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Separation in the Polytope Algebra

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Abstract. The polytope algebra is the universal group for translation invariant valuations on the family of polytopes in a finite dimensional vector space over an ordered field. In an earlier paper, it was shown that the polytope algebra is, in all but one trivial respect, a graded (commutative) algebra over the base field. Also described was a family of separating (group) homomorphisms, called frame functionals. However, various questions relating to the frame functionals were left open, such as what syzygies exist between them, and what the image of a certain closely related mapping is. Here, these questions are settled: essentially, the only restrictions are imposed by the Minkowski relations. In doing this, simpler proofs are also found of some results in that earlier paper. Finally, there are consequences for expressing certain translation invariant valuations in terms of mixed volumes.

MSC 1991: Primary 52B45; dissections and valuations.

1. Introduction

The polytope algebra Π is the universal group for translation invariant valuations in a finite dimensional vector space V over an ordered field $\mathbf{F}$. In the earlier paper [10], it was shown that, in all but one trivial respect, Π is a graded (commutative) algebra over $\mathbf{F}$ (the main structure theorem is given in §2).

Also described in that paper was a family of separating homomorphisms into $\mathbf{F}$, called the frame functionals. Now, it is known that the frame functionals are not independent; linear relations between them (which may involve infinitely many terms) are known as syzygies. However, a question which was left open asked whether the known syzygies are the only ones. A closely connected question involves the determination of the image of a mapping σ related to the frame functionals. In this paper, these questions are settled. It is shown that the only syzygies are those given in [10]; they are basically the Minkowski relations. Further, the image of σ is effectively determined by the Minkowski relations.

The method of proof actually yields more. One interpretation of the result is that a weakly continuous translation invariant valuation on polytopes can be represented, on any given finite set of polytopes, by mixed volumes involving only polytopes.

2. The polytope algebra

Let $\mathbf{F}$ be an ordered field, and let V be a d-dimensional vector space over $\mathbf{F}$. For convenience, we shall assume that V is endowed with a positive definite inner product $\langle\cdot,\cdot\rangle$. This implies that we can define orthogonal projection on a linear subspace of V; note, however, that many other features of euclidean space are absent, since we cannot generally take square roots, and so cannot define norms.

We begin by giving a brief description of the polytope algebra $\Pi = \Pi(V)$. For the general terminology and notation for convex polytopes, we refer to [3,13]. The *polytope algebra* Π is initially an abelian group, with a generator $[P]$, called the *class* of P, for each $P \in \mathcal{P}$, the family of convex polytopes in V; we define $[\emptyset] := 0$. These generators satisfy the relations (V): $[P\cup Q]+[P\cap Q] = [P]+[Q]$ whenever $P, Q \in \mathcal{P}$ are such that $P \cup Q \in \mathcal{P}$ also (this corresponds to the *valuation property*), and (T): $[P + t] = [P]$ when $P \in \mathcal{P}$ and $t \in V$ is a translation vector (this is *translation invariance*). Next, the *multiplication* on Π is given by (M): $[P].[Q] = [P + Q]$, and extended to Π by linearity. Finally, we have the *dilatation*, defined on the generators by (D): $\Delta(\lambda)[P] = [\lambda P]$ for $P \in \mathcal{P}$ and $\lambda \in \mathbf{F}$.

In this context, we recall that the *vector* (or *Minkowski*) *sum* of subsets $X, Y \subseteq V$ is defined by

$$X + Y := \{x + y \mid x \in X, y \in Y\};$$

we also write $X + t := X + \{t\}$ when $t \in V$ for the *translate* of X by t. The *scalar multiple* of X by $\lambda \in \mathbf{F}$ is similarly

$$\lambda X := \{\lambda x \mid x \in X\}.$$

The main structure theorem for Π is the following (see [10]).

Theorem 2.1 *The polytope algebra is almost a graded (commutative) algebra, in the following sense:*

(a) there is a direct sum decomposition $\Pi = \bigoplus_{r=0}^{d} \Xi_r$, *such that* $\Xi_0 \cong \mathbf{Z}$, *and* Ξ_r *is a real vector space for* $r = 1, \ldots, d$ *(with* $\Xi_d \cong \mathbf{F}$*);*

(b) $\Xi_r.\Xi_s = \Xi_{r+s}$ *for each* r, s *(with* $\Xi_r = \{0\}$ *for* $r > d$*);*

(c) if $x, y \in Z_1 := \bigoplus_{r=1}^{d} \Xi_r$ *and* $\lambda \in \mathbf{F}$, *then* $(\lambda x)y = x(\lambda y) = \lambda(xy)$*;*

(d) if $x \in \Xi_r$ *and* $\lambda \geq 0$, *then* $\Delta(\lambda)x = \lambda^r x$ *(with* $\lambda^0 = 1$*).*

We call Ξ_r the *r-th weight space* of Π. The two extreme cases need special mention. First, Ξ_0 is generated by the class $[o] = [t]$ of a point (we write $[t] := [\{t\}]$ for $t \in V$; o denotes the zero vector); we actually write $1 := [o]$, and identify Ξ_0 with $\mathbf{Z}$ in the obvious way. In some respects, it is inconvenient not to have the full algebra properties; however, we can easily impose these, if we replace $\Xi_0 \cong \mathbf{Z}$ by the tensor product $\mathbf{F} \otimes \Xi_0 \cong \mathbf{F}$ (all tensor products are over $\mathbf{Z}$, unless specified otherwise). While this is more satisfying from the algebraic point of view (and it is the convention we shall henceforth adopt), it is perhaps less so from the geometric.

Second, Ξ_d is just volume. Moreover, if L is a linear subspace of V, then we can define the subalgebra $\Pi(L)$ to be generated by the classes $[P]$, such that $P \subseteq L + t$ for some $t \in V$ (we only use this and related notation in this paragraph). If $\dim L = k$, then

$\Xi_k(L) \cong \mathbf{F}$ is just k-dimensional volume (in translates of L), which we denote by vol_k, or vol if no confusion about the dimension is likely. Note that a positive scaling factor is open to choice in the definition of volume (this corresponds to a choice of basis — see [10]); we shall have to face problems which this choice poses.

Let us also make some remarks about the other weight spaces; we first treat Ξ_1. The abelian group $\mathcal{P}_T$ (actually, in an obvious way a vector space over $\mathbf{F}$) consists of the pairs (P, Q) with $P, Q \in \mathcal{P} \setminus \{\emptyset\}$, factored out by the equivalence relation

$$(P, Q) \sim (P', Q') \text{ if and only if } P + Q' = P' + Q + t \text{ for some } t \in V,$$

with addition induced by Minkowski addition, and given by

$$(P, Q) + (P', Q') = (P + P', Q + Q').$$

Note that the identity is $(\{o\}, \{o\})$, and the additive inverse of (P, Q) is (Q, P). We recall that the property

$$Q = \{x \in V \mid P + x \subseteq P + Q\}$$

for $P, Q \in \mathcal{P} \setminus \{\emptyset\}$ implies the cancellation law in the semigroup $\langle \mathcal{P}, + \rangle$. Then we have

Lemma 2.2 $\Xi_1 \cong \mathcal{P}_T$.

If P is a non-empty polytope, we write $[P] =: \sum_{r=0}^{d} [P]_r$, with $[P]_r \in \Xi_r$ its r-*component*. We always have $[P]_0 = 1$. Since Z_1 is nilpotent and $[P] - 1 \in Z_1$, it follows that the *logarithm* $\log P := \log[P]$ is well-defined, and that in fact the 1-component of $[P]$ is $[P]_1 = \log P$. We shall therefore invariably use the notation $\log P$ for the 1-component of the class of P. Observe further that the inverse *exponential* $\exp z$ of an element $z \in Z_1$ is also well defined, and, if $p := \log P$, with $P \in \mathcal{P} \setminus \{\emptyset\}$, then $\exp p = [P]$.

3. Separation

This paper is mainly concerned with separation in Π. A *k-frame* is an orthogonal set $W = (w_1, \ldots, w_k)$ of non-zero vectors. (Note that, while we can talk about orthogonality in V, and even orthogonal projection onto a subspace of V, we cannot generally normalize vectors to be of unit length.) If we denote by Q_w the face of the polytope Q in direction w, that is, the intersection of Q with its support hyperplane whose outer normal vector is w, and define recursively

$$Q_W := (Q_{(w_1, \ldots, w_{k-1})})_{w_k},$$

with W as above, then the mapping $Q \mapsto Q_W$ induces an algebra endomorphism $x \mapsto x_W$ of Π. A *frame functional of type* r is then a mapping f_W defined by

$$f_W(Q) := \mathrm{vol}_r Q_W,$$

where W is a $(d - r)$-frame; this induces a corresponding homomorphism (also denoted f_W) on Π. Observe that only the directions of the vectors in a frame determine the frame functional. The natural convention is to take the frame functional of type d (with empty frame) to be ordinary volume. We then have

Theorem 3.1 *The frame functionals separate* Π.

That is, if $x \in \Pi$ is such that $f_W(x) = 0$ for every frame W, then $x = 0$.

It may be noted that a frame functional of type r will vanish on the weight space Ξ_s unless $s = r$; thus, effectively, the frame functionals of type r separate Ξ_r.

The frame functionals are not independent; relationships between them are called *syzygies*. There are two kinds of syzygy, of which one can be thought of as trivial; it just says that, if two adjacent vectors in a frame are varied in a fixed (2-dimensional) plane with a given orientation, then faces determined by the frames are encountered twice. That is

Lemma 3.2 *Let* (W', W'') *be a fixed frame in* V*, and let* L *be a plane in* V *with* $L \subseteq (W', W'')^\perp$*. Then*

$$\sum_{(u,v)\subseteq L} (f_{(W',u,v,W'')} - f_{(W',u,-v,W'')}) = 0,$$

where the sum extends over all frames (u, v) *in* L *of a given orientation.*

In fact, as is clear, the sum is really only over $u \in L$, since the orientation and $\langle u, v\rangle = 0$ determine the direction of v. The above sum is, despite appearances, finite, since its terms vanish on any particular polytope for all but finitely many u.

The non-trivial syzygies arise from Minkowski's theorem on facet areas. Let L, M be two subspaces of V of the same dimension, with corresponding volumes $\mathrm{vol}_L, \mathrm{vol}_M$, and let Φ_L denote orthogonal projection onto L. Volume is unique up to scaling (see [5]), and so there is a non-negative scalar $\vartheta(L, M)$, such that

$$\mathrm{vol}_L(\Phi_L P) = \vartheta(L, M)\mathrm{vol}_M P$$

for each $P \in \mathcal{P}(M)$. If now W is a fixed frame, and u, v are vectors orthogonal to W, write $L_u := (W, u)^\perp$ (and similarly L_v), and

$$\tau(W, u, v) := \mathrm{sign}\langle u, v\rangle\vartheta(L_u, L_v).$$

By considering the areas of the projections of a polytope in $\mathcal{P}(W^\perp)$ on L_u, we obtain the *Minkowski relations*

Lemma 3.3 *For each frame* W*, and each fixed* $u \in W^\perp$*,*

$$\sum_{v\in W^\perp} \tau(W, u, v) f_{(W,v)} = 0.$$

Again, the sum in Lemma 3.3 is not really infinite, since for a given polytope Q, only finitely many of the terms in the sum will not vanish on Q. Our first question is then whether the syzygies of Lemmas 3.2 and 3.3 are (essentially) the only ones between frame functionals.

When we employ the Minkowski relations, it is usually more convenient to think of them in a slightly different way. It is enough to work in V itself. Let P be a d-polytope in V, with n facets F_j with outer normal vectors u_j and corresponding $(n-1)$-volumes $\mathrm{vol}_{d-1} F_j := \alpha_j$ $(j = 1, \ldots, n)$. Let v be another vector, and suppose P translated to lie in the half-space $H^+(v, 0)$. Let $F'_j := \Phi_H F_j$ with $H := H(v, 0)$ $(= v^\perp)$, and let $\overline{F}_j := \mathrm{conv}(F_j \cup F'_j)$ $(j = 1, \ldots, n)$. Then we have

$$\mathrm{vol} P = \sum_{j=1}^{n} \mathrm{sign}\langle u_j, v\rangle \, \mathrm{vol}\, \overline{F}_j.$$

Now, if $E := \mathrm{conv}\{o, v\}$, then there is a constant $\gamma > 0$ such that, for each $(d-1)$-polytope $G \subseteq H$ and $\lambda \geq 0$,

$$\mathrm{vol}(G + \lambda E) = \lambda\gamma \mathrm{vol}_{d-1} G.$$

Thus, if we replace P by its translate $P + \lambda v$ in the above formula, we obtain

$$\begin{aligned} \mathrm{vol} P &= \sum_{j=1}^{n} \mathrm{sign}\langle u_j, v\rangle \, \mathrm{vol}(\overline{F}_j \cup (F'_j + \lambda E)) \\ &= \sum_{j=1}^{n} \mathrm{sign}\langle u_j, v\rangle \, (\mathrm{vol}\, \overline{F}_j + \lambda\gamma \mathrm{vol}_{d-1} F'_j) \\ &= \sum_{j=1}^{n} \mathrm{sign}\langle u_j, v\rangle \, (\mathrm{vol}\, \overline{F}_j + \lambda\gamma\vartheta(H, H_j) \mathrm{vol}_{d-1} F_j), \end{aligned}$$

where $H_j := u_j^\perp$. Subtracting the previous expression for vol$P$ from this yields the Minkowski relation for P. In other words, the Minkowski relation is equivalent to the translation invariance of volume in V. In §4 below, we shall appeal to this formulation to express the Minkowski relations in a yet different way.

Closely related to the frame functionals is a certain homomorphism on Π. We must first define the cone groups. Let L be a subspace of V, and let $\mathcal{C}(L)$ denote the family of all polyhedral cones in L with apex o. The *cone group* $\widehat{\Sigma}(L)$ is the abelian group with generators $\langle K\rangle$ for $K \in \mathcal{C}(L)$, which satisfy the relations (V) $\langle J \cup K\rangle + \langle J \cap K\rangle = \langle J\rangle + \langle K\rangle$ whenever $J, K, J \cup K \in \mathcal{C}(L)$ and (S) $\langle K\rangle = 0$ if $K \in \mathcal{C}(L)$ satisfies $\dim K < \dim L$. The group structure on $\widehat{\Sigma}(L)$ is thus somewhat exiguous.

The *full cone group* $\widehat{\Sigma}$ is defined to be $\widehat{\Sigma} := \bigoplus_L \widehat{\Sigma}(L)$, the direct sum extending over all subspaces L of V, including $\{o\}$ and V itself. We also write $\widehat{\Sigma}^k := \bigoplus_{\dim L = k} \widehat{\Sigma}(L)$, so that $\widehat{\Sigma} = \bigoplus_{k=0}^{d} \widehat{\Sigma}^k$.

With a non-empty face F of a polytope P is associated its *outer* (or *normal*) *cone* $N(F, P)$, which is the cone of all outer normal vectors to support hyperplanes of P which contain F (we also take $o \in N(F,P)$. Its class is then written $n(F,P) := \langle N(F,P) \rangle$, the class being taken intrinsically, that is, in $\widehat{\Sigma}(\text{lin}N(F,P))$.

Finally, the homomorphism $\sigma : \Pi \to \mathbf{F} \otimes \widehat{\Sigma}$ is defined on $\mathcal{P}$ by

$$\sigma(P) := \sum_F \text{vol}F \otimes n(F,P),$$

the sum extending over all non-empty faces F of P. Here, vol is that appropriate for F; it was observed in [9] that a scaling of volume could be chosen simultaneously for all subspaces of V. Since σ is a translation invariant valuation on $\mathcal{P}$, it induces a homomorphism, denoted by the same symbol, on Π.

The restriction of σ to Ξ_r is denoted by σ_r. Thus, on $\mathcal{P}$,

$$\sigma_r(P) = \sum_{\dim F = r} \text{vol}F \otimes n(F,P).$$

It may help to think of $\sigma_r(P)$ as an abstract version of the r-th intrinsic volume of P (see [6]).

Because the value of a frame functional on $x \in \Pi$ can be determined from $\sigma(x)$ (see below), there follows (see [10])

Theorem 3.4 *The homomorphism* $\sigma : \Pi \to \mathbf{F} \otimes \widehat{\Sigma}$ *is injective.*

The second question we shall address concerns the image of σ, and asks whether this is determined solely by the Minkowski relations. It was observed in [10] that the trivial syzygies are automatically satisfied on $\mathbf{F} \otimes \widehat{\Sigma}$.

In order to interpret the Minkowski relations on $\mathbf{F} \otimes \widehat{\Sigma}$, we must describe how we determine the values of the frame functionals on $x \in \Pi$ from $\sigma(x)$. Let r be fixed. We say that the $(d-r)$-cone K is *adapted* to the $(d-r)$-frame $W = (w_1, \ldots, w_{d-r})$ if $K \subseteq \text{lin}W$, and K has a chain of faces $\{o\} = K_0 \subseteq K_1 \subseteq \cdots \subseteq K_{d-r} = K$, with

$$w_j \in \text{relint}\,(K_j - K_{j-1})$$

for $j = 1, \ldots, d-r$. If $x \in \Xi_r$ is such that

$$\sigma_r(x) = \sum_K \mu_K \otimes \langle K \rangle,$$

where the sum extends over a finite set of $(d-r)$-cones K, then, for each $(d-r)$-frame W, we have

$$f_W(x) = \sum \{\mu_K \mid K \text{ is adapted to } W\}$$

(see [10] for details). The injectivity of the σ_r is thus obvious.

The Minkowski relations then concern those $(d-r)$-cones K which contain a common $(d-r-1)$-cone; we shall not write down any details explicitly, since we shall later describe what is happening in yet a third way.

4. Weights

In this section, we shall rework much of §5 of [11]. As we have already indicated, being unable to take square roots in a general ordered field $\mathbf{F}$ prevents our having natural scalings for volumes in subspaces. This means that we cannot avoid various scaling factors appearing in our formulæ, with the attendant complications in expressing them. However, a compensating advantage is that we can use this section to justify the claim in §15 of [11] that the results of that paper remain valid over a general ordered field.

We shall approach our problems by means of certain finite dimensional subalgebras of Π. For the moment, our definitions will be completely general. Denote by $\mathcal{F}_r(P)$ the family of r-faces of a polytope P. Then an *r-weight* on P is a mapping $w : \mathcal{F}_r(P) \to \mathbf{F}$ which satisfies the Minkowski relations relative to each $(r+1)$-face G of P (we shall be more explicit about these below). The vector space of r-weights on P is denoted by $\Omega_r(P)$.

We shall write $\Pi(P)$ for the subalgebra of Π generated by the classes $[Q]$ of the *summands* Q of P, which are such that $P = Q + Q'$ for some polytope Q'. In fact, $\Pi(P)$ is generated by the classes of polytopes which are *strongly (combinatorially) isomorphic* to P, in the sense that parallel support hyperplanes determine faces of the same dimension; such polytopes are clearly isomorphic to P in the usual sense. Indeed, an even stronger result holds.

Lemma 4.1 *The subalgebra $\Pi(P)$ is generated by the classes of polytopes in a neighbourhood of any polytope in the strong isomorphism class of P.*

We shall largely confine our attention in this section to *simple* d-polytopes P, by which we mean as usual that each vertex of P belongs to exactly d facets of P. We shall assume that P has n facets $F_1, \ldots, F_n$, with corresponding outer normal vectors $u_1, \ldots, u_n$. Note that P can then be expressed in the form

$$P = \{x \in V \mid \langle x, u_j \rangle \leq \eta_j \ (j = 1, \ldots, n)\};$$

when we think of the normal vectors u_j as fixed, we shall call the numbers η_j the *support parameters* of P. It is important to note how the support parameters depend on the u_j; if we multiply u_j by $\lambda_j > 0$, then we multiply the corresponding η_j by λ_j also. If $U := (u_1, \ldots, u_n)$ (it is usual to take U to be an ordered set), we write $\mathcal{P}(U)$ for the family of all polytopes which are representable in the form above; such a polytope need not in general have n facets, or even be full-dimensional.

We need to begin with a remark. If F is an r-face of the simple d-polytope P, then there is a $(d-r)$-frame W such that $F = P_W$. If $x \in \Pi(P)$, we then write $x|_F := x_W$; this reflects the fact that x_W depends on F, rather than on the particular frame W employed. Since the face of a summand of P corresponding to F is a summand of F, it follows that $x|_F \in \Pi(F)$. We call $x \mapsto x|_F$ the *face map* of $\Pi(P)$ to $\Pi(F)$. It should be noted that we do not assert that every polytope strongly isomorphic to F will necessarily occur as the corresponding face of a polytope strongly isomorphic to P. In view of the freedom to apply arbitrary (sufficiently small) displacements to the facets of a simple polytope, we deduce from Lemma 4.1.

Theorem 4.2 *Let P be a simple polytope, and let F be a non-empty face of P. Then the mapping $x \mapsto x|_F$ maps $\Pi(P)$ onto $\Pi(F)$.*

We write $\Xi_r(P) := \Xi_r \cap \Pi(P)$. While the arguments used in [11] do not depend on working over the real field (the representation theory of [7,8] works equally well over any ordered field), we need for other purposes a result from which we can calculate $\dim \Xi_1(P)$. In view of the isomorphism $\Xi_1 \cong \mathcal{P}_T$ of Lemma 2.2, which clearly implies $\Xi_1(P) \cong \mathcal{P}_T(U)$ with the obvious meaning of the latter notation, we can identify an element of $\Xi_1(P)$ with a vector $y = (\eta_1 \ldots, \eta_n)$ of *parameters*, which are differences of support parameters of polytopes in $\mathcal{P}(U)$. (We thus have a linear extension of the cone of support parameter vectors; observe that different simple polytopes in $\mathcal{P}(U)$ will have different such linear extensions.) Because we identify by translations, we have

Lemma 4.3 *Two parameter vectors $(\eta_1, \ldots, \eta_n)$ and $(\zeta_1, \ldots, \zeta_n)$ represent the same element of $\Xi_1(P)$ if and only if $\zeta_j = \eta_j + \langle t, u_j \rangle$ $(j = 1, \ldots, n)$ for some translation vector $t \in V$.*

It follows that $\dim \Xi_1(P) = n - d$, and from that we easily deduce that $\Pi(P)$ is finite-dimensional.

The separation Theorem 3.1 associates each element $x \in \Xi_r(P)$ with a unique r-weight, so that there is a natural embedding $\Xi_r(P) \hookrightarrow \Omega_r(P)$. For this reason, we extend the previous notation, and write $w|_F$ for the restriction of a weight w to a face F. (If w is an r-weight, and F is an r-face, then $w|_F$ is the value of w on F.) We shall prove in this section that, for a simple polytope P, this embedding is an isomorphism. Observe that this will not remain true for a general polytope. For example, if P is a simplicial d-polytope with n facets, then $\Xi_r(P)$ is 1-dimensional for each $r = 0, \ldots, d$; however, we clearly have $\dim\, \Omega_{d-1}(P) = n - d > 1$ (except when P is a simplex).

In [11], we described how multiplication of elements of Π can be extended to multiplication of weights. However, as there, we actually need rather less, and so we shall confine our attention to what is necessary for our purposes.

Throughout the rest of this section, P will be a simple d-polytope in V. The main result is then:

Theorem 4.4 *For each $r = 0, \ldots, d$, the embedding of $\Xi_r(P)$ in $\Omega_r(P)$ is an isomorphism.*

As in [11], we shall prove this result by establishing

Theorem 4.5 *For each $r = 0, \ldots, d$, the weight spaces $\Xi_r(P)$ and $\Xi_{d-r}(P)$ are in duality under the multiplication on $\Pi(P)$.*

Of course, this implies that these spaces have the same dimension.

Our tool is a result which describes how to multiply by elements of $\Xi_1(P)$.

Lemma 4.6 *Multiplying an element of $\Omega_r(P)$ by one of $\Xi_1(P)$ yields an element of $\Omega_{r+1}(P)$.*

We begin with the case $r = d - 1$. We have freedom to choose to scale the normal vectors in U, or the weights on the corresponding facets; in fact, we shall do neither. Instead, we take these scalings as given, so that, for each j, we have a constant $\lambda_j > 0$, such that if $a \in \Omega_{d-1}(P)$ has weight α_j on F_j, and $y \in \Xi_1(P)$ has parameter vector $(\eta_1, \dots, \eta_n)$, then

$$ya = \sum_{j=1}^{n} \lambda_j \eta_j \alpha_j.$$

This formula generalizes the usual (asymmetric) one for mixed volumes, apart from the omission of the constant factor $1/d$ (this is absent, because we are actually evaluating a product); we shall call it the *mixed volume calculation*. Except that α_j is a $(d-1)$-weight, rather than a $(d-1)$-volume, this is just the familiar "volume = base × height".

More generally, the product of y by an element of $\Omega_r(P)$ is calculated analogously, except that the α_j are replaced by weights on the r-faces of an $(r+1)$-face, the η_j by the parameters of y induced on that $(r+1)$-face, and the λ_j by suitable scaling factors. The problem is to show how these latter scaling factors are related. For this, it is enough to consider the case $r = d - 2$. Let $G_{jk} := F_j \cap F_k$ be a non-empty $(d-2)$-face of P. One normal vector to F_j at G_{jk} is the projection of u_k on the orthogonal complement $u_j^{\perp}$ of u_j, namely

$$v_{jk} := u_k - \frac{\langle u_k, u_j \rangle}{\langle u_j, u_j \rangle} u_j.$$

If the induced parameter of $z \in \Xi_1(P)$ relative to v_{jk} in F_j is ζ_{jk}, and if $b \in \Omega_{d-2}(P)$ has weight β_{jk} on G_{jk}, then the scaling factors $\mu_{jk} > 0$ are such that

$$(zb)|_{F_j} = \sum_{k}^{*} \mu_{jk} \zeta_{jk} \beta_{jk},$$

where such a sum runs over those k for which G_{jk} is non-empty.

We can now state the relationship between the scaling factors.

Lemma 4.7 *If $j \neq k$ and $G_{jk} := F_j \cap F_k \neq \emptyset$, then $\lambda_j \mu_{jk} = \lambda_k \mu_{kj}$.*

We note that the usual calculation of the 0-volume of a point is 1, even though we allow a point to carry any element of $\mathbf{F}$ as a weight (subject to the Minkowski relations, which demand that each vertex of P carries the same weight).

We shall prove the lemma by, in a sense, choosing suitable parameters to correspond to the normal vectors themselves. We thus define the following line segments:

$$U_j := \operatorname{conv}\{o, u_j\},$$
$$V_{jk} := \operatorname{conv}\{o, v_{jk}\},$$

for each j, k with (in the latter case) $G_{jk} \neq \emptyset$. We can then think of the term $\lambda_j \eta_j \alpha_j$ in the product calculation as arising from the prism with upright

$$\frac{\eta_j}{\langle u_j, u_j \rangle} U_j$$

and base area α_j in the (linear) hyperplane $H_j := u_j^\perp$ with normal u_j. Let us choose y to correspond to U_j and z to U_k (in each case, the remaining parameters are set equal to 0). Then we have

$$\begin{aligned}\eta_j &:= \langle u_j, u_j\rangle,\\ \zeta_{jk} &:= \langle v_{jk}, v_{jk}\rangle\\ &= \langle u_k, u_k\rangle - \frac{\langle u_j, u_k\rangle^2}{\langle u_j, u_j\rangle}.\end{aligned}$$

We next observe that, if B is a polytope in the linear $(d-2)$-space $H_j \cap H_k = \{u_j, u_k\}^\perp$ parallel to G_{jk}, then

$$\begin{aligned}\mathrm{vol}(U_j + V_{jk} + B) &= \mathrm{vol}(U_j + U_k + B)\\ &= \mathrm{vol}(U_k + V_{kj} + B)\end{aligned}$$

(this uses an elementary dissection argument, which does not depend on any scaling). Now let $b \in \Omega_{d-2}(P)$ have weight β_{jk} on G_{jk}. There is no harm in supposing that $\beta_{jk} > 0$ (change the sign of b if necessary, since $\beta_{jk} = 0$ is of no interest). We now think of replacing the weight β_{jk} by a polytope B as above, with $\mathrm{vol}_{d-2}(B) = \beta_{jk}$. Comparing the volume and product calculations, noting that the latter consists of a single term, and substituting from above, we have

$$\begin{aligned}\mathrm{vol}(U_j + V_{jk} + B) &= yzb\\ &= \lambda_j\eta_j.\mu_{jk}\zeta_{jk}\beta_{jk}\\ &= \lambda_j\mu_{jk}\beta_{jk}\langle u_j, u_j\rangle\left(\langle u_k, u_k\rangle - \frac{\langle u_j, u_k\rangle^2}{\langle u_j, u_j\rangle}\right)\\ &= \lambda_j\mu_{jk}\beta_{jk}\left(\langle u_j, u_j\rangle\langle u_k, u_k\rangle - \langle u_j, u_k\rangle^2\right).\end{aligned}$$

Performing the calculation in the opposite order, as

$$\mathrm{vol}(U_k + V_{kj} + B) = zyb,$$

equating the two expressions, and using the fact that all the factors are positive (with $\beta_{jk} = \beta_{kj}$), yields the result. □

We now resume the proof of Lemma 4.6. As we noted in §3, the Minkowski relations on $\Omega_r(P)$ reflect the translation invariance of volumes, and this ensures that multiplication by an element of $\Xi_1(P)$ (using the mixed volume calculation) is well-defined. Thus, in case $r = d-1$, since by Lemma 4.3 we can replace the parameter η_j of $y \in \Xi_1(P)$ by $\eta_j + \langle t, u_j\rangle$ $(j = 1, \ldots, n)$ for any $t \in V$, we deduce that we can write the Minkowski relation for $a \in \Omega_{d-1}(P)$ as the vector formula

$$\sum_{j=1}^{n} \lambda_j\alpha_j u_j = o.$$

The notation (here and in what follows) is that used previously.

In order to check that the Minkowski relations hold on products, it suffices to consider the case $r = d-2$, that is that $yb \in \Omega_{d-1}(P)$ for each $y \in \Xi_1(P)$ and $b \in \Omega_{d-2}(P)$. The Minkowski relations on F_j say that

$$\sum_k^* \mu_{jk}\beta_{jk}v_{jk} = 0.$$

The induced parameters of y in F_j are

$$\zeta_{jk} = \eta_k - \frac{\langle u_k, u_j\rangle}{\langle u_j, u_j\rangle}\eta_j.$$

Thus the weight of $a := yb$ on F_j is

$$\begin{aligned}\alpha_j &= \sum_k^* \mu_{jk}\zeta_{jk}\beta_{jk} \\ &= \sum_k^* \mu_{jk}\beta_{jk}\left(\eta_k - \frac{\langle u_k, u_j\rangle}{\langle u_j, u_j\rangle}\eta_j\right).\end{aligned}$$

It follows that

$$\begin{aligned}\sum_{j=1}^n \lambda_j\alpha_j u_j &= \sum_{j=1}^n \lambda_j\alpha_j\left(\sum_k^* \mu_{jk}\beta_{jk}\left(\eta_k - \frac{\langle u_k, u_j\rangle}{\langle u_j, u_j\rangle}\eta_j\right)\right)u_j \\ &= \sum_{j=1}^n \lambda_j\alpha_j\eta_j\left(\sum_k^* \mu_{jk}\beta_{jk}\left(u_k - \frac{\langle u_k, u_j\rangle}{\langle u_j, u_j\rangle}u_j\right)\right) \\ &= \sum_{j=1}^n \lambda_j\alpha_j\eta_j\left(\sum_k^* \mu_{jk}\beta_{jk}v_{jk}\right) \\ &= o,\end{aligned}$$

as required. Note that the change of summation is justified because $\lambda_j\mu_{jk} = \lambda_k\mu_{kj}$ by Lemma 4.7.

If we now replace P in this argument by a typical $(r+1)$-face, we see that we have proved Lemma 4.6. □

The core of Theorem 4.5 (and hence Theorem 4.4) is the analogue of Theorem 11 of [10] (or, rather, of its consequence), namely

Lemma 4.8 *Let $0 \le r \le d$, and let $x \in \Omega_r(P)$ be such that $x \ne 0$. Then there exists $y \in \Xi_{d-r}(P)$ such that $xy \ne 0$.*

The extreme case $r = d$ is trivial, since $\Xi_0(P) \cong \mathbf{F}$. The crucial case is $r = d-1$; we shall prove that first. If $a \in \Omega_{d-1}(P)$ with $a \ne 0$, then then there is some facet F_j of P for which the restriction $\alpha_j := a|_{F_j}$ of a to F does not vanish. Let $e_j \in \Xi_1(P)$ be the element with parameters satisfying $\eta_j = 1$ and $\eta_k = 0$ if $k \neq j$; such a choice is valid by Lemma 4.3. We saw above that $e_j a = \lambda_j \alpha_j$, and thus $e_j a \neq 0$ as required.

Now suppose that $r < d-1$; we proceed by induction on d. Since $x \neq 0$, we can find some r-face G of P on which the weight $x|_G$ induced by x does not vanish. Let F be any facet of P which contains G; then $x|_F \neq 0$. By the inductive assumption, and using Theorem 4.2 (which says that the face map $x \mapsto x|_F$ is onto), we can find a $z \in \Xi_{d-r-1}(P)$, such that $(xz)|_F = x|_F z|_F \neq 0$. If $F = F_j$, then with e_j as above and $y = ze_j$, we have $xy = xze_j \neq 0$, which completes the proof. □

Lemma 4.8 says that $\Xi_{d-r}(P)$ separates $\Omega_r(P)$. There follows at once

$$\dim \Xi_r(P) \leq \dim \Omega_r(P) \leq \dim \Xi_{d-r}(P).$$

Interchanging the rôles of r and $d-r$ shows that we have equality throughout, and the two theorems are immediate consequences. □

As we have noted, Theorem 4.5 shows that $\Xi_r(P)$ and $\Xi_{d-r}(P)$ have the same dimension. We can use the method of [11] to find these dimensions, which have considerable combinatorial interest, but we shall not pursue that line of enquiry here. The argument of the first part of Lemma 4.8 can be extended to yield

Theorem 4.9 *Let G be an r-face of P. If $F_1, \ldots, F_{d-r}$ are the facets of P which contain G, and if $e_1, \ldots, e_{d-r}$ are the corresponding elements of $\Xi_1(P)$, then there is a positive constant ν_G such that*

$$x|_G = \nu_G e_1 \ldots e_{d-r} x,$$

for each $x \in \Xi_r(P)$.

Thus, when we work in $\Pi(P)$ (for a fixed simple d-polytope P), we can mimic the effect of frame functionals by multiplication within $\Pi(P)$.

5. Syzygies and the image of σ

We shall next show that the only non-trivial syzygies between the frame functionals are those induced by the Minkowski relations. This will also yield a description of the image of the mapping $\sigma : \Pi \to \mathbf{F} \otimes \widehat{\Sigma}$. In fact, the two results are effectively proved together.

It is reasonable to talk about an element of $\mathbf{F} \otimes \widehat{\Sigma}^{d-r}$ satisfying the Minkowski relations, in view of the way that the corresponding values taken by the frame functionals of type r can be calculated using the same formula as in §3. That is, if $x := \sum_K \mu_K \otimes \langle K \rangle \in \mathbf{F} \otimes \widehat{\Sigma}^{d-r}$ with the sum extending over finitely many $(d-r)$-cones K, and if W is a $(d-r)$-frame, then we define

$$f_W(x) = \sum \{\mu_K \mid K \text{ is adapted to } W\}.$$

An element of $\mathbf{F} \otimes \widehat{\Sigma}^{d-r}$ which satisfies the Minkowski relations between frame functionals of type r is called an *r-weight*; we write Ω_r for the set of all r-weights.

Our key result can now be stated as

Theorem 5.1 *The image of* $\sigma_r : \Xi_r \to \mathbf{F} \otimes \widehat{\Sigma}^{d-r}$ *is* Ω_r.

To prove this, let x as above be an r-weight. We can suppose, by subdividing them if necessary, that the cones K which occur in the expression for x are pointed (have o as their single apex). They have between them only finitely many edges, whose directions thus form a finite set U_1. By adding in (finitely many) more directions if necessary, we can suppose that U_1 spans V positively. There are only finitely many different isomorphism classes of polytopes in $\mathcal{P}(U_1)$ (the combinatorial type of a polytope is determined by which of its vertices are contained in which facets). Indeed, each polytope in $\mathcal{P}(U_1)$ is easily seen to be a limit of simple d-polytopes in $\mathcal{P}(U_1)$, and thus in fact a summand of such a polytope. Now let Q be the sum of one representative of each of the strong isomorphism classes of simple d-polytopes in $\mathcal{P}(U_1)$, and finally let P be a simple polytope of which Q is a summand.

We claim that (in an obvious sense) $x \in \Omega_r(P)$. Indeed, suitable refinements of the cones K are clearly unions of normal cone $N(F, P)$ to r-faces F of P, since such unions contain all possible $(d-r)$-cones with edges in directions in U_1. The Minkowski relations are preserved by such refinements, and hence x is an r-weight on P. But Theorem 4.4 says that $\Omega_r(P) = \sigma_r(\Xi_r(P))$. The theorem is an immediate consequence. □.

Since any syzygies between frame functionals which are not consequences of the Minkowski relations would impose further restrictions on $rmim\,\sigma$, we conclude that no such syzygies can exist.

6. Translation invariant valuations

Let G be an abelian group. A mapping $\varphi : \mathcal{P} \to G$ is called a *valuation* if

$$\varphi(P \cup Q) + \varphi(P \cap Q) = \varphi(P) + \varphi(Q),$$

whenever P, $Q \in \mathcal{P}$ are such that $P \cup Q \in \mathcal{P}$ also. We say that φ is *translation invariant* if

$$\varphi(P + t) = \varphi(P),$$

whenever $P \in \mathcal{P}$ and $t \in V$. There follows at once from the definition of Π:

Theorem 6.1 *A translation invariant valuation on* $\mathcal{P}$ *induces a group homomorphism on* Π, *and conversely.*

However, in view of the structure Theorem 2.1 for Π, homomorphisms of vector spaces are more appropriate objects of study. The corresponding condition on a valuation φ is that it be *dilatation continuous*, meaning that, when $P \in \mathcal{P}$ is fixed, the mapping $\lambda \mapsto \varphi(\lambda P)$ is continuous for $\lambda \geq 0$. As pointed out in [9,12], this is actually equivalent to the formally stronger condition of being *weakly continuous*, which says that, for each set U of normal vectors, the mapping $P(y) \mapsto \varphi(P(y))$ on $\mathcal{P}(U)$ is continuous in the parameter vector y (see also [4]). We thus have

Theorem 6.2 *A dilatation continuous translation invariant valuation on $\mathcal{P}$ induces a vector space homomorphism on Π, and conversely.*

In what follows, in talking about a dilatation continuous translation invariant valuation φ on $\mathcal{P}$, we shall equate it with the vector space homomorphism on Π which it induces. For $r = 0, \ldots, d$, we say that φ is *homogeneous of degree* r if $\varphi(\lambda P) = \lambda^r \varphi(P)$ for $P \in \mathcal{P}$ and $\lambda \geq 0$, which means that φ has domain Ξ_r.

The most interesting case is $\varphi : \Xi_r \to \mathbf{F}$, to which we henceforth confine our attention. We recall that the *mixed volume* (compare [2,3]) of polytopes $P_1, \ldots, P_d \in \mathcal{P}$ admits the expression

$$V(P_1, \ldots, P_d) = \frac{1}{d!} p_1 \ldots p_d,$$

where $p_i := \log P_i$ for $i = 1, \ldots, d$ (see [10]). We adopt the abbreviated notation

$$V_r(P, Q) := V(\underbrace{P, \ldots, P}_{r}, \underbrace{Q, \ldots, Q}_{d-r}).$$

Then we have the following important result.

Theorem 6.3 *Let $\varphi : \mathcal{P} \to \mathbf{F}$ be a dilatation continuous translation invariant valuation which is homogeneous of degree r. If $P_1, \ldots, P_m$ are any finite number of polytopes, then there are polytopes Q_k and signs $\varepsilon_k = \pm 1$ $(k = 1, \ldots, s)$, such that*

$$\varphi(P_i) = \sum_{k=1}^{s} \varepsilon_k V_r(P_i, Q_k)$$

for each $i = 1, \ldots, m$.

As we saw in §5, there exists some simple d-polytope P such that each P_i is a summand of P. Then we can think of φ, or, rather, $d!\varphi$ (because of the scaling factor in the definition of mixed volumes) as a linear functional on $\Xi_r(P)$; in particular, we can write

$$\varphi(Q) = \varphi(q^r),$$

for each summand Q of P, where $q := \log Q$. But Theorem 4.5 says that the dual of $\Xi_r(P)$ is $\Xi_{d-r}(P)$. Further, $\Xi_{d-r}(P)$ is generated, as an abelian group, by the elements of the form q^{d-r}, with $q := \log Q$ for some summand Q of P. It follows that there are $q_k := \log Q_k$ (with Q_k a summand of P) and $\varepsilon_k = \pm 1$ for $k = 1, \ldots, s$, such that for each $x \in \Xi_r(P)$,

$$d!\varphi(x) = \sum_{k=1}^{s} \varepsilon_k x q_k^{d-r}.$$

With $p_i := \log P_i$ for $i = 1, \ldots, m$, this implies

$$d!\varphi(P_i) = \sum_{k=1}^{s} \varepsilon_k p_i^r q_k^{d-r} = d! \sum_{k=1}^{s} \varepsilon_k V_r(P_i, Q_k),$$

as required. □

Let us consider the special case $\mathbf{F} = \mathbf{R}$, with $V = \mathbf{E}^d$ the euclidean space. In $\mathbf{E}^d$, the usual concept of continuity on, for example, polytopes, is with respect to the *Hausdorff distance* $\rho(K_1, K_2)$ between non-empty compact sets K_1 and K_2, defined by

$$\rho(K_1, K_2) := \min\{\rho \geq 0 \mid K_1 \subseteq K_2 + \rho B,\ K_2 \subseteq K_1 + \rho B\},$$

where $B = \{x \in \mathbf{E}^d \mid \|x\| \leq 1\}$ is the unit ball. Continuity clearly implies weak continuity. Let φ be a translation invariant valuation on the space of *convex bodies* (that is, non-empty compact convex sets) which is homogeneous of degree r and continuous with respect to the Hausdorff metric. If $(P_1, P_2, \ldots)$ is an infinite sequence of polytopes in $\mathbf{E}^d$ whose summands form a dense subset of the space of all convex bodies, then we can deduce from the above that, for each m, there are polytopes Q_{mk} and signs ε_{mk} $(k = 1, \ldots, s(m))$, such that

$$\varphi(P) = \sum_{k=1}^{s(m)} \varepsilon_{mk} V_r(P, Q_{mk}),$$

for $P = P_1, \ldots, P_m$. Since the $\varphi(P_i)$ completely determine φ, because of the density properties of the sequence, there follows

Theorem 6.4 *A continuous translation invariant valuation on convex bodies in* $\mathbf{E}^d$ *is, in some sense, a limit of a sum or difference of mixed volumes.*

The real problem here is the sense in which the limit is to be defined. This point proved to be a stumbling block in the different, but analogous, approach to a characterization of such valuations in [1].

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