TOPOLOGY OF THE CONVEX POLYTOPES' MANIFOLDS, THE MANIFOLD OF THE PROJECTIVE CONFIGURATIONS OF A GIVEN COMBINATORIAL TYPE AND REPRESENTATIONS OF LATTICES

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CONTENTS

§ 0. INTRODUCTION

The following intuitive question posed by the author more than 10 years ago was one of the simplest which has stimulated combinatorial-topological topics presented in what follows.

1. Is the set of all m-dimensional simplicial convex polytopes with n verteces of a given combinatorial type connected?

At the first glance reliably supported by the three-dimensional experience (in which case the affirmative answer follows from the famous Steinitz theorem) it seems that the same affirmative answer holds true in the general case. However, we shall see that this is far from being true.

Further analysis and making use of Gale duality immediately lead us to following problem formally more general but in an essence equivalent to above problem, which is interest by itself.

II . Is the set of real rectangular $m \times n$ matrices with fixed signs of all the minors connected ? In an equivalent form (see § 3): is the set of all equally oriented nonsingular ordered configurations of n points of the m-dimensional affine space connected ?

It turned out that this problems are rather complicated. From on side they originated from attempt to use the tools of modern topology and algebraic geometry to such classical combinatorial and geometrical objects as configurations in a projective space, convex polytopes, matroid etc. From the other side some quastions arising in applications lead to similar natural problems. To mention only one of the examples, the concept of the convex polytopes' field (see our paper [V.Ch]) in vector bundle over a smooth manifold is a natural generalization of the vector field and polysystem concepts; the problem on the existence of convex polytopes'fields with a given combinatorial structure on a given manifold arises in optimal control, Patero-theory, convex and differential geometries, linear programming etc. Solving the problem is reduced exactly to investigation of homotopy type of above spaces of the polytopes and specific characterictic classes. The same holds for fields of other combinatorial geometric objects [V.Ch.] . That is why the following general problem arises.

III. What is the homotopy type of space of all convex polytopes' (or cones, configurations, triangulations, arrangements etc) of a given combinatorial type ?

There is one more line which connects these problems with representation theory namely with representations of partially ordered sets (posets). We shall mention in briefly in what follows referring for the detail to publication elsewhere . There is a definite connection with more complicated problems on rigid isotopy (in Rokhlin sense) non-singular real algebraic manifolds. Previous experience may be of some use, it suggests in particular that the problem on rigid isotopy is most probably universal in a sense defined below. It is useful to emphasize importance of the (§ 7) problem not only for the field $\mathbb{R}$ but also for $\mathbb{C}$, for finite characteristic fields etc.

To the best of the author's knowledge there has been no systematic study of topology of the space of configurations and related spaces. Making use of modern tools has to be useful not only for the problems themselves out for topology as well especially for combinatorial and homotopy topologies. Some obtained results support this suggestions.

Let us come back to the problems formulated in the beginning. N.E.Mnev proved in [MI] (see also this volume and §§ 7,8) that the answer to problems I, II is negative in its extreme form: in general these sets are not only disconnected but can be homotopy equivalent to arbitrary algebraic manifold (or even semialgebraic set). For example, one cane fix the signs of 3×19 matrices'minors in such a way that the set of these matrices is disconnected (The minimal n for which $\mathrm{Mat}_{3\times n}\mathbb{R}$ can have the same effect is unknown but $8 \leq n \leq 14$. This situation makes the collection of the above manifolds universal and gives in a sense one more way of representing algebraic manifolds. It is clear apriori that topological invariants of these manifolds are invariants of the combinatorial types themselves, the Mnev's theorem claims that these invariants are meaningful and there are plenty of them. For example, contractibility of the manifolds of a given combinatorial type of simplicial convex polytopes becomes its important invariants. For Gale's polytopes this qustion is studied in a paper by A.Barvinok (see this volume).

There arises new classification of combinatorial problem into "tame and wild" ones depending an universality of the set formed by the manifolds of combinatorial type. Another area of interest is representations of partially ordered sets, it is closely connected with the configuration theory. Here a new "isotopy" principle of classification is suggested - extending usual linear equivalence. Emphasize the importance of what we call Grassmanian approach to this questions (see § 3).

A number of papers of this volume deal with these topics (by N.Mnev A.Barvinok, A.Chernyakov, S.Finashin, V.Gershkovich).We are going to develope this subject in a series of papers elsewhere.

The author while elaborating formulations and approach to the above problems discussed some of them with V.A.Rokhlin who showed a vivid interest towards the problems. This introductory paper is a modest tribute to this outstunding mathematician who has greatly influenced the author.

§ 1. CONFIGURATIONS, COMBINATORIAL TYPE

Combinatorial-geometric objects are usually referred to configurations. In what follows under the term configurations we mean an ordered finite collection objects: points, lines, planes,... hyperplanes of either a vector or an affine or a projective space over a field. A homogeneous configuration consists of objects of the same dimension. We shall consider mainly point configurations. A configuration of hyperplanes is called an arrangement. It is convenient to consider all the objects different through multiplisities appear in a natural way. A configuration is nondegenerate if the space spanned onto all the objects (= hull) coincides with the whole space and their intersection is empty. We shall associated with each subset of configurations objects the upper and lower ranks, i.e. the dimensions of the hulls and intersections respectively. If either the upper rank is less (or the lower rank is greater) than its values in the general case (with taking into consideration the objects dimensions) then the subset is said to be in the upper (lower) incidence. The complete (upper or lower) combinatorial type (or shortly combinatorial type is by definition the list of the all (upper or lower) ranks of all the subsets. (Actually it is enough to give it for all the subset which are in the incidence). Note that the list includes also the dimensions of all the objects. The term"combinatorial type" will be alternatively used for the set of all the configurations with given ranks.

EXAMPLE 1. The combinatorial type of a points configuration $\gamma = (x_1, \ldots, x_m)$, $x_i \in E$ is the set

$$\Gamma_\gamma = \{\gamma' : \gamma' = (x_1', \ldots, x_m'),\ \forall i_1, \ldots, i_k,\ \forall k,\ k = 1, \ldots, m$$

$$\dim \mathcal{L}(x_{i_1}, \ldots, x_{i_k}) = \dim \mathcal{L}(x_{i_1}', \ldots, x_{i_k}')\}$$

Here there is no necessity to give liwer ranks. If E is a plane it is enough to indicate all the subsets formed by incident points. In the general case of points configuration it is enough to indicate all those subsets which consist of all the points belonging to hyperplanes.

EXAMPLE 2. The combinatorial type of a hyperplanes configuration (arrangement) is defined similarly with the hulls replaced by the intersections.

Note that according to the projective or another duality to any configuration of hyperplanes there corresponds a configuration in the dual space and this correspondence is an isomorphism preserving the combinatorial type.

In some problems there is considered an incomplete combinatorial type, i.e. not all the incidences are taking into account. (The meaning of the term "combinatorial type" depends on the problem.)

EXAMPLE 3. The convex combinatorial type of a points configuration (or the combinatorial type of a polytope) in a vector or affine space over a linearly ordered field. Let K be a linearly ordered field and γ - a points configuration in K^m. To define its combinatorial type means to indicate all the subsets - called faces of the highest dimensions - of the points belonging to the supporting hyperplanes (and not to all the hyperplanes as in Example 2 above).

A supporting hyperplane is defined as usual by virtue of linear ordering (see § 4).

This list of examples can be continued. Further we shall define an oriented combinatorial type for ordered fields. The classical concept of configuration (see [H.C]) is included into the set up.

A configuration (and its combinatorial type) is called open, or structural stability, or nonsingular or generic if all subsets of its objects are in the general position. F.e. a point's configuration is generic iff none of $n+1$ points lie on a hyperplane (n is the dimension of the space). A generic convex type is a simplicial(all the faces are simpleces) one of a polytope. The combinatorial genus of a configuration is the set of all configurations having the same list of the ranks of the insident subsets (including the dimension of the objects) but is general having additional incidences. Evidently the genus is composed of some combinatorial types; two genuses can be intersected. The combinatorial genus of a generic configuration is the set of all configurations with a given dimension of the objects.

§ 2. SPACE OF THE CONFIGURATIONS:

In a natural sense the set of all configurations in a given problem with fixed dimensions, of the objects forms an algebraic variety which we shall call the natural space of the configurations (contrary to the grassmanian - see § 3).

In what follows, we shall restrict ourselves with the points confirations the natural space being $E^m \setminus \Delta$, where E is the space which containing the configurations, m is the number of points, Δ - the set of collection with multiple points. The combinatorial types divide the space into the strata consisting of all configurations of a given combinatorial type and are constructuve sets (i.e. sets defined by the conditions $P_i = 0$, $Q_j \neq 0$, $i \in I$, $j \in J$, P_i, Q_j being polynomials over $\mathbb{Q}$ of the point's coordinates. Actually, each incidence is described by an equality of some determinants to zero, and absence of the incidence - by an inequality (see § 6).

Studying the topology of these strata and the partition into the strata is the main problem we pose now.

The generic stratum is open in Zariski topology in the space of the configurations. The closure of any stratum is combinatorial genus, which is an algebraic variety over $\mathbb{Q}$. For the ordered fields every stratum can be divided into oriented types which are semialgebraic sets (see § 4).

§ 3. GRASSMANIAN IDEOLOGY

There is an alternative way of constructing the space of the configurations in a given problem which is very convenient in different respects some of them we shall point out.

Through our considerations are of universal character we shall restrict ourselves only with the points configurations in the real projective space or in other words the lines configurations in the vector space. Let n and m be naturals, $E = \mathbb{R}^n$, $G(n, m+1)$

be the Grassmanian manifold of the $(m+1)$-dimension subspaces in E and let $e_1, \dots, e_n$ be the coordinate lines in E. The standard basis in E defines <u>Cartan subgroup</u> H_n of the group $GL(n, \mathbb{R})$ of the invertible transformations preserving the coordinate axes. If $F \in G(n, m+1)$ then E_F is the orthogonal euclidian projector onto F. The set $E_F e_1, \dots, E_F e_n$ is a configuration of lines in F, or a points configuration in PF (projectivisation of F). It is easy to prove

PROPOSITION 1. Every points'configuration in $P_m \mathbb{R}$ can be obtained up to isometry by this construction.

Thus we have a reason to consider the grassmanian $G(n, m+1)$ as the space of the m-points'configurations in $P_m \mathbb{R}$. We shall denote by $\tau_{n,m} = \tau$ the partition of $G(n, m+1)$ into combinatorial types and call it the <u>configuration projective partition</u>. Let us describe τ explicitly. With this aim in view we note the if $F_1, F_2 \in G(n, m+1)$ belong to the same stratum then there exists an $h \in H_n$ such that F_1 and hF_2 have the same position with respect to the coordinate subspaces. In other words for every coordinate subspace $K_I = \{x \in \mathbb{R}^n : x_i = 0, i \notin I \subset \overline{1,n}\}$ we have

$$\dim(F_1^\perp \cap K_I) = \dim(F_2^\perp \cap K_I)$$

where $F^\perp = \operatorname{Ker} E_F$

In fact every incidence and its rank in a configuration can be expressed in terms of the above dimensions. We have

PROPOSITION 2. A stratum of the projective configuration partition consists of all the subspaces $F \in G(n, m+1)$ for which the values of

$$\dim(hF^\perp \cap K_I), \quad I \subset \overline{1,n}$$

are constant for an $h \in H_n$ depending on F. It is enough to define the value for I with $|I| = n - m + 1$.

The partition τ is invariant under the action of Cartan group. In a similar way the space of the m-points'configurations in $\mathbb{R}^n$ is $G(n,m)$ if one identifying $F \in G(n,m)$ with the configuration formed by the orthogonal projective onto F of the coordinate vectors. The corresponding vector configuration partition is the product of all the partitions into Shubert cells, generated by all the coordinate flags. The projective configuration partition is a factor partition of the above vector partition with respect to the Cartan group action. The affine configuration partition can be defined in the same way.

All the configuration partitions are not stratifications in Whitney sense (that is why we avoid the term "stratification") but each stratum is a constructive set. These partitions are not orbit partitions for any algebraic groups. The stratum in these definitions and in those of § 2 differ only by the action of a projective or a linear group.

In a similar way considering the projection of the standard simplex onto $F \in G(n,m)$ one can consider the grassmanian as the space of all the convex polytopes whose dimensions are not greater than m and the number of verteces is not greater than n. If we take the positive orthant instead of the simplex we obtain the space of the convex cones. We can construct the convex configuration partition $\tau^{sym}_{n,m}$ (or $\tau^{cone}_{n,m}$) of the grassmanian as above by identifying the subspaces with the same convex type of projection of the standard symplex (or the cone).

In any case we deal in fact with the tautological vector bundle over the grassmanian and with fields of the configurations polytopes, cones in the leaves. These fields play the same role in our theory of the fields of configurations polytopes etc as the tautological vector bundle itself does for the theory of the vector bundles and charactetistic classes.

The first advantage of the grassmanian ideology is a presence of a duality.

grassmanian approach.

Recently I.Gel'fand jointly with pupils have considered the partition $\tau_{n,m}$ in a completely different context (in connection with a generalization of the hypergeometric functions) see [Ge] . The paper [G.M] is especialy closed to ours one.

§ 4. ORIENTATION, ORIENTED COMBINATORIAL TYPE

In the case of ordered fields one can introduce a finer partition - namely that into oriented combinatorial types. We begin with the definition of the oriented type of a vector configuration.

Let K be a linearly ordered field, then we can define an orientation on K^m and the orientation of the m -point subsets. The oriented combinatorial type of a m -points'configuration $(n \geqslant m)$ is the set of all the configurations with the same orientations of the corresponding m -subsets. In other words we fix orientations of all the nonincident m -subsets in addition to all the ranks of incidence (see § 1). The partition of the configurations'space into sets with given oriented combinatorial types will be called the vector oriented combinatorial partition.

Reformulate this definition on the grassmanian context. For I, $I', I'' \subset \overline{1,n}$ put

$$Q = \{ x \in K^m : x_i > 0 , i \in I ; x_i = 0 , i \in I' ; x_i < 0 , i \in I'' \}$$

We say that two subspaces $F_1, F_2 \in G(n,m)$ belong to the same oriented combinatorial type iff $\dim (F_1^{\perp} \cap Q) = \dim (F_2^{\perp} \cap Q)$ for all (I, I', I''), $I \cup I' \cup I'' = \overline{1,n}$

The equivalence between two definitions of the oriented combinatorial type follows immediately from the fact that the orientation of

PROPOSITION 3. The usual isomorphism between $G(n,m)$ and $G(n,n-m)$ (the transition to the orthogonal complement) transforms the projective (vector) configuration partition $\tau_{n,m}$ into $\tau_{n,n-m}$

In fact, the vector $\{\dim(F^{\perp}\cap K_I), I\subset \overline{1,n}\}$ is both defined and defines the vector $\{\dim(F\cap K_I), I\subset \overline{1,n}\}$

This duality between the combinatorial types of n -points'configurations in $P_{m-1}\mathbb{R}$ and the ones of n -points'configurations in $P_{n-m-1}\mathbb{R}$, should not be confused with the projective duality in the dual space. We shall call the above duality complementariess.

Another situation appear in the convex case. The isomorphism between $G(n,m)$ and $G(n,n-m)$ maps $\tau^{sym}_{n,m}$ (and $\tau^{cone}_{n,m}$) into a new partition which had been described by Gale in other terms (see [Cr] and §§ 4, 6). We do not discuss here other cases (of non.points'configurations, arrangements unordered configurations etc), in every case we can use the grassmanian as the space of the configurations. For the arrangements another version of the grassmanian approach is more convenient. Namely, the subspace $F\in G(n,m)$ will be identified with the intersection of F and the coordinate hyperplanes (instead of the projections). Replacing the projections with the intersections we get the projective duality between the points'configurations and arrangements. One duality more is obtained by the transition from the projections in F to the intersections in $F^{\perp}$. All these dualities are very useful through they don't make the problems easier. They are very effective for studing the case of small defects between the number of points and the dimension.

Gale-duality

Another advantage of the grassmanian approach is in metric and measure structures: there are orthoinvariant metric and measure on $G(n,m)$, and we can pose the questions on metric and measure properties of the strata. It is especiall important in the asymptotic theory (see [VS]). The Gale's diagrams (arising in the theory of convex polytopes see [Gr]) is a good illustration of the effectiveness of the an m -subset is determined by the intersections of the kernel of the projection and the coordinate orthants. We denote this partition of $G(n,m)$ by $\tau^{+}_{n,m}$. The projective case differs from the vector case above by the factorization with respect Cartan group (as a result we come to the projective oriented combinatorial type).

The oriented type of a configuration on an affine plane is called also the topological type of the configuration because due to the fact that projectively dual configurations to configurations with the some oriented type divide the plane into one and same types of the polytopes.

PROPOSITION 4. The isomorphism between $G(n,m)$ and $G(n,n-m)$ transforms $\tau^{+}_{n,m}$ into $\tau^{+}_{n-m,m}$

In fact, the vector $\{\dim(F\cap Q)\}$ determines the dimension of the cone hull $K(F,Q)$ (by the duality theorem for the convex cones and the latter in its turn determines the vector $\{\dim(F^{\perp}\cap Q)\}$

Thus we have got the complementariness for the oriented configurations.

As we saw for a ordered field we can define the convex combinatorial type (§ 1) and the corresponding partition τ^{conv} (of the grassmanian or natural space (see [VCh]) . Actually it is defined with the help of the orientation because the notion of the supporting hyperplane uses that the notion orientation.

The convex combinatorial partition in any space is not a refinement of the oriented combinatorial partition because we take into account not all the orientations and incidences of the subsets but only of those which lie on the supporting hyperplanes.

Theory of the Gale's diagrams leads us to

Lawrence ?

PROPOSITION 5. For every n,m there exist N,M such that every stratum of $\tau^{+}_{n,m}$ is homeomorphic to some stratum both of $\tau^{conv}_{N,M}$ and $\tau_{N,M}$.

Hence studying the space of the polytopes of a given convex type is equivalent up to a certain construction to studying the oriented configurations (see § 7) and vice versa.

§ 5. SATURATION, REPRESENTATION OF LATTICES, IMPLICATIONS

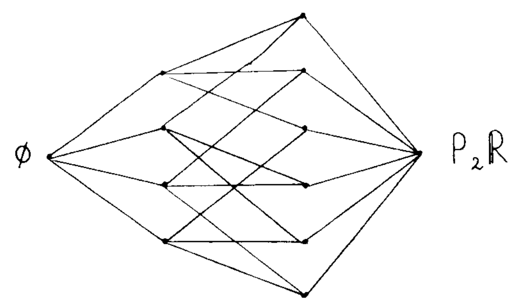
Let us consider an arbitrary configuration γ in a projective space and add to it the hulls of all its subsets. We obtain a new finite configuration (not homogeneous) $\mathcal{U}(\gamma)$ called the upper saturation. As a partially ordered set with respect to inclusion the upper saturation is an upper semilattice (i.e. a partialle ordered set where subset has the supremum). The collection of all the subspaces of a given projective space is a modular (= Dedekind) lattice (see [Bi]) Now the upper saturation becomes the identity upper (i.e. preserving the suprema) representation of some abstract upper lattice into the lattice of all the subspaces. Note that the upper saturation of the points'configuration is a lattice (even geometrical lattice (see [A]) with respect to its partial ordering but the identity representation in general does not preserve the infima. That is why we suggest to introduce to this theory the upper representations. In other words the upper saturation is an upper subsemilattice but in general not sublattice. In the same way we can define the lower saturation of the configurations $\ell(\gamma)$ by adding the intersectionsof all its subsets. All what has been said above on the upper saturations can be repeated word in word on the lower saturation with replacing "supremum" with "infimum" and "upper semilattice" with "lower semilattice" etc.

Let M be an abstract upper (lower) semilattice. We define as usual an upper (lower) representation of M into the lattice of all the subspaces of the projective space. Two representations of M are projectively isomorphic if they can be transformed one to the other by some projective transformation. It means that the configurations which are their images are projectively isomorphic.

Two representations are combinatorially equivalent if the dimensions of the images of the correspondent elements is equal. It is useful to note that the notions of the combinatorial types of the configurations (upper, lower, convex etc) just are connected with a combinatorial type of the identity representation of somenhat lattice associating with configuration (f.e. saturation and so on)

PROPOSITION 6. Two points'configurations are combinatorially equivalent iff their upper saturations are combinatorially equivalent, or iff the two correspondent upper representations of the upper semilattice $\Gamma_{n,m}$ defined below are combinatorially equivalent.

That $\Gamma_{n,m}$ is the lattice which is upper saturation of the generic n -points'configuration in $P_m\mathbb{R}$. F.e. $m=2, n=4$ then $\Gamma_{2,4}$ is as shown



(Whitney's numbers for $m=2$ are $(1, n, C_n^2, 1)$).

A generic points' configuration (or more exactly its upper saturation) is the isomorphic representation of $\Gamma_{n,m}$

Thus we can claim that the theory of configurations is included into the theory of representations of the partially ordered sets . The following theorem is foundation of this theory.

THEOREM 1. Every finite geometric lattice (or finite upper semilattice) has an isomorphic upper representation in a projective space of certain dimension over a given fields.

An important concept is introduced now. Two configurations in $P_m R$ or representations of a partially ordered set are isotopic if they lie in the same connectedness component of a combinatorial type. This equivalence seems more reasonable for this theory then both the projective isomorphism the combinatorial equivalence. The invariants of isotopy has topological nature. We shall discuss it elsewhere.

The upper saturation of a configuration has its lower saturation and conversely. We obtain the increasing sequences of the configurations

$$\gamma \subset u(\gamma) \subset \ell(u(\gamma)) \subset u(\ell(u(\gamma))) \subset \ldots$$

(since $\ell(\gamma) \subset \ell(u(\gamma))$ then the other order of the iterations gives a confinal sequences). They are called implications of the configuration and their union "hull" is in general an infinite configuration which is a lattice in its partial ordering. Moreover the union is a projective space over a certain subfield . This subfield is an invariant of the linear equivalence but not of the isotopic one. The above "hull" is the minimal projective space including the configuration.

The problem on whether or not an abstract lattice (or a matroid) can be realized in a projective space of a given dimension over a given field is solved algorithmically with the help of Tarsky-Zaidenberg theorem. It is important that in Theorem 1 the dimension of the objects cannot be arbitrary and upper representation cannot be replaced with a representation because every sublattice of a modular lattice is modular and that is why it is impossible to imbedd isomorphically an arbitrary lattice into the lattice of the subspaces.

One can define an orientation on lattices and consider the orientation preserving representations. It seems to the author that this has not been considered yet. The most interesting is orientation preserving (neither upper, nor lower) representation of the lattice of all the faces of a convex polytope - shorthly "the representation of the convex polytopes".

§ 6. COORDINATE-FORM FORMULATION

We shall describe briefly some of the previous formulations in matrices terms. Mainly we are interested in the configuration partitions.

Let $M_{n,m}K$ be the space of all the $n \times m$ matrices over K, $M'_{n,m}K$ the subspace of $M_{n,m}K$ whose first m-minor does not vanish.

a) the group $G = GL(m, K)$ acts from the left on both $M_{n,m}K$ and $M'_{n,m}K$, and the factor-süace $G \backslash M'_{n,m}$ is identified with one of the maps of the standard atlas of the grassmanian $G(n,m)$. As usual we can consider $G \backslash M'_{n,m}$as $M_{n,n-m}K$. Since this map consists of some combinatorial types as a whole (see § 3) we may restrict the configuration partition $\tau_{n,m}$ onto the map.

PROPOSITION 7. The restriction of the vector configuration partition on $M_{n-m,m}K$ is a partition into the classes each of them is a set of the matrices with fixed nonvanishing minors. The generic stratum consists of all the matrices whose all the minors don't vanish.

b) if M' is factorized with respect to Cartan group H_n,which acting from the right we get the projective configuration partition.

c) if to imbed $GL(m)$ into the affine group $Aff(m)$,and to add the row $(\underbrace{1, \ldots, 1}_{n})$ to M' and then to factorize the new space with respect to $Aff(m) = \left\{ \left(\begin{array}{c|c} GL(m) & * \\ \hline 0 & 1 \end{array} \right) \right\}$, we obtain the affine configuration partition.

d) finally if K is a linearly ordered field then fixing the signs of nonzero minors results in the oriented configuration partition of $M'_{n,m}$.

Note that all the maps of the standard atlas in $G(n,m)$ are equivalent to each other, hence we may assume that the coordinate description of the configuration partitions of the grassmanian is completed.

The same partitions of $M_{n,m}K$ are obtained if to consider the columns of matrices as vectors in K^m generating a configuration. The above factorization is equivalent to fixing the first m points.

e) the convex configuration partition in matrices terms slightly different from Gale's one is the following. The stratum of this partition consists of the matrices with the same list of "convex submatrices" the lattering such a matrices that there exists a vanishing linear combination of the columns with positive coefficients. (In fact, this is a description of the complementary partition - see §3).

If to omit the details then all the strata of the partitions are sets of matrices subject to certain restriction on the signs or and to the condition of non-vanishing of the minors (see § 9). That is why every stratum is a semialgebraic or constructive set.

Recently A.Barvinok has computed the fundamental group of the main stratum for $P_2\mathbb{C}$. For $P_2\mathbb{R}$ the topology of the main stratum is very complicated and has the primary interest - see Mnev's theorem in § 7. In a similar way one can coordinatize the existence problem of representations of partially ordered sets. Let M be a poset (or a matroid); the existence of an exact representation (upper representation) with a given dimension of the objects over a given field is reduced to solving some algebraic system in the field("the determinant system" formed by equalities and unequalities, see § 9). In principle the problem is solved by Tarsky-Zaidenberg theorem.

The result can be useful in the theory of fields of the convex polytopes and for describing obstructions. Some results are obtained by A.Barvinok and in an other context by A.Chernyakov (see this volume).

§ 7. CLASSIFICATION PROBLEMS. UNIVERSALITY. BASIC THEOREMS

We saw that the set of all the configurations or those with a given combinatorial type or genus is an algebraic set as well as the set of all the representations of other posets or finitely presented groups or algebras a given type. It can be a submanifold of a product of the grassmanians etc. The group of automorhisms of the space acts on that algebraic set or the variety and its orbits form classes of (linear or projective) equivalence. However, this equivalence is very cumbersome for the some problem. From a series of investigations due to Gabriel et al, Gel'fand et al etc (see [Gu],[BGP]) we know that the orbit space is very large for a marjority of sets of posets and quivers (the problem is "wild"). The exception is presented by Dynkin's scheme (in that case the problem is "finite" or "tame"). By its very nature the linear equivalence includes the equivalence of all the implications of the configurations i.e. is a very rigid isomorphism of the projective spaces over some subfields and therefore includes the classifications of the latter.

It seems much more effective <u>the classification which is based on topological principle</u> i.e. studying the topology of the varieties of configurations or representations and their stratification.

The linear classification is in a sense the limit case at the combinatorial classification of the implications. Hardly it is appropriate in a geometric theory to distingnish the configurations differens one from another only by arithmetic properties of points'coordinates.

From the other side the variety of combinatorial type is not an orbit of any algebraic group. Studying the topology of the strata as we know now is a more difficult problem then orbits classification. We turn to describing these results omitting both details and the representations theory for other publications. We emphasize the importance and intrinsic natural character of these problem and topological approach to them.

In order to formulate the results we introduce suitable definition. A problem on the classification of collection of specifically given manifolds (et the manifolds of configurations or representations etc) is homotopically (topologically and so on) universal with respect to an algebraic or an other class of the manifolds if every manifold of this class is homotopic (homeomorphic and so on) to a manifold which occurs in our problem. The presence of universality in a given problem must be considered as the negative answer to the question on effective classification and as a claim on the possibility of presenting (implicitly) all the manifold of the class (algebraic varieties or constructive sets etc). In a sense universality is like NP -completeness (in the complexity theory). We can rather easily prove

THEOREM 2. The classification problem of the manifolds of combinatorial genus of the configurations on the real projective plane is homotopically universal with respect to the class of all the real algebraic varieties over $\mathbb{Q}$.

This means that all of then can be given up to homotopy in a determinant form (see § 9). An algebraic manifold (resp. a semialgebraic set) in the space $M_{n\times m}\,k$ is called a determinant algebraic manifold (resp. a determinant semialgebraic set) if it is determined by the following conditions: a set of minors vanish (resp. a set of minors vanish, some minors are positive and others are negative). Theorems 2 and 3 claim that the class of the determinant manifolds (semialgebraic sets) is homotopically universal.

Much more difficult are the following two theorems proved by N. Mnev (see this volume).

THEOREM 3. The classification problem of the manifolds of the generic oriented combinatorial types on the affine (or projective) real plane is homotopical universal with respect to the real semialgebraic sets over $\mathbb{Q}$:

Following the lines of Theorem 3 (see § 9) one can prove the universality of a large series of the classification problem of lines'configurations, arrangements etc in the projective, affine, vector spaces. This theorem makes clear both nontriviality of the homotopical invariants (f.e. contractibility) in combinatorial problems and meaningfulness of the isotopical classification. Moreover, the generic stratum of on oriented configuration partition can have arbitrary singularities which occur in a real algebraic variety over $\mathbb{Q}$! Nevertheless it is difficult to indicate them explicitly.

But one should keep in mind that the classification problems of certain special classes of the configurations (f.e. with a fixed number of points etc) is not universal and effective answers are feasible.

The techniques of Gale's diagrams and additional arguments allow us to deduce from the previous theorem the second one also due to Mnev which answers one of the questions posed formulated by the author (see § 1).

THEOREM 4. The classification problem of the strata of convex simplicial polytopes of a given combinatorial type is universal in the same sense as in Theorem 3.

There are reasons to believe that this theorem is valid for $\mathbb{R}^4$! (Steinitz theorem allows us to prove the contratibility of all strata in $\mathbb{R}^3$). If it is true strong conclusions for the four-dimensional convex geometry can be done. Now the following fact on the latter representations can be proved, it is similar to the above results.

THEOREM 5. The homotopic classification problem of the sets of indecomposable upper representations with given dimension of the geometric lattices is universal.

However, this problem for a given lattice is a "finite" problem. It is very interesting to advance in the classical problem on nonemptiness of the set of representations over a given number field.

THEOREM 6. For every finite extension of the rationals $\mathbb{Q}$ there exists a combinatorial type of points'configurations (resp. convex polytopes, upper real representations of same lattice) which does not realize over this extension.

Actually to prove Theorem 6 enough to present an algebraic veriety which has no points in a given field and then apply the previous theorems. Of course these configurations are degenerate. This theorem has an old history. The question had arised in a connection with Whitney's matroid theory [W]. For the plane configurations the theorem was announced by Mclane [M]. In the convex geometry it was known the Perle's example (answering a question by Klee - see [Gr]) of a 8-dimension of 12-vertex polytope non-realizable over $\mathbb{Q}$, but realizable over $\mathbb{Q}(\sqrt{5})$. The general case was investigated by Mnev in [M II] in according to a plan suggested by the author and which will be briefly described in § 8 (see also [Gr I]).

It is interesting to study the dimension preserving representations of the non-realizable configurations (nonpappian, nondesargian, Fano's etc).

§ 8. THE PRINCIPLE IDEA OF THE METHOD. JOINT MECHANISMS AND SOLUTIONS OF ALGEBRAIC EQUATIONS

The base of the proof of both universality in all the some theorems and constructing of the examples is the following idea. To realize a given algebraic variety or a semialgebraic set as a configuration stratum we need a configuration which allowing us to solve of a system of the algebraic equations and inequalities. To do this one has construct a peculiar joint mechanism whose nodes'coordinates are subject to given algebraic relations (equalities and inequalities). And vice versa any configuration subject to the same relations has the same combinatorial type. The construction of this type is well known ! It was discovered in fact by von Standt (a Gauss's puple) in his "Geometric der Lage" (1 47) and was called "wurf" (literally "throw", "postponement"). The "Wurf"-method was applied in projective geometry and in nomography,essentially Hilbert used it for the aim of coordinatization of the projective geometry.

On iterating the configurations giving the sums and the products of the reals (see [Ha]), drawing the lines throw the points on the projective plane with two marked lines it is possible to construct a configuration on the plane which "computes" the roots. This is the plan of the proof of Theorems 2 and 6. Constructing the generic configurations is more complicated because one deals with inequelities. Actually the bundle over given manifold with contractible leaves is being constructed. Note that in theseproblems it is impossible to use that induction with respect to the number of points - that is the origin of all the difficulties.

All these ideas can be generalized to of more complicated cases (convex polytopes, arrangements, planes etc). In any case one can construct a configuration mechanism. controlling some variables, running exactly manifold (or some special set). The positions of the mechanism itself form a configuration and their set forms a manifold (or a set). For a good example but rather cumbersome example see Mnev's paper.

§ 9. OPEN PROBLEMS

1. What is the least dimension for which the problem of classification of the strata of convex simplicial polytopes is universal of a given combinatorial type. It seems plansibile that the answer is - four. If it is the case a lot of paradoxical examples within the four-dimensional convex geometry will be added to the well-known ones.

Note that apriori it can happen that in any fixed dimension the problem is not universal but is universal as a whole (see § 7).

2. The classification of the open strata for small defects,(i.e. differeness between the number of points and the dimension) seems quite possible. It is interesting to study it and to compare with the problems of singularities theory (see the end of § 6).

3. To give algebraic-geometric axiomaties of the configuration partitions. Remind that they are not Whitney stratifications. To be more precise we should like to put two problems on stratifications. Let $\{P_i, i=1,\dots,n\}$ be a system of polynomials $P_i \in \mathbb{R}[x_1,\dots,x_N]$, $J=\{J_1, J_2, J_3\}$ being a partition of $\overline{1,n}$. The semialgebraic set $A_J \subset \mathbb{R}^N$, $A_J=\{x, P_j(x)=0\ \forall j\in J_1;\ P_j(x)>0\ \forall j\in J_2;\ P_j(x)<0\ \forall j\in J_3\}$ is generated by J. For what system $\{P_i, i=1,\dots,n\}$ is the partition of $\mathbb{R}^N$ into $\{A_J\}$ (where J runs all the partitions of $\overline{1,n}$) a Whitney stratification ? For what subsets $\mathcal{U}$ of the set of all the partitions of $\overline{1,n}$ is a partition of $\bigcup_{J\in\mathcal{U}} A_J$ into $\{A_J; J\in\mathcal{U}\}$ a Whitney stratification for every system .(F.e. it is so for $\mathcal{U}=\{J : J_3=\emptyset\}$, where we have the classical stratification of a semialgebraic set). In the above cases in this paper $\mathbb{R}^N = \mathrm{Mat}_{n\times k}\mathbb{R}$ and every P_i is some minor of matrices.

4. To describe all the partitions of grassmanian invariant with respect to Cartan group (and Weyl group in particular) which are natural within the category of all the fields or of all the ordered fields. It seems not impossible that all of them are combined from configuration partitions.

5. To study the metric properties of the strata of configuration partitions of the grassmanians. In particular to find the asymptotics of the grassmanian measure of generic strata and their unions. There are connexetions between this problem and Banach geometry and linear

programming (see [VS]). The numerical characteristic for of the typical convex polytope (neighbourhood, the numbers of faces etc) are expressed with the help of metric properties of those partitions.

6. The investigate configuration partitions for the fields of a finite characteristic.

7. To give analogues of all the objects for the series of simple groups different from A_n

8. How to use the results on the field $\mathbb{C}$ to the case of $\mathbb{R}$ (as in the theory of real algebraic curve) ?

9. To compare the Viro's examples of non-homeomorphic configurations of the lines in $P_3\mathbb{R}$ which are not distinguished by linking numbers, with the configurations which have the same oriented combinatorial type but are not isotopic.

10. Is there an analog of the invariants of the Knot theory (such as Alexander's, Jone's polynomial etc) in the theory of configurations?

We don't discuss here especially the new problems of the representation theory of lattice, note only that the realization problem (in a given dimension) for the lattice and matroid includes some topological problems. One can expect that the applications of the previous ideas and making use of Tarsky-Zaidenberg theorem to the representation problem will lead to a new progress in combinatorial topology.

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