > 0$ and $t \in V = \mathbb{F}^d$, then $\Pi_G \cong \pi(\mathbb{F}^{d-1})$, under the projection induced by $(\alpha_1, \alpha_2, \dots, \alpha_d) \mapsto (\alpha_2, \dots, \alpha_d)$.

The most interesting special cases are when $\mathbb{F} = \mathbb{R}$ and G is a group of isometries (with respect to the metric derived from some inner product). Then $G_0 = G/T$ is a group of orthogonal mappings, and G is a subdirect product of G_0 and T. We confine our attention to such cases for the rest of the section.

We write $\hat{\Sigma}_G$ for the quotient group of $\hat{\Sigma}$, obtained by imposing on $\hat{\Sigma}$ the additional relations

$$(\mathrm{G}_0) \quad \langle \Phi K \rangle_G = \langle K \rangle_G \text{ for all } K \in \mathscr{C} \text{ and } \Phi \in G_0.$$

Writing $n_G(F, P)$ for the class of $N(F, P)$ in $\hat{\Sigma}_G$, we see that, if vol is now a G-invariant volume (for example, ordinary volume of the appropriate dimension), then the mapping $\sigma_G : \mathscr{P} \to \mathbb{R} \otimes \hat{\Sigma}_G$, defined by

$$\sigma_G(P) = \sum_F \operatorname{vol} F \otimes n_G(F, P),$$

is a G-invariant valuation, and so induces a homomorphism on Π_G. In view of the fact that the action of G_0 on $\hat{\Sigma}$ is compatible with the action of G on Π, the following is very plausible.

Conjecture 3. *The mapping σ_G is an injection.*

The groups $\hat{\Pi}^d_G$ have received much attention in recent years, because of their connexion with Hilbert's Third Problem (particularly when G is the full group of isometries). We denote by $\hat{\Gamma}_G$ the subgroup of $\hat{\Sigma}_G$ generated by the classes of cones which contain a line, and write $\bar{c}$ for the image of c under the quotient mapping from $\hat{\Sigma}_G$ to $\hat{\Sigma}_G/\hat{\Gamma}_G$. The mapping $\bar{\sigma}_G : \mathscr{P} \to \mathbb{R} \otimes (\hat{\Sigma}_G/\hat{\Gamma}_G)$ is defined by

$$\bar{\sigma}_G(P) = \sum_F \operatorname{vol} F \otimes \overline{n_G(F, P)}.$$

As a natural generalization of Theorem 19, we pose:

Conjecture 4. *The mapping $\bar{\sigma}_G$ induces an injection from $\hat{\Pi}_G$ into $\mathbb{R} \otimes (\hat{\Sigma}_G/\hat{\Gamma}_G)$.*

Of course, $\bar{\sigma}_G$ is a G-invariant simple valuation. Equivalently (compare Lemma 48), one would conjecture that, for $x \in \Pi_G$, if $\sigma_G(x) \in \mathbb{R} \otimes \hat{\Gamma}_G$, then $x \in \Pi^S_G$, the subgroup of Π_G generated by the classes of polytopes of dimension less than d.

The mapping $\bar{\sigma}_G$ differs from the (classical total) Dehn invariant, as defined in [12], only in that it is defined in terms of outer cone classes rather than (intrinsic) inner cone classes. But the existence of the antipodal map on $\hat{\Sigma}_G/\hat{\Gamma}_G$, which is an involutory automorphism closely related to polarity (see [12]), shows that our formulation is actually equivalent. However, our approach perhaps suggests that the use of outer rather than inner cone classes may be more natural.

References

1. H. Groemer, Minkowski addition and mixed volumes, *Geom. Dedicata* **6** (1977), 324–330.
2. B. Grünbaum, "Convex Polytopes," Wiley–Interscience, London/New York/Sydney, 1967.
3. H. Hadwiger, "Vorlesungen über Inhalt, Oberfläche und Isoperimetrie," Springer-Verlag, Berlin/Göttingen/Heidelberg, 1957.
4. B. Jessen and A. Thorup, The algebra of polytopes in affine spaces, *Math. Scand.* **43** (1978), 211–240.
5. P. McMullen, Non-linear angle-sum relations for polyhedral cones and polytopes, *Math. Proc. Cambridge Philos. Soc.* **78** (1975), 247–261.
6. P. McMullen, Valuations and Euler-type relations on certain classes of convex polytopes, *Proc. London Math. Soc.* (1) **35** (1977), 113–135.
7. P. McMullen, Weakly continuous valuations on convex polytopes, *Arch. Math.* **41** (1983), 555–564.
8. P. McMullen, Angle-sum relations for polyhedral sets, *Mathematika* **33** (1986), 173–188.
9. P. McMullen and R. Schneider, Valuations on convex bodies, *in* "Convexity and Its Applications" (P. M. Gruber and J. M. Wills, Eds.), Birkhäuser, Basel/Boston/Stuttgart, 1983.
10. Ch. Meier, Multilinearität bei Polyederaddition, *Arch. Math.* **29** (1977), 210–217.
11. G.-C. Rota, On the foundations of combinatorial theory. I. Theory of Möbius functions, *Z. Wahrsch. Verw. Gebiete* **2** (1964), 340–368.
12. C.-H. Sah, "Hilbert's Third Problem: Scissors Congruence," Pitman, San Francisco/London/Melbourne, 1979.
13. G. T. Sallee, A valuation property of Steiner points, *Mathematika* **13** (1966), 76–82.
14. G. T. Sallee, Polytopes, valuations and the Euler relation, *Canad. J. Math.* **20** (1968), 1412–1424.
15. R. Schneider, Equidecomposable polyhedra, *Colloq. Math. Soc. János Bolyai* **48** (1985), 481–501.
16. H. Tverberg, How to cut a polytope into simplices, *Geom. Dedicata* **3** (1974), 239–240.
17. W. Volland, Ein Fortsetzungssatz für additive Eipolyederfunktionale im euklidischen Raum, *Arch. Math.* **8** (1957), 144–149.
18. D. W. Walkup and R. J.-B. Wets, Lifting projections of convex polyhedra, *Pacific J. Math.* **28** (1969), 465–475.

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