Folding Flat Silhouettes and Wrapping Polyhedral Packages: New Results in Computational Origami

Erik D. Demaine* Martin L. Demaine* Joseph S. B. Mitchell†

Abstract

We show a remarkable fact about folding paper: From a single square of paper, one can fold it into a flat origami that takes the (scaled) shape of *any* connected polygonal region, even if it has holes. This resolves a long-standing open problem in origami design. Our proof is constructive, utilizing tools of computational geometry, resulting in efficient algorithms for achieving the target silhouette.

We show further that if the paper has a different color on each side, we can form any connected polygonal pattern of two colors. Our results apply also to polyhedral surfaces, showing that any polyhedron can be "wrapped" by folding a strip of paper around it. We give three methods for solving these problems: the first uses a thin strip whose area is arbitrarily close to optimal; the second allows wider strips to be used; and the third varies the strip width to make a folding that optimizes the number or length of visible "seams."

1 Introduction

Origami provides a rich field of research questions in geometry. At SoCG'96, Robert Lang's popular talk [16] helped to introduce the computational geometry community to this exciting area of research.

A classic open question in origami mathematics is whether every simple polygon is the silhouette of a flat origami. This question was first formally stated within the algorithms community by Bern and Hayes at SODA'96 [6]. More generally, we might ask whether every polygonal region (polygon with holes) is the silhouette of some flat origami. In this paper, we show that the answer is yes, and we provide constructive methods for achieving such origamis.

A more general problem in origami design is to take a sheet of *bicolor* paper, having a different color on each side, and fold it into a desired pattern of two colors. For example, John Montroll's book *Origami Inside-Out* [20] is entirely about such models. Taichiro Hasegawa [10] has designed an entire alphabet, including lower- and upper-case letters as well as punctuation. One origami designer, Toshikazu Kawasaki, has looked at the special case of *iso-area foldings*, that is, foldings that use equal amounts of both colors [11, pp. 96–97] [12, pp. 26–34]. See Figure 1.

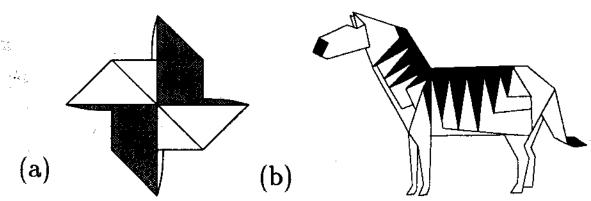

Figure 1: (a) Iso-area pinwheel from [11, p. 97]. (b) Zebra by John Montroll from [19, pp. 94–103].

Formally, we define a *polygonal pattern* $\mathcal{P}$ to be a 2-colored polygonal subdivision of a polygonal region, each subregion of which may have holes. Our most general flat origami question then asks if there exists a flat folding of a sufficiently large piece of bicolor paper such that the top side of the flat origami gives exactly the input 2-color pattern, $\mathcal{P}$.

A more general question asks whether every polyhedron can be *wrapped* with a piece of rectangular paper. This is motivated not only by the problem of constructing three-dimensional origamis, but also the "gift wrapping problem," which was introduced to us by J. Akiyama [3]. We define a *polyhedron* $\mathcal{P}$ very generally to be any connected union of pairwise-interior-disjoint polygonal regions (called faces), each of which lies on a plane in 3-space; we let n denote the number of vertices of $\mathcal{P}$. We consider also polyhedra whose faces are 2-colored. We then ask: Is every polyhedron $\mathcal{P}$ the folding of some sufficiently large rectangular piece of paper? If so, is there a folding of a bicolor sheet of paper that respects the face col-

*Department of Computer Science, University of Waterloo, Waterloo, Ontario N2L 3G1, Canada, email: {eddemaine, mldemaine}@uwaterloo.ca.

†Department of Applied Mathematics and Statistics, State University of New York, Stony Brook, NY 11794-3600, email: jsbm@ams.sunysb.edu.

successive folding under of excess paper decreases the angle by another θ_1, so this process must terminate in at most $1 + (\pi - \theta_1)/\theta_1$ steps. Finally, there is no excess in cones $K_1^{(2)}$, $K_1^{(3)}$, or $K_1^{(4)}$, and we advance to vertex v_2, etc., folding excess under until all edges $e_1, \ldots, e_k$ become boundary edges of the paper.

(Note that multiple foldings of excess corresponding to a single vertex v_i is only an issue if the angle θ_i is very small; in fact, if $\theta_i \geq \pi/2$, then one folding under for each of the lines ℓ_i and ℓ_{i+1} suffices. Of course, C can have at most three angles θ_i less than $\pi/2$.) □

An immediate consequence of this theorem is the following:

Corollary 1 *Given any polygon P and convex polygon Q, such that P can be moved to cover Q, P can be folded into a flat origami whose silhouette is Q.*

2.2 Turning a Strip

A natural tool to fold a paper strip into a desired shape is the ability to *turn* the strip. More formally, we will consider turns of the following sort. Take two infinite strips S and T in the plane, and consider their intersection $I = S \cap T$; see Figure 2. Label the two connected regions of $S - T$ $[T - S]$ by S_1 and S_2 $[T_1$ and $T_2]$. The *turn gadget* must fold a strip so that it covers precisely $U = S_1 \cup (S \cap T) \cup T_1$.

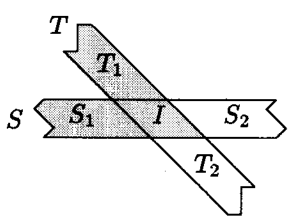

Figure 2: A turn must cover precisely two connected portions of $S - T$ and $T - S$ as well as $I = S \cap T$.

Our turn gadget is shown in Figure 3. The first fold is perpendicular to the edges of S and is incident to the convex vertex of U. The second fold is an angular bisector of the convex angle θ, effecting the turn. If $\theta \geq \pi/2$, these two folds are all that are needed. On the other hand, if $\theta < \pi/2$, they leave a right-angle triangle of excess paper, whose angle incident to the convex vertex is $\pi/2 - \theta$. We can hide this triangle underneath U by "wrapping" it around the angle θ, which requires

$$\left\lceil \frac{\pi/2 - \theta}{\theta} \right\rceil = \left\lceil \frac{\pi}{2\theta} \right\rceil - 1$$

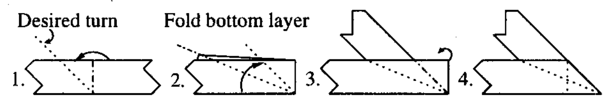

Figure 3: Folding a turn gadget. Step 3 hides the excess paper and is only necessary for $\theta < \pi/2$.

extra folds.

We have thus proved the following lemma.

Lemma 1 *Given two strips S and T in the plane, and given any connected region S_1 $[T_1]$ of $S - T$ $[T - S]$, a strip can be folded into a flat origami whose silhouette is precisely $S_1 \cup (S \cap T) \cup T_1$.*

It turns out that if we apply a sequence of turn gadgets, the first fold of a particular turn gadget (which involves folding through all layers) may destroy the effect of previous turn gadgets, that is, uncover regions that were covered by previous turn gadgets. This can be avoided by using a *generalized turn gadget*, which involves letting the strip go past the turn, making the perpendicular fold once it has gone far enough to avoid destruction, and then bringing the strip back before making the second (turning) fold. See Figure 4. We now obtain a trapezoid of excess of paper, which can be folded underneath by applying Theorem 2.

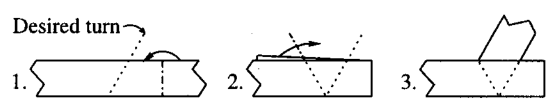

Figure 4: Folding a generalized turn gadget.

Generalized turn gadgets will also be important to produce useful overhang, as we will see in Section 3.

2.3 Color Reversal

We utilize a *color-reversal gadget*, as shown in Figure 5. It consists of three folds: a perpendicular fold, and two 45° folds. The result is a color reversal (that is, an exchange of the showing side of the strip) along the perpendicular edge. Note that the triangle of excess paper underneath the finished gadget can, if necessary, be reduced in size by the gadget of Theorem 2.

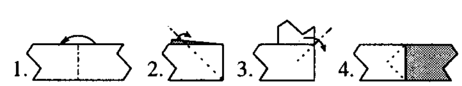

Figure 5: Folding a color-reversal gadget.

the endpoints of e_i; see Figure 9. Thus, we have $A_t = O(h_i w \cot \theta_{min})$, which implies that $A_t = O(wL_i)$, where L_i is the length of the longest side of T_i. Also, we see that $A_o = O(w|e_i|) = O(wL_i)$, since the overlaps between rows need not consume more strip length than twice the longest row (which is roughly of length $|e_i|$). (We have *twice* the longest row because one extra row may be needed to compensate for the round-up from (h_i/w) to $\lceil (h_i/w) \rceil$, while a second extra row may be needed for the parity constraint.) We summarize with

Lemma 2 *The coverage of triangle T_i utilizes a strip of area at most area$(T_i) + O(wL_i)$, where w is the width of the strip and L_i is the length of the longest side of T_i.*

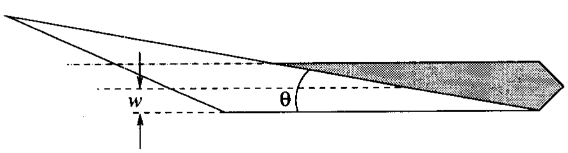

Figure 9: Area estimate for excess paper (shown shaded) that spills over during a turn-around. The area of the shaded region is $w^2 + (2w)(2w) \cot \theta$.

The transition from triangle T_i to T_{i+1} involves turning the strip in such a way that the strip becomes parallel to the edge e_{i+1}, while creating excess that can be folded under $T_i \cup T_{i+1}$. Refer to Figure 11. This can be done using the generalized turn gadget of Section 2.2, but for turn angles of more than $\pi/2$, the amount of excess paper is too large: it grows arbitrary large as the turn angle approaches π. In this case, we use an alternate turn gadget shown in Figure 10. Note that this turn gadget solves a different problem from the one described in Section 2.2 (the corner does not have to be "filled in"), which allows us to reduce the amount of excess paper to $O(w^2)$.

By induction, we can cover all the triangles of $\mathcal{T}'$ (and hence of $\mathcal{T}$) in this way. Note that we can also change which side of the strip is up, as we make the transition between two triangles, using the gadget in Figure 5. Thus, we can control the coverage in such a way that we preserve a given 2-coloring of $\mathcal{T}'$ (which is inherited from a 2-coloring of $\mathcal{T}$ or of the original facets of $\mathcal{P}$).

Since the transition from triangle to triangle uses at most $O(wL)$ excess paper area, where $L = \max_i L_i$, Lemma 2 applied to the $O(n)$ triangles in turn yields the following result:

Lemma 3 *The coverage of $\mathcal{T}$ requires a strip of area at most area$(\mathcal{T}) + O(nwL)$.*

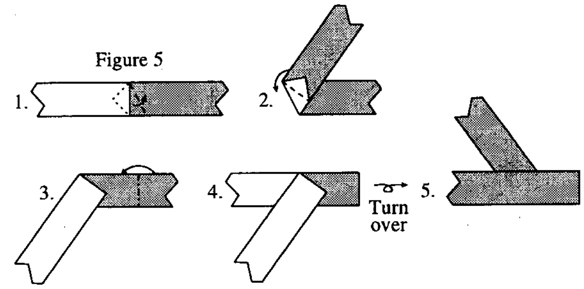

Figure 10: Folding an alternate turn gadget, which reduces the amount of excess paper for turn angles of more than $\pi/2$. Step 3 can be adjusted to produce the desired amount of overhang.

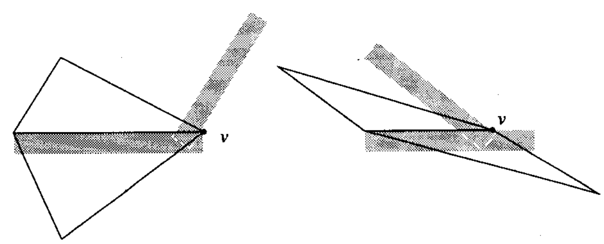

Figure 11: Turning from one triangle to another. Note that the turn must have some overhang to finish covering the triangle.

Using this zig-zagging method with sufficiently narrow strips ($w \to 0$), we obtain, as a consequence of Lemmas 2 and 3, the following result on optimal paper usage.

Theorem 3 *Let A be the surface area of a given 2-colored polyhedron. Then for any $\epsilon > 0$, there is a rectangle R of bicolor paper with area at most $A + \epsilon$ such that R folds into the polyhedron with the desired colors showing.*

Remark. Instead of using a very small width w, our approach also allows one to use a strip with a larger width, up to the smallest altitude of the triangles in $\mathcal{T}'$. Of course, this increases the excess paper that needs to be folded under, and increases the total area of paper required.

4 Ring Method

Our second method is based on covering a polyhedron by a collection of "rings." This method's main advantage is that it allows the strip to have the largest possible width, in the case that the strip width is not allowed to change.

ring. When we traverse a node that we have visited before, we can "walk" around the ring (by constructing part of it) and bring the strip to the desired joining place for an adjacent ring. Hence, we only need to show how to construct a skeleton ring, and how to connect between two skeleton rings with an optional color change.

4.3 Strip Rings

Instead of folding skeleton rings directly, we will cover them by a collection of *strip rings*, that is, rings with the same width as the strip. Strip rings are particularly attractive because they can be constructed simply by folding a sequence of generalized turn gadgets from Section 2.2. (We use *generalized* turn gadgets so that they do not interfere with each other.)

Lemma 5 *Given any ring R of width $|R|$ and a strip of width w, R can be covered by $\lceil |R|/w \rceil$ strip rings, each of which is contained in the current polygonal region.*

Proof: Assume first that $|R| \geq w$. Then one way to build such a cover is as follows. Let $R = (q_0, q')$ be the ring (between walls q_0 and q') that we want to cover. In general, suppose we want to cover a ring $R_i = (q_i, q')$ such that $|R_i| \geq w$, for $i = 0, 1, \ldots$. Shrink or expand the wall q_i to pull it towards the interior of R_i by a perpendicular distance of w. The result is another wall q_{i+1} that is in R_i. Indeed, (q_i, q_{i+1}) is a strip ring.

It remains to cover the subring $R_{i+1} = (q_{i+1}, q')$ of R_i. If $|R_{i+1}| \geq w$, we can recursively apply this procedure. Each iteration decreases the width of the ring to cover by the constant w. Hence, after $k = \lfloor |R|/w \rfloor$ iterations, we are left with a strip $R_k = (q_k, q')$ whose width is less than w. If its width is zero (that is, w evenly divides $|R|$), we stop. Otherwise, we shrink or expand q' to pull it towards the interior of R_k, resulting in a wall q that is outside R_k but inside R. This last strip ring (q, q'), which contains R_k, completes the cover using $\lceil |R|/w \rceil$ strip rings.

Now assume that $|R| < w$, and let $R = (q_1, q_2)$. Consider topologically shrinking or expanding q_1 and q_2 to push them away from R, stopping when we find a ring R' that has the same width as the strip. If a wall hits the boundary of the polygonal region, we stop shrinking/expanding it. Because of the upper bound on the strip's width described in Section 4.4, we cannot have both walls hitting the boundary of the polygonal region. Hence, we obtain a strip ring R' that contains R and is contained in the polygonal region, the desired result. □

It only remains to show how to bridge between two strip rings. Specifically, we need to show how to combine strip rings in two different ways: between overlapping strip rings, and between touching strip rings possibly of different colors. In all cases, we take an arbitrary edge shared by the two strip rings; for overlapping rings, this "edge" has some thickness. We bridge at any joining place along this edge by using the turn-around gadget in Figure 8. The excess paper can be reduced to fit within the two rings by applying Theorem 2. We can also reverse the color of the strip in between the two folds of the turn-around gadget (note that if the two rings have different colors, they do not overlap), using the color-reversal gadget in Section 2.3.

4.4 Strip Width

What are the least possible constraints on the strip's width? If our only building blocks are strip rings (in other words, the width of the strip stays essentially constant), we need the property that at least one strip ring fits inside the polygonal region we are trying to cover. One observation is that the strip's width must be at most the minimum feature size, that is, the minimum distance between two nonincident boundary edges. Indeed, we need a stronger upper bound on the strip's width than the minimum feature size, to ensure that it is possible to turn at every reflex vertex without falling outside the polygonal region.

Consider a reflex vertex v with exterior angle θ and consider the nonincident boundary edge e that is closest to v along the angular bisector at v; refer to Figure 14. To turn at v, a ring turns at a point on the angular bisector of v. Let d denote the distance from v to e along the angular bisector of v. This gives us the maximum allowed "diagonal" width of the strip. This means that the true width of the strip must be at most $d \sin(\theta/2)$. By minimizing this expression over all reflex vertices, we obtain an upper bound on the strip's width for that face.

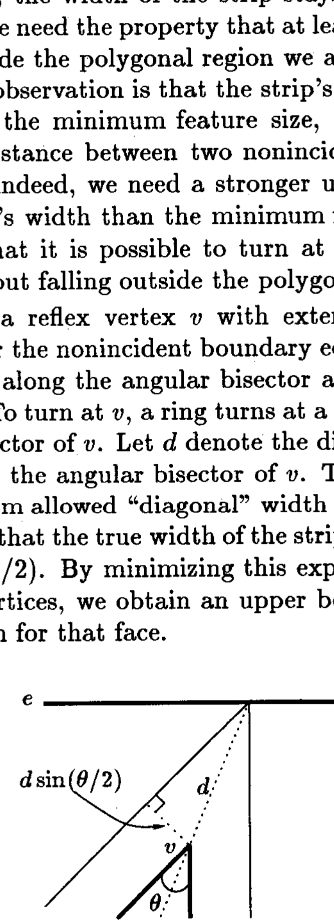

Figure 14: Computing the upper bound on the width of the strip.

The basic gadget is shown in Figure 15. Note that the folding starts with the reverse side of the strip showing, and is flipped back over in Step 4. The first fold is the perpendicular along which we want to change the strip width, and is a valley from this orientation. The second fold is another perpendicular, which is the desired reduction amount away from the first fold. The third fold is a squash fold, which involves folding down the top part of the strip by the desired reduction amount, along a horizontal line; the upper-left corner naturally "squashes" along two 45-degree folds (which are originally right on top of each other). Equivalently, we can squash fold upwards the bottom part of the strip.

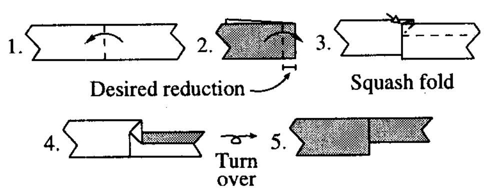

Figure 15: Folding a strip-width gadget.

This gadget can reduce a strip of width w into a strip of width $\alpha \cdot w$ for any $\frac{1}{2} \leq \alpha \leq 1$. By applying the "reverse" of the gadget (that is, flipping the image horizontally), we can also undo any previous reduction. We are now ready to prove the desired theorem:

Theorem 5 *A strip can be repeatedly resized along various perpendicular edges to any width that is at most the original physical width. The number of folds required to change the width from w_1 to w_2 is*

$$O\Big(1 + \big|\log\,(w_1/w_2)\big|\Big).$$

Proof: We maintain the invariant that the strip is the result of several width-halving gadgets (a strip-width gadget with $\alpha = \frac{1}{2}$), possibly followed by a general width-reduction gadget with some α. To achieve a particular strip width, we first fold (if necessary) the reverse strip-width gadget with the same α. Then we apply width-halving or reverse width-halving gadgets until the strip has width within a factor of two of the desired width. Finally, we apply the general width-reduction gadget to obtain the desired strip width. The bound on the number of folds follows immediately. □

Note that any excess paper from strip-width gadgets can be reduced to fit within any desired incident region (namely, the polyhedron face that we are covering), by Theorem 2.

5.2 Approach

We are now in the position to describe a folding that only has seams on the edges of a given convex decomposition of a polyhedron's surface. We define the *diameter* of the convex decomposition to be the largest diameter of any convex polygon in the decomposition, that is, the largest distance between any two points on a common convex polygon. We choose our strip to have this diameter as its physical (initial) width.

The algorithm works as follows. We traverse a spanning tree of the dual of the convex decomposition in a depth-first traversal. The strip always enters a convex polygon P along a subportion of one of its edges, perpendicular to that edge e. Reorient so that e is vertical. At this point of entry along e, we resize the strip to be the vertical extent of P. Note that the resized strip may not have the right vertical positioning to cover all of P; this can be fixed by using the *shift gadget* shown in Figure 16.

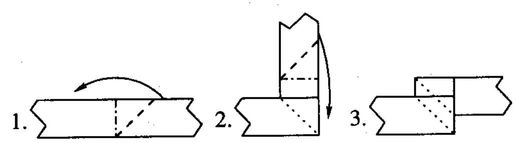

Figure 16: Folding a shift gadget, which is just a sequence of two right-angle turn gadgets from Figure 3.

Next we continue the strip straight until it covers all of P; call this the *completion point.* Let e' denote the edge shared by P and the next polygon P' in the traversal order. If the length of e' is less than the current width of the strip (i.e., the vertical extent of P), then we resize the strip width at the completion point to the length of e'.

It remains to show how to turn the strip to reach e' perpendicularly. In fact, this can be done using a generalized turn gadget (Section 2.2). If e' has positive slope, as in Figure 17(a), the perpendicular fold is right at the completion point. If e' has negative slope, the perpendicular fold may be past the completion point, as in Figure 17(b). In either case, the second fold turns onto the infinite strip perpendicular and incident to e'.

Once we reach the edge e', we immediately reorient so that e' is vertical. We apply Theorem 5 to resize the strip along e' to the vertical extent of P'. Finally, if P and P' have opposite colors, we reverse the strip color along e' by applying Theorem 5.

Folding the excess paper underneath completes the convex-decomposition method, thereby proving Theorem 4: any polyhedron can be wrapped with seams only along the edges of a convex decomposition.

[23] H. Samet. *The Design and Analysis of Spatial Data Structures.* Addison-Wesley, Reading, MA, 1990.