Peter McMullen and Rolf Schneider

Valuations on convex bodies

The investigation of functions on convex bodies which are valuations, or additive in Hadwiger's sense, has always been of interest in particular parts of geometric convexity, and it has seen some progress in recent years. The occurrence of valuations in the theory of convex bodies can be traced back to the notion of volume in two essentially different ways. Firstly, the volume of convex bodies, being the restriction of a measure, is itself a valuation. This valuation property carries over to the functions which are deduced from volume in the Brunn-Minkowski theory, namely to mixed volumes, quermassintegrals, surface area functions, and others. Hadwiger's celebrated characterizations of the quermassintegrals by the valuation and other properties were the culmination of a series of papers on valuations and at the same time the starting point for various subsequent investigations of functionals with similar properties.

A different way from volume to more general valuations was opened by Hilbert's third problem and the solution given to it by Dehn. Motivated by the problem whether the notion of volume for three-dimensional polytopes can be introduced, in analogy to the plane case, by elementary dissection and congruence arguments, without the use of limit processes, Hilbert asked whether two three-dimensional polytopes of equal volume are necessarily equivalent by dissection. Dehn's negative answer was essentially achieved by constructing special valuations which must attain the same value on equidissectable polytopes, and by exhibiting pairs of convex polytopes with equal volume but different values of these functionals. Dehn's set of necessary conditions for equidissectability was proved to be also sufficient only many years later, and in the course of the investigation centring around this and related questions, much information on valuations was gained. Thus the dissection theory of polytopes (which still has to offer some deep open problems) is intimately tied up with valuation theory, and every dissection result has implications on valuations.

A third range for applications of valuations in convexity is seen in questions of combinatorial geometry, where the Euler characteristic on unions of convex bodies is a useful device. The Euler characteristic also plays a role in certain extension procedures for quermassintegrals and other functionals to non-convex sets.

Still another class of valuations arises from the counting of lattice points in convex bodies.

The following survey collects and describes the various examples of valuations on convex bodies that have been treated in the literature, and it presents the known results, mostly without proofs. The emphasis is, first, on the interrelations between simple valuations and dissections, which requires a fairly far-going description of the algebraic arguments on which the progress in equidissectability relies, and second, on characterization theorems for special valuations of geometric interest. Some open problems will also be mentioned at appropriate places.

The section headings are as follows.

§1. Preliminaries

By a *valuation*, or an *additive* functional, on a class $\mathscr{S}$ of sets we understand a function φ on $\mathscr{S}$ satisfying

$$(1.1)\qquad \varphi(K \cup L) + \varphi(K \cap L) = \varphi(K) + \varphi(L)$$

whenever $K, L, K \cup L$ and $K \cap L$ are elements of $\mathscr{S}$. Here we assume that φ takes its values in an abelian group, and we always suppose that $\varphi(\emptyset) = 0$. Often the class $\mathscr{S}$ will be *intersectional*, which means that $K, L \in \mathscr{S}$ implies $K \cap L \in \mathscr{S}$. If $\mathscr{S}$ is an intersectional class, we let $U(\mathscr{S})$ denote the lattice consisting of all finite unions of elements of $\mathscr{S}$.

For the classes $\mathscr{S}$ occurring most frequently in the following, we introduce special notation. Let $\mathbb{E}^d$ be d-dimensional euclidean vector space, with scalar product $\langle \cdot,\cdot \rangle$ and norm $\|\cdot\|$, and let $\Omega = \Omega^{d-1} := \{x \in \mathbb{E}^d : \|x\| = 1\}$ be its unit sphere. By $\mathscr{K}^d$ we denote the class of compact convex subsets of $\mathbb{E}^d$. The elements of $\mathscr{K}^d$ will be called *convex bodies*, which differs slightly from common usage (in particular, the empty set $\emptyset$ is a convex body, which is convenient when valuations are considered). On $\mathscr{K}^d \setminus \{\emptyset\}$ we have the vector or Minkowski addition $+$, defined by

$$K + L := \{x + y : x \in K, y \in L\},$$

and the usual Hausdorff metric ρ, defined by

$$\rho(K,L) := \min\{\rho \geq 0 : K \subseteq L + \rho B,\quad L \subseteq K + \rho B\},$$

with B the unit ball.

The important class $U(\mathscr{K}^d)$ is Hadwiger's *convex-ring*. By $\mathscr{P}^d \subset \mathscr{K}^d$ we denote the class of convex polytopes, and the elements of $U(\mathscr{P}^d)$ are the *polyhedra*. We write $\mathscr{K}^d_d \subset \mathscr{K}^d$ for the subset of d-dimensional convex bodies, and the elements of $\mathscr{P}^d_d := \mathscr{K}^d_d \cap \mathscr{P}^d$ will also be called d-polytopes.

Sometimes it will be convenient to consider also relatively open convex bodies. By a relatively open convex body we understand the relative interior, relint (i.e., the interior with respect to the affine hull) of a convex body. Let $\mathscr{K}^d_{ro}$ denote the set of all relatively open convex bodies in $\mathbb{E}^d$ and $\mathscr{P}^d_{ro}$ the subset of relatively open polytopes, and observe that $\mathscr{P}^d \subset U(\mathscr{P}^d_{ro})$.

For other types of sets to be considered we shall introduce special notation when it seems appropriate. A polytope is called *rational* if its vertices have rational coordinates (with respect to the standard basis of $\mathbb{E}^d$), and it is a *lattice polytope* if its vertices belong to the *integer lattice* $\mathbb{Z}^d$ consisting of all points in $\mathbb{E}^d$ with integer coordinates. A *polyhedral cone* with apex 0 is the intersection of finitely many closed halfspaces each having 0 in its boundary. The intersection of such a cone with the unit sphere Ω^{d-1} is called a *spherical polytope*. Some time we will also mention polytopes in hyperbolic spaces.

If φ is a valuation on $\mathscr{S}$ and $\mathscr{S}$ is a lattice, that is, closed under finite unions and finite intersections, then (1.1) and an easy induction argument yield

(1.2) $$\varphi(K_1 \cup \cdots \cup K_m) = \sum_{r=1}^{m} (-1)^{r-1} \sum_{i_1 < \cdots < i_r} \varphi(K_{i_1} \cap \cdots \cap K_{i_r})$$

for $K_1,\ldots,K_m \in \mathscr{S}$. In general, the function φ defined on an arbitrary class $\mathscr{S}$ is said to satisfy the *inclusion-exclusion principle* if (1.2) holds whenever $K_1,\ldots,K_m, K_1 \cup \cdots \cup K_m, K_{i_1} \cap \cdots \cap K_{i_r} \in \mathscr{S}$. Clearly any valuation on $\mathscr{S}$ which can be extended, as a valuation, to the lattice generated by $\mathscr{S}$, satisfies the inclusion-exclusion principle. We shall consider such extensions in §5.

In the former literature, in particular in the work of Hadwiger (see [1957]), valuations are usually called *additive functionals*. This should not be confused with the notion of Minkowski additivity. A function φ on $\mathscr{K}^d$ or $\mathscr{P}^d$ (with values in an abelian group) is called *Minkowski additive* if $\varphi(\emptyset) = 0$ and

$$\varphi(K + L) = \varphi(K) + \varphi(L) \quad \text{for} \quad \emptyset \neq K, L \in \mathscr{K}^d \text{ resp. } \mathscr{P}^d.$$

Every Minkowski additive function is also a valuation, since

(1.3) $$(K \cup L) + (K \cap L) = K + L$$

if K,L and $K \cup L$ are non-empty convex bodies. This fundamental relation, which appears surprisingly late in the literature (apparently not before Sallee [1966], p. 77; see also Hadwiger [1971]), can also be interpreted as saying that the identical mapping of $\mathscr{K}^d$ into itself is a valuation. (Here we admit a commutative semigroup with cancellation law, namely $\mathscr{K}^d$ with Minkowski addition, as the range of a valuation. This is not an essential difference, since any such semi-group can be embedded in an abelian group.) Since the mapping $\varphi: K \mapsto h(K,\cdot)$, where

$$h(K,u) := \max\{\langle x,u\rangle : x \in K\} \quad \text{for} \quad u \in \Omega^{d-1}$$

defines the *support function* of $K \neq \emptyset$ (restricted to Ω^{d-1}), is Minkowski additive, it is also a valuation, with values in the space of real continuous functions on Ω^{d-1}.

Let C be a fixed convex body. If $K,L \in \mathscr{K}^d$, then

$$(K \cup L) + C = (K + C) \cup (L + C),$$

and if $K \cup L$ is convex, then also

$$(K \cap L) + C = (K + C) \cap (L + C)$$

(see Hadwiger [1957], p. 144). Hence, if φ is a valuation on $\mathscr{K}^d$, then the functio
φ_C defined by

$$\varphi_C(K) := \varphi(K + C) \quad \text{for} \quad K \in \mathscr{K}^d$$

is also a valuation on $\mathscr{K}^d$. Thus the interplay between convexity and Minkowsk
addition yields new valuations from old ones, a remark which will be c
importance in §§3 and 10.

The above passage from φ to φ_C is an example of the following obvious resul

(1.4) **Lemma.** *Let φ be a valuation on $\mathscr{K}^d$, and let $f: \mathscr{K}^d \to \mathscr{K}^d$ be a map whic
satisfies $f(K \cup L) = f(K) \cup f(L)$ and $f(K \cap L) = f(K) \cap f(L)$ if K,L,K ∪ L
$\in \mathscr{K}^d$. Then $\varphi \circ f$ is a valuation.*

In the above example, $f(K) = K + C$. Another example is given by f(K) =
$K \cap C$, where C is a fixed closed convex set. A third one is given by f(K) = α(K
where $\alpha: \mathbb{E}^d \to \mathbb{E}^d$ is an affine map.

The following notion is useful in the investigation of valuations on polytope
For a hyperplane $H \subset \mathbb{E}^d$ let H^+ and H^- be the two closed halfspaces bounded b
H. A function φ on $\mathscr{P}^d$ or $\mathscr{K}^d$ is called a *weak valuation* if $\varphi(\emptyset) = 0$ and

$$\varphi(K) + \varphi(K \cap H) = \varphi(K \cap H^+) + \varphi(K \cap H^-)$$

for every hyperplane H and every K in the domain of φ. Sallee [1968] show
(among related and more general results) that every weak valuation on $\mathscr{P}^d$ is
valuation; see also Groemer [1978] for the case where φ takes its values in a re
vector space. The following example (due to Groemer, private communicatio
shows that a weak valuation on $\mathscr{K}^d$ need not be a valuation. For $K \in \mathscr{K}^2$, defi
$\varphi(K) = 1$ if 0 (the origin of $\mathbb{E}^2$) lies in the boundary of K and is a one-sided, bu
not a two-sided, limit of singular points of K, let $\varphi(K) = 2$ if 0 is a two-sided lim
of singular points of K, and $\varphi(K) = 0$ otherwise. Clearly φ is a weak valuation o
$\mathscr{K}^2$. But it is not a valuation, since one easily finds $K_1,K_2 \in \mathscr{K}^2$ with $\varphi(K_1)$ =
$\varphi(K_2) = \varphi(K_1 \cap K_2) = 1$ and $\varphi(K_1 \cup K_2) = 0$. We do not know an example of
rigid motion invariant weak valuation which is not a valuation.

A different view on valuations is often useful. It is motivated by the well-know
procedure of integration theory which transposes additivity (of set function
into linearity (on vector spaces). Let $\mathscr{S}$ be a class of subsets of some set S. Th
characteristic function of an element $K \in \mathscr{S}$ will be denoted by K^*, that is,

$$K^*(x) = \begin{cases} 1 & \text{if} \quad x \in K \\ 0 & \text{if} \quad x \in S \setminus K. \end{cases}$$

By $V(\mathscr{S})$ we denote the real vector space which is generated by the functio

$K^*, K \in \mathscr{S}$. If $K, L, K \cup L$ and $K \cap L \in \mathscr{S}$, then

$$(K \cup L)^* + (K \cap L)^* = K^* + L^*,$$

thus the map $K \mapsto K^*$ is a valuation. In particular, if $\mathscr{S}$ is intersectional, then (1.2) shows that $K^* \in V(\mathscr{S})$ for all $K \in U(\mathscr{S})$.

Now suppose that $\bar{\varphi}$ is a map from $V(\mathscr{S})$ into an abelian group which satisfies

$$(1.5) \quad \bar{\varphi}(K^* + L^*) = \bar{\varphi}(K^*) + \bar{\varphi}(L^*).$$

Defining $\varphi(K) := \bar{\varphi}(K^*)$ for those $K \subset S$ for which $K^* \in V(\mathscr{S})$, we get

$$\begin{aligned}\varphi(K \cup L) + \varphi(K \cap L) &= \bar{\varphi}((K \cup L)^*) + \bar{\varphi}((K \cap L)^*) \\ &= \bar{\varphi}((K \cup L)^* + (K \cap L)^*) = \bar{\varphi}(K^* + L^*) \\ &= \bar{\varphi}(K^*) + \bar{\varphi}(L^*) = \varphi(K) + \varphi(L),\end{aligned}$$

provided that $\bar{\varphi}$ is defined in each case. Thus φ is a valuation. In particular, if $\mathscr{S}$ is intersectional, then this yields a valuation on $U(\mathscr{S})$.

Vice versa, if a valuation φ on $\mathscr{S}$ with values in some real vector space is given, one might try to define

$$(1.6) \quad \bar{\varphi}(f) := \sum \alpha_i \varphi(K_i) \quad \text{for} \quad f = \sum \alpha_i K_i^* \in V(\mathscr{S}) \quad (\alpha_i \in \mathbb{R})$$

as in integration theory (where usually φ is a measure defined on a ring of subsets). If this is possible, then $\bar{\varphi}$ thus defined clearly satisfies (1.5). But, in general, the right-hand side of (1.6) does not only depend on the function f, but on its special representation. We shall return to these questions in §§2 and 5.

We conclude these preliminaries with a most important definition. A valuation φ on a class of convex subsets of either $\mathbb{E}^d$ or Ω^d is called *simple* if $\varphi(K) = 0$ whenever $\dim K < d$.

I. Classical examples and general results

§2. The Euler characteristic

Before treating valuations from a general point of view, it seems appropriate to review the more familiar classical examples occurring in the theory of convex bodies. The simplest (non-zero) valuation on $\mathscr{K}^d$ is clearly the function χ defined by

$$(2.1) \quad \chi(K) := \begin{cases} 1 & \text{if} \quad K \neq \emptyset \\ 0 & \text{if} \quad K = \emptyset \end{cases} \quad \text{for} \quad K \in \mathscr{K}^d.$$

While this function is of no interest when restricted to convex bodies, it is a non-trivial question whether χ can be extended, as a valuation, to the convex-ring $U(\mathscr{K}^d)$. The answer is in the affirmative, since the Euler characteristic as defined, for example, in singular homology theory, is a valuation satisfying (2.1). In a well-known and influential paper, Hadwiger [1955a] gave an entirely elementary existence proof which is independent of topology (the uniqueness is trivial by (1.2)). His construction proceeds by induction with respect to the dimension. The existence for $d = 0$ being trivial, suppose that the existence of χ has been proved

in dimension $d - 1$. For a unit vector $u \in \Omega^{d-1}$ and for $\lambda \in \mathbb{R}$, let

$$(2.2) \quad H_{u,\lambda} := \{x \in \mathbb{E}^d : \langle x, u \rangle = \lambda\}$$

be the hyperplane through λu orthogonal to u. Then put

$$(2.3) \quad \chi(K) := \sum_{\lambda \in \mathbb{R}} \left[\chi(K \cap H_{u,\lambda}) - \lim_{\mu \downarrow \lambda} \chi(K \cap H_{u,\mu}) \right] \quad \text{for} \quad K \in U(\mathscr{K}^d),$$

where on the right-hand side χ denotes the (unique) Euler characteristic which by the inductive assumption exists in $(d - 1)$-dimensional affine spaces. The sum is finite, and χ thus defined turns out to be a valuation.

By an easy argument, Hadwiger [1955a] also deduced the existence of a valuation χ on the finite unions of closed spherically convex subsets of Ω^{d-1} with $\chi(A) = 1$ if A is spherically convex and contained in an open hemisphere.

The above existence proof for the Euler characteristic on $U(\mathscr{K}^d)$ is reproduced in Hadwiger [1957]. Different variants of the construction are found in Hadwiger [1959], [1968b], [1969c], Hadwiger-Mani [1972].

We also mention an article of Hadwiger [1974a] which contains an elementary treatment of the Euler characteristic for polygons in the plane.

Once the existence of the Euler characteristic on the convex-ring is known, it is, mainly through exploitation of the inclusion-exclusion principle (1.2), a useful tool in combinatorial geometry, see Hadwiger [1947], [1955a], [1968b], Klee [1963]. Klee's paper put the Euler characteristic in a lattice-theoretic setting. This general treatment of valuations and the Euler characteristic in combinatorial and algebraic terms has been further developed by Rota [1971], see also [1964]. Rota [1971], p. 231, makes a very interesting (though somewhat vague and perhaps too optimistic) remark on a conceivable connexion between valuations on $U(\mathscr{P}^d)$ and the problem of finding necessary and sufficient conditions for a lattice to be isomorphic to the face lattice of a convex polytope.

If an element K of the convex-ring $U(\mathscr{K}^d)$ is represented as the union of k convex sets, then (1.2) gives trivial lower and upper bounds for the value $\chi(K)$ of the Euler characteristic in terms of d and k alone. The problem of finding sharp bounds has been posed by Hadwiger-Mani [1974], and they have treated a related problem. A complete solution was given by Eckhoff [1980].

The recursive definition (2.3) works equally well for relatively open convex bodies, and this is often convenient, especially when polyhedra are considered. If we use (2.3) for $K \in U(\mathscr{P}^d_{ro})$, then this yields a valuation χ on the unions of relatively open convex polytopes which evidently satisfies

$$(2.4) \quad \chi(P) = (-1)^{\dim P} \quad \text{for} \quad P \in \mathscr{P}^d_{ro}.$$

This extended Euler characteristic was considered by Lenz [1970] and Groemer [1972], and in special cases also by Hadwiger [1969c], [1973], who, however, preferred to use a different sign for odd-dimensional P, so that he did not get a valuation on $U(\mathscr{P}^d_{ro})$. It appears that the insistence on prescribing $\chi(P) = 1$ for every relatively open, non-empty polytope P, slightly complicates the investigation in Hadwiger [1973] and also in Hadwiger-Mani [1972].

Since every polytope is the disjoint union of the relative interiors of its faces, (2.4) and the additivity of χ immediately yield the well-known Euler relation.

Variants of the recursion formula (2.3) can also be used to extend the Euler characteristic to a linear functional on a vector space, as described in §1. Let us first consider the real vector space $V(\mathscr{P}^d)$ consisting of the finite linear combinations of indicator functions of convex polytopes. We are to show the existence of a real linear functional $\bar{\chi}$ on $V(\mathscr{P}^d)$ such that $\bar{\chi}(\mathscr{P}^*) = 1$ for $\emptyset \neq P \in \mathscr{P}^d$ (the uniqueness is clear). For $d = 1$, such a linear functional $\bar{\chi}_1$ is evidently given by

$$(2.5) \quad \bar{\chi}_1(f) := \sum_{\lambda \in \mathbb{R}} \left[f(\lambda) - \lim_{\mu \downarrow \lambda} f(\mu) \right], \quad f \in V(\mathscr{P}^1).$$

Let $d \geq 2$, suppose that the existence has already been proved in dimension $d - 1$, and call this functional $\bar{\chi}_{d-1}$. Consider $\mathbb{E}^{d-1}$ as a linear subspace of $\mathbb{E}^d$, and let $u \in \mathbb{E}^d$ be a unit vector not in $\mathbb{E}^{d-1}$. Let $f \in V(\mathscr{P}^d)$ be given and define

$$\tilde{f}(x,\lambda) := f(x + \lambda u) \quad \text{for} \quad x \in \mathbb{E}^{d-1}, \quad \lambda \in \mathbb{R}.$$

Two types of induction are possible:

(a) Define the projection $\pi_1 f$ of f on to $\mathbb{E}^{d-1}$ by

$$(\pi_1 f)(x) := \bar{\chi}_1(\tilde{f}(x,\cdot)) \quad \text{for} \quad x \in \mathbb{E}^{d-1},$$

and then put

$$\bar{\chi}_d(f) := \bar{\chi}_{d-1}(\pi_1 f).$$

(b) Define

$$(\pi_2 f)(\lambda) := \bar{\chi}_{d-1}(\tilde{f}(\cdot,\lambda)) \quad \text{for} \quad \lambda \in \mathbb{R}$$

and put

$$\bar{\chi}_d(f) := \bar{\chi}_1(\pi_2 f).$$

In each case it is easy to see that $\bar{\chi}_d$ thus defined has the desired properties.

A procedure equivalent to method (a) was employed by Hadwiger [1960] and also by Groemer [1972], who apparently did not know Hadwiger's paper. Method (b), again in a different but equivalent form, was used by Lenz [1970]. He generalized it as follows. By the basis theorem of linear algebra, the linear functional $\bar{\chi}_1$ on $V(P^1)$ has a linear extension, also called $\bar{\chi}_1$, to the vector space $\mathbb{R}^{\mathbb{R}}$ of all real functions on $\mathbb{R}$. Choose a basis $e_1,\ldots,e_d$ of $\mathbb{E}^d$ and then identify $\mathbb{E}^k$ with the subspace spanned by $e_1,\ldots,e_k$. If now method (b) is applied, one gets a linear functional $\bar{\chi}_d$ on the vector space of all real functions on $\mathbb{E}^d$ which satisfies $\bar{\chi}_d(K^*) = 1$ for $\emptyset \neq K \in \mathscr{K}^d$. Thus the definition $\chi(A) := \bar{\chi}_d(A^*)$ for $A \subset \mathbb{E}^d$ extends the Euler characteristic, as a valuation, from $\mathscr{K}^d$ to the system of all subsets of $\mathbb{E}^d$. This extension, of course, which depends on the extension of $\bar{\chi}_1$ and the choice of the basis, is highly arbitrary and therefore of little geometric interest, the more so since, as Lenz shows, it cannot be translation invariant.

The essential point of Groemer's [1972] paper is the introduction of a vector space A^d of real functions on $\mathbb{E}^d$ with a pseudonorm such that A^d contains $V(\mathscr{P}^d)$ as a proper dense subspace, and $\bar{\chi}$ has a unique continuous linear extension to A^d. The elements of A^d are called "approximable" functions. The system $\mathscr{S}_A$ of subsets of $\mathbb{E}^d$ whose characteristic functions are approximable contains the convex-ring $U(\mathscr{K}^d)$, and, for instance, the relative interiors of convex bodies and thus also the boundaries of convex bodies. Unfortunately, $\mathscr{S}_A$ is not intersectional. Some properties of the extended Euler characteristic on A^d are proved in Groemer [1972], and further invariance properties in Groemer [1973].

The fact that the Euler characteristic has a unique linear extension to the vector space generated by the characteristic functions of convex bodies and of their relative interiors, has been utilized by Groemer [1975] to show the existence and some properties of an Euler characteristic on certain systems of subsets of convex surfaces. This generalizes earlier work of Hadwiger-Mani [1972] which is concerned with spherical polyhedra.

More general results on Euler characteristics for subsets of convex surfaces could also be deduced, as Groemer remarks, from the following elegant result of Groemer [1974]. Let $\mathscr{S}$ be a system of subsets of a set S. The class $\mathscr{S}$ is called *separable* if to any two disjoint sets $A,B \in \mathscr{S}$ there exists a pair $X,Y \subset S$ such that: $X \cap C \in \mathscr{S}$ and $Y \cap C \in \mathscr{S}$ for every $C \in \mathscr{S}$, $A \subset X$, $A \cap Y = \emptyset$, $B \subset Y$, $B \cap X = \emptyset$, $X \cup Y = S$, and $Z \cap X \neq \emptyset$, $Z \cap Y \neq \emptyset$ for $Z \in \mathscr{S}$ only if $Z \cap X \cap Y \neq \emptyset$. Then Groemer shows:

(2.6) **Theorem.** *Let S be a set and let $\mathscr{S}$ be a separable intersectional class of subsets of S. There exists exactly one linear functional on the vector space $V(\mathscr{S})$ such that $\chi(C^*) = 1$ for every nonempty set C of $\mathscr{S}$.*

An example of a class $\mathscr{S}$ satisfying the assumptions of the theorem is the system of compact convex subsets of a locally convex topological vector space. But it should be noted that Groemer's theorem and its proof are purely combinatorial.

§3. Volume and valuations derived from it

Any measure on a ring of subsets of $\mathbb{E}^d$ containing $\mathscr{K}^d$ which is finite on $\mathscr{K}^d$ yields a real valued valuation on $\mathscr{K}^d$. In particular, restriction of the Lebesgue measure gives the volume V, the most familiar example of a simple valuation. In an axiomatic treatment of euclidean (or noneuclidean) geometry one might prefer, instead of taking Lebesgue measure for granted, to introduce the notion of volume for simple geometric figures, like polytopes, in an elementary geometric way. (For the plane case, compare Hilbert [1899], chap. IV.) As mentioned in the introduction, the attempts to do this have initiated a deeper study of simple valuations in general. For a description of these geometric approaches to volume and the difficulties involved, we refer the reader to the books by Hadwiger [1957], Boltianskii [1978], Böhm-Hertel [1980].

We first fix some more notation. By $B = B^d$ we denote the unit ball $\{x \in \mathbb{E}^d : \|x\| \leq 1\}$ of $\mathbb{E}^d$ and by $\varkappa(d)$ its volume. The ordinary spherical Lebesgue measure on the unit sphere $\Omega = \Omega^{d-1}$ is denoted by σ, thus $\sigma(\Omega^{d-1}) = d\varkappa(d)$.

In the theory of convex bodies, the valuation property of volume carries over to a series of other functions which are derived from volume in a natural way. The source of this is the fact, a special case of lemma (1.4), that for any fixed convex body C, the function $K \mapsto V(K + C)$ is also a valuation on $\mathscr{K}^d$. This leads at once to a valuation property of mixed volumes. As is well known, the volume of a linear combination $\lambda_1 K_1 + \cdots + \lambda_k K_k$ of convex bodies $K_1,\ldots,K_k \in \mathscr{K}^d$ with real coefficients $\lambda_1,\dots,\lambda_k \geq 0$ can be expressed as a polynomial

(3.1) $V(\lambda_1 K_1 + \cdots + \lambda_k K_k) = \sum \lambda_{i_1} \cdots \lambda_{i_d} V(K_{i_1},\dots,K_{i_d})$

with $V(K_{i_1},\dots,K_{i_d})$ symmetric in the indices and depending only on $K_{i_1},\dots,K_{i_d}$. It is often convenient to write (3.1) in the form

(3.2) $V(\lambda_1 K_1 + \cdots + \lambda_k K_k) = \sum \binom{d}{r_1 \cdots r_k} \lambda_1^{r_1} \cdots \lambda_k^{r_k} V(K_1,r_1;\dots;K_k,r_k),$

where

$$\binom{d}{r_1 \cdots r_k} := \begin{cases} \dfrac{d!}{r_1! \cdots r_k!} & \text{if } \sum_{j=1}^{k} r_j = d, \quad r_j \geq 0, \\ 0, & \text{otherwise.} \end{cases}$$

Here, as in other cases, we use the abbreviation

(3.3) $f(K_1,r_1;\dots;K_k,r_k) := f(\underbrace{K_1,\dots,K_1}_{r_1 \text{ times}},\dots,\underbrace{K_k,\dots,K_k}_{r_k \text{ times}})$

with $r_1 + \cdots + r_k = m$ for any function f of m variables, and we also write

(3.4) $f(K_1,\dots,K_p,\mathscr{C}) := f(K_1,\dots,K_p,L_{p+1},\dots,L_m)$

where $\mathscr{C}$ stands for the $(m-p)$-tuple $(L_{p+1},\dots,L_m)$. With this notation, (3.2) implies

(3.5) $V(\lambda_1 K_1 + \cdots + \lambda_k K_k,p;\mathscr{C}) = \sum \binom{p}{r_1 \cdots r_k} \lambda_1^{r_1} \cdots \lambda_k^{r_k} V(K_1,r_1;\dots;K_k,r_k;\mathscr{C}).$

Now if $p \in \{1,\dots,d\}$ and a $(d-p)$-tuple $\mathscr{C} = (K_{p+1},\dots,K_d)$ of convex bodies is fixed, then the function φ defined by

(3.6) $\varphi(K) := V(K,p;\mathscr{C}) \quad \text{for} \quad K \in \mathscr{K}^d$

is a valuation. This follows immediately from the fact that the function

$$K \mapsto V(\lambda K + \lambda_{p+1} K_{p+1} + \cdots + \lambda_d K_d)$$

is a valuation on $\mathscr{K}^d$, and that $(d!/p!)\varphi$ is the coefficient of $\lambda^p \lambda_{p+1} \cdots \lambda_d$ in the polynomial expansion of the latter expression.

In particular, this applies to the quermassintegrals $W_0,\dots,W_d$, or the intrinsic volumes $V_0,\dots,V_d$, respectively, defined by

(3.7) $W_r(K) := V(K,d-r;B,r) =: \dfrac{\varkappa(r)}{\binom{d}{r}} V_{d-r}(K).$

Since $W_d(K) = \varkappa(d)$ if $K \neq \emptyset$, this gives $V_0(K) = 1 (= \chi(K))$.

It appears that the valuation property of the quermassintegrals was first pointed out by Blaschke [1937], §43. Later it played an important role in the work of Hadwiger, to be reviewed later. Curiously, the valuation property of the general mixed volume (3.6) is not mentioned in the standard textbooks treating mixed volumes.

As a special consequence of the foregoing, we mention an identity for mixed volumes. Let $\mathscr{C}$ be a $(d-2)$-tuple of convex bodies, and let $K,L \in \mathscr{K}^d$ be such

that $K \cup L$ is convex. Writing $V(K,K,\mathscr{C}) =: v(K)$ and $V(K,L,\mathscr{C}) =: v(K$
for the moment and using the valuation property of v, the expansion (3
the identity (1.3) and again the expansion (3.5), we get

$$\begin{aligned} v(K) + v(L) + 2v(K \cup L,K \cap L) &= v(K \cup L) + v(K \cap L) \\ &\quad + 2v(K \cup L,K \cap L) \\ &= v((K \cup L) + (K \cap L)) = v(K + L) \\ &= v(K) + v(L) + 2v(K,L), \end{aligned}$$

thus

(3.8) $V(K,L,\mathscr{C}) = V(K \cup L,K \cap L,\mathscr{C}).$

Identity (3.8) was first observed by Groemer [1977a] p. 160, who proved it i
more indirect way.

Above, the quermassintegrals were defined as specialized mixed volumes
equivalently, by means of the so-called *Steiner formula*

(3.9) $V(K_\rho) = \sum_{j=0}^{d} \binom{d}{j} \rho^j W_j(K) = \sum_{i=0}^{d} \varkappa(d-i) \rho^{d-i} V_i(K),$

where $K_\rho = K + \rho B$ is the outer parallel body of K at distance $\rho \geq 0$. A differe
approach comes from integral geometry. Let SO_d denote the rotation group of
and ν its invariant measure, suitably normalized. For given $r \in \{0,\dots,d\}$, choo
an r-dimensional linear subspace F_r of $\mathbb{E}^d$, and let δF_r denote its image under t
rotation $\delta \in SO_d$. Further, let $\mathscr{E}_r^d$ be the space of r-flats in $\mathbb{E}^d$ with rigid motio
invariant measure μ_r, also suitably normalized. Then for $K \in \mathscr{K}^d$ the formula

(3.10) $W_{d-r}(K) = a_{d,r} \int_{SO_d} V_r(K|\delta F_r)\, d\nu(\delta),$

where $K|\delta F_r$ denotes the image of K under orthogonal projection on to δF_r, an

(3.11) $W_r(K) = b_{d,r} \int_{\mathscr{E}_r^d} V_0(K \cap E_r)\, d\mu_r(E_r)$

are valid. Here $a_{d,r}$ and $b_{d,r}$ are positive constants depending only on d and r. Fo
proofs and generalizations of (3.10), (3.11) (and for explicit values of th
constants) the reader may consult Hadwiger [1957], chap. 6, or Santaló [1976
§§13, 14. Note that V_r in (3.10) is just r-dimensional volume, while V_0 in (3.11)
the Euler characteristic. Thus in either formula the quermassintegrals are derive
from a more elementary valuation. Hadwiger [1957] uses formulae similar t
(3.10) to give a recursive definition for the quermassintegrals and later prove
(3.9).

We remark that any measure λ on SO_d and any valuation φ on $\mathscr{K}^d$ for whic
each function $\delta \mapsto \varphi(K|\delta F_r)$ is λ-integrable ($K \in \mathscr{K}^d$), yields a new valuation ψ
by means of the definition

(3.12) $\psi_r(K) := \int_{SO_d} \varphi(K|\delta F_r)\, d\lambda(\delta), \quad K \in \mathscr{K}^d.$

The valuation property carries over because of Lemma (1.4) (see the thir
example given there). Similarly, a measure λ_r on $\mathscr{E}_r^d$ and a valuation φ on $\mathscr{K}^d$ fo

which each function $E_r \mapsto \varphi(K \cap E_r)$ is λ_r-integrable ($K \in \mathscr{K}^d$), gives the new valuation φ_r defined by

$$(3.13)\quad \varphi_r(K) := \int_{\mathscr{E}_r^d} \varphi(K \cap E_r)\, d\lambda_r(E_r), \quad K \in \mathscr{K}^d.$$

For $\lambda_r = \mu_r$ and for continuous valuations φ, these associated valuations φ_r play an important role in Hadwiger's generalization of the principal kinematic formula of integral geometry, see Hadwiger [1956], [1957], p. 241.

Let us return to mixed volumes. Using their properties (Minkowski additivity and uniform continuity in each argument), one easily deduces from the Riesz representation theorem that, for given convex bodies $K_1,\dots,K_{d-1} \in \mathscr{K}^d$, there exists a unique (positive) measure $S(K_1,\dots,K_{d-1};\cdot)$ on the Borel sets of the unit sphere Ω of $\mathbb{E}^d$ such that

$$(3.14)\quad V(K,K_1,\dots,K_{d-1}) = \frac{1}{d}\int_{\Omega} h(K,u)\, dS(K_1,\dots,K_{d-1};u) \quad \text{for} \quad K \in \mathscr{K}^d,$$

where $h(K,\cdot)$ denotes the support function of K. This measure, which is called the *mixed area function* of $K_1,\dots,K_{d-1}$, was introduced independently by Fenchel-Jessen [1938] and Aleksandrov [1937], see also Busemann [1958]. In particular, one writes (with the same notation as in (3.3))

$$S_p(K;\cdot) := S(K,p;B,d-1-p;\cdot)$$

for $p = 0,\dots,d-1$ and calls this the *p-th order area function* of K. Clearly we have

$$(3.15)\quad S_p(K;\Omega) = dW_{d-p}(K).$$

$S_{d-1}(K;\cdot)$ has a simple geometric meaning: For a Borel set $\omega \subset \Omega$, $S_{d-1}(K;\omega)$ is the surface area ((d − 1)-dimensional Hausdorff measure) of the set of boundary points of K at which there exists an outer unit normal vector falling in ω. From this special measure, one gets back the general mixed area function by means of the polynomial expansion

$$(3.16)\quad S_{d-1}(\lambda_1 K_1 + \dots + \lambda_{d-1}K_{d-1};\cdot) = \sum \lambda_{i_1}\cdots\lambda_{i_{d-1}} S(K_{i_1},\dots,K_{i_{d-1}};\cdot).$$

Either from this representation (and the obvious valuation property of S_{d-1}), or from (3.14) and the valuation property of mixed volumes, it is clear that each function

$$K \mapsto S(K,p;K_{p+1},\dots,K_{d-1};\cdot),$$

and in particular each S_p, is a valuation on $\mathscr{K}^d$ (with values in the vector space of signed Borel measures on Ω). Apparently the valuation property of the S_p was first pointed out and used by Schneider [1975a].

The area functions of order p can also be obtained from a local version of the Steiner formula (3.9). For $K \in \mathscr{K}^d$ let $p(K,\cdot)$: $\mathbb{E}^d \to K$ denote the metric projection, that is, $p(K,x)$ is the point in K nearest to x. Then for a Borel set $\omega \subset \Omega$ and for $\rho > 0$ consider the "local parallel set"

$$(3.17)\quad B_\rho(K,\omega) := \left\{ x \in \mathbb{E}^d : 0 < \|x - p(K,x)\| \le \rho \quad \text{and} \quad \frac{x - p(K,x)}{\|x - p(K,x)\|} \in \omega \right\}.$$

If V denotes Lebesgue measure, then

$$(3.18)\quad V(B_\rho(K,\omega)) = \frac{1}{d}\sum_{i=0}^{d-1}\binom{d}{i}\rho^{d-i} S_i(K;\omega).$$

In particular, the map $K \mapsto V(B_\rho(K,\cdot))$ is a valuation.

Similarly, if one defines, for a Borel set β in $\mathbb{E}^d$ and for $\rho > 0$,

$$(3.19)\quad A_\rho(K,\beta) := \{x \in \mathbb{E}^d : 0 < \|x - p(K,x)\| \le \rho \quad \text{and} \quad p(K,x) \in \beta\},$$

then $V(A_\rho(K,\cdot))$ is a measure and one has a polynomial expansion

$$(3.20)\quad V(A_\rho(K,\beta)) = \frac{1}{d}\sum_{i=0}^{d-1}\binom{d}{i}\rho^{d-i} C_i(K,\beta).$$

If $I_\rho(K,\beta,\cdot)$ denotes the characteristic function (on $\mathbb{E}^d$) of the set $A_\rho(K,\beta)$, th the map $K \mapsto I_\rho(K,\beta,\cdot)$ is a valuation (see Schneider [1978], p. 106). It follows th $K \mapsto V(A_\rho(K,\cdot))$ and hence each function $K \mapsto C_i(K,\cdot)$ is a valuation with valu in the vector space of signed Borel measures (with compact support) on $\mathbb{E}^d$. T measures $C_0(K,\cdot),\dots,C_{d-1}(K,\cdot)$ are Federer's curvature measures. They we introduced (for more general sets than convex bodies) by Federer [1959]. For unified treatment of the measures S_i and C_i on $\mathscr{K}^d$ along the lines sketched abov see Schneider [1978]; further references are contained in the survey artic Schneider [1979]. Far-reaching generalizations of the curvature measures ar the area functions, in the form of measures on the Borel subsets of $\mathscr{K}^d$, ha recently been proposed and investigated by Wieacker [1982].

In much the same way as the notion of volume, combined with Minkows addition, leads to mixed volumes, the notion of centroid is the source of a seri of vector valued functionals. For $K \in \mathscr{K}^d$, let

$$z(K) := \int_K x\, dV(x),$$

so that, for dim K = d, the point $z(K)/V(K)$ is the centre of gravity of K. Th vector z(K) will be called the *moment vector* of K. We have a polynomi expansion

$$(3.21)\quad z(\lambda_1 K_1 + \dots + \lambda_k K_k) = \sum \lambda_{i_1}\cdots\lambda_{i_{d+1}} z(K_{i_1},\dots,K_{i_{d+1}}),$$

where the vector valued coefficients are assumed symmetric in their indices. Thes coefficients will be called *mixed moment vectors*. The expansion (3.21) was (fo d = 3) already noticed by Minkowski [1911], §23. A more thorough study o mixed moment vectors was undertaken by Schneider [1972a,b]. In the sam way as for mixed volumes, one shows that each function

$$K \mapsto z(K,p;\mathscr{C})$$

($p \in \{1,\dots,d+1\}$ and the $(d+1-p)$-tuple $\mathscr{C}$ of convex bodies fixed) is valuation on $\mathscr{K}^d$. Also the other properties of mixed moment vectors ar analogous to those of mixed volumes, but observe that, while the mixed volume i invariant under translation of any of its arguments, one has

$$(3.22)\quad z(K_1 + t,K_2,\dots,K_{d+1}) = z(K_1,K_2,\dots,K_{d+1}) + \frac{1}{d+1} V(K_2,\dots,K_{d+1})t.$$

Specialization of mixed moment vectors yields the so-called *quermassvectors* defined by

$$(3.23)\quad q_r(K) := \frac{d+1}{d+1-r} z(K,d+1-r;B,r)$$

for $r = 0,\ldots,d$. Note that (3.22) implies

$$(3.24)\quad q_r(K+t) = q_r(K) + W_r(K)t,$$

and that a Steiner formula,

$$(3.25)\quad z(K+\rho B) = \sum_{j=0}^{d} \binom{d}{j} \rho^j q_j(K),$$

is valid. Using Federer's curvature measures defined above, one has an integral representation

$$(3.26)\quad q_r(K) = \frac{1}{d} \int_{\partial K} x \, dC_{d-r}(K,x), \quad r = 1,\ldots,d$$

(compare Schneider [1972a], p. 123 and the remark on p. 129), which shows that $q_r(K)/W_r(K)$ is the centroid of the mass distribution on ∂K defined by the curvature measure $C_{d-1}(K;\cdot)$ In particular, q_1 is the area centroid, and

$$(3.27)\quad s(K) := \frac{q_d(K)}{\varkappa(d)}$$

is the so-called *Steiner point* of K. It can also be represented by

$$(3.28)\quad s(K) = \frac{1}{\varkappa(d)} \int_{\Omega} h(K,u)u \, d\sigma(u).$$

This shows that s is a Minkowski additive function on $\mathscr{K}^d$. References concerning this remarkable point can be found in Schneider [1972a] p. 128–129; others will be given in §13.

The quermassvectors satisfy integral geometric relations analogous to (3.11). Vice versa, this yields an alternative approach to these valuations: One may define s directly by (3.28) and then q_r, $0 \le r \le d$, by means of

$$(3.29)\quad q_r(K) = b_{d,r} \int_{\mathscr{E}_r^d} s(K \cap E_r) \, d\mu_r(E_r)$$

(with $b_{d,r}$ as in (3.11)). The valuation property of q_r is then obvious from (3.28), (3.29) and Lemma (1.4). In this way, the quermassvectors were introduced by Hadwiger-Schneider [1971].

In the above discussion of classical valuations we have stressed the existence of polynomial expansions, since this will be an essential point in the investigation of general valuations in later chapters. We have also mentioned a few more general constructions for valuations with a view to the problem of representing general, and characterizing special, valuations which will be the topic of chapter IV.

We conclude this section with a look at spherical space. Valuations on spherical polytopes, besides being interesting in themselves, also enter the

investigation of valuations on euclidean polytopes. The spherical volum Ω^{d-1}, which we denote by σ, yields a simple valuation on the spherically co polytopes, say. It gives rise to the angle functions, which will play an impo role in later discussions. For a (convex) polyhedral set P in $\mathbb{E}^d$ and non-e faces $F \subset G$ of P, we denote by $\beta(F,G)$ and $\gamma(F,G)$ the internal and ext angles, respectively, of G at its face F, measured in aff G and normalized so the total angle is 1 (see, e.g., Grünbaum [1967], p. 297 and p. 308, for definiti We also define $\beta(F,F) = 1 = \gamma(F,F)$ and $\beta(F,G) = 0 = \gamma(F,G)$ if $F \not\subseteq G$.

The Steiner formula (3.9), applied to a polytope $P \in \mathscr{P}^d$, yields an ex representation of the quermassintegrals involving external angles, namely

$$(3.30)\quad \frac{\binom{d}{r}}{\kappa(d-r)} W_{d-r}(P) = V_r(P) = \sum_{F^r} \gamma(F^r, P) V_r(F^r)$$

for $r = 0,1,\ldots,d$, where the sum extends over the r-dimensional faces F^r of P; that $V_r(F^r)$ is just the r-dimensional volume of F^r. Similar formulae exist fo r-th order area functions and for the curvature measures (see Schne [1978], (4.9) and (3.7)), as well as for the quermassvectors (Schneider [197 p. 125).

Now let us consider a spherically convex polytope $P \subset \Omega = \Omega^{d-1}$. It is intersection of Ω with a convex polyhedral cone C with apex 0. We define

$$(3.31)\quad \varphi_r(P) = \varphi_r(C) := \sum_{F^r} \beta(A,F^r)\gamma(F^r,C)$$

for $r = 0,\ldots,d$, where the sum extends over all r-faces F^r of C and A denotes face of apices of C. By definition of the angles, $\varphi_d(P)$ is the normalized spher volume of P, while $\varphi_0(P)$ is the normalized spherical volume of the polar set (the intersection of Ω with the polar cone of C). If $A = \{0\}$ (so that P lies in open hemisphere) and $r \ge 1$, then $\beta(A,F^r) = \beta(0,F^r)$ is the normalized (r – dimensional spherical volume of the $(r-1)$-face $\Omega \cap F^r$ of P. Thus φ_r is spherical analogue of the intrinsic $(r-1)$-volume V_{r-1} for a euclidean polyt in $\mathbb{E}^{d-1}$. In fact, the functions $\varphi_1,\ldots,\varphi_d$ can also be obtained from a Stei formula in Ω^{d-1} analogous to (3.9) (but with the powers of ρ replaced by ot functions, see Allendoerfer [1948] in the smooth case). This approach can also used to extend the definition of the φ_r to general spherically convex sets and prove some of their properties. In particular, one gets rotation invari continuous valuations. For polytopes, the valuation property can also deduced directly from (3.31). The local Steiner formula (3.20) and the definiti of the curvature measures also carry over. One could proceed similarly hyperbolic space.

However, the analogy to euclidean space breaks down in several respec While the euclidean quermassintegrals are monotonically increasing with resp to set inclusion, this is not generally true for the spherical φ_r. Clearly φ_d increasing, and it can be shown that φ_{d-1} is increasing (e.g., Shephard [1968 (32)). By duality (considering polar sets) it follows that φ_0 and φ_1 are decreasi For $2 \le r \le d-2$, φ_r is neither increasing nor decreasing. To see this, let denote a half j-space. If $2 \le r \le d-2$, we can arrange that $D_{r-1} \subset D_r \subset D_r$

$\subset D_{r+2}$ and φ_r successively takes the values 0, 1/2, 1/2, 0. Then we approximate the halfspaces by pointed d-dimensional polyhedral cones obeying the corresponding inclusion relations. The result follows by continuity.

Another difference to the euclidean case occurs when we consider the integral geometric approach (3.11). The spherical analogue to that formula is

$$(3.32) \quad \int \chi(P \cap L_r)\,dL_r = 2 \sum_{m \geq 0} \varphi_{d+1-r+2m}(P) =: 2\psi_r(P)$$

for r = 1,...,d and spherically convex polytopes P (this can be generalized); χ is the Euler characteristic, and the integral is over the Grassmannian of all r-flats L_r through 0, with the invariant measure normalized to total measure 1 (see Santaló [1976], p. 310, with different terminology). Thus the ψ_r are also spherical analogues of the quermassintegrals, and perhaps the better analogues, since ψ_r is increasing for each r.

§4. The lattice point enumerator

Among the valuations derived from a measure, the next natural one after volume is perhaps the lattice point enumerator G defined by

$$G(K) := \operatorname{card}(K \cap \mathbb{Z}^d),$$

where $\mathbb{Z}^d$ is the integer lattice in $\mathbb{E}^d$. This function, of course, has its important place in geometry of numbers, and we refer the reader to the survey article of Gruber [1979]. An equally useful review of the known relations between the lattice point enumerator and other functionals on convex bodies has been given by Betke-Wills [1979]. Here we are only concerned with the valuation aspect of G. From this point of view, it turns out that most of the relations for G which are found in the literature are special cases of results which hold for more general valuations. Although these will be discussed in §§10, 12, we give a few references to the special results, since they added to the motivation for developing a general theory.

By $\mathscr{P}^d_L$ we denote the set of convex lattice polytopes in $\mathbb{E}^d$, and by $\mathbb{N}$ the set of positive integers. Ehrhart [1967a] proved the existence of a polynomial expansion

$$(4.1) \quad G(nP) = \sum_{i=0}^{d} n^i G_i(P) \quad \text{for } P \in \mathscr{P}^d_L, \quad n \in \mathbb{N},$$

where the coefficients G_i depend only on P, and he made applications of it to various counting problems. Ehrhart [1967b] also discovered and proved the so-called "reciprocity law"

$$(4.2) \quad G(\operatorname{relint} nP) = (-1)^{\dim P} \sum_{i=0}^{d} (-n)^i G_i(P) \quad \text{for} \quad P \in \mathscr{P}^d_L, \quad n \in \mathbb{N}.$$

Equality (4.1) has a generalization similar to the polynomial expansion (3.2), namely

$$(4.3) \quad G(n_1P_1 + \cdots + n_kP_k) = \sum n_1^{r_1} \cdots n_k^{r_k} G(P_1,r_1;\ldots;P_k,r_k)$$

for $P_1,\ldots,P_k \in \mathscr{P}^d_L$ and $n_1,\ldots,n_k \in \mathbb{N}$; the sum extends over the nonnega
integers $r_1,\ldots,r_k$ with $r_1 + \cdots + r_k \leq d$. Expansion (4.3) for the lattice po
enumerator was obtained by Bernštein [1976] at about the same time that it
discovered to hold for more general valuations, see §10.

Besides G, weighted lattice point numbers have been considered; these
simple valuations. Macdonald [1963], [1971] defined A(P) as the number wh
results when each lattice point in P is counted with weight $V(P \cap B)/V(B)$ fo
sufficiently small ball B centred at the point. He proved an expansion similar
(4.1) and obtained some information on the coefficients. Hadwiger [195
[1957], p. 69, took lattice-oriented cubes instead of balls and used the result
simple valuation, applied to translates of lattice polytopes, in an equidissec
bility criterion with respect to lattice translations.

§5. Extension problems

Since the definition of a valuation involves unions and intersections of sets
seems preferable that valuations be defined on set systems which are closed und
(finite) unions and intersections. Thus for the valuations considered here whi
are, in the first instance, defined on the set $\mathscr{K}^d$ of convex bodies or the set $\mathscr{P}^d$
convex polytopes, there arises the problem of extending them, as valuations,
the lattices $U(\mathscr{K}^d)$ or $U(\mathscr{P}^d)$, respectively. This extension problem is not only
formal interest. For instance, in integral geometry and its applications, one wa
the formulae involving specific valuations to hold not only for convex sets, but
least for the elements of the convex-ring. A second reason for investigati
extensions is technical: Even if only information on certain valuations on conv
bodies is the aim, it may be necessary in the course of the proof first to extend t
valuations to a broader class of sets.

For the classical valuations considered in the foregoing sections, one kno
special constructions for extending them to $U(\mathscr{K}^d)$. Although the existence
such extensions would often also follow from more general theorems to
reviewed later, the explicit definitions are nevertheless of interest, since th
exhibit the geometric meaning of the extended valuations, and sometimes perm
the deduction of additional information.

There is, of course, no problem with volume or the lattice point enumerato
since they are measures, and hence valuations, on a ring containing $U(\mathscr{K}^d)$. F
the Euler characteristic, the explicit construction of extensions was described
§2. Once the Euler characteristic χ is available on $U(\mathscr{K}^d)$, one can define (
Hadwiger [1957] does)

$$(5.1) \quad W_r(K) := b_{d,r} \int_{\mathscr{E}^d_r} \chi(K \cap E_r)\,d\mu_r(E_r)$$

for $K \in U(\mathscr{K}^d)$. For $K \in \mathscr{K}^d$ this gives, by (3.11), the r-th quermassintegral of K
and since the valuation property of χ carries over to W_r, this yields an additiv
extension of the quermassintegrals to the convex-ring.

In a similar way one can proceed with the quermassvectors. In analogy to (2.
one first defines (following Mani [1971]) for $K \in U(\mathscr{K}^d)$, $u \in \Omega^{d-1}$, and $\lambda \in$
(with $H_{u,\lambda}$ as in (2.2))

$$(5.2) \quad h(K,u) := \sum_{\lambda \in \mathbb{R}} \lambda \left[\chi(K \cap H_{u,\lambda}) - \lim_{\mu \downarrow \lambda} \chi(K \cap H_{u,\mu}) \right],$$

where the limit exists and the sum is finite since K is a union of finitely many convex bodies. If K is convex and non-empty, then the right-hand side of (5.2) is the value of the support function of K at u, thus the definition is consistent. On the other hand, the valuation property of χ carries over to $h(\cdot,u)$, so that (5.2) extends the support function, as a valuation, to $U(\mathscr{K}^d)$. Observe that the support function of $K \in U(\mathscr{K}^d)$ no longer determines K uniquely, and that

(5.3) $h(K + t,u) = h(K,u) + \chi(K)\langle t,u\rangle$

for $t \in \mathbb{E}^d$.

Now equation (3.28) may serve as definition of the Steiner point s(K) for $K \in U(\mathscr{K}^d)$; then s is a valuation on $U(\mathscr{K}^d)$ extending the classical Steiner point. Equation (5.3) implies the translation covariance property

(5.4) $s(K + t) = s(K) + \chi(K)t.$

As proposed by Hadwiger-Schneider [1971], one may now define the quermassvectors $q_r(K)$ of $K \in U(\mathscr{K}^d)$ by equation (3.29). Then q_r is a valuation on $U(\mathscr{K}^d)$, and by (5.4) and (3.11) one has

(5.5) $q_r(K + t) = q_r(K) + W_r(K)t.$

The Euler characteristic on $U(\mathscr{K}^d)$ can also be used to obtain an additive extension of the area functions and the curvature measures. We briefly explain this method of Schneider [1980]. For $z \in \mathbb{E}^d$ and $\rho > 0$, let $B(z,\rho) \subset \mathbb{E}^d$ denote the closed ball with centre z and radius ρ. Then for $K \in U(\mathscr{K}^d)$ and $q,x \in \mathbb{E}^d$, one defines the *index* of K at q with respect to x by

(5.6) $$j(K,q,x) := \begin{cases} 1 - \lim\limits_{\delta\downarrow 0}\lim\limits_{\varepsilon\downarrow 0} \chi(K \cap B(x,\|x-q\|-\varepsilon) \cap B(q,\delta)) & \text{if} \quad q \in K \\ 0 & \text{if} \quad q \notin K. \end{cases}$$

If K is convex, then $j(K,q,x) = 1$ if $q = p(K,x)$ (where $p(K,\cdot)$ is the metric projection on to K), and $j(K,q,x) = 0$ otherwise. The additivity of the Euler characteristic implies that $j(\cdot, q,x)$ is a valuation on $U(\mathscr{K}^d)$. Now for $K \in U(\mathscr{K}^d)$, $\rho > 0$, a Borel set $\omega \subset \Omega$, and $x \in \mathbb{E}^d$, one defines

(5.7) $$s_\rho(K,\omega,x) := \sum_{\substack{q \in \mathbb{E}^d\setminus\{x\} \\ (x-q)_0 \in \omega}} j(K \cap B(x,\rho),q,x),$$

where $(x - q)_0 := (x - q)/\|x - q\|$. If K is convex, then $s_\rho(K,\omega,\cdot)$ is the characteristic function of the local parallel set $B_\rho(K,\omega)$ defined by (3.17), hence the Lebesgue integral of this function satisfies the Steiner formula (3.18). Since $s_\rho(\cdot, \omega,x)$ is a valuation on $U(\mathscr{K}^d)$, it is easy to see that the polynomial expansion extends, and one has

(5.8) $$\int_{\mathbb{E}^d} s_\rho(K,\omega,x)\,dx = \frac{1}{d}\sum_{i=0}^{d-1}\binom{d}{i}\rho^{d-i}S_i(K;\omega)$$

for $K \in U(\mathscr{K}^d)$. This defines signed Borel measures S_i on Ω which are the additive extensions to $U(\mathscr{K}^d)$ of the area functions.

Similarly, for a Borel set β in $\mathbb{E}^d$ one defines

(5.9) $$c_\rho(K,\beta,x) := \sum_{q \in \beta\setminus\{x\}} j(K \cap B(x,\rho),q,x);$$

then $K \mapsto c_\rho(K,\beta,\cdot)$ is a valuation on $U(\mathscr{K}^d)$ which extends the indicator function of the local parallel set $A_\rho(K,\beta)$ defined by (3.19). The Steiner formula (3.20) extends to give

(5.10) $$\int_{\mathbb{E}^d} c_\rho(K,\beta,x)\,dx = \frac{1}{d}\sum_{i=0}^{d-1}\binom{d}{i}\rho^{d-i}C_i(K,\beta)$$

for $K \in U(\mathscr{K}^d)$ and thus signed Borel measures C_i on $\mathbb{E}^d$ which additively extend the curvature measures.

If the point x in (5.6) "tends to infinity", one gets a different notion of index (see Schneider [1977a]). For $K \in U(\mathscr{K}^d)$, $q \in \mathbb{E}^d$, and $u \in \Omega^{d-1}$, we write

(5.11) $$i(K,q,u) := \begin{cases} 1 - \lim\limits_{\delta\downarrow 0}\lim\limits_{\varepsilon\downarrow 0} \chi(K \cap H_{u,<q,u>+\varepsilon} \cap B(q,\delta)) & \text{if} \quad q \in K \\ 0 & \text{if} \quad q \notin K. \end{cases}$$

If K is convex, then $i(K,q,u) = 1$ if u is an exterior normal vector to K at q, and $i(K,q,u) = 0$ otherwise. Again, $i(\cdot,q,u)$ is a valuation on $U(\mathscr{K}^d)$. In analogy to a differential geometric notion, one may call q a *critical point* of K with respect to the height function $\langle \cdot,u\rangle$ if $i(K,p,u) \neq 0$. This analogy extends in so far as the "critical point theorem"

(5.12) $$\sum_{q \in K} i(K,q,u) = \chi(K)$$

is valid for σ-almost all directions u.

For $K \in \mathscr{K}^d$ and a Borel set $\beta \subset \mathbb{E}^d$ it is easy to see that

(5.13) $$C_0(K,\beta) = \int_\Omega \sum_{q \in \beta} i(K,q,u)\,d\sigma(u),$$

saying that $C_0(K,\beta)$ measures the area of the spherical image of $\partial K \cap \beta$. By additivity, (5.13) holds for arbitrary $K \in U(\mathscr{K}^d)$, thus yielding a geometric interpretation of C_0 as measuring the spherical image "with multiplicity". A particular consequence of (5.12) and (5.13) is the equality

(5.14) $C_0(K,\mathbb{E}^d) = \sigma(\Omega^{d-1})\chi(K).$

Although this seems trivial, since it is evident for convex K and both sides are valuations in K, it has the nontrivial interpretation of expressing that the Euler characteristic can be obtained by adding up the local information provided by the curvature measure C_0. Thus (5.14) is an analogue of the Gauss-Bonnet theorem of differential geometry. For a polyhedron $P \in U(\mathscr{P}^d)$, Hadwiger [1969b] gave a different, though equivalent, definition for the curvature $C_0(P,\{p\})$ and proved (5.14).

If $P \in U(\mathscr{P}^d)$ is the point set of a cell complex of which Δ^k is the set of k-cells, then the index defined above satisfies

(5.15) $$i(P,q,u) = \sum_{k=0}^{d}(-1)^k \sum_{Z \in \Delta^k} i(Z,q,-u).$$

For the special case of the boundary complex of a convex polytope P, this equality, which is related to the Euler-type theorems to be discussed in §12, was proved by Shephard [1968c], Lemma (13). Due to the valuation property of the index, (5.15) can be extended to the general case by means of an argument by Perles-Sallee [1970]. A definition equivalent to (5.15) was used by Banchoff [1967], [1970], who discussed critical point theory, curvature, and the Gauss-Bonnet theorem for polyhedra. Finally we mention that for polyhedral cell complexes a purely combinatorial (and very elementary) analogue of the Gauss-Bonnet theorem was proposed by Schneider [1977b].

As we have seen, some of the classical valuations derived from volume have natural additive extensions to the sets of the convex-ring $U(\mathscr{K}^d)$. There remains the question whether there exist more general sets to which these functions can be additively extended in a reasonable and useful way. Since, for instance, the integral geometric formulae involving quermassintegrals have applications in geometric probability theory and stereology and thus to practical problems, one would like to have those formulae available for fairly general sets which might serve as approximate models for material bodies occurring in reality. This motivation led Hadwiger [1959] to the introduction of his so-called normal bodies. Roughly speaking, their definition is chosen in such a way that Hadwiger's inductive definition (2.3) of the Euler characteristic can be carried over, and then the quermassintegrals of normal bodies can be defined by means of (5.1). As Lenz [1970] remarks, it seems difficult to prove that point sets which "normally" occur (he mentions the example of compact solution sets of finitely many analytic inequalities) are normal in Hadwiger's sense.

In a different direction, the quermassintegrals have been extended beyond the convex-ring by Groemer [1972]. He considers his vector space A^d of approximable functions, which was mentioned in §2. Since the Euler characteristic and projections of functions in A^d are available, it is possible to use an analogue of Kubota's recursion formula for the quermassintegrals to extend the latter from $\mathscr{K}^d$ to continuous linear functionals on A^d. In particular, this yields an additive extension W_i of the quermassintegral to the class $\mathscr{S}_A$ of subsets of $\mathbb{E}^d$ whose characteristic function belongs to A^d. Among other results, Groemer shows that

(5.16) $W_i(\text{relint } K) = (-1)^{d-i+\dim K} W_i(K)$

for $K \in \mathscr{K}^d$. However, it seems difficult to describe the sets of $\mathscr{S}_A$ geometrically.

A class of point-sets with an easy intuitive definition was considered in this context by Federer [1959]. A subset K of $\mathbb{E}^d$ is called *of positive reach* if there exists a number $\varepsilon > 0$ such that, for each $x \in \mathbb{E}^d$ with distance less than ε from K, there is a unique point $p(K,x)$ in M nearest to x. For such sets, Federer was able to show that (3.20) holds (for $\rho < \varepsilon$) and yields signed measures $C_0(K,\cdot),\ldots,C_{d-1}(K,\cdot)$, the curvature measures of K. These are valuations, for compact K the Gauss-Bonnet formula (5.14) (where the topological definition of χ is used) holds, and the curvature measures satisfy generalizations of the kinematic and Crofton formulae of integral geometry. A special case is the principal kinematic formula for compact sets K,L of positive reach,

(5.17) $\int \chi(K \cap gL)\, d\mu(g) = \sum_{k=0}^{d} c_{d,k} W_k(K) W_{d-k}(L),$

where the integration is over the group of rigid motions of $\mathbb{E}^d$ with Haar measure μ, the $c_{d,k}$ are certain positive numbers, and $dW_k(K) = C_{d-k}(K,K)$ for $k = 1,\ldots,d, W_0(K) = V(K)$. From the point of view explained above, a class of subsets of $\mathbb{E}^d$ containing $\mathscr{K}^d$ to which the Euler characteristic and the quermassintegrals can be additively extended, gains in interest if (5.17) can be proved for K,L in this class. For the class of normal bodies and for the class $\mathscr{S}_A$ of sets with approximable characteristic functions, such a result is not known. To be sure, neither of these classes nor the class of sets of positive reach is intersectional, but in (5.17), $\chi(K \cap gL)$ need only be defined μ-almost everywhere on the motion group. In fact, Federer [1959] shows that for K,L of positive reach, $K \cap gL$ is of positive reach for almost all g.

On the other hand, the class of sets of positive reach has the disadvantage that it does not contain the convex-ring $U(\mathscr{K}^d)$. Perhaps the system of all finite unions of (compact) sets of positive reach is a good class of point sets to consider in integral geometry. However, it seems unknown whether Federer's curvature measures admit additive extensions to this class.

We mention that certain generalizations of Federer's curvature measures to arbitrary closed sets have been constructed by Stachó [1979], but these do not have the valuation property. Kuiper [1971] has used singular homology to define a sequence of curvature measures including a generalization of C_0. In either case it seems difficult to work with these measures except in the special cases already known.

Let us now turn to the extension problem for the mixed volume. This is the function, also denoted by V, on $(\mathscr{K}^d)^d$ which sends the d-tuple $(K_1,\ldots,K_d)$ of convex bodies into $V(K_1,\ldots,K_d)$, the coefficient of $\lambda_1 \cdots \lambda_d$ in (3.1) for $k = d$. Since the mixed volume is Minkowski additive in each argument, only such extensions to more general domains are of interest for which this property is maintained in some sense. Now Minkowski or vector addition is defined for arbitrary subsets of $\mathbb{E}^d$. But, as Groemer [1977a] (p. 141) remarks, "this generalized concept of addition is not suitable as a basis for a theory of mixed volumes that applies to a reasonably large class of non-convex sets". In a strong sense, this was made clear by Weil [1975a] who showed the following. Suppose that $\mathscr{K}'$ is a class of compact subsets of $\mathbb{E}^d$ containing $\mathscr{K}^d$ and closed under vector addition. If there exists a function $V:(\mathscr{K}')^d \to \mathbb{R}$ which is Minkowski additive in each variable and for which $V(K,\ldots,K)$ is the Lebesgue measure of $K \in \mathscr{K}'$, then $\mathscr{K}' = \mathscr{K}^d$. Thus it appears that a theory of mixed volumes of point sets which is satisfactory from a geometric point of view is only possible for convex sets. But extensions of the concept of mixed volumes beyond the field of convexity are possible if functions instead of point sets are considered.

The following such extension is particularly useful. Clearly the mixed volume V can be viewed as a function defined on a subset of $C(\Omega)^d$, where $C(\Omega)$ denotes the real vector space of continuous real functions (with the maximum norm) on Ω, just by identifying a convex body with its support function on Ω. Then V can be linearly extended to differences of support functions. By use of uniform continuity, in one argument a further linear extension to continuous functions is possible (as mentioned already in connexion with (3.14)). This procedure occurs, as a technical device, already in the important work of Aleksandrov [1937a,b]. Later the extension of mixed volumes (and a similar one for mixed area functions) was exploited in two papers by Weil [1974a,b]. One might ask (compare Weil [1974a], pp. 355–356) whether a further extension of the mixed volume to a continuous d-linear functional on $C(\Omega)^d$ is possible, but this has been answered in the negative by Meier [1982]. Yet further progress along these lines is possible, as shown by Weil [1981], whose starting point was the representation (3.14). That equality exhibits the linearity of V in its first argument, but it does not reflect the symmetry of V in its arguments. Therefore, one might ask whether there exists a completely symmetric analytic representation of the mixed volume involving each of the support functions linearly. Weil [1981] was able to show that there exists a distribution T on $(\Omega^{d-1})^d$ such that

$$V(K_1,\dots,K_d) = T\big(h(K_1,\cdot)\otimes\cdots\otimes h(K_d,\cdot)\big),$$

where $\otimes$ denotes the tensor product. As a consequence, there exists a sequence $(f_j)_{j\in\mathbb{N}}$ of C^∞-functions on $(\Omega^{d-1})^d$ such that

$$V(K_1,\dots,K_d) = \lim_{j\to\infty}\int_{\Omega^{d-1}}\cdots\int_{\Omega^{d-1}} h(K_1,u_1)\cdots h(K_d,u_d)f_j(u_1,\dots,u_d)\,d\sigma(u_1)\cdots d\sigma(u_d)$$

uniformly for all $K_1,\dots,K_d$ in some fixed ball.

Quite a different extension problem arises if convex bodies are identified with their characteristic functions. An interesting theory of mixed volumes on $V(\mathscr{K}^d)^d$ was developed by Groemer [1977a]. He first shows that there exists a unique bilinear map ψ of $V(\mathscr{K}^d)\oplus V(\mathscr{K}^d)$ into $V(\mathscr{K}^d)$ such that

$$\psi(K^*,L^*) = (K+L)^* \quad \text{for} \quad K,L\in\mathscr{K}^d,$$

and that the vector space $V(\mathscr{K}^d)$ together with the multiplication $\times$ defined by $f\times g := \psi(f,g)$ is a commutative algebra over $\mathbb{R}$ with unit element. The Euler characteristic $\bar\chi$ on $V(\mathscr{K}^d)$ (as defined in §2) is an algebra homomorphism. To every affine map $a:\mathbb{E}^d\to\mathbb{E}^d$ there exists a unique linear map $\alpha: V(\mathscr{K}^d)\to V(\mathscr{K}^d)$ such that $\alpha(K^*) = (aK)^*$ for $K\in\mathscr{K}^d$. In the special case $ax = \lambda x$ $(x\in\mathbb{E}^d)$ with $\lambda\in\mathbb{R}^+$ one writes $\alpha(f) =: \lambda\circ f$ for $f\in V(\mathscr{K}^d)$. Clearly there is a unique linear functional $\bar V$ on $V(\mathscr{K}^d)$ such that $\bar V(K^*)$ is the volume of $K\in\mathscr{K}^d$. Groemer shows that (3.2) can be generalized as follows. Let $k_1,\dots,k_d$ be nonnegative integers so that $k_1+\cdots+k_d = d$ and write $k=(k_1,\dots,k_d)$. There exists exactly one d-linear mapping v_k of $V(\mathscr{K}^d)^d$ into $\mathbb{R}$ so that

$$v_k(K_1^*,\dots,K_d^*) = V(K_1,k_1;\dots;K_d,k_d) \quad \text{for} \quad K_1,\dots,K_d\in\mathscr{K}^d.$$

For $f_1,\dots,f_d\in V(\mathscr{K}^d)$ and $\lambda_1,\dots,\lambda_d\geq 0$ one has

$$\bar V((\lambda_1\circ f_1)\times\cdots\times(\lambda_d\circ f_d)) = \sum\binom{d}{k_1\cdots k_d}\lambda_1^{k_1}\cdots\lambda_d^{k_d}\,v_{(k_1,\dots,k_d)}(f_1,\dots,f_d).$$

The functions $v_{(k_1,\dots,k_d)}$ are called mixed volumes, and Groemer investigates their properties carefully. We remark that mixed area functions could also be extended similarly and that some parts of the convex theory, for instance certain integral geometric formulae for mixed area functions, carry ever (see Schneider [1980], Bemerkung 9). Similar extensions of some integral geometric formulae for mixed volumes have been given by Groemer [1977b]. But, on the whole, the theory seems rather algebraic in character, and it loses much of its elegance if one

tries to go back to point sets whose characteristic functions belong to $V(\mathscr{K}^d)$, and to the generalized mixed volumes of such sets.

So much for the extensions of special valuations derived from the notion of volume. Now we consider the general problem of extending an arbitrary valuation on a class $\mathscr{S}$ (of subsets of some set S) to a valuation on $U(\mathscr{S})$ or to a linear functional on $V(\mathscr{S})$ as described in §1.

Volland [1957] has proved that every valuation on the class $\mathscr{P}^d$ of convex polytopes admits a unique additive extension to the class $U(\mathscr{P}^d)$ of polyhedra (related but simpler extension results can be found in Hadwiger [1957], p. 81, Böhm-Hertel [1980], p. 47). Volland first uses induction to prove the following result.

(5.18) **Theorem.** *Every valuation on $\mathscr{P}^d$ satisfies the inclusion-exclusion principle.*

The rest of Volland's proof is not restricted to the special geometric situation. Together with the obvious remark made in §1 after (1.2), it shows the following.

(5.19) **Theorem.** *Let φ be a valuation on an intersectional class $\mathscr{S}$. Then φ has an additive extension to $U(\mathscr{S})$ if and only if φ satisfies the inclusion-exclusion principle. The extension is unique.*

Volland's theorem was rediscovered by Perles-Sallee [1970]. They prove (5.19) (in its abstract form) in essentially the same way, and for the assertion of (5.18) they refer to Sallee [1966], where the corresponding result was obtained for the Steiner point. The proof (which is similar to Volland's) holds for arbitrary valuations on $\mathscr{P}^d$. Sallee [1966] subsequently used the continuity of the Steiner point to show that the Steiner point satisfies the inclusion-exclusion principle on $\mathscr{K}^d$. Spiegel [1976b] noticed that here one does not really need the continuity; he proved that any Minkowski additive function from $\mathscr{K}^d$ into $\mathbb{E}^d$ which vanishes on centrally symmetric bodies satisfies the inclusion-exclusion principle. This can be generalized:

(5.20) **Theorem.** *Every Minkowski additive function on $\mathscr{K}^d$ (with values in an abelian group) satisfies the inclusion-exclusion principle, hence by (5.19) it can be extended to a valuation on the convex-ring $U(\mathscr{K}^d)$.*

The proof is very simple. By formula (5.2) the support function was extended, as a valuation, to $U(\mathscr{K}^d)$, hence it satisfies (1.2). Thus for $K = K_1\cup\cdots\cup K_m$ with $K,K_i\in\mathscr{K}^d$ $(i=1,\dots,m)$ we have

$$h(K,\cdot) = \sum_{r=1}^{m}(-1)^{r-1}\sum_{i_1<\cdots<i_r} h(K_{i_1}\cap\dots\cap K_{i_r},\cdot).$$

Observing that $h(\emptyset,\cdot) = 0$ by (5.2), we can write this in the form

$$K + \sum_{r\ \text{even}}\ \sum_{i_1<\cdots<i_r}{}' K_{i_1}\cap\cdots\cap K_{i_r} = \sum_{r\ \text{odd}}\ \sum_{i_1<\cdots<i_r}{}' K_{i_1}\cap\cdots\cap K_{i_r},$$

where the sums $\sum'$ extend only over those $K_{i_1}\cap\cdots\cap K_{i_r}$ which are non-empty. If now φ is a Minkowski additive function on $\mathscr{K}^d$, we may apply φ to either side of this equality and deduce that φ satisfies (1.2). This completes the proof of (5.20).

We turn to the question of linear extensions to the real vector space $V(\mathscr{S})$ spanned by the characteristic functions A^* (defined on S) of the sets $A \in \mathscr{S}$. Here we assume that φ is a function on $\mathscr{S}$ with values in some real vector space $\mathscr{X}$. By a *linear extension* of φ to $V(\mathscr{S})$ we understand a linear map $\bar{\varphi}$ from $V(\mathscr{S})$ into $\mathscr{X}$ for which $\bar{\varphi}(A^*) = \varphi(A)$ for $A \in \mathscr{S}$. If φ admits such an extension, it must be a valuation (see §1). The following was proved by Groemer [1978]:

(5.21) **Theorem.** *Let φ be a function from the intersectional class $\mathscr{S}$ into a real vector space so that $\varphi(\emptyset) = 0$. Then the following statements are equivalent:*

(a) *φ has an additive extension* to $U(\mathscr{S})$,
(b) *φ has a linear extension* to $V(\mathscr{S})$,
(c) $\alpha_1 K_1^* + \cdots + \alpha_m K_m^* = 0$ *with* $K_i \in \mathscr{S}, \alpha_i \in \mathbb{R} (i = 1,\ldots,m)$

implies $\alpha_1\varphi(K_1) + \cdots + \alpha_m\varphi(K_m) = 0$.

The implications (b) $\Rightarrow$ (a) and (c) $\Rightarrow$ (b) are easy and were already discussed in §1, so the essentially new result is the fact that (a) (or, what is equivalent by (5.19), the inclusion-exclusion principle for φ on $\mathscr{S}$) implies (c). Using (5.21), Groemer also proved an extension theorem for valuations on $\mathscr{K}^d$. Let us say that the function φ from $\mathscr{K}^d$ into a topological (Hausdorff) vector space is *σ-continuous* if for every decreasing sequence $(K_i)_{i \in \mathbb{N}}$ in $\mathscr{K}^d$ one has

$$\lim_{i\to\infty} \varphi(K_i) = \varphi\left(\bigcap_{i\in\mathbb{N}} K_i\right).$$

Clearly continuity with respect to the usual Hausdorff metric on $\mathscr{K}^d$ implies σ-continuity. Groemer [1978] proved:

(5.22) **Theorem.** *Every σ-continuous weak valuation on $\mathscr{K}^d$ (with values in some topological vector space) has an additive extension to $U(\mathscr{K}^d)$.*

Apparently it is not known whether every valuation on $\mathscr{K}^d$ has an additive extension to $U(\mathscr{K}^d)$. About valuations on classes which are not intersectional, only a recent result of Stein [1982] is known. He showed that every valuation on $\mathscr{P}^d_L$, the class of convex lattice polytopes in $\mathbb{E}^d$, satisfies the inclusion-exclusion principle if it is invariant under lattice translations. The latter restriction has recently been removed by Betke.

We conclude this section with a remark concerning valuations on polytopes. Let φ be a valuation on $\mathscr{P}^d$ with values in a real vector space (this could be generalized). By (5.18), (5.19), (5.21) there is a linear function $\bar{\varphi}$ on $V(\mathscr{P}^d)$ such that $\bar{\varphi}(P^*) = \varphi(P)$ for $P \in \mathscr{P}^d$. We extend the definition of φ by means of $\varphi(P) := \bar{\varphi}(P^*)$ for each set P for which $P^* \in V(\mathscr{P}^d)$. It is not difficult to see that these sets are precisely the finite unions of relatively open convex polytopes. Thus φ as just defined is a valuation on $U(\mathscr{P}^d_{ro})$. The value of φ on a relatively open polytope is

given by

(5.23) $\varphi^0(P) := \varphi(\text{relint } P) = \sum_F (-1)^{\dim P - \dim F} \varphi(F)$

for $P \in \mathscr{P}^d$, where the sum extends over the faces F of P (including P). The proo[f is] easily done by induction with respect to dim P if one uses the Euler relation[s for] faces $F \subseteq G$ of a polytope. Equivalently, (5.23) results from the obvious form[ula]

$$\varphi(P) = \sum_F \varphi(\text{relint } P)$$

(which holds since φ is a valuation on $U(\mathscr{P}^d_{ro})$) by applying the Möbius invers[ion] formula (see Rota [1964], p. 344, and observe that $\mu(F,P) = (-1)^{\dim P - \dim F}$). [For] a given valuation φ on $\mathscr{P}^d$, Sallee [1968] defined

(5.24) $\varphi^*(P) := \sum_F (-1)^{\dim F} \varphi(F)$ for $P \in \mathscr{P}^d$.

Formula (5.23) exhibits the meaning of φ^*, namely

(5.25) $\varphi^*(P) = (-1)^{\dim P} \varphi(\text{relint } P)$.

Using the additivity of φ on $U(\mathscr{P}^d_{ro})$ it is easy to see that φ^* is a valuation on [$\mathscr{P}^d$]. Sallee proved this in a different way.

The derived valuation φ^* will be taken up in §12.

II. Dissections and simple valuations

Valuations, and particularly simple valuations, which are invariant under a giv[en] group G of transformations acting on $\mathbb{E}^d$, are intimately connected wi[th] equidissectability of polytopes with respect to the same group G. We are going [to] review the more recent developments in this area. The reader will find usef[ul] introductions and extensive references to the older literature in the books [of] Hadwiger [1957] and Boltianskii [1978] (see also Boltianskii [1963]). For speci[al] aspects also the shorter survey articles by Hadwiger [1968c], [1975] and Hert[el] [1977] may be helpful.

We adopt the convention here that "valuation", without specification of t[he] range, means real valued valuation.

§6. The algebra of polytopes

Let us define our terms here. Let P and Q be two d-polytopes, or, more generall[y] elements of $U(\mathscr{P}^d)$. A *dissection* of P is an expression of P in the form $P =$ $P_1 \cup \cdots \cup P_k$, where $P_1,\ldots,P_k \in \mathscr{P}^d$ (or $U(\mathscr{P}^d)$) satisfy $\text{int}(P_i \cap P_j) = \emptyset$ fo[r] $i \neq j$. We write this as

$$P = P_1 \dot{\cup} \cdots \dot{\cup} P_k \quad \text{or} \quad P = \dot{\bigcup}_{i=1}^k P_i.$$

We say that P and Q are *equidissectable with respect to* G, or *G-equidissectable*, i[f] there are dissections

$$P = P_1 \dot{\cup} \cdots \dot{\cup} P_k, \quad Q = Q_1 \dot{\cup} \cdots \dot{\cup} Q_k,$$

such that $Q_i = g_i(P_i)$ for some $g_i \in G$ ($i = 1,\dots,k$); we write this as $P \approx_G Q$. We call P and Q *equicomplementable with respect to* G, or G-*equicomplementable*, if there are P', P'', Q', Q'', such that $P'' = P \mathbin{\dot\cup} P'$, $Q'' = Q \mathbin{\dot\cup} Q'$, and $P' \approx_G Q'$, $P'' \approx_G Q''$; we write this as $P \sim_G Q$. (The corresponding terms in Hadwiger [1957], where most of the early work on this topic is collected, are "zerlegungsgleich" and "ergänzungsgleich", respectively. A variety of terms have been used in English; we have chosen to use those above because they more exactly convey the meaning of their definitions, and, equally importantly, do not have any other meaning within convexity.)

As examples of groups G commonly encountered, we have the full group T of translations, the group TH of translations and reflexions in points (half turns in $\mathbb{E}^2$), the group D of isometries of $\mathbb{E}^d$, its subgroup SD of direct isometries or rigid motions, the full affine group A, and the group EA of equiaffine mappings (determinant $= \pm 1$). (We also refer later to the orthogonal group O and the rotation subgroup SO.) In what follows, we shall always assume G to have T at least as a subgroup.

Before we go any further, let us mention two results by Hadwiger [1957].

(6.1) **Lemma.** *Let* $P, Q \in \mathscr{P}^d$. *Then* $P \approx_G Q$ *if and only if* $P \sim_G Q$.

The validity of this result depends upon the field $\mathbb{R}$ being archimedean.

Let φ be a simple valuation on $\mathscr{P}^d$. We say that φ is G-*invariant* if $\varphi(gP) = \varphi(P)$ for all $g \in G$. Then we have

(6.2) **Theorem.** *Let* $P, Q \in \mathscr{P}^d$. *Then* $P \approx_G Q$ *if and only if* $\varphi(P) = \varphi(Q)$ *for every real valued* G-*invariant simple* φ.

Unfortunately, Hadwiger's proof is highly non-constructive, depending as it does on using the axiom of choice to pick a basis of the polytope group Π_G, which we are about to define below. In fact, the problem which will concern us here and in §§8 and 9 can be succinctly phrased as:

(6.3) **Problem.** *Given a group* G, *find a "nice" subfamily of* G-*invariant simple valuations which separates* Π_G.

In other words, we would wish to find an easily describable family $\mathscr{V}$ of such valuations φ, so that $P \approx_G Q$ if and only if $\varphi(P) = \varphi(Q)$ for all $\varphi \in \mathscr{V}$.

We define the polytope group Π_G as follows. Let $\mathbb{Z}\mathscr{P}^d$ be the free abelian group with the d-polytopes as basis elements; we write $\boldsymbol{+}$ and $\boldsymbol{-}$ for the group operations, to distinguish them from the usual + and − in $\mathbb{E}^d$. Let $\mathscr{T} = \mathscr{T}_G$ be the subgroup generated by all elements of the form $P \boldsymbol{-} P_1 \boldsymbol{-} P_2$ and $P \boldsymbol{-} gP$, where $P = P_1 \mathbin{\dot\cup} P_2$ and $g \in G$. In other words, $\mathscr{T}$ is generated by all $P \boldsymbol{-} Q$, with $P \approx_G Q$. The quotient group

$$\Pi_G^d = \mathbb{Z}\mathscr{P}^d / \mathscr{T}_G$$

is called the *polytope group with respect to* G. We also use notations such as $\Pi_G(\mathbb{E}^d)$, as we shall wish later to write $\Pi(L) = \Pi_T(L)$ with the obvious meaning for linear subspaces L of $\mathbb{E}^d$. Until we wish to emphasize the dimension d, we shall write Π_G instead of Π_G^d.

We now make an obvious remark.

(6.4) *Let* $\mathscr{X}$ *be a group. The (group) homomorphisms* $\varphi: \Pi_G \to \mathscr{X}$ *are precisely those mappings induced by the* G-*invariant simple valuations* $\varphi: \mathscr{P}^d \to \mathscr{X}$.

In such cases, we shall often use the same letter to denote two such related mappings.

For a number of reasons, the basic polytope group Π_G we need to investigate is $\Pi = \Pi_T$. For one thing, the familiar group isomorphism theorem gives:

(6.5) *Let* G *be a group containing* T, *and let* $\Sigma_G = \mathscr{T}_G / \mathscr{T}_T$ *be the subgroup of* Π *corresponding to* $\mathscr{T}_G$. *Then* $\Pi / \Sigma_G \cong \Pi_G$.

That is, every polytope group Π_G is a suitable quotient of Π.

After these preliminaries, we are now ready to embark upon our investigation of Π. We shall give the main results, always without proof, following the treatments of Jessen-Thorup [1978] and Sah [1979]. Our notation will be closer to that of Jessen-Thorup; Sah's is more comprehensive, but also more complicated.

An important role is played in these investigations by the *dilatation operator* m, which is induced by the corresponding operator μ on $\mathscr{P}^d$, defined by $\mu(\lambda)P = \lambda P = \{\lambda x \mid x \in P\}$, for $\lambda > 0$. That is, $m(\lambda)[P] = [\mu(\lambda)P]$, where $[P]$ is the equivalence class of P under $\approx \; = \; \approx_T$. For $\lambda < 0$, we define $m(\lambda) = (-1)^d m(-\lambda)$, and also $m(0) = 0$. (The choice of $(-1)^d$ is due to our considering unoriented polytopes. Further justification comes from Theorem (7.2); compare also (6.6) below.)

We call a polytope P a (basic) r-cylinder, if there are independent linear subspaces $L_1,\dots,L_r$ of $\mathbb{E}^d$, whose dimensions $d_i = \dim L_i$ are positive and satisfy $d_1 + \dots + d_r = d$, and d_i-polytopes P_i in L_i, with $P = P_1 + \dots + P_r$. We denote by $\mathfrak{Z}_r$ the family of all r-cylinders in $\mathscr{P}^d$, and by

$$Z_r = (\mathbb{Z}\mathfrak{Z}_r \boldsymbol{+} \mathscr{T}_T)/\mathscr{T}_T$$

the corresponding subgroup of Π. Thus we have

$$\Pi = Z_1 \supseteq Z_2 \supseteq \dots \supseteq Z_d \supseteq Z_{d+1} = \{0\}.$$

In fact, as will be clearer later, all these inclusions are strict. We notice as well that we have a natural embedding

$$\Pi(L_1) \otimes \dots \otimes \Pi(L_r) \mapsto \Pi;$$

we shall write $x_1 \times \dots \times x_r$ for the image of $(x_1,\dots,x_r)$ under this embedding (the tensor product is, as yet, only over $\mathbb{Z}$).

We write $[a_1, a_2, \dots, a_d]$ to denote the equivalence class of the simplex with vertices $a_0, a_0 + a_1, a_0 + a_1 + a_2, \dots, a_0 + a_1 + \dots + a_d$, for $a_0 \in \mathbb{E}^d$, where $\{a_1,\dots,a_d\}$ is linearly independent. We have the two canonical simplex dissections:

(6.6) **Theorem.** $m(\lambda + \mu)[a_1,\dots,a_d]$

$$= \sum_{j=0}^{d} \{m(\lambda)[a_1,\dots,a_j] \times m(\mu)[a_{j+1},\dots,a_d]\}.$$

(6.7) **Theorem.** *Let* n *be a nonnegative integer. Then*

$$m(n)[a_1,\ldots,a_d] = \sum_{r=1}^{d} \binom{n}{r} z_r,$$

where $z_r = \sum_{1 \leq j_1 \leq \cdots \leq j_{r-1} < d} [a_1,\ldots,a_{j_1}] \times \cdots \times [a_{j_{r-1}+1},\ldots,a_d] \in Z_r$.

The general term is $[a_{j_s+1},\ldots,a_{j_{s+1}}]$.

For future reference, it is helpful to take note of the actual translations involved in (6.7). In a different notation, let T be the d-simplex $T = \text{conv}\{x_0,\ldots,x_d\}$, and write

$$T(j_1,\ldots,j_{r-1}) = \text{conv}\{x_0,\ldots,x_{j_1}\} + \cdots + \text{conv}\{x_{j_{r-1}},\ldots,x_d\}.$$

Here the general term is $\text{conv}\{x_{j_s},\ldots,x_{j_{s+1}}\}$. Then we have (compare McMullen [1977])

$$\mu(n)T = \dot{\bigcup}_{r=1}^{d} \dot{\bigcup}_{0<j_1<\cdots<j_{r-1}<d} \dot{\bigcup}_{k_i \geq 0, \Sigma k_i = n-r} \left(T(j_1,\ldots,j_{r-1}) + \sum_{i=0}^{r} k_i x_{j_i} \right).$$

Next, we have a fundamental result, from which much else follows.

(6.8) **Theorem.** *Let* $\mathbb{E}^d = L_1 \oplus L_2$ *(direct sum), and let* $x_i \in \Pi(L_i)$ $(i = 1,2)$. *Then, for all* $\lambda,\mu \in \mathbb{R}$,

$$x_1 * x_2 = m(\lambda)x_1 \times m(\mu)x_2 - m(\mu)x_1 \times m(\lambda)x_2 \in Z_3.$$

From (6.6) follows directly

$$\sum_{j=1}^{d-1} [a_1,\ldots,a_j] * [a_{j+1},\ldots,a_d] \in Z_3.$$

(Curiously, Sah fails to notice that it is a trivial consequence.) Jessen-Thorup then deduce (6.8) from this by a geometric lemma due to Thorup, while Sah uses a more algebraic argument to give the same deduction.

The next results concern various properties of the map $m: \mathbb{R} \to \text{End}\ \Pi$. We denote by $E \subseteq \text{End}\ \Pi$ the subring of endomorphisms of Π generated by the $m(\lambda)$. It is clear that m is multiplicative:

$$m(\lambda\mu) = m(\lambda)m(\mu).$$

Also $m(0) = 0$, $m(1) = 1$. Our aim is to show that $\mathbb{R}$ is embedded in E, so that Π is a real vector space. Our first step is to embed $\mathbb{Q}$ (the rationals) in E.

We define the operation $\Delta_\alpha (\alpha \in \mathbb{R})$ by

$$\Delta_\alpha m(\lambda) = m(\lambda + \alpha) - m(\lambda).$$

Then $\Delta_\alpha \Delta_\beta = \Delta_\beta \Delta_\alpha$. We write $\Delta_\alpha^s = \Delta_\alpha \cdots \Delta_\alpha$ (s times). It easily follows by an inductive argument that

$$m(q\alpha) = \sum_{p=0}^{d} \binom{q}{p} \Delta_\alpha^p m(0)$$

for every $\alpha \in \mathbb{R}$ and $q \in \mathbb{Z}$ (we must interpret $\binom{q}{p}$ here as $q(q-1)\cdots(q-p+1)/p!$). There follows:

(6.9) **Lemma.** *If* $k \in \mathbb{Z}\setminus\{0\}$, *then* $1/k \in E$. *Thus* E *is divisible, and* $\mathbb{Q} \subseteq E$.

For,

$$\frac{1}{k} = \frac{1}{k} m(1) = \sum_{p=1}^{d} \frac{d!}{p} \binom{kd!-1}{p-1} \Delta_{1/kd!}^{p} m(0).$$

Now, by induction from (6.6) follows

$$\Delta_{\alpha_p} \cdots \Delta_{\alpha_1} m(\lambda) x_{0d} = \sum_{0<i_1 \cdots < i_p \leq d} m(\alpha_1)x_{0i_1} \times \cdots \times m(\alpha_p)x_{i_{p-1}i_p} \times m(\lambda)x_{i_p d},$$

where we write $x_{ij} = [a_{i+1},\ldots,a_j]$. In particular,

$$\Delta_{\alpha_d} \cdots \Delta_{\alpha_0} m(\lambda) = 0$$

for all $\lambda,\alpha_0,\ldots,\alpha_d$. This means that $m(\lambda)$ exhibits polynomial like behaviour (which, using (6.9), is already clear from (6.7) for positive integral λ). We now define

$$\tilde{m}_d(\alpha_1,\ldots,\alpha_d) = \frac{1}{d!} \Delta_{\alpha_1} \cdots \Delta_{\alpha_d} m(\lambda)$$

(this is independent of λ, by the above), and for $r < d$,

$$\tilde{m}_r(\alpha_1,\ldots,\alpha_r) = \frac{1}{r!} \Delta_{\alpha_1} \cdots \Delta_{\alpha_r}(m - m_d - \cdots - m_{r+1})(\lambda),$$

where $m_s(\lambda) = \tilde{m}_s(\lambda,\ldots,\lambda)$. We also write $\xi_s = m_s(1)$, $\Xi_s = \xi_s \Pi$ and

$$\iota(\lambda) = \tilde{m}_1(\lambda) + \tilde{m}_2(1,\lambda) + \cdots + \tilde{m}_d(1,\ldots,1,\lambda).$$

We can conclude the description of the polytope group Π as follows:

(6.10) **Theorem.** *The dilatation operator* m *has the following properties:*

(i) $\xi_1,\ldots,\xi_d$ *are orthogonal idempotents, with* $1 = \xi_1 + \cdots + \xi_d$;
(ii) $\tilde{m}_s(\lambda_1,\ldots,\lambda_s) = \iota(\lambda_1 \ldots \lambda_s)\xi_s$;
(iii) $\iota: \mathbb{R} \to E$ *is a (non-trivial) ring homomorphism;*
(iv) E *is isomorphic to the product ring* $\mathbb{R}^d$, *the isomorphism being given by* $(\lambda_1,\ldots,\lambda_d) \mapsto \tilde{m}_1(\lambda_1)\xi_1 + \cdots + \tilde{m}_d(1,\ldots,1,\lambda_d)\xi_d$.

(6.11) **Theorem.** *The polytope group* Π *has the following properties:*

(i) $Z_r = \Xi_r \oplus \Xi_{r+1} \oplus \cdots \oplus \Xi_d$ $(r = 1,\ldots,d)$;
(ii) *each* Ξ_s *is a non-trivial real vector space, generated by the elements* $x_1 \times \cdots \times x_s$, *with* $x_i \in \Xi_1(L_i)$ $(i = 1,\ldots,s)$, *and* $\mathbb{E}^d = L_1 \oplus \cdots \oplus L_s$;
(iii) *the scalar multiplication in* Ξ_s *is given by* $\lambda(x_1 \times \cdots \times x_s) = (m(\lambda)x_1) \times \cdots \times x_s = \iota(\lambda)(x_1 \times \cdots \times x_s)$.

(iv) *the dilatation operator* m *acts on* Ξ_s *by*
$m(\lambda)x = \iota(\lambda^s)x, \quad x \in \Xi_s$.

Of course, in (6.11)(iii), to which factor x_j we apply $m(\lambda)$ is immaterial. We may remark that Ξ_d is isomorphic to $\mathbb{R}$ under the volume map.

It is now relatively easy to find a suitable family of functionals which separates Π. Let $\mathscr{U}^k$ be the Stiefel manifold of k-frames $U = (u_1,\ldots,u_k)$ of orthonormal sets in $\mathbb{E}^d$. If P is a polytope, we denote by P_u the face of P in direction $u \in \Omega$; that is, the face of P lying in the supporting hyperplane of P with outer normal u. We define $P_U(U = (u_1,\ldots,u_k))$ inductively by

$$P_{(u_1,\ldots,u_k)} = (P_{(u_1,\ldots,u_{k-1})})_{u_k}.$$

A *basic Hadwiger functional of type* r is a function φ_U defined by

$$\varphi_U(P) = \sum_{\varepsilon_i = \pm 1} \varepsilon_1 \cdots \varepsilon_{d-r} V_r(P_{(\varepsilon_1 u_1,\ldots,\varepsilon_{d-r} u_{d-r})})$$

where $U = (u_1,\ldots,u_{d-r}) \in \mathscr{U}^{d-r}$, and V_r is ordinary r-dimensional volume. (In case r = d and $U = \varnothing$, we have $\varphi_\varnothing = V$, ordinary volume.) We observe that $\varphi_U(P) = 0$ unless, for some choice $\varepsilon_i = \pm 1$,

$$\dim P_{(\varepsilon_1 u_1,\ldots,\varepsilon_j u_j)} = d - j$$

for $j = 1,\ldots,d-r$. Thus an unrestricted "linear" combination $\sum_U \varphi_U c_U$ ($c_U \in \mathscr{X}$, where $\mathscr{X}$ is now a real vector space) of basic Hadwiger functionals is always finite-valued on $\mathscr{P}^d$; such a linear combination is called a *Hadwiger functional*. We write $\mathscr{H}_r$ for the real linear space spanned by the basic Hadwiger functionals of type r.

It is easy to verify that a (basic) Hadwiger functional is a simple translation invariant valuation, so that it induces a linear mapping on Π, denoted by the same symbol.

(6.12) **Theorem.** *The Hadwiger functionals separate* Π.

The theorem is trivial for d = 1 (the only Hadwiger functionals are multiples of length). One now proceeds by induction. The proof is in two steps. Let H be a (linear) hyperplane, and $e \notin H$ a vector. Let Z be the subgroup of Π generated by the "prisms" $m(\lambda)[e] \times y$, where $y \in \Pi(H)$. If L is any hyperplane not containing e, we can construct homomorphisms $\rho_L : \Pi \to \Pi(L)$ and $\pi_L : \Pi(L) \to \Pi/Z$ as follows. We define ρ_L on $\mathscr{P}^d$ first. Let u be a fixed normal vector to L (say that lying on the same side of L as e); for $P \in \mathscr{P}^d$, define $\rho_L(P) = P_u - P_{-u}$. If $x \in \Pi$ is such that $\varphi(x) = 0$ for all Hadwiger functionals φ, then (by the induction assumption) $\rho_L(x) = 0$ for all L.

We again define π_L on $\mathscr{P}^{d-1}(L)$ first. Let $F \in \mathscr{P}^{d-1}(L)$. Translate F so that it lies in the open half space bounded by H containing e. Let $\bar{F}$ be the convex hull of F and its projection on to H in direction e. Then $\pi_L(F)$ is the equivalence class of $\bar{F}$ in Π modulo Z. It is easy to see that π_L is well defined.

For the equivalence class of $P \in \mathscr{P}^d$ modulo Z, we have

$$[P] = \sum_F \varepsilon\pi(F) \quad (\varepsilon = \pm 1),$$

where $\pi(F) = \pi_L(F)$ if F is parallel to L, and the sum is over the facets of P
parallel to e. We can write this as

$$[P] = \sum_L \pi_L(\rho_L(P)).$$

So, if $x \in \Pi$ is such that $\rho_L(x) = 0$ for all L, we have $x \in Z$ (since its equivale
class modulo Z vanishes).

Now Z is generated by all

$$m(\lambda)[e] \times y_1 \times \cdots \times y_t = [e] \times m(\lambda)y_1 \times \cdots \times y_t,$$

where $H = H_1 \oplus \cdots \oplus H_t$ and $y_i \in \Xi_1(H_i)(i = 1,\ldots,t)$. That is, if $x \in Z$, then
$[e] \times x^*$ for some $x^* \in \Pi(H)$. But now $\varphi(x) = 0$ for all Hadwiger functional
on Π is easily seen to imply $\varphi^*(x^*) = 0$ for all Hadwiger functionals φ^* on Π(
and so, by our induction assumption, $x^* = 0$, and hence $x = 0$. This then pro
Theorem (6.12).

Jessen and Thorup [1978] also give further results. For example:

(6.13) **Theorem.** *Let* $x \in \Pi$. *Then* $x \in Z_s$ *if and only if* $\varphi_U(x) = 0$ *for all b*
Hadwiger functionals φ_U *of type* r *with* r < s.

In fact, if $x \in \Xi_s$ and φ_U is of type r, then $\varphi_U(x) = 0$ unless r = s. In particu
this implies that each Ξ_s (s = 1,...,d) is non-trivial.

We can also show:

(6.14) **Theorem.** *Let* $\mathscr{X}$ *be any real vector space. Then all linear mappi*
$\varphi : \Pi \to \mathscr{X}$ *are of the form*

$$\varphi = \sum_U \varphi_U c_U$$

where $U \mapsto c_U$ *is an arbitrary function from frames into* $\mathscr{X}$.

Jessen and Thorup [1978] appeal to a "well-known theorem on vector spac
to prove this, but it is not clear to us what this theorem is. Sah [1979] explici
constructs bases in duality of Ξ_1 and $\mathscr{H}_1$, and then remarks that these can be u
inductively to find bases of Ξ_r and $\mathscr{H}_r$, not now in duality, but neverthel
showing that every (non-trivial) linear function on Ξ_r is in $\mathscr{H}_r$.

To conclude this section, we mention some other results on equidissectabili
If G is any group of transformations of $\mathbb{E}^d$ containing T, let $\mathscr{H}(G)$ denote
family of real valued G-invariant Hadwiger functions; that is, $\varphi \in \mathscr{H}(G)$
$\varphi(P) = \varphi(g(P))$ for each $P \in \mathscr{P}^d$ and $g \in G$. Then we have (Jessen and Thor
[1978]):

(6.15) **Theorem.** *Let* $x \in \Pi_G$. *Then* $x = 0$ *if and only if* $\varphi(x) = 0$ *for each*
$\mathscr{H}(G)$.

In most cases of interest, (6.15) is as useful as (6.2). However, in two cases, (6.
can be applied.

(6.16) **Theorem.** (a) *Two polytopes* P *and* Q *are* TH-*equidissectable if and onl*
$\varphi(P) = \varphi(Q)$ *for each* $\varphi \in \mathscr{H}_d + \mathscr{H}_{d-2} + \cdots$

(b) *Let* $\lambda \notin \{0,1,-1\}$, *and let* $G(\lambda)$ *be the subgroup generated by* T *and a dilatation of ratio* λ. *Then* $\Pi_{G(\lambda)} = \{0\}$.

In (a), TH is the group of translations and reflexions in points, and $\mathscr{H}_d + \mathscr{H}_{d-2} + \cdots$ is the sum of the $\mathscr{H}_r$ with d − r even (compare Hadwiger [1952e], see also Harazišvili [1978]).
(b) is due for $\lambda > 0$ to Meier [1972]; it says that any two d-polytopes are $G(\lambda)$-equivalent (see also Zylev [1968], Debrunner [1969], Hadwiger [1974b] and Harazišvili [1977]).

(6.16) (b) clearly implies that any two d-polytopes are A-equidissectable. It is also easy to prove:

(6.17) **Theorem.** *Two* d-*polytopes* P *and* Q *are* EA-*equidissectable if and only if* $V(P) = V(Q)$.

Jessen and Thorup [1978] and Sah [1979] base their description of Π on the fact that a general polytope can be dissected into simplices. Tverberg [1974] gives a particularly nice way of doing this: if a polytope P is not a simplex, it can be cut into two polytopes P_1 and P_2 by a hyperplane spanned by a (d − 2)-face and a vertex of P; P_1 and P_2 are then treated similarly, and after a finite number of such cuts, P is dissected into simplices. Such a dissection avoids the need to appeal to the inclusion-exclusion principle. Another method of finding a dissection into simplices is given immediately below.

Meier [1977] has a theory of "mixed polytopes", to generalize that of mixed volumes, but it appears that his proof is flawed at one point. We describe here an alternative approach, which in fact employs a more elementary method.

We use the "lifting theorem" of Walkup-Wets [1969], which (for our purposes) states that if a polyhedron Q is the image of a polyhedron P under an affine map Φ, then there is a subcomplex $\mathscr{C}$ of faces of P, such that Φ is one-to-one on set $\mathscr{C}$, and $\Phi(\text{set } \mathscr{C}) = Q$. This same lifting theorem can be used to show that, if X is a finite set in $\mathbb{E}^d$, then the polytope P = conv X admits a dissection into a simplicial complex, whose 0-cells are just the points of X; this can be proved by more direct methods.

We now apply the lifting theorem to the affine map

$$P_1 \times \cdots \times P_k \mapsto \lambda_1 P_1 + \cdots + \lambda_k P_k,$$

where the term on the right is a Minkowski linear combination of the polytopes P_i with non-negative coefficients λ_i. Since the d-faces of $P_1 \times \cdots \times P_k$ are of the form $F_1 \times \cdots \times F_k$, with F_i a face of $P_i (i = 1,\ldots,k)$ and $\sum_{i=1}^k \dim F_i = d$, we see that $\lambda_1 P_1 + \cdots + \lambda_k P_k$ admits a dissection into cylinders $\lambda_1 F_1 + \cdots + \lambda_k F_k$ (note that some F_i may be vertices, so these are not necessarily k-cylinders). Passing to the polytope group Π, expanding the terms $[\lambda_i F_i] = m(\lambda_i)[F_i]$ into their homogeneous components (in the appropriate polytope groups), and collecting together terms of the same degree, we now see that we have an expression

$$[\lambda_1 P_1 + \cdots + \lambda_k P_k] = \sum_{r_i \geq 0,\, 1 \leq r_1 + \cdots + r_k \leq d} \lambda_1^{r_1} \cdots \lambda_k^{r_k} \binom{r_1 + \cdots + r_k}{r_1 \cdots r_k} \cdot p(P_1,r_1;\cdots;P_k,r_k).$$

The *mixed polytope* $p(P_1,r_1;\ldots;P_k,r_k)$ is positive homogeneous of degree r_i in (and hence independent of P_i if $r_i = 0$); we have taken out the multinom coefficient as a factor, so that if $P_1 = \cdots = P_k = P$ and $r_1 + \cdots + r_k = r$, th $p(P_1,r_1;\ldots;P_k,r_k)$ is just the r-th homogeneous component (in Ξ_r) of [P]. (T notation is again as in (3.3).)

However, it must be emphasized that the mapping

$$(P_1,\ldots,P_k) \mapsto p(P_1,r_1;\ldots;P_k,r_k)$$

is not particularly nice, in that it does not induce a multi-linear mapping $\Pi \times \cdots \times \Pi$ (k-times). Indeed, if we replace P_j by $-P_j$, we obtain only Eul type relations, of the kind we discuss in §12 below.

We saw in (6.14) that the linear functionals on Π are precisely the unrestrict linear combinations of the Hadwiger functionals φ_U. It is thus a natural questi to ask about (linear) relations between the Hadwiger functionals, or, in oth words, about *syzygies* between them. The complete picture is, as yet, unclear, b Sah [1979] gives some partial results.

We first dismiss, with the barest mention, the trivial relationships between φ_U when the U's are obtained from each other merely by changing signs.

For the next relations, we recall that if the d-polytope P has n facets with u outer normal vectors u_i and areas A_i $(i = 1,\ldots,n)$, then $\sum_{i=1}^n u_i A_i = 0$. Applyi this to the facets of an (r + 1)-face, we deduce:

(6.18) **Theorem.** *Let* $U = (u_1,\ldots,u_{d-r-1})$ *be fixed. Then*

$$\sum_u u\, \varphi_{(U,u)} = 0,$$

where the sum extends over all u *orthogonal to* $u_1,\ldots,u_{d-r-1}$.

A (d − 2)-face of a d-polytope P is of the form $P_{(u,v)}$ for exactly two (u,v)'s, w u normal to a facet of P, and these pairs span the same 2-dimensional subspa but have opposite orientation. There easily follows:

(6.19) **Theorem.** *Let* $(u_1,\ldots,u_{s-1},u_{s+2},\ldots,u_{d-r})$ *be fixed. Then*

$$\sum\nolimits^* \varphi_U = 0,$$

where the sum extends over all $U = (u_1,\ldots,u_{d-r})$, *such that* (u_s,u_{s+1}) *i basis of given orientation of a given 2-dimensional subspace orthogonal the remaining* u_i*'s.*

There are no other known syzygies. Sah further discusses this topic in a m general (algebraic) context, but we shall refrain from following his example, a instead refer the interested reader to Sah [1979], §5.2–5.8.

§7. Simple valuations

Let φ be a simple valuation, taking values (for the moment) in a divisible abel group ($\mathbb{Q}$-module) $\mathscr{X}$. There follows directly from (6.7) and the subsequ remarks:

(7.1) **Theorem.** (a) *If* φ *is translation invariant, then for* $P \in \mathscr{P}^d$ *and rational* $\lambda \geq 0$, *there is a polynomial expansion*

$$\varphi(\lambda P) = \sum_{r=1}^{d} \lambda^r \varphi_r(P).$$

(b) *If* φ *is translation covariant, then for such* P *and* λ, *there is a polynomial expansion of degree* d + 1.
In each case, the function φ_r *is a translation invariant or covariant valuation on* $\mathscr{P}^d$, *which is (non-negative, rational) homogeneous of degree* r.

Translation covariance will be discussed in §10, where the term is defined.
We need some kind of continuity condition to extend the range of validity of (7.1) beyond the rationals, as the following example illustrates. Let $\theta: \mathbb{R} \to \mathbb{R}$ be a Cauchy-Hamel function:

$$\theta(\alpha + \beta) = \theta(\alpha) + \theta(\beta), \quad \theta(1) = 0;$$

if the axiom of choice is used to construct a basis for $\mathbb{R}$ over $\mathbb{Q}$, non-trivial Cauchy-Hamel functions can be found. Then, with V being as usual volume, $\varphi = \theta \circ V(\varphi(P) = \theta(V(P)))$ is a rational homogeneous translation invariant valuation which is not real homogeneous.
In this section, we wish only to note one additional result, which justifies our definition in §6 of $m(\lambda)$ for $\lambda < 0$.

(7.2) **Theorem.** *Let* φ *be a translation invariant simple valuation on* $\mathscr{P}^d$, *which is homogeneous of degree* r. *Then*

$$\varphi(-P) = (-1)^{d-r}\varphi(P).$$

In view of the description of the polytope group Π in §6, it follows that the behaviour of φ is completely determined by its behaviour on r-cylinders: $\varphi \equiv 0$ if and only if $\varphi(P) = 0$ for all $P \in \mathfrak{Z}^d_r$. If $\mathbb{E}^d = L_1 \oplus \cdots \oplus L_r$, with $d_i = \dim L_i$ $(i = 1,\ldots,r)$, then for $P = P_1 + \cdots + P_r$, $\varphi(P) = \varphi(P_1 + \cdots + P_r) = \tilde{\varphi}(P_1,\ldots,P_r)$ induces a simple valuation on each $\mathscr{P}(L_i)$, which is homogeneous of degree 1. So, $\varphi(-P) = (-1)^{d-r}\varphi(P)$ will follow from $\tilde{\varphi}(\ldots,-P_i,\ldots) = (-1)^{d_i-1}\tilde{\varphi}(\ldots,P_i,\ldots)$.
The case r = 1 is established by induction on d. For d = 1 it is trivial (−P is a translate of P). For $d \geq 2$, φ vanishes on $\mathfrak{Z}^d_2$. We now use an argument similar to that of (6.12). Let H be a linear hyperplane, and e a vector not in H. Let L be a general hyperplane not parallel to H. We define φ_L as follows. Translate $F \in \mathscr{P}^{d-1}(L)$ so that it lies on the positive side of H relative to e, let F′ be the image of F under parallel projection onto H in direction e, and define $\bar{F} = \operatorname{conv}(F \cup F')$. Since two such $\bar{F}$ differ only by a prism with upright e, $\varphi_L(F) = \varphi(\bar{F})$ is well-defined. Then for $P \in \mathscr{P}^d$, $\varphi(P) = \sum \varepsilon\varphi_L(F)$, where $\varepsilon = \pm 1$ as F is a positive or negative face, and from

$$\varphi(-P) = \sum(-\varepsilon)\varphi_L(-F) = (-1)(-1)^{d-2}\sum \varepsilon\varphi_L(F)$$
$$= (-1)^{d-1}\varphi(P)$$

(7.2) now follows.
Theorem (7.2) also turns out to be at the basis of the Euler-type theorems which we shall discuss in §12.
Theorems (7.1) (with non-negative integer coefficients) and (7.2) apply equally to valuations on lattice polytopes which are invariant under integer translations. In McMullen [1975b, 1977], where these generalizations were first proved, i seemed necessary to assume that the valuation φ satisfied the inclusion-exclusion principle, but Stein [1982] has recently shown how to remove this assumption.
Of course, (7.1) and (7.2) also apply directly to valuations on rational polytope invariant under rational translations. The extension to such valuations invarian only under integer translations is discussed in McMullen [1978, 1982b].
Theorems (7.1) and (7.2) are due, as are (6.6) and (6.7) on which they depend, to series of papers culminating in Hadwiger [1957]. However, Hadwiger usuall also imposes certain continuity or monotoneity conditions; for a discussion o their implications, see §11 below.

§8. Spherical dissections

It is natural to pose the same questions about equidissectability of polytopes i any spaces in which the question is reasonable; in particular, the spherical (c elliptic) and hyperbolic spaces. What little we shall say about the hyperbolic cas we postpone to the end of this section. However, it has become clear that a understanding of the spherical case is a necessary prerequisite for the unde standing of the general euclidean case.
We write $\Omega = \Omega^{d-1}$ for the unit sphere in $\mathbb{E}^d$. According to §1, a *spheric polytope* in Ω is an intersection $P = \Omega \cap K$ of Ω with a polyhedral cone K wi apex 0. The dimension of P is thus $\dim P = \dim K - 1$. We often identify P wi the corresponding cone K; as we shall see, for many reasons, it is more convenie to work with the cones K.
The only groups acting on Ω which we should naturally wish to consider a the full orthogonal group $O = O_d$, and its subgroup $SO = SO_d$ of rotations.
The concepts of equidissectability and O- or SO-equivalence of spheric (d − 1)-polytopes are defined in the obvious way, and lead immediately to t *spherical polytope group* Σ^d. For d = 0, it is convenient to define Σ^0 $\mathbb{Z} = \mathbb{Z}.[\varnothing]$. Clearly, $\Sigma^1 \cong \mathbb{Z}$ also. In what follows, we shall sketch a descripti of what is currently known about Σ^d; this is largely taken from Sah [1979].

(8.1) **Theorem.** (a) Σ^2 *is divisible;*
(b) Σ^d *is 2-divisible for* $d \geq 3$.

Part (a) is obvious. It is enough to prove (b) for a simplex T. Let T have fac $F_1,\ldots,F_d$, and let $G_{ij} = F_i \cap F_j$ $(i \neq j)$. Let the insphere of T have centre p, a meet F_i in q_i $(i = 1,\ldots,d)$, and for $i \neq j$, define

$$T_{ij} = \operatorname{conv}\{G_{ij}, q_i, q_j, p\}.$$

Then $T = \dot{\bigcup}_{1 \leq i \leq j \leq d} T_{ij}$, and T_{ij} is symmetric in the plane spanned by G_{ij} anc This proves firstly that T is equidissectable under SO with any of its images un

O (justifying our not mentioning which group we used to define Σ), and consequently that T is itself 2-divisible, since each T_{ij} is.

Whether Σ^d is divisible for $d \geq 4$ is an open and apparently rather deep question (we shall treat the case $d = 3$ shortly).

While the openness of this question clearly prevents our assuming even a $\mathbb{Q}$-linear structure for Σ^d, we do (in a certain sense) have a reduction from odd dimension d to the even dimension next below. This is a consequence of the first of two dissection theorems we now describe.

Let K be any polyhedral set in $\mathbb{E}^d$ (for example, a cone or a polytope), and F a non-empty face of K. Translate K so that $0 \in \operatorname{relint} F$; then the positive hull pos K of K, which is the cone generated by K with apex 0, is a polyhedral cone, which we shall call the *angle cone* of K at F, and denote by A(F,K).

Associated with K is another convex cone, its *recession cone* or *characteristic cone* rec K, which is defined by

$$\operatorname{rec} K = \{x \in \mathbb{E}^d | x + y \in K \quad \text{for all} \quad y \in K\}.$$

Thus if K is a polytope, $\operatorname{rec} K = \{0\}$, while if K is itself a cone with apex a, rec $K = K - a$.

Using $[\cdot]$ to denote corresponding equivalence classes in Σ^d, we then have the following recently proved (McMullen [1982a]) generalization and abstraction of the theorems of Brianchon-Gram (see Brianchon [1837], Gram [1874], and, for a proof in the present spirit, Shephard [1967]) and Sommerville [1927]:

(8.2) **Theorem.** $\sum_F (-1)^{\dim F}[A(F, K)] = (-1)^d[\operatorname{rec}(-K)]$.

We have kept rec $(-K)$ instead of rec K to emphasize the geometric nature of the dissection, which follows by considering the orthogonal projections of K onto arbitrary hyperplanes.

In the case of cones (Sommerville's theorem), we can apply (8.1) to (8.2) to obtain

(8.3) **Theorem.** *For* d *odd, if* K *is a pointed polyhedral cone in* $\mathbb{E}^d$,

$$[K] = \frac{1}{2} \sum_{F \neq \{0\}} (-1)^{\dim F - 1}[A(F,K)].$$

Let Σ be the direct sum of the Σ^d ($d \geq 0$). Then Σ admits a product, induced by the cartesian product of cones lying in orthogonal subspaces. We shall denote this product by $*$, so that $[K_1] * [K_2] = [K_1 \times \sigma(K_2)]$, where σ is a suitable rotation taking K_2 into a subspace orthogonal to that carrying K_1. With this product (and the carefully chosen definition of Σ^0), Σ becomes a graded $\mathbb{Z}$-algebra, Σ^d being assigned the degree d.

An r-*fold* (*orthogonal*) *join* is just an r-fold product

$$[K_1] * \cdots * [K_r],$$

with each $\dim K_i \geq 1$; we write Σ_r^d for the subgroup of Σ^d generated by the r-fold

joins, and $\Sigma_r = \bigcup_{d \geq 0} \Sigma_r^d$. We then have

$$\Sigma = \Sigma_1 \supseteq \Sigma_2 \supseteq \cdots, \quad \Sigma_{r_1}^{d_1} * \Sigma_{r_2}^{d_2} \subseteq \Sigma_{r_1 + r_2}^{d_1 + d_2}.$$

We may now observe that each term on the right of (8.3) is a join. Indeed, let us write (with the convention introduced above) $B(F,K) = A(F,K) \cap L$, where $L \subseteq \mathbb{E}^d$ is the orthogonal complement of lin F; we call B(F,K) the *intrinsic inner cone* of K at F. Then for $\dim F \geq 1$,

$$A(F,K) = B(F,K) \times \operatorname{lin} F$$

is a non-trivial cartesian product. Yet a third angle function is also useful:

$$\tilde{A}(F,K) = B(F,K) \times \mathbb{E}^{\dim F - 1}$$

for $\dim F \geq 1$, which is the corresponding interior angle of the spherical polytope $K \cap \Omega$ at its face $F \cap \Omega$. The mapping $e: \Sigma^d \to \Sigma^{d-1}$ given by

$$e[K] = \sum_{F \neq \{0\}} (-1)^{\dim F - 1}[\tilde{A}(F,K)]$$

is called by Sah (for reasons which are not entirely convincing) the *Gauss-Bonnet map*. Thus, for d odd, (8.3) can be written

$$[K] = p * e([K]),$$

where (illogically) we write $p = [\text{point}]$ for the class of a point of Ω, or a ray of some $\mathbb{E}^d$ (we here use the 2-divisibility of Σ). If d is even, Sah [1981] shows that $e([K]) = 0$. Since it subsequently assumes some importance, let us write $\Gamma^d = p * \Sigma^{d-1}$ ($d \geq 1$). Then the foregoing can be summarized as:

(8.4) **Theorem.** *For* $j \geq 0$, $\Sigma^{2j+1} = \Gamma^{2j+1} = p * \Sigma^{2j}$. *The map* $e: \Sigma^{2j+1} \to$ Σ^{2j} *is an isomorphism inverse to* $x \mapsto p * x$.

In particular, writing $\Gamma = p * \Sigma$ for the direct sum of the subrings Γ^d ($d \geq 1$), we see that Σ/Γ is evenly graded by degree.

In order to give these results a more concrete interpretation, we introduce the graded volume map. Sah follows Schläfli in normalizing so that the volume of Ω^{d-1} is 2^d. For reasons that will become clearer below, we prefer a different normalization. The easiest way to introduce this is to define

$$\operatorname{vol} K = \int_K \exp(-\pi \|x\|^2)\, dx,$$

where dx is ordinary Lebesgue measure in the linear subspace lin K spanned by the polyhedral cone K with apex 0. We then define the graded volume by

$$\operatorname{gr.vol}[K] = \operatorname{vol} K . T^{\dim K},$$

where T is an indeterminate, and extend to Σ by linearity. The normalization is such that $\operatorname{vol} L\ (= \operatorname{vol}(\Omega \cap L)) = 1$ for all linear subspaces L.

Using volume, we see that (8.3) becomes

$$\operatorname{vol} K = \frac{1}{2} \sum_{F \neq \{0\}} (-1)^{\dim F - 1} \beta(F,K),$$

where we write $\beta(F,K) = \text{vol } A(F,K) = \text{vol } B(F,K)$; the latter follows from the elementary observation

$$\text{vol } [K_1] * [K_2] = \text{vol } [K_1] . \text{vol } [K_2].$$

In case $d = \dim K = 3$, this gives the well-known formula expressing the spherical area of a spherical polygon in terms of the angles at its vertices.

We now come to our dissection result. To describe this, we need to introduce another important concept. The *polar* K^0 of a polyhedral cone K with apex 0 in $\mathbb{E}^d$ is defined, as usual, by

$$K^0 = \{x \in \mathbb{E}^d | \langle x,y \rangle \leq 0 \quad \text{for all} \quad y \in K\}.$$

Then $K^{00} = K$. We shall discuss properties of the polarity correspondence (or, rather, a suitable modification of it) as it applies to the group Σ a little later. For the moment, let us define the *normal cone* $N(F,K)$ of a polyhedral set K at its face F to be $(\text{pos } K)^0 = K^0$, where (as previously) we have taken $0 \in \text{relint } F$.

(8.5) **Theorem.** *Let* K *be a polyhedral cone with apex* 0 *in* $\mathbb{E}^d$. *Then the cones* F *and* $N(F,K)$ (F *a face of* K) *are orthogonal, and* $\mathbb{E}^d$ *is dissected into the cones* $F \times N(F,K)$.

In terms of Σ, this gives

$$[\mathbb{E}^d] = \sum_F [F] * [N(F,K)].$$

The proof of (8.5) is immediate, on noticing that an arbitrary point $z \in \mathbb{E}^d$ can be uniquely represented in the form $z = x + y$, where, for some face F of K (possibly K itself) $x \in \text{relint } F$ and $y \in N(F,K)$; x is the (unique) nearest point of K to z.

There are two other equidissection results that we shall want to use later, at least on the level of (graded) volume. To prove these results, it is convenient to introduce a little more notation.

We shall write $b(F,K) = [B(F,K)]$, and adopt a similar convention for other such cones. The *intrinsic outer cone* $C(F,K)$ to K at F is just $C(F,K) = N(F,K) \cap \text{lin } K$; then $C(F,K)$ is the polar of $B(F,K)$ with respect to its linear hull. We let $Z(F,K)$ be the orthogonal complement of lin F in lin K, and (with $z(F,K) = [Z(F,K)]$ as above) we define

$$m(F,K) = (-1)^{\dim Z(F,K)} z(F,K).$$

As we shall shortly see, z and m are Σ-valued analogues of the zeta and Möbius functions of Rota [1964] and McMullen [1975b]. Indeed, $*$ induces a multiplication on the Σ-valued functions f on pairs of cones, which are such that $f(F,G) = 0$ unless F is a face of G, by

$$f * g(F,G) = \sum_J f(F,J) * g(J,G),$$

the sum extending over all faces J of G. We then have from Euler's theorem:

(8.6) **Theorem.** $m * z = i = z * m$, *where* i *is the identity function; that is,* $i(F,G) = 0$ *if* $F \neq G$, *and* $1\ (\in \mathbb{Z})$ *if* $F = G$.

We can now rephrase (8.2) (for cones) and (8.5) as

(8.7) **Theorem.** (a) $m * b = \tilde{b}$, *where* $\tilde{b}(F,G) = (-1)^{\dim G - \dim F} b(F,G)$;
(b) $b * c = z$.

Next, we can paraphrase the arguments of McMullen [1975b], to prove new equidissectability results by purely algebraic means. For, let us similarly write $\tilde{c}(F,G) = (-1)^{\dim G - \dim F} c(F,G)$. Then we have:

(8.8) **Theorem.** (a) $\tilde{b} * c = i = b * \tilde{c}$;
(b) $c * \tilde{b} = i\ (= \tilde{c} * b)$.

For, (8.7)(a) and polarity yields $\tilde{c} = c * m$. (8.8)(a) follows at once from (8.7)(b), since $\tilde{b} * c = m * b * c = m * z = i$; the other relation is proved similarly, or by another appeal to polarity. (8.8)(b) needs a further remark. Confining attention to pairs of faces of a fixed polyhedral cone, we see that $\tilde{b}$ and c correspond to triangular matrices with values in the ring Σ, whose diagonal entries are 1. So, $\tilde{b}$ and c are both invertible. But since $\tilde{b} * c = i$, we have $c * \tilde{b} = i$ also. (The other relation of (8.8)(b) is redundant, since $\tilde{c} * b = c * m * b = c * \tilde{b}$.)

Note that (8.8) (a) implies

$$0 = \sum_F (-1)^{\dim F} [F] * [N(F,K)]$$

whenever K is not a subspace. By adding and subtracting this and the relation of (8.5), we obtain generalizations of Sah [1979], Proposition 6.3.5. In odd dimensions, we obtain another way of expressing a cone K in terms of product cones, while in even dimensions, we get a relation between K, K^0 and product cones, whose components are also even dimensional.

We shall need the metrical consequences of the relation of (8.5), which can be written $b * c = z$, and those of (8.8), in §10 below.

(8.9) **Theorem.** *Let* K *be a pointed polyhedral* d*-cone with apex* 0 ($d \geq 1$). Then,

(a) $\sum_F \beta(0,F)\gamma(F,K) = 1$;

(b) $\sum_F (-1)^{\dim F} \beta(0,F)\gamma(F,K) = 0$;

(c) $\sum_F (-1)^{\dim K - \dim F} \gamma(0,F)\beta(F,K) = 0$.

Here, $\gamma(F,G) = \text{vol } c(F,G)$ is the normalized *external angle* of the polyhedral set G at its face F (as used in §3).

Polarity plays a further role in Σ. To begin our discussion, we remark:

(8.10) **Lemma.** *Let* K_1, K_2 *be polyhedral cones in* $\mathbb{E}^d$ *with apex* 0. *Then*

$$(K_1 \cap K_2)^0 = K_1^0 + K_2^0,$$

where the sum is Minkowski addition.

Now, if $K_1 \cup K_2$ is also convex, we have $K_1 \cup K_2 = K_1 + K_2$. Further, if $\dim K < d$, then K^0 has the orthogonal complement $(\mathrm{lin}\, K)^\perp$ as its non-trivial face of apices, and so $[K^0] \in \Gamma^d$; the converse also holds. In other words:

(8.11) **Theorem.** *The polarity correspondence induces an involutory automorphism* § *of* Σ^d/Γ^d, *defined by*

$$[K]^{\S} = [K^0].$$

This automorphism, which for reasons explained below we call the *antipodal map*, extends to Σ/Γ. The algebra structure on Σ, with multiplication $*$, induces an algebra structure on Σ/Γ. To complete the picture, we now describe a coalgebra structure on Σ/Γ.

In fact, we begin with something more general. For $x \in \Sigma$, write $\bar{x} \in \Sigma/\Gamma$ for its representative. The *total spherical Dehn invariant* of a pointed polyhedral cone K (or corresponding spherical polytope) is

$$\Psi_S(K) = \sum_F^{*} [F] \otimes \overline{[B(F,K)]} \in \Sigma \otimes (\Sigma/\Gamma),$$

where the sum extends over all faces $F \neq \{0\}$ of K with $\dim K - \dim F$ even. (In fact, the terms with $\dim K - \dim F$ odd automatically drop out, due to the even grading of Σ/Γ. The term $F = \{0\}$, in case $\dim K$ is even, is not needed, since the information contained in the first term $K \otimes \overline{[\emptyset]}$ includes that contained in $[\emptyset] \otimes \overline{[K]}$.)

The map Ψ_S then induces a map $\bar{\Psi}_S : \Sigma/\Gamma \to (\Sigma/\Gamma) \otimes (\Sigma/\Gamma)$; defined on a generator $\overline{[K]}$ of Σ/Γ, its effect is

$$\bar{\Psi}_S \overline{[K]} = \sum_F^{*} \overline{[F]} \otimes \overline{[B(F,K)]}.$$

This is the *comultiplication on* Σ/Γ. The *counit* or *augmentation* (which is dual to the unit in an algebra) is the natural mapping whose kernel is the set of elements of Σ/Γ of positive degree. With these algebra and coalgebra structures, and the antipodal map, Σ/Γ then becomes a Hopf algebra (see Sah [1979]).

We are now in a position to discuss equidissectability in general. As with the case of the group Π^d, we look for a suitable separating family of functions on Σ^d. The map Ψ_S does separate Σ^d, but in a rather trivial way, since it is obviously injective. More to the point is the *total classical spherical Dehn invariant*

$$\Phi_S = (\mathrm{gr.\,vol} \otimes \mathrm{id}) \circ \Psi_S,$$

so that

$$\Phi_S(K) = \sum_F^{*} (\mathrm{vol}\, F \cdot T^{\dim F}) \otimes \overline{[B(F,K)]} \in \mathbb{R}[T] \otimes (\Sigma/\Gamma).$$

The summation convention is that used in defining Ψ_S above.

Unfortunately, the general equidissectability problem is far from solved.

(8.12) **Theorem.** Φ_S *separates* Σ^d *for* $d = 2$ *and* 3.

All this says is that arc length (in case $d = 2$) or area (for $d = 3$) characterize spherical polytopes up to equidissectability.

For $d \geq 4$, we encounter severe problems; the case $d = 4$ already illustrates most of these. For, we observe that, if K has rational dihedral angles, then $\Phi_S(K) = \mathrm{vol}\, K . T^4$, and the lower degree terms vanish identically. So, if Φ_S does separate Σ^4, then such K should be equidissectable with product cones. However, this is far from obviously true in general, although Dupont and Sah [1982] have recently shown that it is true in one important case.

(8.13) **Theorem.** *A fundamental polyhedral cone for a finite orthogonal group in* $\mathbb{E}^d$ *is equidissectable with a* $(d-2)$*-fold product cone over a planar cone.*

In fact, what Dupont and Sah prove is that fundamental polyhedral cones for two finite orthogonal groups in $\mathbb{E}^d$ of the same order are equidissectable. The result ultimately reduces to proving this for the special case of p-groups (Sylow subgroups) of the same order.

But the general problem remains unsolved; indeed, polyhedral cones with rational dihedral angles are a possible source of torsion in Σ^4 (Sah [1979]).

As we said at the beginning, there is also some interest in the hyperbolic case. As with the spherical case, in the hyperbolic line or plane, equality of length or area is a necessary and sufficient condition for equidissectability. Once again, the real problems begin in the next dimension.

In general, as with the spherical case, there is a difference between odd and even dimensional hyperbolic spaces H^d. The "Gauss-Bonnet" map can be defined equally well for hyperbolic polytopes, and associates with an element x of the polytope group in H^d (with respect, of course, to the group of all hyperbolic motions) an element e(x) of the polytope group in Ω^{d-1}. On the level of volume, we have

$$\mathrm{vol}_{d-1}(e(x)) = (-1)^{d/2}\chi(\Omega^d)\mathrm{vol}_d(x).$$

where $\chi(\Omega^d) = 1 + (-1)^d$ is the Euler characteristic of Ω^d (in the spherical case, the factor $(-1)^{d/2}$ is omitted). It is possible that there are deeper connexions, but these have not been much investigated.

Of course, if d is odd, the above formula yields no information. In recent years, much attention has been payed to the first non-trivial case $d = 3$, but even so, it seems not to be near complete solution.

One can further consider equidissectability in $\bar{H}^d = H^d \cup \partial H^d$, obtained by adding to H^d the ideal points (at infinity) forming ∂H^d. However, we are going a little far from our topic, so we shall merely refer the reader to Sah [1979; 1981], Dupont-Sah [1982], and the bibliographies contained therein.

§9. Hilbert's third problem

Discussion of the spherical problem was a prerequisite for consideration of the euclidean equidissectability problem, as we shall see. From an historical point of view, of course, we have things back to front, as the euclidean problem was investigated earlier. The case of dimension $d = 1$ is trivial, and the case $d = 2$ is nearly as easy. As Gerwien [1833a] and F. Bólyai observed, a triangle is D- (or

SD- or even TH-) equidissectable with a parallelogram of the same area. As a consequence, and using the results of §6:

(9.1) **Theorem.** *Two polygons in* $\mathbb{E}^2$ *are* D*-equidissectable if and only if they have the same area.*

In a later extension of this result, Hadwiger-Glur [1951] showed that, if G is a group such that two polygons with the same area are always G-equidissectable, then $G \supseteq TH$.

However, it was early recognized that the situation in $\mathbb{E}^3$ was likely to be different, and in 1900 Hilbert [1900] formally posed the problem of finding two 3-polytopes of the same volume (specifically, pyramids with the same height on the same base) which were not D-equidissectable. Modifying an earlier incorrect attempt of Bricard [1896], Dehn [1900], [1902] found a suitable pair of examples in the same year the problem was posed.

Before going further into Dehn's examples, it will be helpful to make some remarks about the appropriate polytope group Π_D^d. (Once more, we distinguish the dimension d.) Firstly, the fact that reflected polytopes are SD-equidissectable shows that $\Pi_D^d = \Pi_{SD}^d$. Secondly, dilatation commutes with isometry (at least, modulo translation), so that Π_D^d as a quotient space of Π_T^d also admits a grading

$$\Pi_D^d = \Xi_1^d \oplus \cdots \oplus \Xi_d^d.$$

However, the presence of scaling by -1 ensures that $\Xi_r \equiv 0$ unless $d - r \equiv 0 \pmod 2$.

Writing $\Pi_D = \Sigma_{d\geq 0}\Pi_D^d$ (with $\Pi_D^0 = \mathbb{R}$), we see that we have a natural product structure induced by orthogonal cartesian product, which we denote by $\times$.

We now describe what are conjectured to be a separating family of functionals on Π_D. These are the euclidean Dehn invariants, which are exact analogues of the spherical Dehn invariants. The *total euclidean Dehn invariant* of a polytope P is

$$\Psi_E(P) = \sum_F^* [F] \otimes \overline{[B(F,P)]} \in \Pi_D \otimes (\Sigma/\Gamma),$$

where the sum extends over all faces F of P with (dim F > 0 and) dim P − dim F even. Similarly, the *classical total euclidean Dehn invariant* is just

$$\Phi_E = (\mathrm{gr.vol} \otimes \overline{\mathrm{id}}) \circ \Psi_E,$$

so that

$$\Phi_E(P) = \sum_F^* \mathrm{vol}\, F . T^{\dim F} \otimes \overline{[B(F,P)]}.$$

Theorem (9.1) shows that Φ_E separates Π_D^d for $d \leq 2$. The considerable achievement of Sydler [1965] was to extend this to d = 3; Jessen [1972] was then able to use Sydler's result further to extend this to d = 4. In fact, Jessen [1968] was also able to simplify Sydler's original proof, essentially by using the language of the algebra of polytopes.

It would be inappropriate to give full details of these proofs here, but we can point out some of the salient features.

(9.2) **Theorem.** Φ_E *separates* Π_D^3.

It is easy to see that the group Π_D^d is generated by the (equivalence classes *orthogonal* simplices $[a_1,\ldots,a_d]$, which are such that $\{a_1,\ldots,a_d\}$ is an orthogo set of vectors. In the particular case d = 3, Sydler's proof of Theorem (9 depends upon considering those particular orthogonal simplices $T(\xi,\eta)$ $[a_1,a_2,a_3]$, where

$$\|a_1\| = \left(\frac{1-\xi}{\xi}\right)^{1/2}, \quad \|a_3\| = \left(\frac{1-\eta}{\eta}\right)^{1/2}, \quad \|a_2\| = \|a_1\|\,\|a_3\|,$$

with $\xi,\eta \in]0,1[$. The more important and less obvious of Sydler's results is

(9.3) **Lemma.** *For* $\xi,\eta_1,\eta_2 \in]0,1[$,

$$T(\xi,\eta_1) + T(\xi\eta_1,\eta_2) = T(\xi,\eta_2) + T(\xi\eta_2,\eta_1).$$

The proof of this result relies on equicomplementability implying equidissec bility. Another result, which follows from dissecting one tetrahedron in two wa is much easier.

(9.4) **Lemma.** *For* $\xi,\eta,\zeta > 0$,

$$\mu(\xi)T\left(\frac{\xi+\eta}{\xi+\eta+\zeta},\frac{\xi}{\xi+\eta}\right) + \mu(\eta)T\left(\frac{\xi+\eta}{\xi+\eta+\zeta},\frac{\eta}{\xi+\eta}\right)$$
$$= \mu(\xi)T\left(\frac{\xi+\zeta}{\xi+\eta+\zeta},\frac{\xi}{\xi+\zeta}\right) + \mu(\xi)T\left(\frac{\xi+\zeta}{\xi+\eta+\zeta},\frac{\zeta}{\xi+\zeta}\right).$$

Finally, there is a further easy result.

(9.5) **Lemma.** *Let* $\alpha,\beta,\gamma \in]0,\frac{1}{2}\pi[$, *with* $\alpha+\beta+\gamma=\pi$. *Then there is a rect gular parallelepiped, which when dissected by the three planes containin fixed diagonal and pairs of opposite edges of the box, yields orthogo tetrahedra whose dihedral angles at the diagonal are* $\alpha,\beta,\gamma,\alpha,\beta,\gamma$.

Now Theorem (9.2) reduces to the following assertion.

(9.6) **Lemma.** *Let* $\varphi : \Xi_1^3 \to Y$ *be a linear map into a real linear space* Y. *T there is a linear map* $\vartheta : \mathbb{R}\otimes(\mathbb{R}/\mathbb{Z}) \to Y$, *such that*

$$\varphi = \vartheta \circ \Phi^{(1)},$$

where $\Phi_E = \Phi^{(3)}T^3 + \Phi^{(1)}T$ *on* Π_D^3.

In particular, with $Y = \Xi_1^3$, this shows that $\Phi^{(1)}$ is a monomorphism. Note t the real linear structure of $\mathbb{R}\otimes(\mathbb{R}/\mathbb{Z})$ is inherited from its first component.

If we write $[T(\xi,\eta)] = t_3(\xi,\eta) + t_1(\xi,\eta)$ with $t_r(\xi,\eta) \in \Xi_r$ (r = 1,3), and $\varphi(\xi,\eta)$ $\varphi(t_1(\xi,\eta))$, then (9.3) and (9.4) yield corresponding functional relations for φ. Fr

(9.3), there is an $f:]0,1[\to Y$, such that

$$\varphi(\xi,\eta) = f(\xi) + f(\eta) - f(\xi\eta).$$

(Curiously, the proof of this fact, and that to follow, need the axiom of choice.) From f, one defines $G:\mathbb{R}_+ \times \mathbb{R}_+ \to Y$ by

$$G(\xi,\eta) = \xi f\left(\frac{\xi}{\xi+\eta}\right) + \eta f\left(\frac{\eta}{\xi+\eta}\right),$$

so that (9.4) and its definition yield

$$G(\xi,\eta) = G(\eta,\xi), \quad G(\lambda\xi,\lambda\eta) = \lambda G(\xi,\eta) \quad (\lambda > 0)$$
$$G(\xi,\eta) + G(\xi+\eta,\zeta) = G(\xi+\zeta,\eta) + G(\xi,\zeta).$$

G can be extended to $\mathbb{R} \times \mathbb{R}$ satisfying the same functional equations. Now we find $g_1:\mathbb{R}_+ \to Y$, such that

$$g_1(\xi\eta) = \eta g_1(\xi) + \xi g_1(\eta),$$

and

$$G(\xi,\eta) = g_1(\xi) + g_1(\eta) - g_1(\xi+\eta).$$

Writing $g_1(\xi) = \xi g(\xi)$ $(\xi > 0)$, we have $g:\mathbb{R}_+ \to Y$, such that

$$g(\xi\eta) = g(\xi) + g(\eta),$$
$$G(\xi,\eta) = \xi g(\xi) + \eta g(\eta) - (\xi+\eta)g(\xi+\eta).$$

We note that $g(1) = 0$. Now, if $\xi,\eta > 0$ with $\xi + \eta = 1$,

$$\xi f(\xi) + \eta f(\eta) = G(\xi,\eta) = \xi g(\xi) + \eta g(\eta).$$

If $h = f - g:]0,1[\to Y$, then

$$\varphi(\xi,\eta) = h(\xi) + h(\eta) - h(\xi\eta),$$

with

$$\xi h(\xi) + \eta h(\eta) = 0 \quad \text{if} \quad \xi,\eta > 0, \quad \xi + \eta = 1.$$

We complete the proof by defining $\sigma:\mathbb{R} \to Y$, where σ is periodic with period $\frac{1}{2}\pi$, by

$$\sigma(\alpha) = \tan\alpha\, h(\sin^2\alpha), \quad 0 < \alpha < \tfrac{1}{2}\pi$$

with $\sigma(\frac{1}{2}\pi) = 0$. That σ is additive follows from the above and (9.5), which gives

$$\sigma(\alpha) + \sigma(\beta) + \sigma(\gamma) = 0, \quad 0 < \alpha,\beta,\gamma < \tfrac{1}{2}\pi, \quad \alpha + \beta + \gamma = \pi.$$

From its definition follows

$$\varphi(\xi,\eta) = \cot\alpha\,\sigma(\alpha) + \cot\beta\,\sigma(\beta) - \cot(\alpha * \beta)\sigma(\alpha * \beta),$$

where

$$\xi = \sin^2\alpha, \quad \eta = \sin^2\beta$$

and

$$\sin(\alpha * \beta) = \sin\alpha\sin\beta, \quad 0 < \alpha,\beta,\alpha * \beta < \tfrac{1}{2}\pi.$$

Finally, if we define $\theta:\mathbb{R} \otimes (\mathbb{R}/\mathbb{Z}) \to Y$ by

$$\theta(\xi \otimes \omega) = \xi\sigma(\tfrac{1}{2}\pi\omega),$$

then θ is the required linear map.
The extension by Jessen [1972] easily follows.

(9.7) **Theorem.** Φ_E *separates* Π_D^4.

We have $\Pi_D^4 = \Xi_2^4 \oplus \Xi_4^4$, so that every $x \in \Pi_D^4$ is equivalent to a prism e ×
where e is a unit segment. (Jessen [1972] gives a direct proof of this). The resu
now follows directly from (9.2).
We end this section by making a few remarks, and stating some problems.
fact we have used above is that the Dehn invariant Φ_E is compatible with t
product structure. Indeed,

$$\Psi_E(P \times Q) = \Psi_E(P)\Psi_E(Q),$$

since the angle $P \times Q$ at its face $F \times G$ is

$$B(F \times G, P \times Q) = B(F,P) \times B(G,Q).$$

A general question raised by the proof of (9.7) is:

(9.8) **Problem.** *Is every even-dimensional polytope equivalent to a direct sum products of odd-dimensional components?*

For example, $\Pi_D^2 = \Xi_2^2 \cong \Xi_1^1 \otimes \Xi_1^1$. Further, $\Pi_D^4 = \Xi_2^4 \oplus \Xi_4^4$. Now Ξ_4^4
$\otimes^4 \Xi_1^1$, and Ξ_2^4 is generated by $\Xi_1^1 \otimes \Xi_1^3$ and $\Xi_1^2 \otimes \Xi_1^2$, the latter term vanishin
so the result holds here also. Certainly, the space of indecomposable elements
Π_D is the sum of the spaces Ξ_1^{2s+1}. So, a variant of (9.8) is:

(9.9) **Problem.** *Is Π_D isomorphic to a symmetric algebra based on the space indecomposable elements?*

In particular:

(9.10) **Problem.** *Is Π_D an integral domain? Is Π_D a Hopf algebra?*

Finally, related to (9.6), there is a question about Dehn invariants.

(9.11) **Problem.** *Is $\Phi^{(1)}:\Xi_1^3 \to \mathbb{R} \otimes (\mathbb{R}/\mathbb{Z})$ an isomorphism? More generally, wh are the images of the Dehn invariants?*

Those readers who, after their excursion into dissection theory, might wis
some recreation, are referred to the amusing book of Lindgren [1972].

III. General valuations

§10. Polynomial expansions for general valuations

While the polynomial expansion of $\varphi(\lambda P)$ for a simple (translation invariant) valuation φ dates back, as we have seen, to Hadwiger [1957], the question of the existence of such polynomial expansions for general valuations was settled much more recently. The question was posed (in the context of continuous or monotone valuations, with real coefficients) by McMullen at the Oberwolfach meeting on Convex Bodies in 1974, and settled by him the same year (see McMullen [1975a; 1977]). Somewhat later, but independently, Spiegel [1978] and Meier [1977] gave proofs using different approaches; in retrospect, their proofs should have been available earlier, since they involve fairly elementary modifications of the ideas of §7. However, McMullen's approach yields considerably more, and so we shall largely follow that here.

The basic idea is to use the angle-sum relations described in §8, to relate a given valuation φ (which for simplicity of exposition will always take values in a real vector space $\mathscr{X}$) to a family of simple valuations. Let $\mathscr{A}$ be the class of translates of a given flat A in $\mathbb{E}^d$ (possibly $A = \mathbb{E}^d$ itself), and write $\mathscr{P}(\mathscr{A})$ for the family of polytopes P such that aff P is a subflat of some flat in $\mathscr{A}$.

(10.1) **Lemma.** *Let φ be a translation invariant valuation on $\mathscr{P}^d$. Let the function $\psi_{\mathscr{A}}$ be defined by*

$$\psi_{\mathscr{A}}(P) = \begin{cases} \sum_F (-1)^{\dim P - \dim F}\beta(F,P)\varphi(F), & \textit{if} \quad \text{aff } P \in \mathscr{A}, \\ 0, & \textit{otherwise.} \end{cases}$$

The $\psi_{\mathscr{A}}$ is a simple translation invariant valuation on $\mathscr{P}(\mathscr{A})$.

We can use (8.9) to invert (10.1); the converse result was first proved by Hadwiger [1953b; 1957].

(10.2) **Lemma.** *For each translation class $\mathscr{A}$ of flats in $\mathbb{E}^d$, let $\psi_{\mathscr{A}}$ be a simple translation invariant valuation on $\mathscr{P}(\mathscr{A})$. Define $\psi(P) = \psi_{[\text{aff } P]}(P)$, where [A] is the translation class of A, and*

$$\varphi(P) = \sum_F \gamma(F,P)\psi(F).$$

Then $\psi_{\mathscr{A}}$ is a simple translation invariant valuation on $\mathscr{P}(\mathscr{A})$.

The proofs of (10.1) and (10.2) are straightforward but tedious.

From (7.1a), (10.1) and (10.2) lead at once to

(10.3) **Theorem.** *Let φ be a translation invariant valuation on $\mathscr{P}^d$. Then for rational $\lambda \geq 0$,*

$$\varphi(\lambda P) = \sum_{r=0}^{d} \lambda^r \varphi_r(P).$$

The coefficient $\varphi_r(P)$ (which is independent of λ) is a translation invari[...] valuation on $\mathscr{P}^d$, which is homogeneous of degree r.

In particular, $\varphi_0(P) = \varphi(\{x\})$ is the value taken by φ on a point.

We extend (10.3) to general linear combinations by means of (1.4). Let φ b[...] translation invariant valuation on $\mathscr{P}^d$, and let $Q \in \mathscr{P}^d$ be fixed. Then by (1.4) t[...] function ψ defined by $\psi(P) = \varphi(P + Q)$ is also a translation invariant valuati[...] on $\mathscr{P}^d$.

We deduce immediately the general result:

(10.4) **Theorem.** *Let φ be a translation invariant valuation on $\mathscr{P}^d$. Then [...] $P_1,\ldots,P_k \in \mathscr{P}^d$ and rational $\lambda_1,\ldots,\lambda_k \geq 0$, $\varphi(\lambda_1 P_1 + \cdots + \lambda_k P_k)$ is a po[...] nomial in $\lambda_1,\ldots,\lambda_k$ of total degree at most* d. *The coefficient of $\lambda_1^{r_1} \cdots \lambda_k^{r_k}$ i[...] translation invariant valuation in P_j which is homogeneous of degree [...] ($j = 1,\ldots,k$).*

We call these coefficients *mixed valuations.* If φ_r is the homogeneous valuati[...] occurring in (10.3), so that $\varphi = \sum_{r=0}^{d} \varphi_r$, we may write

$$\varphi_r(\lambda_1 P_1 + \cdots + \lambda_k P_k) = \sum \binom{r}{r_1 \ldots r_k} \lambda_1^{r_1} \ldots \lambda_k^{r_k} \varphi_r(P_1,r_1;\ldots;P_k,r_k),$$

in analogy to (3.2).

Let us briefly survey the other two approaches. Spiegel [1978] uses t[...] canonical simplex dissection, the inclusion-exclusion principle and induction [...] the dimension to obtain a direct proof. Meier [1977], which we have mention[...] earlier, uses his mixed polytopes and the inclusion-exclusion principle to obta[...] the same result. (In fact, these approaches cover valuations taking values in[...] rational vector space, while McMullen's needs a real vector space; the modific[...] tion of Meier's argument outlined in §6 possibly shows this most easily.)

The discussion of covariant valuations proceeds on very similar lines. We s[...] that a valuation $\varphi: \mathscr{P}^d \to \mathscr{X}$ is *translation covariant* if there exists an *associat[...]* function $\Phi: \mathscr{P}^d \to \mathrm{Hom}_{\mathbb{Z}}(\mathbb{E}^d, \mathscr{X}) = \mathrm{Hom}_{\mathbb{Q}}(\mathbb{E}^d, \mathscr{X})$, such that

$$\varphi(P + t) - \varphi(P) = \Phi(P)t,$$

for $t \in \mathbb{E}^d$ and $P \in \mathscr{P}^d$.

(10.5) **Lemma.** *Let φ be a translation covariant valuation on $\mathscr{P}^d$. Then t[...] associated function Φ is a translation invariant valuation on $\mathscr{P}^d$.*

Lemmas (10.1) and (10.2) carry over at once to covariant valuations. So, all th[...] is needed is to appeal to the expression after (6.7) for the specific translatio[...] involved in the canonical simplex dissection, in order to generalize (10.3) ar[...] (10.4), and show:

(10.6) **Theorem.** *Let φ be a translation covariant valuation on $\mathscr{P}^d$. Then f[...] $P_1,\ldots,P_k \in \mathscr{P}^d$ and rational $\lambda_1,\ldots,\lambda_k \geq 0$, $\varphi(\lambda_1 P_1 + \cdots + \lambda_k P_k)$ is a pol[...]*

nomial in $\lambda_1,\ldots,\lambda_k$ of total degree at most d + 1. *The coefficient of $\lambda_1^{r_1}\cdots\lambda_k^{r_k}$ is a translation covariant valuation in* P_j *which is homogeneous of degree* r_j $(j = 1,\ldots,k)$.

The degree d + 1 (rather than d, as in (10.4)) arises from the translations.

The results discussed above can be generalized in various ways. The main way is by restricting the translations which are allowed. For several reasons, the most interesting cases concern the translation subgroups $\mathbb{Q}^d$ and $\mathbb{Z}^d$ (or, of course, isomorphic subgroups). It might be thought that the case $\mathbb{F}^d$, for a general subfield $\mathbb{F}$ of $\mathbb{R}$, would also be of interest; however, analogues of the weak continuity condition to be discussed in §11 are needed to extend $\mathbb{Q}$-linearity to $\mathbb{F}$-linearity.

The case of the translation group $\mathbb{Q}^d$ and rational polytopes needs no further comment; the results above carry over with no change of language. When the translation group is $\mathbb{Z}^d$ and the polytopes are lattice polytopes, (10.4) and (10.6) remain valid with integers λ_i (McMullen [1977], and using Stein [1982]; see also Bernštein [1976] for the special case of the lattice point enumerator). More generally, though, some changes are needed.

Let P be a rational polytope. The r-*index* $\mathrm{ind}_r(P)$ is the smallest positive integer m, such that, for every r-face F of P, aff (mF) is spanned by lattice points (or, equivalently, contains a lattice point). Then we have:

(10.7) **Theorem.** *Let φ be a lattice translation invariant valuation on rational polytopes in $\mathbb{E}^d$. Then for rational polytopes* P *and integer* $n \geq 0$, *there is an expression*

$$\varphi(nP) = \sum_{r=0}^{d} n^r \varphi_r(P,n),$$

where $\varphi_r(P,n)$ depends only on the congruence class of n modulo $\mathrm{ind}_r(P)$.

Such an expression is called a *near-polynomial* in McMullen [1978], where this theorem is proved (and "polynome mixte" by Ehrhart [1967a] in the case of the lattice point enumerator G). The coefficient φ_r is *near-homogeneous* of degree r, in that

$$\varphi_r(mP,n) = m^r \varphi_r(P,mn)$$

for all integer $m,n \geq 0$. There are analogous expressions for integer combinations of rational polytopes; similarly, (10.7) and its generalizations are valid for lattice translation covariant valuations (McMullen [1982b]).

The proof of (10.7) depends upon a specific representation of the valuation, and so we defer further discussion until §17.

In view of (10.4) and (10.6), a translation invariant or covariant valuation on $\mathscr{P}^d$ which is homogeneous of degree 1 is also Minkowski additive:

$$\varphi(P + Q) = \varphi(P) + \varphi(Q);$$

this was earlier proved (for a special case of covariance) by Spiegel [1976a]. We have already remarked on the converse in (1.3).

§11. Additional properties

We now complete the discussion of the original question of McMullen, by considering various continuity conditions. As usual, continuity of functions on $\mathscr{P}^d$ or $\mathscr{K}^d$ will be with respect to the Hausdorff metric.

A different concept of continuity is due to Hadwiger [1952e]. Let $U = (u_1,\ldots,u_n)$ be a (for the moment) fixed set of unit outer normal vectors, and write $\mathscr{P}^d(U)$ for the family of polytopes of the form

$$P(y) = \{x \in \mathbb{E}^d \mid \langle x,u_i\rangle \leq \eta_i \quad (i = 1,\ldots,n)\},$$

where $y = (\eta_1,\ldots,\eta_n)$. We call a function φ on $\mathscr{P}^d$ *weakly continuous* if, for each such U, the function φ_U defined by $\varphi_U(y) = \varphi(P(y))$ is continuous. Clearly we have:

(11.1) **Lemma.** *A continuous function on $\mathscr{P}^d$ is weakly continuous.*

It turns out that, to extend (10.4) and (10.6) to real (rather than rational) coefficients, all that is needed is weak continuity. In fact, we have:

(11.2) **Theorem.** *The following conditions on a translation invariant or covariant valuation φ are equivalent:*

(a) *φ is weakly continuous;*
(b) *for all polytopes $P_1,\ldots,P_k$ and all real numbers $\lambda_1,\ldots,\lambda_k \geq 0$, $\varphi(\lambda_1 P_1 + \cdots + \lambda_k P_k)$ is a polynomial in $\lambda_1,\ldots,\lambda_k$.*
(c) *for each* U, *the one-sided partial derivatives of φ_U exist.*
(d) *φ is continuous under dilatations; that is, the mapping Θ_P on $\mathbb{R}$ defined by $\Theta_P(\lambda) = \varphi(\lambda P)$ is continuous for each* P.

In fact, condition (c) can be replaced by
(c′) *for each* U *and each* $a \in \mathbb{E}^d$, *the (one-sided) derivative of φ_U in direction* a *exists.*

The equivalence of (a), (b) and (c) is due to McMullen [1977]. The equivalence of (a) and (d), which was left as an open problem by Hadwiger [1952e], follows from the inversion formulae (10.1) and (10.2), and the fact that Π^d is a real vector space (see also (11.4) below).

As far as (weakly) continuous valuations are concerned, one general remark is often useful.

(11.4) **Theorem.** *The mixed valuations associated with a (weakly) continuous translation invariant or covariant valuation are (weakly) continuous in each of their arguments.*

To what extent this result extends to monotone valuations (which we shall next consider) is unknown. We call a function φ on $\mathscr{P}^d$ or $\mathscr{K}^d$ taking values in a partially ordered real linear space *monotone* if $\varphi(P) \leq \varphi(Q)$ whenever $P \subseteq Q$.

Then we have the following result of McMullen [1977] (for d = 2, Hadwiger [1951b], §5):

(11.5) **Theorem.** *A monotone translation invariant valuation φ is continuous.*

It is enough to prove this on $\mathscr{P}^d$; any extension to $\mathscr{K}^d$ is fairly easy. Let $(P_j) \subset \mathscr{P}^d$ be a sequence with $\lim P_j = P$. If $\dim P = d$, assume $0 \in \operatorname{int} P$; from $(1-\varepsilon)P \subseteq P_j \subseteq (1+\varepsilon)P$ for each rational $\varepsilon > 0$ and all large enough j, $\lim \varphi(P_j) = \varphi(P)$ follows from the polynomial expansion of φ. If $\dim P < d$, we consider instead $\varphi(P_j + nQ)$, where Q is a fixed d-polytope and $n > 0$ a positive integer; $\varphi(P_j)$ is the constant term in the expansion of $\varphi(P_j + nQ)$ as a polynomial in n, and the result follows from the previous case.

§12. Valuations and Euler-type relations

If φ is a function defined on $\mathscr{P}^d$ (with values in, for the moment, an abelian group), we can define a new function φ^* on $\mathscr{P}^d$ by means of

$$\varphi^*(P) = \sum_F (-1)^{\dim F}\varphi(F).$$

(Our usual convention for such sums prevails.) It is a consequence of Euler's theorem that $\varphi^{**}(=(\varphi^*)^*) = \varphi$. As shown by Sallee [1968],

(12.1) **Theorem.** *If φ is a valuation on $\mathscr{P}^d$, then so is φ^*.*

Sallee's proof is direct, and needs the same kind of considerations which prove (10.1) and (10.2); a different version of the proof was indicated at the end of §5, and an alternative one as a consequence of (10.1) and (10.2) is indicated below.

As an extension of the definition due to Sallee, let us say that φ satisfies an *Euler-type relation of kind* (ε,η) if, for all $P \in \mathscr{P}^d$,

$$\varphi^*(P) = \varepsilon\varphi(\eta P).$$

Evidently, from $\varphi^{**} = \varphi$ follows $\varphi(P) = \varepsilon^2\varphi(\eta^2 P)$. Sallee only considers the case $\eta = 1$, so that $\varepsilon = \pm 1$ (assuming φ non-trivial); more generally, we shall see that $\eta = \pm 1$ is usual, so that $\varepsilon = \pm 1$ also holds (we know of no valuation satisfying an Euler-type relation with $\eta^2 \neq 1$).

If φ is any function on $\mathscr{P}^d$, now taking values in a rational vector space, and we define $\varphi_\pm = \frac{1}{2}(\varphi \pm \varphi^*)$, then $\varphi = \varphi_+ + \varphi_-$, and $\varphi_+^* = \varphi_+$, $\varphi_-^* = -\varphi_-$. Thus φ is always the sum of functions satisfying Euler-type relations. This might seem to make the concept of an Euler-type relation of little significance, were it not for the rather deep connexions between them and valuations which we shall now describe.

We first discuss results of Sallee [1968].

(12.2) **Theorem.** *Let φ be a continuous function on $\mathscr{P}^d$ which satisfies an Euler-type relation $\varphi^* = (-1)^{d-1}\varphi$. Then φ is a valuation.*

(12.2) depends upon several results. Firstly

(12.3) **Lemma.** *If φ is a continuous function on $\mathscr{P}^d$, satisfying $\varphi(P \cup Q) + \varphi(P \cap Q) = \varphi(P) + \varphi(Q)$ whenever P,Q, $P \cup Q$ are d-polytopes, then φ is a valuation.*

Secondly, a function φ satisfying the condition of (12.2) yields expressions of the form

$$\varphi(P) = \frac{1}{2}\sum_{F, \dim F < d} (-1)^{d-1-\dim F}\varphi(F)$$

for d-polytopes P, enabling $\varphi(P)$ to be calculated from its values on lower dimensional faces.

Thirdly, using the continuity of φ, to verify that φ is a valuation, it is enough to consider P and Q of (12.3) satisfying $\dim(P \cap Q) = d - 1$, such that each face of $P \cup Q$ is a face of P or Q. Checking the valuation property in this case is straightforward.

Sallee appears to claim a more general result than (12.2), involving continuous functions satisfying $\varphi^* = \varepsilon\varphi$ for any $\varepsilon (= \pm 1$ by previous remarks). But his φ has domain $\mathscr{P} = \cup\mathscr{P}^d$ (presumably in $\mathbb{E}^\infty = \cup\mathbb{E}^d$). In fact, (12.2) does not extend to $\varepsilon = (-1)^d$; a counterexample is $\varphi(P) = V(P_{-\rho})$, where $P_{-\rho}$ is the inner parallel body of P at any fixed positive distance ρ. (The crucial feature of this example is that $\varphi(P) = 0$ if $\dim P < d$, the example can clearly be modified to give functions φ positive on all d-polytopes.)

The results in the other direction are more general and interesting. Various individual cases of translation invariant or covariant valuations satisfying Euler-type relations were shown by Shephard: the Steiner point [1966]; the mean width [1968a]; mixed volumes [1968c]. This last result, although its proof did not admit of generalization, gave a pointer to the general result, due to McMullen [1975a, 1977].

(12.4) **Theorem.** *Let φ be a translation invariant or covariant valuation on $\mathscr{P}^d$ which is homogeneous of degree r. Then $\varphi^*(P) = (-1)^r\varphi(-P)$ for all $P \in \mathscr{P}^d$.*

For translation invariant valuations, (12.4) reduces to (7.2) in view of

(12.5) **Lemma.** *For each translation class of flats $\mathscr{A}$, let $\psi_{\mathscr{A}}$ be the simple valuation corresponding to φ. Then $\psi_{\mathscr{A}}^*$ corresponds to φ^*.*

Since $\psi_{\mathscr{A}}^*(P) = (-1)^{\dim \mathscr{A}}\psi_{\mathscr{A}}(P) = (-1)^r\psi_{\mathscr{A}}(-P)$ for aff $P \in \mathscr{A}$, by (7.2), and $\gamma(F,P) = \gamma(-F,-P)$, (12.4) clearly follows.

For translation covariant valuations, we consider the valuation $\bar\varphi$ defined by $\bar\varphi(P) = \varphi(P) - (-1)^r\varphi^*(-P)$. It is easy to verify that the associated valuation $\bar\Phi$ vanishes (by the already proved cases of (12.4)); hence $\bar\varphi$ is translation invariant, so $\bar\varphi^*(P) = (-1)^r\bar\varphi(-P)$ (again, by the previously proved cases). Thus $\bar\varphi = 0$, as required.

Nothing we have said above needs to be altered if we confine our attention to lattice polytopes and valuations on them which are invariant or covariant under lattice translations. In the particularly interesting case of the lattice point enumerator G, the Euler-type relation

$$G_r^*(P) = (-1)^r G_r(-P) = (-1)^r G_r(P)$$

yields the *reciprocity law* of Ehrhart [1967b]:

(12.6) **Theorem.** *For a lattice polytope* P, *let* $G^0(P) = G(\text{relint } P)$. *Then*

$$G^0(nP) = (-1)^{\dim P} \sum_{r \geq 0} (-n)^r G_r(P),$$

for integer $n \geq 1$.

For,

$$G^0(P) = (-1)^{\dim P} G^*(P)$$

by (5.24).

Erhart [1967c], [1968] has also investigated G(P) for rational polytopes P (that is, vert $P \subseteq \mathbb{Q}^d$), and has proved an analogous formula to (12.6). More generally, we have (with the conventions and terminology of §10):

(12.7) **Theorem.** *Let* φ *be a lattice translation invariant or covariant valuation on rational polytopes, which is near-homogeneous of degree* r. *Then for all rational polytopes* P *and all integer* n,

$$\varphi^*(P,n) = (-1)^r \varphi(-P,-n).$$

The proof of (12.7) depends upon a specific characterization of such valuations φ, for which see §17 below, and McMullen [1978; 1982b].

For the extension of some Euler-type relations to systems larger than $\mathscr{P}^d$, see Perles-Sallee [1970], Groemer [1972], Hadwiger [1973].

IV. Characterization theorems

The purpose of this last chapter is a survey over the existing results, and some open problems, that concern the classification of valuations. For the most part, we shall deal with characterizations of the classical valuations described in §§3, 4 by some of their properties, where invariance under some group plays a crucial role. Further, a few results will be given on the explicit representation of more general classes of valuations.

§13. Minkowski additive functions

Before considering more general valuations, it seems appropriate to study those special valuations φ on $\mathscr{K}^d$ or $\mathscr{P}^d$ which are Minkowski additive, that is, satisfy

(13.1) $\varphi(K + L) = \varphi(K) + \varphi(L)$

for K,L in the domain of φ. Here K + L is the Minkowski sum of K and L. The valuations deserve special interest at least for two reasons: Due to the stro assumptions, more specific results are available; on the other hand, some proo for more general characterization results make essential use of certain i formation on Minkowski additive functions.

The Minkowski additive functions which we consider will have their values either $\mathbb{R}$, $\mathbb{E}^d$, or $\mathscr{K}^d$. It is clear that (13.1) then implies $\varphi(\lambda K) = \lambda\varphi(K)$ for ration $\lambda \geq 0$, and if φ is continuous, then this holds for all real $\lambda \geq 0$.

A familiar example of a real-valued Minkowski additive function on $\mathscr{K}^d$ is th mean width $\bar{b}$,

$$\bar{b}(K) = \frac{2}{\sigma(\Omega)} \int_\Omega h(K,u)\, d\sigma(u) \quad \text{for} \quad K \in \mathscr{K}^d,$$

which is a constant multiple of the quermassintegral W_{d-1}. It is continuous an rigid motion invariant, and it is essentially the only function with all thes properties:

(13.2) **Theorem.** *If* $\varphi: \mathscr{K}^d \to \mathbb{R}$ *is Minkowski additive, continuous, and invarian under rigid motions, then* $\varphi = \alpha W_{d-1}$ *with some real constant* α.

Hadwiger's proof ([1957], p. 213) uses a rotation averaging process and show more. Suppose that $\varphi: \mathscr{K}^d \to \mathbb{R}$ is Minkowski additive and rigid motio invariant. Let $K \in \mathscr{K}^d$ be given. If $K' = \lambda_1 g_1 K + \cdots + \lambda_r g_r K$ with positiv rational numbers $\lambda_1, \ldots, \lambda_r$ and rotations $g_1, \ldots, g_r \in SO_d$, then $\varphi(K') = (\lambda_1 + \cdots +$ $\lambda_r)\varphi(K)$. Since the same holds for the mean width, we have $\varphi(K)/\bar{b}(K) =$ $\varphi(K')/\bar{b}(K')$ assuming that $\bar{b}(K) \neq 0$. Now one knows that the λ's and g's can b chosen such that K' is arbitrarily close to the unit ball B. If φ is continuous at B then it follows that $\varphi(K)/\bar{b}(K) = \varphi(B)/2$ is independent of K. Thus in (13.2) i suffices to assume continuity of φ only at the unit ball.

On the other hand, this proof does not work if φ is only defined on polytopes Since (13.2) is a tool in the proof of later characterization theorems, this defec restricts the generality of those results in a similar way. Of course, if φ is locally uniformly continuous (with respect to the Hausdorff metric) on $\mathscr{P}^d$, then it has a unique continuous extension to $\mathscr{K}^d$, and the additivity and invariance properties would carry over. (We say that a function is *locally uniformly continuous* or *locally bounded*, if it is uniformly continuous, respectively bounded, on the elements of its domain inside any fixed ball.)

(13.3) **Problem.** *If* $\varphi: \mathscr{P}^d \to \mathbb{R}$ *is Minkowski additive, invariant under rigid motions, and either continuous or locally bounded, must it be a constant multiple of the mean width?*

As a vector valued counterpart to mean width we have the Steiner point s which can be defined by

$$s(K) = \frac{1}{\varkappa(d)} \int_\Omega h(K,u)\, u\, d\sigma(u)$$

and is obviously Minkowski additive. We say that a map f from $\mathscr{K}^d$ or $\mathscr{P}^d$ into $\mathbb{E}^d$ or $\mathscr{K}^d$ is *rigid motion equivariant* (*translation equivariant*) if $f(gK) = gf(K)$ for every rigid motion g of $\mathbb{E}^d$ ($f(K + t) = f(K) + t$ for $t \in \mathbb{E}^d$, respectively) and for all K. The following analogue of (13.2) holds:

(13.4) **Theorem.** *If* $f:\mathscr{K}^d \to \mathbb{E}^d$ *is Minkowski additive, continuous, and rigid motion equivariant, then* f(K) *is the Steiner point of* $K \in \mathscr{K}^d$.

Again, it suffices to assume that f is merely continuous at the unit ball. A proof for this case, which also uses rotation averaging, though in a more subtle way, was given by Posicel'skiĭ [1973]. The problem of characterizing the Steiner point by some of its properties was first posed by Grünbaum [1963], p. 239, who asked whether Minkowski additivity and similarity equivariance are sufficient to characterize the Steiner point. This is not the case, as can be shown by counter-examples. The first example to this effect was constructed by Sallee [1971]. A different example which is easier to describe, was mentioned by Schneider [1974a], p. 76. The first author to add the continuity assumption to Grünbaum's conditions was Shephard [1968b]. He proved (13.4) for d = 2, using Fourier series. His method was extended, though not in a straightforward way, to $d \geq 3$ by Schneider [1971], who proved (13.4) by making use of the fact that certain representations of the rotation group in spaces of spherical harmonics are irreducible. Although this method is less elementary and needs stronger assumptions than Posicel'skii's proof, the application of spherical harmonics seems to be a proper tool in this context, since it has proved useful in treating similar questions to be discussed below. Before Schneider's [1971] proof of (13.4), a slightly weaker version was obtained by Meyer [1970], who assumed uniform continuity. Two attempts to prove (13.4) (Schmitt [1968], Hadwiger [1969a]) contained errors. For the two-dimensional case, interesting elementary proofs were given by Hadwiger [1971] and Berg [1971]. The latter author obtained additional results for polytopes. To describe them, define

(13.5) $$s_\varphi(P) := \sum_{i=1}^{f_0(P)} \varphi(N(v_i,P) \cap \Omega) v_i \quad \text{for} \quad P \in \mathscr{P}^d,$$

where the sum extends over the vertices $v_1,\ldots,v_{f_0(P)}$ of the convex polytope P, $N(v_i,P)$ denotes the cone of exterior normal vectors of P at v_i (see §8) and φ is a simple valuation on spherical polytopes. If $\varphi = \sigma/\sigma(\Omega)$, where σ is spherical volume, then s_φ is the Steiner point s; this representation is a special case of (3.26). The proof for the Minkowski additivity of s which uses this representation (see Grünbaum [1967], p. 309) extends to yield that s_φ as defined above is Minkowski additive. Clearly $s_\varphi(\lambda P) = \lambda s_\varphi(P)$ for $\lambda \geq 0$, and if $\varphi(\Omega) = 1$, then s_φ is translation equivariant. If $f:\mathscr{P}^d \to \mathbb{E}^d$ is Minkowski additive and commutes with all similarities (including improper ones), then Berg [1971] calls this map an *abstract Steiner point*. Formula (13.5) yields an abstract Steiner point if φ is invariant under rotations and reflections and is normalized to $\varphi(\Omega) = 1$. Whether every abstract Steiner point is obtained in this way is not known, but Berg [1971] showed that this is true for d = 2,3. In these dimensions one then also has $\varphi = \theta \circ (\sigma/\sigma(\Omega))$ with some function $\theta:[0,1] \to \mathbb{R}$ satisfying $\theta(1) = 1$ and $\theta(u + v) = \theta(u) + \theta(v)$ for $u,v,u + v \in [0,1]$. Berg deduced that every abstract Steiner point on $\mathscr{P}^d$, d = 2,3, which is locally bounded, is the usual Steiner point.

When Minkowski additive maps on $\mathscr{K}^d$ are considered, it appears particularly natural to investigate those ones whose range is also $\mathscr{K}^d$. A Minkowski additive map $\Phi:\mathscr{K}^d \to \mathscr{K}^d$ which is continuous and rigid motion equivariant has been called an *endomorphism* of $\mathscr{K}^d$ in Schneider [1974a], since such a map is compatible with the most natural and geometrically important structures which one usually associates with $\mathscr{K}^d$. Special questions concerning endomorphisms of $\mathscr{K}^d$ were apparently first posed by Grünbaum ([1963], p. 239, [1967], p. 315; answers are given by Schneider [1974a], p. 54 and pp. 55–56). The investigation of endomorphisms of $\mathscr{K}^d$ shows different features in dimensions d = 2 and d > 2, due to the commutativity of the rotation group in dimension two. Let us first consider the two-dimensional case.

A particular example of an endomorphism $\Phi:\mathscr{K}^2 \to \mathscr{K}^2$ is given by

(13.6) $$\Phi(K) := \lambda_1 g_1[K - s(K)] + \cdots + \lambda_r g_r[K - s(K)] + s(K)$$

for $K \in \mathscr{K}^2$, where $\lambda_1,\ldots,\lambda_r \geq 0$ are real numbers and $g_1,\ldots,g_r \in SO_2$ are rotations. It turns out that the general endomorphism is a limit of such rotation averages. To formulate a precise result, we choose an orthonormal basis e_1,e_2 of $\mathbb{E}^2$ and write $u(\alpha) := (\cos\alpha)e_1 + (\sin\alpha)e_2$ for $\alpha \in [0,2\pi)$; further, for $K \in \mathscr{K}^2$ we write $h(K,\alpha)$ instead of $h(K,u(\alpha))$.

(13.7) **Theorem.** *Let* Φ *be an endomorphism of* $\mathscr{K}^2$. *Then there exists a (positive) measure* ν *on the Borel subsets of* $[0,2\pi)$ *such that*

(13.8) $$h(\Phi(K),\alpha) = \int_0^{2\pi} h(K - s(K),\alpha + \beta)\,d\nu(\beta) + \langle s(K),u(\alpha)\rangle$$

for $\alpha \in [0,2\pi)$ *and all* $K \in \mathscr{K}^2$.

This was proved by Schneider [1974b]. Conversely, any Borel measure ν on $[0,2\pi)$ defines an endomorphism Φ by means of (13.8). If Φ is given, the measure ν in (13.8) is unique up to the indefinite integral of $(a_1\cos\alpha + a_2\sin\alpha)$ with constants a_1,a_2. From (13.8) it can be deduced that any endomorphism Φ of $\mathscr{K}^2$ whose image contains a polygon (with more than one point) is of the form (13.6). Moreover, every endomorphism which maps $\mathscr{K}^2$ onto all of $\mathscr{K}^2$ is of the form $\Phi(K) = \lambda g[K - s(K)] + s(K)$ with $\lambda > 0$ and $g \in SO_2$. These are also precisely the extreme endomorphisms, if the set of all endomorphisms of $\mathscr{K}^2$ is made into a convex cone in a natural way (loc. cit., p. 310). Another consequence of (13.7) is the fact that some properties which Inzinger [1949] has proved for a certain very special class of endomorphisms, are shared by all endomorphisms, at least after suitable normalizations.

The proof of Theorem (13.7) uses a characterization, essentially due to Hadwiger [1951b], of the Minkowski additive, continuous, and translation invariant real functions on $\mathscr{K}^2$ (see §16). This result first yields an integral representation of the form

(13.9) $$h(\Phi(K) - s(K),\alpha) = \int_0^{2\pi} g(\beta - \alpha)\,dS_1(K,\alpha)$$

for $\alpha \in [0,2\pi)$ and $K \in \mathscr{K}^2$, where g is a continuous real function determined by the endomorphism Φ and $S_1(K,\cdot)$ is the first order area function of K, considered as a measure on $[0,2\pi)$ instead of Ω^1. The passage from (13.9) to (13.8) uses mainly analytic arguments.

For $d \geq 3$, a comparatively complete description of the endomorphisms of $\mathscr{K}^d$ is not known, but a series of partial results were obtained by Schneider [1974a]. Non-trivial examples of endomorphisms can be obtained as follows. We consider the support function $h(K,u) := \max\{\langle x,u\rangle : x \in K\}$ of $K \in \mathscr{K}^d$ for arbitrary $u \in \mathbb{E}^d$. Let $q:[0,\infty) \to [0,\infty)$ be a function for which all the following integrals exist and are finite. Let $K \in \mathscr{K}^d$ be given. It can be shown that $x \mapsto \int_{\mathbb{E}^d} h(K, x - \|x\|z)q(\|z\|)dz$ is a support function, hence there exists a unique convex body $\Phi_q(K)$ for which

(13.10) $$h(\Phi_q(K),x) = \int_{\mathbb{E}^d} h(K - s(K), x - \|x\|z)q(\|z\|)\,dz + \langle s(K),x\rangle$$

for $x \in \mathbb{E}^d$. It is then easy to see that the map $\Phi_q : \mathscr{K}^d \to \mathscr{K}^d$ defined by (13.10) is an endomorphism of $\mathscr{K}^d$. We remark that this map Φ_q, with special choices for q, is particularly useful in the treatment of certain approximation problems for convex bodies, see Berg [1969], Weil [1975b]. Further constructions for endomorphisms of $\mathscr{K}^d$ were described in Schneider [1974a]. The main theme of that paper was the investigation of additional assumptions by which, from the variety of endomorphisms, those with a simple geometric meaning could be singled out. The following results were obtained. Here we assume $d \geq 3$.

(13.11) **Theorem.** (a) *Every endomorphism of $\mathscr{K}^d$ is uniquely determined by the image of one suitably chosen convex body, for example, a triangle with at least one irrational* angle.

(b) *Let $\Phi : \mathscr{K}^d_d \to \mathscr{K}^d$ be a Minkowski additive and continuous map such that $\Phi(aK) = a\Phi(K)$ for every nonsingular affine transformation* a *of* $\mathbb{E}^d$. *Then*

$$\Phi(K) = K + \lambda[K + (-K)] \quad \textit{for} \quad K \in \mathscr{K}^d_d,$$

where $\lambda \geq 0$ is a real constant.

(c) *Let Φ be an endomorphism of $\mathscr{K}^d$. If the image under Φ of some at least one-dimensional convex body is a point, then $\Phi(K) = \{s(K)\}$ for $K \in \mathscr{K}^d$. If the image under Φ of some convex body is a segment, then*

$$\Phi(K) = \lambda[K - s(K)] + \mu[-K + s(K)] + s(K) \quad \textit{for} \quad K \in \mathscr{K}^d$$

with real numbers $\lambda,\mu \geq 0$, $\lambda + \mu > 0$.

(d) *The only surjective endomorphisms of $\mathscr{K}^d$ are given by*

$$\Phi(K) = \lambda[K - s(K)] + s(K) \quad \textit{for} \quad K \in \mathscr{K}^d$$

with $\lambda \neq 0$.

(e) *If Φ is an endomorphism of $\mathscr{K}^d$ satisfying $W_k(\Phi(K)) = W_k(K)$ for some $k \in \{0,1,\ldots,d-2\}$ and all $K \in \mathscr{K}^d$, then*

$$\Phi(K) = \varepsilon[K - s(K)] + s(K) \quad \textit{for} \quad K \in \mathscr{K}^d$$

where $\varepsilon \in \{1,-1\}$.

The proof uses a combination of elementary facts from harmonic analysis (for the rotation group acting on the sphere Ω) with convexity arguments.

So far the invariance or equivariance with respect to rigid motions has played an important role. It is clear that, in absence of such an assumption, particular Minkowski additive functions can only be characterized if fairly strong assumptions are imposed. Here is an example for such a result, due to Schneider [1974c]:

(13.2) **Theorem.** *Let $T : \mathscr{K}^d \to \mathscr{K}^d$ ($d \geq 2$) be a Minkowski additive map satisfying $V(T(K)) = V(K)$ for all $K \in \mathscr{K}^d$. Then there exists a volume preserving affine map $a : \mathbb{E}^d \to \mathbb{E}^d$ such that $T(K)$ is a translate of aK for each $K \in \mathscr{K}^d$.*

Finally we mention that Valette [1974] has studied the continuous maps $F : \mathscr{K}^d \to \mathscr{K}^d$ which commute with affine maps and, instead of being Minkowski additive, only satisfy $F(K_1 + K_2) \supseteq F(K_1) + F(K_2)$ for $K_1, K_2 \in \mathscr{K}^d$.

§14. Volume and centroid

From now on, it seems convenient to define that "valuation", without further specification of the range, means real-valued valuation.

We shall now review the characterization theorems for volume. The essential uniqueness of Haar measure, applied to the additive group of $\mathbb{E}^d$, gives the following well-known uniqueness theorem: If φ is a translation invariant (positive) measure on the Borel sets of $\mathbb{E}^d$ with $\varphi(C) = 1$ for some fixed unit cube C, then φ is the volume (Lebesgue measure). From a geometric point of view one would, of course, not want to assume from the beginning that φ is defined on the Borel sets and is σ-additive, but rather that φ is a simple valuation defined on $\mathscr{P}^d$ or $\mathscr{K}^d$. We shall often consider only $\mathscr{P}^d$ as domain, since the assumptions which we have to impose in order to guarantee uniqueness will afterwards also give uniqueness if $\mathscr{K}^d$ was the original domain. If φ is a simple valuation on $\mathscr{P}^d$, then we know from (5.19) and (5.20) (or from the simpler version in Hadwiger [1957], p. 81) that φ has an additive (simple) extension to the set $U(\mathscr{P}^d)$ of polyhedra, and if φ was invariant under translations or rigid motions, then the extension also is. If φ is nonnegative, then the extension is obviously monotone, that is, it satisfies $\varphi(A) \leq \varphi(B)$ for $A \subseteq B$ ($A,B \in U(\mathscr{P}^d)$), and hence also nonnegative. Thus for simple valuations it makes no difference whether they are defined on $\mathscr{P}^d$ or on $U(\mathscr{P}^d)$, and whether they are assumed nonnegative or monotone. For these valuations we have:

(14.1) **Theorem.** *Let φ be a translation invariant, nonnegative, simple valuation on $\mathscr{P}^d$. Then $\varphi = \alpha V$ with some real constant $\alpha \geq 0$.*

The easy proof is well-known from analysis text books, see, e.g., Maak [1960], §46, Satz 8; see also Hadwiger [1955b], p. 47. These and similar proofs for the characterization of elementary volume cannot be considered as strictly elementary from a geometric point of view, since they use infinite processes for simple geometric figures, like "exhaustion" or polyhedral approximation. By Dehn's theorem, no proof can work with finite dissections alone, without some limit process, even if translation invariance is sharpened to rigid motion invariance. But one can give a proof where the unavoidable limit process is only used to establish the essential uniqueness of a monotone solution of Cauchy's functional equation, while the geometric part uses only finite dissections of polytopes. A proof which is elementary in this sense was given by Hadwiger [1950] (for d = 3 also [1949a]), [1957], 2.1.3.

We digress a bit and consider the corresponding problem in spherical and hyperbolic space. For the notion of convex polytope in these spaces the reader may consult Böhm-Hertel [1980]. These authors also discuss at length the problem of an elementary theory of volume in those spaces, and they give such a proof of existence and uniqueness in dimension two. The uniqueness problem in general is the following: Suppose that φ is a nonnegative simple valuation on the convex polytopes of R^d (this stands for either euclidean, spherical, or hyperbolic space of dimension d) which is invariant under the motion group of R^d. Is φ a constant multiple of the usual volume? All the methods of proof mentioned above for $R^d = \mathbb{E}^d$ make essential use of the vector space structure of $\mathbb{E}^d$ and hence do not extend to non-euclidean spaces. An affirmative answer for spherical space was given by Schneider [1978], Th. (6.2). The proof, which can be generalized to yield an abstract version of the result for compact homogeneous spaces (Schneider [1981]), relies heavily on compactness. For hyperbolic space, too, the answer is in the affirmative, although no proof seems to be known which is as directly geometric as Hadwiger's in the euclidean case. We sketch a less elementary proof which works for all three types of spaces.

Suppose that $\varphi : \mathscr{P}(R^d) \to \mathbb{R}$ (where $\mathscr{P}(R^d)$ denotes the set of convex polytopes in R^d) is a nonnegative simple valuation, which is invariant under the motions of R^d. For an arbitrary subset $A \subset R^d$, define $\varphi^*(A) := \inf \sum \varphi(P_n)$, where the infimum extends over all sequences $(P_n)_{n \in \mathbb{N}}$ of convex polytopes in R^d which cover A. Then φ^* is an outer measure on R^d (see, e.g., Munroe [1953], ch. II). If $A, B \subset R^d$ are such that their distance (with respect to the usual metric on R^d) is positive, then it is easy to see (using coverings by convex polytopes with sufficiently small diameters) that $\varphi^*(A \cup B) = \varphi^*(A) + \varphi^*(B)$. Thus φ^* is a metric outer measure and hence all Borel sets of R^d are φ^*-measurable (Munroe, loc. cit.). Since φ is a simple valuation, it has an additive extension to the ring generated by the convex polytopes. We assert that the extension, also denoted by φ, is a premeasure (i.e., countably additive). This can be shown as in the usual construction of Lebesgue measure on $\mathbb{E}^d$ (e.g., Bauer [1978], p. 29), as soon as the following has been proved: (*) To any convex polytope $P \subset R^d$ and any $\varepsilon > 0$, there exists a convex polytope $P' \subset R^d$ such that $P' \subset \operatorname{int} P$ and $\varphi(P) - \varphi(P') \leq \varepsilon$. If P and ε are given, let P_1 be obtained from P by "pushing a facet" of P towards the interior of P. For all P_1 sufficiently close to P we have $\varphi(P \setminus P_1) < \varepsilon/n$, where n is the number of facets of P. Otherwise, the nonnegativity and motion invariance of the extension φ would easily yield a contradiction (since a large number of congruent copies of a "sufficiently flat" $P \setminus P_1$ can be packed into some fixed polytope). Continuing in this way, (*) is proved. After that, it follows that the restriction $\bar{\varphi}$ of φ^* to the φ^*-measurable sets is a measure which extends φ (e.g., Bauer [1978], §5). Since by construction it is invariant, its restriction to the Borel sets must be a constant multiple of Haar measure, from which the assertion follows.

We return to euclidean space and consider characterizations of volume t use assumptions different from those of (14.1). Hadwiger [1957], p. 79, pro that a valuation (not necessarily simple or nonnegative) on $\mathscr{P}^d$ which translation invariant and homogeneous of degree d, must be a constant multi of volume. A different invariance property was supposed by Hadwiger [1970] the course of proving a more general theorem he showed: A nonnegative sim valuation on $\mathscr{P}^d$ which is invariant under volume preserving linear maps of (keeping the origin fixed) must be a constant multiple of volume.

More important, with a view to the extension to quermassintegrals (see §15) a characterization of volume where the nonnegativity is replaced by continu with respect to the usual Hausdorff metric for convex bodies. The followi theorem is due to Hadwiger [1952d], [1957], p. 221.

(14.2) **Theorem.** *Let φ be a rigid motion invariant, continuous, simple valuation $\mathscr{K}^d$. Then φ is a constant multiple of volume.*

It seems to be unknown whether here $\mathscr{K}^d$ can be replaced by $\mathscr{P}^d$. Hadwige proof works with polytopes for a long while, but at the end it essentially us (13.2). An affirmative answer to Problem (13.3) would, therefore, yield corresponding generalization of (14.2). We remark that in (14.2) it would not sufficient to assume translation invariance instead of rigid motion invariance, s §16.

An apparently hard problem is the analogue of (14.2) in non-euclidean spac evidently the proof does not carry over. We restate Problem 74 of Grube Schneider [1979]:

(14.3) **Problem.** *Let φ be a rotation invariant, continuous, simple valuation on t spherically convex polytopes or convex bodies in Ω^{d-1}. Is φ a consta multiple of spherical volume?*

Partial information can be obtained from §8. Instead of considering spherical convex polytopes, it is convenient to use the convex polyhedral cones with apex which they generate. Theorem (8.8) and the remarks following it imply:

(14.4) **Theorem.** *If d is odd, then a rotation invariant simple valuation o polyhedral cones can be expressed as a linear combination of the valuatio on proper product cones.*

In particular, this can be used to give an affirmative answer to Problem (14.3 for d = 3. But whether (14.4) is helpful for the general case remains open.

We conclude this paragraph with a view to vector valued valuations. Some o the characterizations of volume in euclidean space have counterparts for th centroid (centre of gravity). For a bounded measurable set $K \subset \mathbb{E}^d$ with $V(K) \neq$ (the centroid is defined by

$$c(K) := \frac{1}{V(K)} \int_K x \, dV(x)$$

(so that, in the terminology of §3, $V(K)c(K) = z(K) = q_0(K)$ for $K \in \mathscr{K}^d$). Clearly Vc is a simple valuation with values in $\mathbb{E}^d$. We recall that $\mathscr{P}^d_d \subset \mathscr{P}^d$ ($\mathscr{K}^d_d \subset \mathscr{K}^d$) is the subset of polytopes (resp. convex bodies) with interior points. The following theorem of Schneider [1973] may be viewed as a counterpart to (14.1); the proof is also similar.

(14.5) **Theorem.** *Let* $f: U(\mathscr{P}^d_d) \to \mathbb{E}^d$ *be a translation equivariant function such that* $f(P)$ *lies in the convex hull of* P *and* Vf *is a simple valuation. Then* $f(P)$ *is the centroid of* P *for* $P \in U(\mathscr{P}^d_d)$.

And the following result of Schneider [1972b] (p. 211) is a counterpart to (14.2):

(14.6) **Theorem.** *Let* $f: \mathscr{K}^d_d \to \mathbb{E}^d$ *be a rigid motion equivariant continuous function such that* Vf *is a valuation. Then* $f(K)$ *is the centroid of* K *for* $K \in \mathscr{K}^d_d$.

§15. Quermassintegrals, mixed volumes, moment vectors, curvature measures

The following famous theorem of Hadwiger, which characterizes the linear combinations of quermassintegrals, is certainly the central result in a theory of valuations on convex bodies.

(15.1) **Theorem.** *If* φ *is a continuous and rigid motion invariant valuation on* $\mathscr{K}^d$, *then*

(15.2) $$\varphi(K) = \sum_{i=0}^{d} c_i W_i(K) \quad \textit{for} \quad K \in \mathscr{K}^d$$

with real constants $c_0, \ldots, c_d$.

There is a companion to (15.1) with continuity replaced by monotoneity (with respect to set inclusion):

(15.3) **Theorem.** *If* φ *is an increasing and rigid motion invariant valuation on* $\mathscr{K}^d$, *then* (15.2) *holds with nonnegative real constants* $c_0, \ldots, c_d$.

By means of (11.5), which was proved by McMullen [1977], it has later become clear that (15.3) can be deduced from (15.1).

Results of this type were first considered by Blaschke [1937], §43. He investigated the rigid motion invariant, locally bounded valuations on $U(\mathscr{P}^3)$, but in order to obtain a representation of type (15.2), he had to impose an additional assumption, namely that the volume part be invariant under volume preserving affinities. Since the other quermassintegrals do not have this property, the assumption, which is dictated by the method of proof, seems artificial; moreover, it can only be formulated in the course of the proof, since a "volume part" of the valuation is not defined from the beginning. Hadwiger proved (15.1) for $d = 3$ in [1951a] (see also [1955b], §16), and for general d in [1952d]. The proof of (15.3) was then given in Hadwiger [1953a]. Both proofs were reproduced in Hadwiger [1957], 6.1.10 (see also Leichtweiß [1980], §17). It may be remarked that the proof needs only the weak valuation property (without first having to

deduce the valuation property). Some (obvious) supplementary remarks to (15. were published by Müller [1967].

Hadwiger's proof of (15.1) uses induction with respect to the dimension a relies on (14.2) and thus on (13.2). It is, therefore, necessary to consider valuatio on $\mathscr{K}^d$, while the proof would not work if $\mathscr{P}^d$ were the domain. This is also tr for Hadwiger's proof of (15.3). The following seems to be open.

(15.4) **Problem.** *Let* φ *be a rigid motion invariant valuation on* $\mathscr{P}^d$ *which is eith continuous, or increasing, or locally bounded, or nonnegative. Is it true th* $\varphi(P) = \sum c_i W_i(P)$ *for* $P \in \mathscr{P}^d$ *with constants* $c_0, \ldots, c_d$?

Questions of this type were already suggested in the work of Hadwiger. T assumption of nonnegativity, known to imply monotoneity only in the simp case, was supposed by Spiegel. Both assumptions, local boundedness a nonnegativity, appear particularly natural in the case of polytopes. They a however, not appropriate for general convex bodies: For $K \in \mathscr{K}^d$, let $\varphi(K)$ be t sum (finite or infinite, but obviously well defined) of the $(d-1)$-volumes of t $(d-1)$-dimensional faces of K (and twice the $(d-1)$-volume if K is of dimensi at most $d-1$). Then φ is a rigid motion invariant valuation which is local bounded and nonnegative, but not a linear combination of quermassintegrals

The characterization theorem (15.1) has important applications in integr geometry. The principal idea also goes back to Blaschke [1937]. To demonstra this method in a simple case, consider formula (3.11). The integral on the righ hand side, considered as a function of K, clearly defines a rigid motion invaria continuous valuation on $\mathscr{K}^d$, hence it can be expressed as a linear combination quermassintegrals. Choosing for K a ball with variable radius, one then easi calculates the coefficients and thus proves (3.11). Hadwiger [1950d], [1955b [1956], [1957] used this approach systematically for the derivation of several o and new integral geometric formulae. An interesting application of (15.1) of different kind appears in Matheron's [1975] work on random sets.

We mention two variants of (15.1). As described in §5, Groemer [1972] h defined extensions of the quermassintegrals in the form of continuous line functions on a certain vector space A^d of so-called approximable functions on $\mathbb{E}$ From Hadwiger's theorem he could then deduce a corresponding characte ization theorem for the extended quermassintegrals. Baddeley [1980], motivat by requirements of stereology, developed an integral geometric theory of certa absolute curvature integrals, and he proved a characterization result analogo to, and motivated by, Hadwiger's theorem.

Results analogous to Theorems (15.1) and (15.3) should be expected to hold i noneuclidean spaces. Let us state this as a problem for spherical space. Fo spherically convex polytopes we defined the functionals φ_r by (3.31), and w mentioned that they have continuous extensions to general spherically conv sets. The functionals ψ_r defined by (3.32) are increasing.

(15.5) **Problem.** *Let* φ *be a rotation invariant valuation on the spherically conv sets (or polytopes) in the sphere* Ω. *If* φ *is continuous, is it a line combination of the* φ_r *with constant coefficients? If* φ *is increasing, is it linear combination of the* ψ_r *with nonnegative coefficients?*

The first part could be answered in the affirmative if (14.3) were true, since the induction part of Hadwiger's argument easily carries over.

We turn back to euclidean space and consider analogues of Hadwiger's theorem for the other particular valuations described in §3. A result which closely parallels (15.1) exists for the quermassvectors q_r defined by (3.23). These are continuous, rotation equivariant $\mathbb{E}^d$-valued valuations ($\mathbb{E}^d$-valuations, for short), as is clear from the definition. The behaviour under translations is exhibited by (3.24), namely

$$q_r(K + t) = q_r(K) + W_r(K)t \quad \text{for} \quad K \in \mathscr{K}^d \quad \text{and} \quad t \in \mathbb{E}^d.$$

From the characterization (13.4) of the Steiner point $s = q_d/\varkappa_d$ one can deduce the following stronger version, where Minkowski additivity is replaced by the valuation property (Schneider [1972b]). If one assumes that f commutes also with similarities, then a simpler reduction to (13.4) is possible, see Hadwiger [1971].

(15.6) **Theorem.** *Let* f *be a rigid motion equivariant continuous* $\mathbb{E}^d$*-valuation on* $\mathscr{K}^d$. *Then* f(K) *is the Steiner point of* K *for* $K \in \mathscr{K}^d$.

From this result and Hadwiger's theorem (15.1) it is not difficult to conclude the following.

(15.7) **Theorem.** *Let* f *be a rotation equivariant continuous* $\mathbb{E}^d$*-valuation on* $\mathscr{K}^d$ *such that* $f(K + t) - f(K)$ *is always parallel to* t. *Then*

$$f(K) = \sum_{i=0}^{d} c_i q_i(K) \quad \textit{for} \quad K \in \mathscr{K}^d$$

with real constants $c_0, \ldots, c_d$.

The proof may be found in Hadwiger-Schneider [1971] and Schneider [1972b]; these papers also contain applications to integral geometric formulae for the quermassvectors.

Also the area functions and curvature measures satisfy characterization theorems which may be compared with Hadwiger's theorem (15.1). These were proved by Schneider [1975a], [1975b], [1978] and were also applied to the derivation of integral geometric formulae. Since these results have already been reviewed in Schneider [1979] (§§6,7), we refer the reader to that survey. A new application of the characterization theorem for area functions to a problem on geometric probabilities was recently made by Vogiatzaki.

Since the quermassintegrals are special mixed volumes, one may ask whether more general mixed volumes can be characterized in a similar way. Only a very few special results in this direction are known. Fáry [1961] characterized, for a given convex body U, the functionals

$$\varphi : K \mapsto \sum_{i=0}^{d} c_i V(K,i;U,d-i)$$

with $c_0, \ldots, c_d \in \mathbb{R}$ as the translation invariant, continuous valuations φ satisfying

$\varphi(K) = \varphi(L)$ whenever $V(K,i;U,d-i) = V(L,i;U,d-i)$ for $i = 0,\ldots,d$. Clea an assumption of this kind cannot be omitted, but it makes the wh characterization theorem appear slightly artificial. Firey [1976] replaced valuation property by Minkowski linearity and he showed that an increas function $\varphi : \mathscr{K}^d \to \mathbb{R}$ with this property which is zero on one-pointed sets must of the form

$$\varphi(K) = V(K,1;\bar{K},p-1;s_{p+1},\ldots,s_d)$$

with an essentially unique convex body $\bar{K}$ and pairwise orthogonal segme $s_{p+1},\ldots,s_d$ of unit length which span the orthogonal complement of the affine h of $\bar{K}$. Obviously the assumptions of this theorem are quite strong, implying particular, that φ is translation invariant, nonnegative, and continuous; togetl with the linearity this opens the way to an application of the Riesz representati theorem.

Further characterization results related to mixed volumes appear in §16.

§16. Translation invariant valuations

Hadwiger's theorem (15.1) characterizes the (linear combinations of) querma integrals as those real valuations on $\mathscr{K}^d$ which are continuous and invaria under rigid motions. Among the various possibilities of relaxing the assumptio the condition of translation instead of motion invariance seems both natural a important. Translation invariant or covariant valuations are the topic of t present section. Theorem (14.1) is an example where translation invarianc together with some other conditions, is still sufficient to characterize a particul valuation. But without such strong additional assumptions, translation invaria valuations abound, and a classification seems difficult.

Let us first consider some examples. Generally speaking, any representation the quermassintegrals, suitably modified, will yield more general translatic invariant valuations. For polytopes $P \in \mathscr{P}^d$ we have

(16.1) $V_r(P) = \sum_{F^r} \gamma(F^r,P)V_r(F^r)$

by (3.30). Two modifications offer themselves. We may replace V_r on the righ hand side by a real function which, restricted to the polytopes in any dimensional affine subspace, yields a simple valuation in that space. The resultin function is a valuation on $\mathscr{P}^d$. In fact, every valuation on $\mathscr{P}^d$ (not necessaril enjoying an invariance property) can be represented as a finite sum of suc valuations (Hadwiger [1953b]). In the translation invariant case, a special suc representation was described and used in §10. On the other hand, the externa angle $\gamma(F^r,P)$ in (16.1) was defined via the notion of spherical volume. We ma replace this by other simple valuations on spherical polytopes and obtain translation invariant valuation on $\mathscr{P}^d$. For the characterization of suc valuations, see Theorem (16.6) below.

Now we consider general convex bodies. Definition (3.7) of the quermass integrals as special mixed volumes generalizes to

(16.1) $\varphi(K) := V(K,p;\mathscr{C}) \quad \text{for} \quad K \in \mathscr{K}^d,$

where $p \in \{1,\ldots,d-1\}$ and $\mathscr{C} = (K_{p+1},\ldots,K_d)$ is a fixed $(d-p)$-tuple of convex bodies. This yields a translation invariant valuation φ on $\mathscr{K}^d$ which is increasing and (hence) continuous. The integral geometric approach to quermassintegrals motivates the definitions (3.12) and (3.13) of valuations on $\mathscr{K}^d$ which are translation invariant if φ is. Finally, the mixed area functions defined by (3.14) can be used to generalize (16.2) by

(16.3) $$\varphi(K) := \int_\Omega g(u)\, dS(K,p;\mathscr{C};u) \quad \text{for} \quad K \in \mathscr{K}^d,$$

where $p \in \{1,\ldots,d-1\}, \mathscr{C} = (K_{p+1},\ldots,K_{d-1})$ is a fixed $(d-p-1)$-tuple of convex bodies, and g is a fixed continuous real function on Ω. It follows from the properties of the mixed area functions that φ is a continuous translation invariant valuation on $\mathscr{K}^d$ which is homogeneous of degree p. The special case

(16.4) $$\varphi(K) := \int_\Omega g(u)\, dS_{d-1}(K;u) \quad \text{for} \quad K \in \mathscr{K}^d$$

with an odd continuous function g yields a simple valuation. This shows that in (14.2) the rigid motion invariance is indispensable (and also that the last sentence on p. 387 of Hadwiger [1952e] is erroneous).

For each of these special classes of translation invariant valuations it would be interesting to have an axiomatic characterization. Let us now describe the known results.

First we consider the translation invariant valuations on $\mathscr{P}^d$ which are weakly continuous (see §11). Here a complete description is available. The case of simple valuations goes back to Hadwiger [1952e]. It extends, without change of proof, to valuations with values in a real topological vector space $\mathscr{X}$. To describe the result, recall that $\mathscr{U}^s$ is the Stiefel manifold of s-frames $U = (u_1,\ldots,u_s)$, that is, ordered orthonormal s-tuples of vectors, in $\mathbb{E}^d$. We call a function $\eta: \mathscr{U}^s \to \mathscr{X}$ *odd* if

$$\eta(\varepsilon_1 u_1,\ldots,\varepsilon_s u_s) = \varepsilon_1 \cdots \varepsilon_s \eta(u_1,\ldots,u_s)$$

whenever $\varepsilon_i = \pm 1$ $(i = 1,\ldots,s)$. For $p \in \mathscr{P}^d$ and $U \in \mathscr{U}^s$, the face P_U was defined in §6; by convention, $P_\varnothing = P$. Then we have (Hadwiger [1952e]):

(16.5) **Theorem.** *A function* $\varphi: \mathscr{P}^d \to \mathscr{X}$ *is a weakly continuous translation invariant simple valuation if and only if there is an expression*

$$\varphi(P) = \sum_{r=0}^{d} \sum_{U \in \mathscr{U}^{d-s}} V_r(P_U)\eta_r(U) \quad \text{for} \quad P \in \mathscr{P}^d,$$

where $\eta_r: \mathscr{U}^{d-r} \to \mathscr{X}$ *is an odd function* $(r = 0,\ldots,d-1;$ η_d *a constant).*

Clearly the sums occurring in (16.5) are finite. Note that the term for $r = 0$ vanishes identically when $d \geq 1$.

Essentially by employing the method used in §10, McMullen [1982b] extended Hadwiger's result to non-simple valuations:

(16.6) **Theorem.** *A function* $\varphi: \mathscr{P}^d \to \mathscr{X}$ *is a weakly continuous translation invariant valuation if and only if there is an expression*

$$\varphi(P) = \sum_{r=0}^{d} \sum_{F \in \mathscr{F}^r(P)} V_r(F)\lambda_r(F,P) \quad \textit{for} \quad P \in \mathscr{P}^d.$$

Here, $\mathscr{F}^r$ *denotes the family of* r*-faces of* P, λ_r *is a simple valuation o* $(d-r)$*-cones, and* $\lambda_r(F,P) = \lambda_r(N(F,P))$, *where* N(F,P) *is the cone of a outer normal vectors to* P *at* F.

Further extension is possible to translation covariant valuations (McMullen loc. cit.). According to §11, a valuation $\varphi: \mathscr{P}^d \to \mathscr{X}$ is translation covariant if ther exists a map $\Phi: \mathscr{P}^d \to \mathrm{Hom}_{\mathbb{Q}}(\mathbb{E}^d,\mathscr{X})$ such that $\varphi(P+t) - \varphi(P) = \Phi(P)t$ for $t \in \mathbb{E}$ and $P \in \mathscr{P}^d$. If φ is weakly continuous, then the rational linearity of Φ extend to real linearity, so that $\Phi(P) \in \mathrm{Hom}_{\mathbb{R}}(\mathbb{E}^d,\mathscr{X}) = \mathrm{Hom}(\mathbb{E}^d,\mathscr{X})$. Moreover, Φ is weakly continuous translation invariant valuation, and this permits to deduc the following from (16.5) and (16.6). Here the moment vector $m_{r+1}(P)$ of an r dimensional polytope P in $\mathbb{E}^d$ is defined by

$$m_{r+1}(P) := \int_P x\, dx,$$

where the integration is with respect to r-dimensional Lebesgue measure in aff P

(16.7) **Theorem.** *A function* $\varphi: \mathscr{P}^d \to \mathscr{X}$ *is a weakly continuous translatio covariant simple valuation if and only if there is an expression*

$$\varphi(P) = \sum_{r=0}^{d} \sum_{U \in \mathscr{U}^{d-r}} [H_r(U)m_{r+1}(P_U) + V_r(P_U)\eta_r(U)],$$

where $H_r: \mathscr{U}^{d-r} \to \mathrm{Hom}(\mathbb{E}^d,\mathscr{X})$ *and* $\eta_r: \mathscr{U}^{d-r} \to \mathscr{X}$ *are odd functions.*

(16.8) **Theorem.** *A function* $\varphi: \mathscr{P}^d \to \mathscr{X}$ *is a weakly continuous translatio covariant valuation if and only if there is an expression*

$$\varphi(P) = \sum_{r=0}^{d} \sum_{F \in \mathscr{F}^r(P)} [\Lambda_r(F,P)m_{r+1}(F) + V_r(F)\lambda_r(F,P)],$$

where λ_r *is a simple* $\mathscr{X}$*-valued valuation on normal cones of dimension* d − *and* Λ_r *is a simple* $\mathrm{Hom}(\mathbb{E}^d,\mathscr{X})$*-valued such valuation.*

Let us now turn to translation invariant (real valued) valuations on $\mathscr{K}^d$. Th problem of characterizing the translation invariant continuous valuations o $\mathscr{K}^d$ is open, but one has some partial results. A complete explicit representation i known in the two-dimensional case. We recall from §3 that $S_p(K;\cdot)$ is the p-th order area function of $K \in \mathscr{K}^d$, it is a positive measure on $\Omega = \Omega^{d-1}$. For $K \in \mathscr{P}^2$ or $\mathscr{K}^2$ let

(16.9) $$\varphi(K) = a + \int_{\Omega^1} g(u)\, dS_1(K;u) + bV_2(K),$$

where a,b are real constants and g is a real function on Ω^1 so that the integra exists for all K. Then φ is a translation invariant valuation.

(16.10) **Theorem.** *If* φ *is a locally bounded translation invariant valuation on* $\mathscr{P}^2$ *then constants* a,b *and a bounded function* g *exist so that* (16.9) *holds fo* $K \in \mathscr{P}^2$.

If φ is a continuous translation invariant valuation on $\mathscr{K}^2$, then constants a,b *and a continuous function* g *exist so that* (16.9) *holds for* $K \in \mathscr{K}^2$.

This was proved by Hadwiger [1949b], [1951b]. Actually, he did not use the area function $S_1(K;\cdot)$, but his results are easily seen to be equivalent to the above. The function g is uniquely determined by the valuation φ up to a summand of the form $\langle v,\cdot\rangle$ with a constant vector v. If φ in the second part of (16.10) is even Minkowski additive, then (16.9) holds with $a = b = 0$. This result was used to obtain (13.9), in the course of the proof of Theorem (13.7).

For $d \geq 3$, no such explicit representation is known. If φ is a continuous translation invariant valuation on $\mathscr{K}^d$, then it follows from §§10, 11 that $\varphi = \sum_{r=0}^{d} \varphi_r$, where φ_r is a continuous translation invariant valuation on $\mathscr{K}^d$ which is homogeneous of degree r. By Hadwiger [1957], p. 79, φ_d is a constant multiple of volume, and clearly φ_0 is constant. Thus there remains the problem of determining the continuous translation invariant valuations on $\mathscr{K}^d$ which are homogeneous of degree $r \in \{1,\ldots,d-1\}$. For $r = d-1$ the following solution was given by McMullen [1980].

(16.11) **Theorem.** *Let φ be a continuous translation invariant valuation on $\mathscr{K}^d$ which is homogeneous of degree* $d-1$. *Then there is a continuous function* g *on the unit sphere Ω such that*

$$\varphi(K) = \int_{\Omega} g(u)\, dS_{d-1}(K;u) \quad \textit{for} \quad K \in \mathscr{K}^d.$$

Thus (16.3), with $p = d-1$, describes the general translation invariant valuation which is continuous and homogeneous of degree $d-1$. If one prefers an expression in terms of mixed volumes, that is, valuations of type (16.2), one can deduce from (16.11) (see McMullen, loc. cit.) that to any such valuation φ correspond sequences $(L_i)_{i\in\mathbb{N}}$, $(M_i)_{i\in\mathbb{N}}$ in $\mathscr{K}^d$ such that

$$\varphi(K) = \lim_{i\to\infty} [V(K,d-1;L_i) - V(K,d-1;M_i)]$$

for $K \in \mathscr{K}^d$.

For degrees $p \in \{1,\ldots,d-2\}$, no analogue of (16.11) is known. Clearly any finite linear combination of functions of type (16.3) leads to a continuous translation invariant valuation on $\mathscr{K}^d$ which is homogeneous of degree p, but one cannot obtain the general such valuation in this way. Also, it seems difficult to draw any further conclusions from Theorem (16.6) in case φ is continuous.

Only for $p = 1$ and under a stronger continuity assumption has one a result (McMullen [1980]; compare also Schneider [1974b], p. 306). If φ is a uniformly continuous translation invariant valuation on $\mathscr{K}^d$ which is homogeneous of degree 1, then one easily deduces from the Riesz representation theorem that there exists a signed Borel measure μ on Ω such that

$$\varphi(K) = \int_{\Omega} h(K,u)\, d\mu(u) \quad \text{for} \quad K \in \mathscr{K}^d.$$

Hence, there exist $L, M \in \mathscr{K}^d$ such that

$$\varphi(K) = V(K;L,d-1) - V(K;M,d-1).$$

But the assumption is fairly strong, and there exist continuous translatio invariant valuations on $\mathscr{K}^d$, homogeneous of degree 1, which are not uniforml continuous; for an example, see Schneider [1974b], p. 306.

§17. Lattice invariant valuations

A function on subsets of $\mathbb{E}^d$ which is invariant under the translations of th integer lattice $\mathbb{Z}^d$, will briefly be called *lattice invariant*. The lattice poin enumerator G, the functionals G_i appearing in (4.1), and the weighted lattic point numbers are natural examples of lattice invariant valuations. In this sectio we consider, under the aspects of characterization and representation, the lattic invariant valuations on $\mathscr{P}^d_L$, the class of lattice polytopes in $\mathbb{E}^d$, and on $\mathscr{P}^d_{\mathbb{Q}}$, th class of polytopes with vertices in $\mathbb{Q}$.

It is an interesting challenge to prove characterization theorems in the spirit of §§16, 17 for valuations on $\mathscr{P}^d_L$. The following result of Betke [1979; 1982] may b viewed as an analogue of Hadwiger's theorem (15.1).

(17.1) **Theorem.** *Let φ be a real valuation on $\mathscr{P}^d_L$ which satisfies the inclusion-exclusion principle and is invariant under unimodular transformations. Then*

$$\varphi(P) = \sum_{i=0}^{d} a_i G_i(P) \quad \textit{for} \quad P \in \mathscr{P}^d_L$$

with real constants $a_0,\ldots,a_d$.

A unimodular transformation of $\mathbb{E}^d$ is a volume preserving affine map of $\mathbb{E}^d$ into itself which leaves the lattice $\mathbb{Z}^d$ invariant. Due to Stein's [1982] result, the assumption that the valuation φ satisfy the inclusion-exclusion principle can be omitted.

We turn to lattice invariant valuations on $\mathscr{P}^d_{\mathbb{Q}}$. Some of the structural results of §16 have analogues for this case; these were used to prove Theorems (10.7) and (12.7). The following can be proved along the lines of Hadwiger's characterization of weakly continuous translation invariant simple valuations.

(17.2) **Theorem.** *A function $\varphi : \mathscr{P}^d_{\mathbb{Q}} \to \mathbb{R}$ is a lattice invariant simple valuation if and only if there is an expression*

$$\varphi(P) = \sum_{r=0}^{d} \sum_{U \in \mathscr{U}^{d-r}} V_r(P_U)\eta_r(U,P_U) \quad \textit{for} \quad P \in \mathscr{P}^d_{\mathbb{Q}},$$

where $\eta_r(U,P_U)$ is odd in its first argument and depends only upon the translation class modulo $\mathbb{Z}^d$ of the translate of the subspace orthogonal to the frame U *which contains the face* P_U.

This was proved by McMullen [1978]. As a consequence, one has (10.7) for simple valuations, and then the methods of §10 can be adapted to this case to yield (10.7) in general.

Inspection of the proof of (17.2) shows that the range $\mathbb{R}$ of the valuation φ may be replaced by an arbitrary $\mathbb{Z}$-module $\mathscr{X}$. As remarked in McMullen [1982b], it is further possible to extend Theorems (16.6), (16.7) and (16.8) to lattice invariant or covariant valuations on $\mathscr{P}^d_{\mathbb{Q}}$, as long as the coefficients λ_r,Λ_r,H_r and η_r exhibit the same additional dependence as the η_r in Theorem (17.2); further, $\mathrm{Hom}(\mathbb{E}^d,\mathscr{X})$ is replaced by $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}^d,\mathscr{X})$.

References

Numbers in square brackets at the end of a reference denote the sections of this report in which that reference is quoted. However, we have also included a number of references of which no specific mention is made in the text (we have attempted a certain degree of completeness in the references, but we make no claim to success).

Aleksandrov, A.D.
1937a Zur Theorie der gemischten Volumina von konvexen Körpern. I. Verallgemeinerung einiger Begriffe der Theorie der konvexen Körper. (Russian) Mat. Sbornik N.S. **2** (1937), 947–972. Zbl 17–426. [5]
1937b Zur Theorie der gemischten Volumina von konvexen Körpern. II: Neue Ungleichungen zwischen den gemischten Volumina und ihre Anwendungen. (Russian) Mat. Sbornik N.S. **2** (1937), 1205–1238. Zbl 18–276. [5]

Allendoerfer, C.B.
1948 Steiner's formulae on a general S^{n+1}. Bull. Amer. Math. Soc. **54** (1948), 128–135. MR 9–607. [3]

Baddeley, A.
1980 Absolute curvatures in integral geometry. Math. Proc. Camb. Phil. Soc. **88** (1980), 45–58. MR. 81k:52011. [15]

Banchoff, T.
1967 Critical points and curvature for embedded polyhedra. J. Diff. Geom. **1** (1967), 245–256. MR 37 # 921. [5]
1970 Critical points and curvature for embedded polyhedral surfaces. Amer. Math. Monthly **77** (1970), 475–485. MR 41 # 4444. [5]

Bauer, H.
1978 Wahrscheinlichkeitstheorie und Grundzüge der Maßtheorie. 3rd ed., W. de Gruyter, Berlin-New York 1978. MR 39 # 983, MR 80b: 60001. [14].

Berg, C.
1969 Corps convexes et potentiels sphériques. Danske Vid. Selsk. Mat.-Fys. Medd. **37, 6** (1969). MR 40 # 7996. [13]
1971 Abstract Steiner points for convex polytopes. J. London Math. Soc. (2) **4** (1971), 176–180. MR 45 # 7593. [13]

Bernštein (Bernshtein), D.N.
1976 The number of integral points in integral polyhedra. (Russian) Funkcional. Anal. i Priložen. **10** (1976), no. 3, 72–73. English translation: Functional Anal. Appl. **10** (1976), 293–294. MR 54 # 8451. [4, 10]

Betke, U.
1979 Gitterpunkte und Gitterpunktfunktionale. Habilitationsschrift, Siegen 1979. [17]
1982 Ein Funktionalsatz für Gitterpolytope. (to appear) [17]

Betke, U., J.M. Wills
1979 Stetige und diskrete Funktionale konvexer Körper. In: Contributions to Geometry, ed. by J. Tölke and J.M. Wills. Birkhäuser Verlag, Basel etc. 1979, pp. 226–237. MR 81m: 10062. [4]

Blaschke, W.
1937 Vorlesungen über Integralgeometrie. 3rd ed., VEB Deutsch. Verl. d. Wiss., Berlin 1955 (First edition: 1937). [3, 15]

Böhm, J., E. Hertel
1980 Polyedergeometrie in n-dimensionalen Räumen konstanter Krümmung. VEB Deutsch. Verl. d. Wiss., Berlin 1980. Zbl 464.51009 + 466.52001. [3, 5, 14]

Boltianskii, V.G.
1956 Equivalent and equidecomposable figures. D.C. Heath Comp., Boston 1963 (Russian original: 1956). MR 37 # 5761. [6]
1958 Zerlegungsgleichheit ebener Polygone. Bul. Inst. Politehnic Jaşi **4** (8), 33–38. MR 21 # 2197.
1966 Equidecomposability of polygons and polyhedra. (Russian) Encyclopedia of Elementary Mathematics, vol. V, Moscow 1966, pp. 142–180.
1969 On Hilbert's third problem (Russian). Hilbert's problems (Russian). Izdat. "Nauka", Moscow, 1969, 92–94. MR 40 # 7920.
1976 Decomposition equivalence of polyhedra and groups of motions. Dokl. Akad. Nauk SSSR **231** (1976), 788–790. English translation: Soviet Math. Dokl. **17** (1976), 1628–1631. MR 54 # 13703.
1978 Hilbert's third problem. John Wiley & Sons, New York etc. 1978. MR 57 # 7346, MR 58 # 18074. [3, 6]

Brianchon, C.J.
1837 Théorème nouveau sur les polyèdres. J. Ecole (Royale) Polytechnique **15** (1837), 317–319. [8]

Bricard, R.
1896 Sur une question de géométrie relative aux polyèdres. Nouv. Ann. Math. **15** (1896), 331–334. [9]

Busemann, H.
1958 Convex surfaces. Interscience Publishers, New York 1958. MR 21 # 3900. [3]

Debrunner, H.
1952 Translative Zerlegungsgleichheit von Würfeln. Arch. Math. **3** (1952), 479–480. MR 14–1115.
1969 Zerlegungsähnlichkeit von Polyedern. Elem. Math. **24** (1969), 1–6. MR 38 # 6432. [6]
1978 Zerlegungsrelationen zwischen regulären Polyedern des E^d. Arch. Math. **30** (1978), 656–660. MR 50 # 18089.
1980 Über Zerlegungsgleichheit von Pflasterpolytopen mit Würfeln. Arch. Math. **35** (1980), 583–587. MR 82e: 52012.

Dehn, M.
1900 Über raumgleiche Polyeder. Nachr. Akad. Wiss. Göttingen, Math.-Phys. Kl. 1900, 345–354. [9]
1902 Über den Rauminhalt. Math. Ann. **55** (1902), 465–478. [9]
1905 Über den Inhalt sphärischer Dreiecke. Math. Ann. **60** (1905), 166–174.

Dupont, J.L., C.-H. Sah
1982 Scissors congruences, II. J. Pure Appl. Algebra **25** (1982), 159–195. [8]

Eckhoff, J.
1980 Die Euler-Charakteristik von Vereinigungen konvexer Mengen im $\mathbb{R}^d$. Abh. Math. Sem. Univ. Hamburg **50** (1980), 133–144. MR 82d: 52008. [2]

Ehrhart, E.
1967a Sur un problème de géométrie diophantienne linéaire, I. J. reine angew. Math. **226** (1967), 1–29. MR 35 # 4184. [4, 10]
1967b Démonstration de la loi de réciprocité pour un polyèdre entier. C.R. Acad. Sc. Paris, Sér. A **265** (1967), 5–7. MR 39 # 6826. [4]
1967c Démonstration de la loi de réciprocité du polyèdre rationnel. C.R. Acad. Sc. Paris, Sér. A **265** (1967), 91–94. MR 36 # 1393. [12]
1968 Sur la loi de réciprocité des polyèdres rationnels. C.R. Acad. Sc. Paris, Ser. A **266** (1968), 696–697. MR 38 # 3233. [12]

Emch, A.
1946 Endlichgleiche Zerschneidung von Parallelotopen in gewöhnlichen und höheren euklidischen Räumen. Comment. Math. Helvet. **18** (1945), 224–231.
MR 8-83.
Fáry, I.
1961 Functionals related to mixed volumes. Illinois J. Math. **5** (1961), 425–430.
MR 24 # A811. [15]
Federer, H.
1959 Curvature measures. Trans. Amer. Math. Soc. **93** (1959), 418–491.
MR 22 # 961. [5]
Fenchel, W., B. Jessen
1938 Mengenfunktionen und konvexe Körper. Danske Vid. Selskab. Mat.-fys. Medd. **16**, 3 (1938).
Zbl 18, p. 424. [3]
Firey, W.J.
1976 A functional characterization of certain mixed volumes. Israel J. Math. **24** (1976), 274–281.
MR 58 # 30744. [15]
Gerwien, P.
1833a Zerschneidung jeder beliebigen Anzahl von gleichen geradlinigen Figuren in dieselben Stücke. J. reine angew. Math. **10** (1833), 228–234. [9]
1833b Zerschneidung jeder beliebigen Menge verschieden gestalteter Figuren von gleichem Inhalt auf der Kugelfläche in dieselben Stücke. J. reine angew. Math. **10** (1833), 235–240.
Gram, J.P.
1874 Om Rumvinklerne i et Polyeder. Tidsskr. Math. (Copenhagen) (3) **4** (1874), 161–163. [8]
Groemer, H.
1972 Eulersche Charakteristik, Projektionen und Quermaßintegrale. Math. Ann. **198** (1972), 23–56.
MR 48 # 7116. [2, 5, 12, 15]
1973 Über einige Invarianzeigenschaften der Eulerschen Charakteristik. Comment. Math. Helvet. **48** (1973), 87–99.
MR 49 # 7967. [2]
1974 On the Euler characteristic in spaces with a separability property. Math. Ann. **211** (1974), 315–321.
MR 50 # 5635. [2]
1975 The Euler characteristic and related functionals on convex surfaces. Geom. Dedicata **4** (1975), 91–104.
MR 56 # 6574. [2]
1977a Minkowski addition and mixed volumes. Geom. Dedicata **6** (1977), 141–163.
MR 57 # 1386. [3, 5]
1977b On translative integral geometry. Arch. Math. **29** (1977), 324–330.
MR 58 # 12872. [5]
1978 On the extension of additive functionals on classes of convex sets. Pacific J. Math. **75** (1978), 397–410.
MR 58 # 24003. [1, 5]
Gruber, P.M.
1979 Geometry of numbers. In: Contributions to Geometry, ed. by J. Tölke and J.M. Wills. Birkhäuser Verlag, Basel etc. 1979, pp. 186–225.
MR 81h: 10044. [4]
Gruber, P.M., R. Schneider
1979 Problems in geometric convexity. In: Contributions to Geometry, ed. by J. Tölke and J. M. Wills. Birkhäuser Verlag, Basel etc. 1979, pp. 255–278.
MR. 82d: 52001. [14]
Grünbaum, B.
1963 Measures of symmetry for convex bodies. Proc. Symp. Pure Math. **7** (1963), 223–270; Amer. Math. Soc., Providence, R.I.
MR 27 # 6187. [13]
1967 Convex polytopes. Interscience Publishers, London etc. 1967.
MR 37 # 2085. [3, 13]

Hadwiger, H.
1947 Über eine symbolisch-topologische Formel. Elem. Math. **2** (1947), 35–41. Portugues translation: Gazeta Mat. **35** (1948), 6–9.
MR 8-484. [2]
1949a Bemerkung zur elementaren Inhaltslehre des Raumes. Elem. Math. **4** (1949), 3–7.
MR 10-564. [14]
1949b Über beschränkte additive Funktionale konvexer Polygone. Publ. Math. Debrecen **1** (1949) 104–108.
MR 12-124. [16]
1949c Zerlegungsgleichheit und additive Polyederfunktionale. Arch. Math. **1** (1949), 468–472.
MR 11-165.
1950a Zum Problem der Zerlegungsgleichheit der Polyeder. Arch. Math. **2** (1950), 441–444.
MR 13-377.
1950b Zerlegungsgleichheit und additive Polyederfunktionale. Comment. Math. Helvet. **24** (1950), 204–218.
MR 12-526.
1950c Zur Inhaltstheorie der Polyeder. Collect. Math **3** (1950), 137–158.
MR 13-154. [14]
1950d Einige Anwendungen eines Funktionalsatzes für konvexe Körper in der räumlichen Integralgeometrie. Monatsh. Math. **54** (1950), 345–353.
MR 12-734. [15]
1950e Translative Zerlegungsgleichheit k-dimensionaler Parallelotope. Collect. Math. **3** (1950), 11–23.
MR 13-768.
1951a Beweis eines Funktionalsatzes für konvexe Körper. Abh. Math. Sem. Univ. Hamburg **17** (1951), 69–76.
MR 12-851. [15]
1951b Translationsinvariante, additive und stetige Eibereichfunktionale. Publ. Math. Debrecen **2** (1951), 81–94.
MR!3-971. [11, 13, 16]
1952a Ergänzungsgleichheit k-dimensionaler Polyeder. Math. Z. **55** (1952), 292–298.
MR 14-309.
1952b Über addierbare Intervallfunktionale. Tôhoku Math. J. **4** (1952), 32–37.
MR 14-28.
1952c Mittelpunktspolyeder und translative Zerlegungsgleichheit. Math. Nachr. **8** (1952), 53–58.
MR 14-309.
1952d Additive Funktionale k-dimensionaler Eikörper. I. Arch. Math. **3** (1952), 470–478.
MR 14-1114. [14, 15]
1952e Translationsinvariante, additive und schwachstetige Polyederfunktionale. Arch. Math. **3** (1952), 387–394.
MR 14-964. [11, 15]
1953a Additive Funktionale k-dimensionaler Eikörper. II. Arch. Math. **4** (1953), 374–379.
MR 15-551.
1953b Über additive Funktionale k-dimensionaler Eipolyeder. Publ. Math. Debrecen **3** (1953), 87–94.
MR 16-345. [10, 16]
1953c Lineare additive Polyederfunktionale und Zerlegungsgleichheit. Math. Z. **58** (1953), 4–14.
MR 14-964.
1953d Über Gitter und Polyeder. Monatsh. Math. **57** (1953), 246–254.
MR 15-551. [4]
1954a Zum Problem der Zerlegungsgleichheit k-dimensionaler Polyeder. Math. Ann. **127** (1954), 170–174.
MR 15-691.
1954b Zur Zerlegungstheorie euklidischer Polyeder. Ann. Mat. pura appl. (IV) **36** (1954), 315–334.
MR 16-396.
1955a Eulers Charakteristik und kombinatorische Geometrie. J. reine angew. Math. **194** (1955), 101–110.
MR 17-402. [2]

1955b Altes und Neues über konvexe Körper. Birkhäuser Verlag, Basel und Stuttgart 1955.
MR 17-401. [14, 15]
1956 Integralsätze im Konvexring. Abh. Math. Sem. Univ. Hamburg **20** (1956), 136–154.
MR 18-503. [3, 15]
1957 Vorlesungen über Inhalt, Oberfläche und Isoperimetrie. Springer-Verlag, Berlin etc. 1957.
MR 21 # 1561. [1–7, 10, 13–16]
1959 Normale Körper im euklidischen Raum und ihre topologischen und metrischen Eigenschaften. Math. Z. **71** (1959), 124–140.
MR 23 # A1008. [2, 5]
1960 Zur Eulerschen Charakteristik euklidischer Polyeder. Monatsh. Math. **64** (1960), 49–60.
MR 23 # A992. [2]
1963 Ungelöstes Problem Nr. 45. Elem. Math. **18** (1963), 29–31.
1968a Translative Zerlegungsgleichheit der Polyeder des gewöhnlichen Raumes. J. reine angew. Math. **233** (1968), 200–212.
MR 38 # 6454.
1968b Eine Schnittrekursion für die Eulersche Charakteristik euklidischer Polyeder mit Anwendungen innerhalb der kombinatorischen Geometrie. Elem. Math. **23** (1968), 121–132.
MR 38 # 5112. [2]
1968c Neuere Ergebnisse innerhalb der Zerlegungstheorie euklidischer Polyeder. Jber. Deutsche Math. Ver. **70** (1968), 167–176.
MR 39 # 869. [6]
1969a Zur axiomatischen Charakterisierung des Steinerpunktes konvexer Körper. Israel J. Math. **7** (1969), 168–176.
MR 40 # 4863. [13]
1969b Eckenkrümmung beliebiger kompakter euklidischer Polyeder und Charakteristik von Euler-Poincaré. L'Enseignement Math. **15** (1969), 147–151.
MR 40 # 864. [5]
1969c Notiz zur Eulerschen Charakteristik offener und abgeschlossener Polyeder. Studia Sci. Math. Hung. **4** (1969), 385–387.
MR 40 # 1888. [2]
1970 Zentralaffine Kennzeichnung des Jordanschen Inhaltes. Elem. Math. **25** (1970), 25–27.
MR 55 # 8322. [14]
1971 Zur axiomatischen Charakterisierung des Steinerpunktes konvexer Körper; Berichtigung und Nachtrag. Israel J. Math. **9** (1971), 466–472.
MR 43 # 3919. [1, 13, 15]
1972 Polytopes and translative equidecomposability. Amer. Math. Monthly **79** (1972), 275–276.
1973 Erweiterter Polyedersatz und Euler-Shephardsche Additionstheoreme. Abh. Math. Sem. Univ. Hamburg **39** (1973), 120–129.
MR 48 # 9555. [2, 12]
1974a Begründung der Eulerschen Charakteristik innerhalb der ebenen Elementargeometrie. L'Enseignement Math. **20** (1974), 33–43.
MR 50 # 14484. [2]
1974b Homothetieinvariante und additive Polyederfunktionen. Arch. Math. **25** (1974), 203–205.
MR 49 # 11387. [6]
1975 Zerlegungsgleichheit euklidischer Polyeder bezüglich passender Abbildungsgruppen und invariante Funktionale. Math.-Phys. Sem. Ber. **22** (1975), 125–133.
MR 51 # 6586. [6]

Hadwiger, H., P. Glur
1951 Zerlegungsgleichheit ebener Polygone. Elem. Math. **6** (1951), 97–106.
MR 13-576. [9]

Hadwiger, H., P. Mani
1972 On the Euler characteristic of spherical polyhedra and the Euler relation. Mathematika **19** (1972), 139–143.
MR 48 # 1040. [2]
1974 On polyhedra with extremal Euler characteristic. J. Combinat. Theory, Ser. A **17** (1974), 345–349.
MR 50 # 11032. [2]

Hadwiger, H., R. Schneider
1971 Vektorielle Integralgeometrie. Elem. Math. **26** (1971), 49–57.
MR 44 # 967. [3, 5, 15]

Harazišvili, A.B.
1977 Equicomposition of polyhedra relative to the group of homotheties and translation (Russian) Dokl. Akad. Nauk SSSR **236** (1977), 552–555. English translation: Soviet Matl Dokl. **18** (1977), 1246–1249.
MR 58 # 12726. [6]
1978 On an invariant defined on the class of three-dimensional polytopes. (Russian) Bull. Acac Sci. Georgian SSR **91** (1978), 33–36.
MR 81b: 52006 [6]

Hertel, E.
1969 Über Intervallpolygone. Wiss. Z. Friedrich-Schiller-Univ. Jena, Math.-Nat. Reihe **18** (1969 299–303.
MR 42 # 955.
1971 Über Intervallpolyeder im R_n. Beitr. Algebra Geom. **1** (1971), 77–83.
MR 46 # 2550.
1973 Zur translativen Zerlegungsgleichheit n-dimensionaler Polyeder. Publ. Math. Debrecen 2 (1973), 133–140.
MR 48 # 7117.
1974a Ein Subtraktionssatz der Polyederalgebra. Beitr. Algebra Geom. **2** (1974), 83–86.
MR 53 # 11498.
1974b Mittelpunktspolyeder im E^4. Elem. Math. **29** (1974), 59–64.
MR 49 # 7920.
1974c Polyederstrukturen. Math. Nachr. **62** (1974), 57–63.
MR 50 # 14494.
1977 Neuere Ergebnisse und Richtungen der Zerlegungstheorie von Polyedern. Mitt. Math. Ges DDR 1977, Heft 4, 5–22.
Zbl 401. 52004. [6]

Hertel, E., H.E. Debrunner
1980 Zur Rolle von Subtraktionssatz und Divisionssatz in Zerlegungsstrukturen. Beitr. Algebr Geom. **10** (1980), 145–148.
Zbl 469. 52004.

Hilbert, D.
1899 Grundlagen der Geometrie. Teubner, Leipzig 1899. [3]
1900 Mathematische Probleme. Nachr. Königl. Ges. Wiss. Göttingen, Math.-Phys. Kl. 1900, 253-297. English translation: Bull. Amer. Math. Soc. **8** (1902), 437–479. [9]

Inzinger, R.
1949 Über eine lineare Transformation in den Mengen der konvexen und der stützbaren Bereich einer Ebene. Monatsh. Math. **53** (1949), 227–250.
MR 11-455. [13]

Jessen, B.
1939 Om Polyedres Rumfang. Mat. Tidsskrift A 1939, 35–44.
MR 1-156.
1941 En Bemaerkning om Polyedres Volumen. Mat. Tidsskrift B 1941, 59–65.
MR 7-68.
1946 Om Aekvivalens af Aggregater af regulaere Polyedre. Mat. Tidsskrift B 1946, 145–148.
MR 7-386.
1967 Orthogonal icosahedra. Nordisk Mat. Tidsskrift **15** (1967), 90–96.
MR 37 # 2088.
1968 The algebra of polyhedra and the Dehn-Sydler theorem. Math. Scand. **22** (1968), 241 256.
MR 40 # 4860. [9]
1972 Zur Algebra der Polytope. Nachr. Akad. Wiss. Göttingen. II. Math.-phys. Kl. 1972, 47–53
MR 50 # 5636. [9]
1978 Einige Bemerkungen zur Algebra der Polyeder in nicht-euklidischen Räumen. Comment Math. Helvet. **53** (1978), 525–528.
MR 80b: 51010.

Jessen, B., J. Karpf, A. Thorup
1968 Some functional equations in groups and rings. Math. Scand. **22** (1968), 257–265.
MR 44 # 1963.
Jessen, B., A. Thorup
1978 The algebra of polytopes in affine spaces. Math. Scand. **43** (1978), 211–240.
MR 80i: 52008. [6]
Kagan, W.F.
1903 Über die Transformation der Polyeder. Math. Ann. **57** (1903), 421–424.
Kirsch, A.
1978 Polyederfunktionale, die nicht translationsinvariant, aber injektiv sind. Elem. Math. **33** (1978), 105–107.
MR 80a: 52009.
Klee, V.
1963 The Euler characteristic in combinatorial geometry. Amer. Math. Monthly **70** (1963), 119–127.
MR 26 # 3627. [2]
Kuiper, N.H.
1971 Morse relations for curvature and tightness. In: Proc. Liverpool Singularities Symp. II, ed. by C.T.C. Wall, pp. 77–89. Lecture Notes in Math. **209**, Springer, Berlin etc. 1971.
MR 49 # 8039. [5]
Kummer, H.
1956 Translative Zerlegungsgleichheit k-dimensionaler Parallelotope. Arch. Math. **7** (1956), 219–220.
MR 18-228.
Lebesgue, H.
1938 Sur l'équivalence des polyèdres, en particulier des polyèdres réguliers, et sur la dissection des polyèdres réguliers en polyèdres réguliers. Ann. Soc. Polon. Math. **17** (1938), 193–226.
1945 Sur l'équivalence des polyèdres. Ann. Soc. Polon. Math. **18** (1945), 1–3.
MR 8-218.
Leichtweiss, K.
1980 Konvexe Mengen. VEB Deutsch. Verl. d. Wiss., Berlin 1980.
MR 81b: 52001. [15]
Lenz, H.
1970 Mengenalgebra und Eulersche Charakteristik. Abh. Math. Sem. Univ. Hamburg **34** (1970), 135–147.
MR 42 # 3674. [2, 5]
Lindgren, H.
1972 Recreational problems in geometric dissections and how to solve them. Revised and enlarged by Greg Frederickson. Dover Publ., New York 1972.
Zbl 261. 50006.
Maak, W.
1960 Differential- und Integralrechnung. 2nd ed., Vandenhoeck und Ruprecht, Göttingen 1960.
MR 22 # 3764. [14]
Macdonald, I.G.
1963 The volume of a lattice polyhedron. Proc. Camb. Phil. Soc. **59** (1963), 719–726.
MR 27 # 4139. [4]
1971 Polynomials associated with finite cell-complexes. J. London Math. Soc. **4** (1971), 181–192.
MR 45 # 7594. [4]
Mani, P.
1971 On angle sums and Steiner points of polyhedra. Israel J. Math. **9** (1971), 380–388.
MR 43 # 3913. [5]
Matheron, G.
1975 Random sets and integral geometry. Wiley, New York etc. 1975.
MR 52 # 6828. [15]
McMullen, P.
1975a Metrical and combinatorial properties of convex polytopes. Proc. Internat. Congr. Math. Vancouver 1974. (1975), pp. 491–495.
MR 55 # 3974. [10, 12]

1975b Non-linear angle-sum relations for polyhedral cones and polytopes. Math. Proc. Cam Phil. Soc. **78** (1975), 247–261.
MR 52 # 15238. [7, 8]
1977 Valuations and Euler-type relations on certain classes of convex polytopes. Proc. Londo Math. Soc. **35** (1977), 113–135.
MR 56 # 6548. [6, 7, 10–12, 15]
1978 Lattice invariant valuations on rational polytopes. Arch. Math. **31** (1978), 509–516.
MR 80d: 52011. [10, 12, 17]
1980 Continuous translation invariant valuations on the space of compact convex sets. Arc Math. **34** (1980), 377–384.
MR 81m: 52013. [16]
1982a The Brianchon-Gram theorem for polyhedral sets. (to appear) [8]
1982b Weakly continuous valuations on convex polytopes. (to appear) [6, 7, 10, 12, 16, 17]
Meier, C.
1972 Zerlegungsähnlichkeit von Polyedern. J. reine angew. Math. **253** (1972), 193–202.
MR 46 # 792. [6]
1977 Multilinearität bei Polyederaddition. Arch. Math. **29** (1977), 210–217.
MR 57 # 4005. [6, 10]
1982 Ein Skalarprodukt für konvexe Körper. (in preparation) [5]
Meyer, W.J.
1970 Characterization of the Steiner point. Pacific J. Math. **35** (1970), 717–725.
MR 43 # 3917. [13]
Minkowski, H.
1911 Theorie der konvexen Körper, insbesondere Begründung ihres Oberflächenbegriffs. Ges Abh., vol. II, pp. 131–229. B.G. Teubner, Leipzig etc. 1911. [3]
Müller, H.R.
1967 Zur axiomatischen Begründung der Eikörperfunktionale. Monatsh. Math. **71** (1967), 338–343.
MR 36 # 5818. [15]
Munroe, M.E.
1953 Introduction to measure and integration. Addison-Wesley Publ. Comp., Reading, Mass 1953.
MR 14-734. [14]
Mürner, P.
1974 Zwei Beispiele zur Zerlegungsgleichheit 4-dimensionaler Polytope. Elem. Math. **29** (1974) 132–135.
MR 50 # 8303.
1975 Translative Parkettierungspolyeder und Zerlegungsgleichheit. Elem. Math. **30** (1975), 25–27.
MR 51 # 11282.
1977 Translative Zerlegungsgleichheit von Polytopen. Arch. Math. **29** (1977), 218–224.
MR 57 # 4006.
Nicoletti, O.
1914 Sulla equivalenza dei poliedri. Rend. Circ. Mat. Palermo **37** (1914), 47–75.
1915 Sulla equivalenza dei poliedri. Rend. Circ. Mat. Palermo **40** (1915), 194–210.
Perles, M.A., Sallee, G.T.
1970 Cell complexes, valuations, and the Euler relation. Canad. J. Math. **22** (1970), 235–241.
MR 41 # 7530. [5, 12]
Posicel'skiĭ (Positsel'skii), E.D.
1973 Characterization of Steiner points. (Russian) Mat. Zametki **14** (1973), 243–247. English translation: Math. Notes **14** (1973), 698–700.
MR 48 # 4916. [13]
Rota, G.-C.
1964 On the foundations of combinatorial theory. I. Theory of Möbius functions. Z. Wahrscheinlichkeitstheorie verw. Geb. **2** (1964), 340–368.
MR 30 # 4688. [2, 5, 8]
1971 On the combinatorics of the Euler characteristic. Studies in Pure Mathematics (Papers presented to Richard Rado), pp. 221–233. Academic Press, London 1971.
MR 44 # 126. [2]

Sah, C.-H.

1979 Hilbert's third problem: Scissors congruence. Pitman Advanced Publishing Program, San Francisco etc. 1979.
MR 81g: 51011. [6, 8]

1981 Scissors congruences. I: The Gauss-Bonnet map. Math. Scand. **49** (1981), 181–210. [8]

Sallee, G.T.

1966 A valuation property of Steiner points. Mathematika **13** (1966), 76–82.
MR 33 # 7937. [1, 5]

1968 Polytopes, valuations, and the Euler relation. Canad. J. Math. **20** (1968), 1412–1424.
MR 38 # 605. [1, 5, 12]

1971 A non-continuous "Steiner point". Israel J. Math. **10** (1971), 1–5.
MR 45 # 5873. [13]

Santaló, L.A.

1976 Integral geometry and geometric probability. Addison-Wesley Publ. Comp., Reading, Mass. 1976.
MR 55 # 6340. [3]

Scherk, P.

1969 Über eine Klasse von Polyederfunktionalen. Comment. Math. Helvet. **44** (1969), 191–201.
MR 39 # 6189.

Schmitt, K.A.

1968 Kennzeichnung des Steinerpunktes konvexer Körper. Math. Z. **105** (1968), 387–392.
MR 37 # 6834. [13]

Schneider, R.

1971 On Steiner points of convex bodies. Israel J. Math. **9** (1971), 241–249.
MR 43 # 3918. [13]

1972a Krümmungsschwerpunkte konvexer Körper, I. Abh. Math. Sem. Univ. Hamburg **37** (1972), 112–132.
MR 46 # 6160. [3]

1972b Krümmungsschwerpunkte konvexer Körper, II. Abh. Math. Sem. Univ. Hamburg **37** (1972), 204–217.
MR 48 # 9554. [3, 14, 15]

1973 Volumen und Schwerpunkt von Polyedern. Elem. Math. **28** (1973), 137–141.
MR 48 # 12305. [14]

1974a Equivariant endomorphisms of the space of convex bodies. Trans. Amer. Math. Soc. **194** (1974), 53–78.
MR 50 # 5633. [13]

1974b Bewegungsäquivariante, additive und stetige Transformationen konvexer Bereiche. Arch. Math. **25** (1974), 303–312.
MR 49 # 9738. [13, 16]

1974c Additive Transformationen konvexer Körper. Geom. Dedicata **3** (1974), 221–228.
MR 50 # 8299. [13]

1975a Kinematische Berührmaße für konvexe Körper. Abh. Math. Sem. Univ. Hamburg **44** (1975), 12–23.
MR 52 # 15249. [3, 15]

1975b Kinematische Berührmaße für konvexe Körper und Integralrelationen für Oberflächenmaße. Math. Ann. **218** (1975), 253–267.
MR 52 # 806. [15]

1977a Kritische Punkte und Krümmung für die Mengen des Konvexringes. L'Enseignement Math. **23** (1977), 1–6.
MR 57 # 7474. [5]

1977b Ein kombinatorisches Analogon zum Satz von Gauss-Bonnet. Elem. Math. **32** (1977), 105–108.
MR 58 # 13057. [5]

1978 Curvature measures of convex bodies. Ann. Mat. pura appl. **116** (1978), 101–134.
MR 80d: 52012. [3, 14, 15]

1979 Boundary structure and curvature of convex bodies. In: Contributions to Geometry, ed. by J. Tölke and J.M. Wills. Birkhäuser Verlag, Basel etc. 1979, pp. 13–59.
MR 81i: 52001. [3, 15]

1980 Parallelmengen mit Vielfachheit und Steiner-Formeln. Geom. Dedicata **9** (1980), 111–12
MR 81f: 52006. [5]

1981 A uniqueness theorem for finitely additive invariant measures on a compact homogeneou space. Rend. Circ. Mat. Palermo (Ser. II) **30** (1981), 341–344. [14]

Shephard, G.C.

1966 The Steiner point of a convex polytope. Canad. J. Math. **18** (1966), 1294–1300.
MR 35 # 4814. [12]

1967 An elementary proof of Gram's theorem for convex polytopes. Canad. J. Math. **19** (1967 1214–1217.
MR 37 # 822. [8]

1968a The mean width of a convex polytope. J. London Math. Soc. **43** (1968), 207–209.
MR 37 # 2087. [12]

1968b A uniqueness theorem for the Steiner point of a convex region. J. London Math. Soc. 4 (1968), 439–444.
MR 37 # 3447. [13]

1968c Euler-type relations for convex polytopes. Proc. London Math. Soc. (3) **18** (1968), 597–60
MR 38 # 606. [5, 12]

1968d Angle deficiencies of convex polytopes. J. London Math. Soc. **43** (1968), 325–336.
MR 37 # 5784. [3]

Sommerville, D.M.Y.

1927 The relations connecting the angle-sum and volume of a polytope in space of *n* dimension Proc. Roy. Soc. London Ser. A **115** (1927), 103–119. [8]

Spiegel, W.

1976a Zur Minkowski-Additivität bestimmter Eikörperabbildungen. J. reine angew. Math **286/287** (1976), 164–168.
MR 55 # 3969. [10]

1976b Ein Zerlegungssatz für spezielle Eikörperabbildungen in den euklidischen Raum. J. rein angew. Math. **283/284** (1976), 282–286.
MR 53 # 6422. [5]

1978 Ein Beitrag über additive, translationsinvariante, stetige Eikörperfunktionale. Geom Dedicata **7** (1978), 9–19.
MR 57 # 13699. [10]

Stachó, L.L.

1979 On curvature measures. Acta Sci. Math. **41** (1979), 191–207.
MR 80j: 53065. [5]

Stein, R.

1982 Additivität und Einschließungs-Ausschließungsprinzip für Funktionale von Gitterpolytopen. Dissertation, Dortmund 1982.
[7, 10, 17]

Strambach, K.

1966 Über die Zerlegungsgleichheit von Polygonen bezüglich Untergruppen nichteuklidische Bewegungsgruppen. Math. Z. **93** (1966), 276–288.
MR 34 # 660.

Sydler, J.-P.

1965 Conditions nécessaires et suffisantes pour l'équivalence des polyèdres de l'espace euclidien à trois dimensions. Comment. Math. Helvet. **40** (1965), 43–80.
MR 33 # 632. [9]

Tverberg, H.

1974 How to cut a convex polytope into simplices. Geom. Dedicata **3** (1974), 239–240.
MR 50 # 1127. [6]

Valette, G.

1974 Subadditive affine-invariant transformations of convex bodies. Geom. Dedicata **2** (1974), 461–465.
MR 51 # 11279. [13]

Volland, W.

1957 Ein Fortsetzungssatz für additive Eipolyederfunktionale im euklidischen Raum. Arch. Math. **8** (1957), 144–149.
MR 19-1074. [5]

Walkup, D.W., R.J. Wets
1969 Lifting projections of convex polyhedra. Pacific J. Math. **28** (1969), 465–475.
MR 39 # 3390. [6]
Weil, W.
1974a Über den Vektorraum der Differenzen von Stützfunktionen konvexer Körper. Math. Nachr. **59** (1974), 353–369.
MR 49 # 6033. [5]
1974b Decomposition of convex bodies. Mathematika **21** (1974), 19–25.
MR 51 # 1611. [5]
1975a On mixed volumes of nonconvex sets. Proc. Amer. Math. Soc. **53** (1975), 191–194.
MR 52 # 4130. [5]
1975b Einschachtelung konvexer Körper. Arch. Math. **26** (1975), 666–669.
MR 53 # 9036. [13]
1981 Das gemischte Volumen als Distribution. Manuscripta math. **36** (1981), 1–18. [5]
Wieacker, J.A.
1982 Translative stochastische Geometrie der konvexen Körper. Dissertation, Freiburg 1982. [3]
Zylev, V.B.
1965 Equicomposability of equicomplementable polyhedra. (Russian) Dokl. Akad. Nauk. SSSR **161** (1965), 515–516. English translation: Soviet Math. Dokl. **6** (1965), 453–455.
MR 31 # 1610.
1968 G-composedness and G-complementability (Russian) Dokl. Akad. Nauk. SSSR **179** (1968), 529–530. English translation: Soviet Math. Dokl. **9** (1968), 403–404.
MR 37 # 2069. [6]

Remarks and further references (added in proof)

An application of the Euler characteristic of Hadwiger's normal bodies, which were mentioned in §5, to a question in probability theory may be found in

Adler, R. J., Hasofer, A. M., Level crossings for random fields. Ann. Prob. **4** (1976), 1–12.
Adler, R. J., The geometry of random fields. Wiley, Chichester, etc. 1981.

An additive extension of Federer's curvature measures to certain (but not all) finite unions of sets of positive reach was recently studied by

Zähle, M., Curvature measures and random sets I (to appear).

The expression of Weil [1981] in §5 has been extended (and the proof corrected) by

Goodey, P. R., W. Weil, Distributions and valuations (to appear).

They show that, if $\varphi:(\mathscr{K}^d)^r \to \mathbb{R}$ is any continuous multilinear function, then there is a distribution T on $(\Omega^{d-1})^r$, such that

$$\varphi(K_1,\dots,K_r) = T(h(K_1,\cdot) \times \cdots \times h(K_r,\cdot))$$

The problem of finding the syzygies between the Hadwiger functionals, discussed at the end of §6, is in effect settled in Proposition 3.16 of

Dupont, J. L., Algebra of polytopes and homology of flag complexes. Osaka J. Math. **19** (1982), 599–641.

The recently published volume "Convexity and related combinatorial geometry", ed. by D. C. Kay and M. Breen, Marcel Dekker, New York etc. 1982, contains two articles which are concerned with valuations on polytopes:

Sallee, G. T., Euler's theorem and where it led; pp. 45–55.
Spiegel, W., Nonnegative, motion-invariant valuations of convex polytopes; pp. 67–72.

In the discussion in §10, it was necessary in following McMullen's approach to assume that $\mathscr{X}$ was a real vector space. It is worth remarking, though, that a rational vector space or abelian group (regarded as a $\mathbb{Z}$-module) $\mathscr{X}$ can be embedded in a real vector space $\mathscr{Y} = \mathbb{R}\mathscr{X}_{\mathbb{Z}}\mathscr{X}$; $\mathscr{Y}$ inherits its real vector space structure from the first component of the tensor product, and $\mathscr{X}$ itself can be identified

with $\{1\} \times \mathscr{X}$. We can now pass from the general valuation φ to the simple valuations $\psi_{\mathscr{A}}$ and back using the angle-sum relations. Analysis of the proof of (7.1) (compare (6.7)) shows that we obtain expressions of the form

$$\varphi(nP) = \sum_{r=0}^{d} \binom{n}{r} \tilde{\varphi}_r(P),$$

where each $\tilde{\varphi}_r\,\mathscr{P}^d \to \mathscr{X}$ is a continuous translation invariant valuation. If $\mathscr{X}$ is a rational vector space, then we obtain the required rational polynomial expansion of (10.3).

Considerable progress with some of the problems mentioned in §15, 16 has been achieved recently. In the work of Goodey and Weil quoted above (for $r = 1$) and by U. Betke and Goodey (in preparation, for $r \in \{2,\dots,d-2\}$), it is shown that, if φ is a continuous translation invariant valuation on $\mathscr{K}^d$, which is homogeneous of degree r, then there exist sequences $(L_i)_{i\in\mathbb{N}}$, $(M_i)_{i\in\mathbb{N}}$ in $\mathscr{K}^d$, such that

$$\varphi(K) = \lim\,[V(K,r;B,d-1-r;L_i) - V(K,r;B,d-1-r;M_i)],$$

uniformly for K in a compact subset of $\mathscr{K}^d$. This exactly generalizes the reformulation of (16.11) (immediately following), and proves a suitably modified conjecture of McMullen [1980]. Betke and Goodey make use of (16.6) to show that, if a polytope P is identified with its r-th order area function $S_r(P;\cdot)$, then α induces a continuous linear mapping on the space spanned by such functions. The method of Betke and Goodey also provides a new approach to Hadwiger's characterization theorem (15.1), as well as an affirmative solution to the continuous case of Problem (15.4).

P. McMullen
Department of Mathematics
University College London
Gower Street
London WC1E 6BT (England)

R. Schneider
Mathematisches Institut
Universität Freiburg
Albertstrasse 236
D-7800 Freiburg (FRG)