Representing Graph Properties by Polynomial Ideals

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Abstract. Computer algebra provides means to obtain a diversity of results in many areas of science. In this paper we explore the possibility of representing properties of graphs by polynomial ideals. We show that several properties (emptiness, colorability) admit such representation, but we also work out limitations of this approach.

1 Introduction

Many areas of science benefit from the ability to describe their objects in various ways using formalisms from fields which are seemingly far away from each other. In most cases, this does not introduce a mere redundancy, but frequently opens new views and many times provides for unforseen insights.

Motivated by the work of Yu. Matiyasevich ([8]), we are going to describe an algebraic method to study properties of graphs. This method is developed to provide means to solve the decision problem whether a given graph G has some property P. Especially, we do *not* intend to design an all-embracing general solution but rather adapt some concepts from algebra to graph theory where these naturally fit.

Given a decision problem on graphs, we will transform each instance of it into an instance of the polynomial ideal membership problem by capturing the essence of the property P in a polynomial ideal and encoding the graph G as a homogeneous polynomial. Then, ideally, the graph G satisfies the property P if and only if the polynomial corresponding to G is contained in the ideal corresponding to P.

What are the supposed merits of such approach? First, numerous methods for solving the polynomial ideal membership problem are available, some of them have efficient implementation in software. Second, polynomial ideals are in most cases not "flat" but rather "structured" objects. This structure can possibly be transformed back into the graph theory where it could provide new insights. And, finally, even though graph properties and polynomial ideals are used to select (basically) equal subsets of the set of all graphs, these two concepts stress different aspects. Hence, it is not unreasonable to assume that there will be classes of graphs which can be described and/or studied using polynomial ideals more easily than using logical formulas.

The approach we will be using here is as follows: Let G be a graph with n vertices. We assign each vertex of G a variable. The graph G will then be associated with a polynomial $f_G \in k[x_1, \ldots, x_n]$, and we construct an ideal $\mathfrak{I}$ in $k[x_1, \ldots, x_n]$ (depending on the property under consideration P, the number n, and usually on some other parameters) such that an assertion of the type

$$P(G) \Longleftrightarrow f_G \in \mathfrak{I} \tag{1}$$

is valid. However, we will see that this situation is not reachable under all circumstances. We will even prove that in some interesting cases the above relationship is impossible.

The next chapter gives a short introduction to polynomial ideals. After some general considerations regarding the properties which are suitable to be described by polynomial ideals we will start with representing the property of a graph to be nonempty. After showing that the colorability with a fixed number of colors is a property which can be described by a relationship as in (1), we will finally consider the property of a graph to contain a complete subgraph. This property is tightly connected with the colorability and the expectation is that they will behave similarly. However, it turns out that the containment of complete graphs cannot be expressed by means of polynomial ideals.

In Section 3 the notion of a Gröbner basis is extensively used. Readers not familiar with Gröbner bases may consult some of various text books such as [4].

2 Basic Notions and Concepts

A *graph* $G = (V, E)$ is a pair of finite sets V (vertices) and E (edges) with $E \subseteq \{\{u, v\} \mid u \neq v,\ u, v \in V\}$, i.e. all considered graphs are undirected with no loops or multiple edges. For the sake of simplicity we assume that the vertices of G are named $\{1, \ldots, n\}$ where $n = |V|$ unless explicitly stated otherwise. We write $[n]$ to denote the set $\{1, \ldots, n\}$.

The following definition associates every graph with a polynomial. It is originally motivated by the wish to express whether the vertices of a graph can be colored in such a way that no two adjacent vertices get the same color. However, same objects will prove useful also for deciding other properties, like the emptiness or whether a graph contains a complete subgraph.

Definition 1. *Let $G = (V, E)$ be a graph on n vertices $\{1, \ldots, n\}$. With each vertex $i \in V$ we associate the variable x_i. Let $\preceq$ be a connex partial order on the set of variables $\{x_1, \ldots, x_n\}$ (i.e. a reflexive, transitive, and antisymmetric order with $x_i \preceq x_j$ or $x_j \preceq x_i$ for any i, j). The* graph polynomial f_G *of G is given by*

$$f_G(x_1, \ldots, x_n) = \prod (x_i - x_j)$$

Note that the polynomial f_G is a homogeneous polynomial of degree $|E|$.

Remark 2. There are several ways to associate a graph with a polynomial. All of them aim at encoding graph properties. The approaches differ in the way which parts of the polynomial are used to encode the desired information. One approach is to use values of the polynomial at certain distinguished points – examples are the *Penrose* and the *Tutte* polynomial (cf. [10], [1]). Another way is to use polynomials as generating functions and to use coefficients – examples for this kind of encoding is the classical *characteristic* polynomial, the *matching* polynomial, and others (cf. [9], [5], [2]).

The graph polynomial defined above carries the information in its coefficients. This makes it possible to vary the ground field according to particular needs. Section 3.2 provides an example where the freedom of choosing the ground field makes solving a problem easier.

Throughout the paper we will denote by k an arbitrary field, and by $k[x_1, \ldots, x_n]$ the ring of polynomials over k. Moreover, we assume that the variables are ordered $x_1 \preceq x_2 \preceq \ldots \preceq x_n$. At this moment it may not be clear why an ordering on variables is needed – except for eliminating redundant factors from the graph polynomial. The reason becomes apparent when Gröbner bases come into play later in the paper.

As we already mentioned, the properties will be encoded in *polynomial ideals.* The structure of an ideal makes it possible to circumvent the problem which arises from the assignment of variables to vertices. This step virtually introduces labels – something we want to avoid. On the other hand, ideals constitute a framework where efficient computations can be performed and whose structure is to a large extent compatible with problems we intend to solve.

Definition 3. *Let R be a commutative ring. An* ideal $\mathfrak{a}$ *in R is an additive subgroup of R such that $\mathfrak{a}R \subseteq \mathfrak{a}$.*

Notation. *Given a (finite) set of elements $A = \{a_1, \ldots, a_n\} \subseteq R$ we denote by $(a_1, \ldots, a_n)$ the smallest ideal of R containing A, i.e. the set of all finite sums $\sum a_i r_i$, $r_i \in R$.*

After having defined basic ingredients we are going to clarify the precise meaning of the relationship (1). There are some questions to answer:

- In order to be able to define a graph polynomial we introduced an assignment of variables to vertices of G. Since this assignment is arbitrary, it should not have any influence on the result, and we require that the defined ideals do not depend on it.
- Since we are working with polynomials with a fixed number of variables, a property P usually cannot be captured in a single ideal. It will be described

Definition 4. *Let P be a predicate (property) defined on the set of all unlabeled graphs. We say that the sequence of ideals $(\mathfrak{I}_n)_{n\in\mathbb{N}}$ represents P if for all $n \in \mathbb{N}$ and for all unlabeled graphs G with n vertices the following holds*

$$P(G) \iff f_{\pi(G)} \in \mathfrak{I}_n \text{ for all permutations } \pi \in S_n,$$

where $\pi(G)$ corresponds to G where each vertex i has been renamed to $\pi(i)$, and S_n denotes the permutation group on n elements.

Whenever it is clear (or irrelevant) what the parameter n is and how the sequence $\mathfrak{I}_n$ is constructed, we will omit the reference to n saying that "the ideal $\mathfrak{I}$" describes the property P.

Before we proceed, let us see what implications arise from the decision to choose ideals as the underlying structure. Suppose, the ideal $\mathfrak{I} \subseteq k[x_1, \ldots, x_n]$ represents a property P. This means that the graph G satisfies P if and only if the graph polynomial f_G lies in $\mathfrak{I}$. Now, let $g \in k[x_1, \ldots, x_n]$ be a polynomial such that the product gf_G turns out to be a graph polynomial (i.e. is a square-free product of differences of variables) of some graph G' (which contains G). Since $\mathfrak{I}$ is an ideal, the product gf_G belongs to $\mathfrak{I}$ and thus G' has to satisfy P too. Since this holds for arbitrary polynomials g, all properties P to be represented by ideals must be *monotone*, i.e. they satisfy the following condition:

> Let G and G' be two graphs such that G is a subgraph of G'. If P holds for G then it holds also for G'.

Fortunately, quite some properties, which we are interested in, are monotone. Some of them will be studied in the next section.

3 Special Graph Properties

In this section we consider three properties of graphs for which we would like to obtain an algebraic description in the sense of Definition 4:

(1) the graph contains at least one edge;
(2) the vertices of a graph are not properly colorable by a fixed number of colors;
(3) the graph contains a complete subgraph.

Next, we will construct ideals representing the first two properties, and we will prove that the third one does not admit such representation.

3.1 Non-Emptiness

We start with a simple property of a graph to contain at least one edge. For any n we want to find an ideal $\mathfrak{E}_n$ such that the following holds: Let G be a graph with n vertices and f the graph polynomial corresponding to some assignment

degree at least 1 and vice versa. Hence, we are looking for an ideal containing all polynomials of degree at least 1.

Obviously, $(x_1, \ldots, x_n) \subseteq k[x_1, \ldots, x_n]$ has the required property. If a graph contains at least one edge, its graph polynomial is a homogeneous polynomial of degree at least one and hence lying in $(x_1, \ldots, x_n)$. On the other hand, if a graph has no edges its graph polynomial is 1 which is not contained in the ideal $(x_1, \ldots, x_n)$. Otherwise we could write 1 as a linear combination of $\{x_1, \ldots, x_n\}$ with polynomial coefficients. Setting all variables to zero would yield a contradiction.

We note that this ideal – and, in general, all ideals which represent some property – is not unique as it can be replaced for example by $(x_n - x_1, x_n - x_2, \ldots, x_n - x_{n-1})$. This ideal represents the non-emptiness of a graph as well.

3.2 Coloring of Graphs

In 1974 Yu. Matiyasevich ([8]) discovered a way to describe proper vertex coloring of graphs in terms of properties of coefficients of graph polynomials (see [8]). In this section we use these ideas to construct an ideal which represents the property of a graph to be *not* colorable with a fixed number of colors.

Let $G = (V, E)$ be a graph with n vertices and let $1 \leq r \leq n$ be an integer. A *proper coloring* of vertices of G with colors $\{c_1, \ldots, c_r\}$ is an assignment $\varphi : V \to \{c_1, \ldots, c_r\}$ of colors to vertices of G such that no two vertices connected with an edge are assigned the same color, i.e. $(\forall i \neq j)(\{v_i, v_j\} \in E \Rightarrow \varphi(v_i) \neq \varphi(v_j))$. Let us denote by $\bar{C}_r^n(G)$ the property that a graph G with n vertices has *no* proper coloring with at most r colors. Since a graph which can be properly colored with a single color has no edges, we assume from now on that $r \geq 2$.

Theorem 5. *Let*

$$U_r^n := (x_1^r - 1, \ldots, x_n^r - 1) \subseteq k[x_1, \ldots, x_n].$$

The ideal U_r^n represents the property of a graph to be not properly colorable by at most r colors.

Proof. Let G be a graph with n vertices such that

$$\bar{C}_r^n(G) \text{ holds,} \tag{2}$$

i.e. there is no proper coloring of G with at most r colors. We will show that (2) is true if and only if its graph polynomial lies in U_r^n.

Let F denote some field. If the colors c_i's are elements from F, the property (2) means that for all subsets $\Gamma \subseteq F$ with at most r elements and for all assignments $\psi : \{x_1, \ldots, x_n\} \to \Gamma$ of colors to variables we obtain

$$f_G(\psi(x_1), \ldots, \psi(x_n)) = 0, \tag{3}$$

assertion is equivalent to the fact that (3) is true for *some* set Γ with r pairwise different elements and for all assignments $\psi : \{x_1, \dots, x_n\} \to \Gamma$. Especially we can choose Γ to contain only nonzero elements of F. Thus (2) is equivalent to

$$f_G(x_1, \dots, x_n) \equiv 0 \text{ on } \Gamma^n. \tag{4}$$

Now we are going to specify precisely how the colors $c_1, \dots, c_r$ in Γ are selected.

Let q be a natural number with $q \equiv 1 \bmod r$ which is a power of some prime p. (If p is chosen so that it does not divide r, the cyclic subgroup of the multiplicative group $\mathbb{Z}_r^*$ generated by p is obviously finite. It is easy to see that we may set q to be $p^{\mathrm{ord}(p)}$, where $\mathrm{ord}(p) > 0$ is the *order* of p in $\mathbb{Z}_r^*$.) Then there is a finite field $\mathrm{GF}(q)$ with q elements (for a summary of properties of finite fields see e.g. [7]). Now, the key idea is to select the colors $\{c_1, \dots, c_r\}$ to be certain distinguished elements of $\mathrm{GF}(q)$.

We recall that every finite field $\mathrm{GF}(q)$ has a primitive element, i.e. an $\alpha \in \mathrm{GF}(q)$ such that any nonzero element $a \in \mathrm{GF}(q)$ can be written as $a = \alpha^i$ for some integer i. Moreover, such elements satisfy the equation

$$X^{q-1} = 1. \tag{5}$$

Let m be such that $q = rm + 1$. We set

$$c_i := \alpha^{(i-1)m}, \text{ for } 1 \le i \le r.$$

Since α is primitive, the set

$$\Gamma := \{1, \alpha^m, \alpha^{2m}, \dots, \alpha^{(r-1)m}\} \tag{6}$$

consists of r pairwise different elements of $\mathrm{GF}(q)$.

Now, if the graph polynomial f_G of G is fully reduced modulo U_r^n (i.e. every occurrence of x_i^r in f_G is replaced by 1), every variable has degree at most $r-1$. The reduction process corresponds to subtraction of elements of U_r^n from f_G. Hence, f_G can be written as

$$f_G = \tilde{f}_G + u, \tag{7}$$

for some $u \in U_r^n$. We will prove that $\tilde{f}_G$ must be zero, i.e. all coefficients of $\tilde{f}_G$ vanish.

We assumed that G has no proper coloring with (at most) r colors. As we saw above, this is equivalent to the fact that f_G vanishes in all points of Γ^n. As $u \in U_r^n$, it is easily seen that the same is true for the polynomial u. Hence, the equation (7) shows that also $\tilde{f}_G$ vanishes on Γ^n. However, the following lemma (see e.g.[3]) shows that this is impossible unless it is identically zero.

Lemma 6. *Let $f(x_1, \dots, x_n) \in k[x_1, \dots, x_n]$ be a polynomial where the degree $\deg_{x_i}(f)$ of the variable x_i is bounded by t_i and in which every variable x_i occurs* [illegible] *with nonzero coefficient. Let $\{S_i\}^n$ be a set of subsets of k*

Proof. If $n = 1$, the assertion follows from the well known fact that a univariate nonzero polynomial of degree t has at most t roots. The general case is inferred by induction. □

Now, let $\{x_{i_1}, \dots, x_{i_s}\}$ be the set of variables which effectively occur in $\tilde{f}_G$. Since the degree of any variable in $\tilde{f}_G$ is at most $r - 1$ and this polynomial vanishes on Γ^n, the previous lemma restricted to $\{x_{i_1}, \dots, x_{i_s}\}$ yields that $\tilde{f}_G$ must vanish identically. Then (7) finally implies

$$f_G \in (x_1^r - 1, \dots, x_n^r - 1). \tag{8}$$

On the other hand, if (8) holds then there is obviously no proper coloring of G with at most r colors. □

To conclude, we proved that for a graph $G = (V, E)$ on n vertices and for any assignment $\psi : V \to \{x_1, \dots, x_n\}$ of variables to vertices the graph G has no proper coloring with at most r colors if and only if the corresponding graph polynomial f_G lies in the ideal U_r^n:

$$\bar{C}_r^n(G) \iff f_G \in U_r^n. \tag{9}$$

3.3 Representation of Subgraphs

After we have affirmatively answered the question about representing non-emptiness and colorability of graphs by ideals, we will consider another interesting property. Let G and H be two graphs with n and r vertices, resp. We are looking for an ideal which would represent the property of H being a "subgraph" of G (as an unlabeled structure). To be more precise, we require G to contain an isomorphic copy of H. Let us elaborate this point.

Both G and H are given by sets of vertices and edges: $G = (V, E)$, $H = (W, F)$. Then, by definition, H is a *subgraph* of G if $W \subseteq V$ and $F \subseteq E$. However, here the names of vertices play an important role. We want to represent a similar property without referring to these names.

Definition 7. *Let G and H be as above. The graph G* contains an isomorphic copy *of H if there is an injective map $\rho : W \to V$ such that $\rho(H)$ is a subgraph of G (by $\rho(H)$ we denote the graph H after renaming each vertex $i \in W$ to $\rho(i) \in V$). This property will be denoted by $H \lesssim G$.*

Remark 8. To keep the notation simple we will use the term "subgraph" in the more general sense of "contains an isomorphic copy".

How should an ideal $\mathfrak{J}$ corresponding to graphs containing H as a subgraph look like? If each vertex $i \in W$ is assigned a variable x_i we obtain a graph polynomial $f_H \in k[x_1, \dots, x_r]$. The polynomial f_H has to lie in $\mathfrak{J}$. In Definition 4 we stipulated that the construction is to be independent of a particular naming of vertices. Hence, $\mathfrak{J}$ has to contain $f_{\pi(H)}$ for all permutations $\pi \in S_r$ on $[r]$

Proof. Let $\preceq$ denote the degree lexicographical ordering. A Gröbner basis of an ideal $\mathfrak{I}$ is a set of polynomials such that the leading terms of these polynomials generate a monomial ideal which is equal to $\mathrm{lt}(\mathfrak{I})$ (w.r.t. some term ordering), where the *leading term* of a polynomial is its largest term (w.r.t. $\preceq$) and

$$\mathrm{lt}(\mathfrak{I}) := \{\mathrm{lt}(f) \mid f \in \mathfrak{I}\}.$$

From this it is clear that any set of polynomials containing a Gröbner basis is again a Gröbner basis. We will show that the set

$$\mathcal{G}^n_{K_r} := \{f_H \mid H = (\{1\} \cup W', F),\ W' \subseteq \{2, \dots, n\},\ H \cong K_r\} \subseteq \mathcal{B}^n_{K_r}, \qquad (11)$$

i.e. the set of graph polynomials of all labelings of K_r containing the vertex 1 is a reduced Gröbner basis of $\mathfrak{K}^n_r$ with respect to the degree lexicographic ordering $\preceq$.

Without loss of generality we assume $x_1 \preceq x_2 \preceq \dots \preceq x_n$. In the sequel we will index the elements of $\mathcal{B}_{K_r}$ by natural numbers instead of graphs. We also assume that all notions which require an ordering refer to the degree lexicographical order. Let $\mathcal{G}^n_{K_r} = \{g_1, \dots, g_t\}$. We will show

$$\mathrm{lt}(\mathfrak{K}^n_r) = (\mathrm{lt}(g_1), \dots, \mathrm{lt}(g_t)). \qquad (12)$$

Since $\mathcal{G}^n_{K_r} \subseteq \mathcal{B}^n_{K_r}$ the inclusion $(\mathrm{lt}(g_1), \dots, \mathrm{lt}(g_t)) \subseteq \mathrm{lt}(\mathfrak{K}^n_r)$ follows immediately. To show the equality, let $f = \sum_{i=1}^{s} p_i f_i \in \mathfrak{K}^n_r$, $p_i \in k[x_1, \dots, x_n]$, and $f_i \in \mathcal{B}^n_{K_r}$. We will prove that there exists an i such that $\mathrm{lt}(g_i) \mid \mathrm{lt}(f)$. Then $\mathrm{lt}(\mathfrak{K}^n_r) \subseteq (\mathrm{lt}(g_1), \dots, \mathrm{lt}(g_t))$ and the equality (12) follows. Since all polynomials are defined over a field, for the sake of simplicity we will work with leading *monomials* (product of powers of variables without the coefficient) instead of leading *terms*.

How do the leading monomials of polynomials from $\mathcal{B}^n_{K_r}$ look like? First, since a graph polynomial is a product of $\binom{r}{2}$ linear factors, it is homogeneous. If a polynomial f involves variables $x_{i_r}, \dots, x_{i_1}$, where $i_r > i_{r-1} > \dots > i_1$, its leading monomial is

$$x_{i_r}^{r-1} x_{i_{r-1}}^{r-2} \cdot \ldots \cdot x_{i_3}^2 x_{i_2}$$

(note that $i_2 > 1$). Let $\mathrm{lm}^{(i)}(f)$ denotes the i-th largest monomial of f. Then, e.g.,

$$\mathrm{lm}^{(2)}(f) = x_{i_r}^{r-1} x_{i_{r-1}}^{r-2} \cdot \ldots \cdot x_{i_3}^2 x_{i_1}.$$

If we consider the $r-1$ exponents of variables in each term of f as a vector in $\mathbb{N}^{r-1}$, the set E of such vectors is a subset of $\mathbb{N}^{r-1}$ consisting of all vectors $(e_{k_1}, \dots, e_{k_{r-1}})$ with

$$\sum_{l=1}^{r-1} e_{k_l} = \binom{r}{2} \quad \text{and} \quad e_{k_l} \leq r-1, \text{ for all } 1 \leq l \leq r-1.$$

The terms of f are sorted in the decreasing lexicographical order of their exponent vectors.

To see that for $f = \sum_{i=1}^{s} p_i f_i \in \mathfrak{K}_r^n$ the leading monomial $\mathrm{lm}(f)$ is divisible by some $\mathrm{lm}(g_i)$, we distinguish two cases depending on whether $\mathrm{lm}(f)$ stems from the leading monomial of a single term $p_i f_i$ or whether it arouse by canceling leading monomials of some terms.

Case 1. ($\mathrm{lm}(f) = \mathrm{lm}(p_i f_i)$ for some i) In this case $\mathrm{lm}(f_i) | \mathrm{lm}(f)$ and since $\mathrm{lm}(f_i)$ is equal to some $\mathrm{lm}(g_j)$ the assertion follows immediately.

Case 2. ($\mathrm{lm}(f)$ is different from all $\mathrm{lm}(p_i f_i)$) Then there are indices $i < j$ such that $\mathrm{lm}(p_i f_i) = \mathrm{lm}(p_j f_j)$ (there may be more than two indices with this property). Let $\{i_1, \dots, i_r\}$ and $\{j_1, \dots, j_r\}$ be the variables involved in $\mathrm{lm}(p_i f_i)$ and $\mathrm{lm}(p_j f_j)$, resp. We assume $i_r > i_{r-1} > \dots > i_1$ and $j_r > j_{r-1} > \dots > j_1$. Then we have for some monomials Y and Y'

$$\mathrm{lm}(p_i f_i) = Y x_{i_r}^{r-1} x_{i_{r-1}}^{r-2} \cdot \ldots \cdot x_{i_3}^2 x_{i_2}$$
$$\mathrm{lm}(p_j f_j) = Y' x_{j_r}^{r-1} x_{j_{r-1}}^{r-2} \cdot \ldots \cdot x_{j_3}^2 x_{j_2}.$$

Let $l \geq 1$ be the least index such that $i_l \neq j_l$ and $i_k = j_k$, for $1 \leq k < l$. Without loss of generality we assume $i_l > j_l$, i.e. $x_{i_l} \succeq x_{j_l}$. The leading monomials of $p_i f_i$ and $p_j f_j$ have the form

$$\mathrm{lm}(p_i f_i) = \mathrm{lm}(p_j f_j) = U x_{i_l}^{l-1} x_{j_l}^{l-1} \cdot \prod_{k=1}^{l-1} x_{i_k}^{k-1},$$

where U is divisible by

$$\prod_{t \in \{\{i_{l+1}, \dots, i_r\} \cup \{j_{l+1}, \dots, j_r\}} x_t^{\epsilon_t}, \tag{13}$$

with $\epsilon_t = k - 1$ if $t = i_k$ or $t = j_k$.

Let l be as above and let

$$\sigma := |\{(e_1, \dots, e_{l-1}) \subseteq \mathbb{N}^{l-1} \,|\, e_i \leq r-1, \sum_{s=1}^{l-1} e_s = \binom{l}{2}\}|.$$

It is not hard to see that the σ largest monomials of $p_i f_i$ and $p_j f_j$ will be identical and thus will not appear in f. Moreover,

$$\mathrm{lm}^{(\sigma+1)}(p_i f_i) = U x_{i_l}^{l-2} x_{j_l}^{l-1} x_{i_{l-1}} \cdot \prod_{k=1}^{l-1} x_{i_k}^{k-1}$$

Thus $\mathrm{lm}^{(\sigma+1)}(p_j f_j)$ becomes the leading monomial of f. Using (13) we see that $\mathrm{lm}(f)$ is divisible by

$$\mathrm{lm}(f_i) = \prod_{k=1}^{r} x_{i_k}^{k-1}$$

and hence by $\mathrm{lm}(g_i)$ for some $g_i \in \mathcal{G}_{K_r}$.

□

The Case K_2 and K_3. In the special case $r = 2$, the basis $\mathcal{B}_{K_2}^n$ of the ideal $\mathfrak{I}_{K_2}^n$ consists of graph polynomials of all edges $\{\{i,j\} |\ i,j \in [n], i \neq j\}$. This is the ideal $\mathfrak{I}_{P_1}^n$ considered in the Example 10. Hence, the ideal $\mathfrak{I}_{K_2}^n$ correctly represents the property of a graph on n vertices to contain an edge.

Now, in general, does $\mathfrak{K}_r^n$ represent the property of a graph with n vertices to contain an isomorphic copy of a complete graph K_r? We will show that the answer is – unfortunately – *No*. The counterexample yields already the ideal $\mathfrak{K}_3^n$, generated by all graph polynomials of triangles.

First, it is clear that if a graph contains a triangle then its graph polynomial (regardless of naming of vertices) is in $\mathfrak{K}_3^n$. However, the reverse direction is not true.

Theorem 12. *Let C_m denote the circle with $m \geq 3$ vertices, i.e. a graph with edges $\{\{i, i+1\} |\ 1 \leq i \leq m\} \cup \{\{1, m\}\}$, and $\pi \in S_m$ a permutation on the set $[m]$. The ideal $\mathfrak{K}_3^n$ contains the graph polynomial of all $\pi(C_m)$ for any odd $m \leq n$.*

Proof. The assertion will be proved by induction on m.

When $m = 3$ the statement is trivial since the graph polynomials of all $\pi(C_3)$ are equal to $f_{C_3} \in \mathcal{B}_{K_3}^n$.

Let $5 \leq m \leq n$ be an odd positive integer and $\pi \in S_m$ a permutation. Then

$$f_{C_m} = \underbrace{(x_m - x_{m-1})(x_m - x_1)(x_{m-1} - x_{m-2}) \cdot \ldots \cdot (x_5 - x_4)}_{=:\sigma} \cdot (x_4 - x_3)(x_3 - x_2)(x_2 - x_1).$$

We obtain

$$\begin{aligned} & f_{C_m} + \sigma f_{C_3} \\ &= \sigma(x_4 - x_3)(x_3 - x_2)(x_2 - x_1) + \sigma(x_3 - x_2)(x_3 - x_1)(x_2 - x_1) \\ &= \sigma(x_4 - x_1)(x_3 - x_2)(x_2 - x_1) \\ &= f_{\pi'(C_{m-2})}(x_3 - x_2)(x_2 - x_1), \end{aligned}$$

for some $\pi' \in S_{m-2}$. Thus $f_{C_m} + \sigma f_{C_3}$ corresponds to a graph polynomial which is a multiple of $f_{\pi'(C_{m-2})}$ for some $\pi' \in S_{m-2}$. Since both f_{C_m} and $f_{\pi'(C_{m-2})}$ are

This is a bad news as it destroys any hope to success by "massaging" the ideal $\mathfrak{K}_3^n$. We immediately obtain:

Proposition 13. *There exists no ideal representing the property of graphs to contain a complete subgraph* K_3.

Proof. If such ideal existed it would contain graph polynomials of all triangles (with vertex names taken from $[n]$ for some n) and hence the whole $\mathfrak{K}_3^n$. Consequently, graph polynomials of circles of odd length would be contained in it as well. □

Despite of this negative statement, Theorem 12 provides for a somewhat surprising conclusion.

Corollary 14. *The ideal* $\mathfrak{K}_3^n$ *contains graph polynomials of all graphs on* n *vertices which are* not *bipartite, i.e. which cannot be properly colored with two colors.*

Proof. Let G be a graph with n vertices which is not bipartite. To keep the proof simple, we borrow a theorem from graph theory saying that every non-bipartite graph contains a circle with an odd number of edges (see e.g. [6]). Then the graph polynomial f_G is a multiple of $f_{\pi(C_m)}$ for some odd m and some $\pi \in S_m$. Thus $f_G \in \mathfrak{K}_3^n$. □

On the other hand, $\mathfrak{K}_3^n$ is generated by polynomials of the form $(x_i - x_j)(x_i - x_k)(x_j - x_k)$, for some $1 \leq k < j < i \leq n$. Taking the normal form with respect to the ideal U_2^n (see Section 3.2) – which is zero – we immediately see that every such polynomial is in U_2^n and hence

$$\mathfrak{K}_3^n \subseteq U_2^n. \tag{14}$$

Thus if G is a graph with n vertices such that $f_G \in \mathfrak{K}_3^n$ then $f_G \in U_2^n$. In Section 3.2 we proved that in this case G cannot be properly colored with two colors – it is not bipartite.

What we got is the representation of not bipartite graphs! The same property which was considered in Section 3.2.

Theorem 15. *The ideal* $\mathfrak{K}_3^n$ *represents the property of a graph to be not bipartite.*

To summarize this subsection, we saw that the ideal $\mathfrak{K}_2^n$ represents the property of a graph to contain K_2 (an edge). However, it proved to be impossible to extend this statement to the general case due to the fact that $\mathfrak{K}_3^n$ contains circles of odd length. Finally, we were able to prove that $\mathfrak{K}_3^n$ represents the same property as U_2^n – the property of a graph to be not bipartite.

What graphs containing complete subgraphs K_r, for

lies in the ideal but which do not contain K_r. The nature of these exceptional graphs is diverse and does not allow a simple description as in the case $r = 3$.

Computations showed that there are e.g. 72 graphs with 6 nodes not containing K_4 but whose polynomials lie in $\mathfrak{K}_4^6$. Similarly, there are 252 graphs with 7 nodes not containing K_5 whose polynomials lie in $\mathfrak{K}_5^7$. These graphs do not fall into any single graph theoretic category. However, we saw that there is a connection between them and the corresponding complete graphs – the defining ideal. This matter requires further investigation to see whether it may be exploited to obtain something useful.

4 Conclusion

In this paper we were exploring the possibility of representing graph properties by polynomial ideals. We showed that some properties (non-emptiness, coloring) are suitable to be treated by this approach, but in some interesting cases a representation (in the sense of Definition 4) is impossible – the ideals proved to be too coarse to capture the desired property exactly. Nevertheless, they may serve as an "approximate" representation.

The proposed approach has many positive aspects, but also some limitations. On the other hand, the fact that it cannot mimic graph theory exactly is not necessarily a failure. It merely leads to new classes of graphs characterized by common properties which cannot be efficiently described in the language of graph theory.

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Parametric G^1–Blending of Several Surfaces*

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Abstract. In this paper we present a symbolic algorithm for blending parametrically with G^1–continuity a collection of rational surfaces with rational clipping intersections. The method provides a family of parametrizations that depends on a set of parameters, which number can be controlled in advance, and that can be used further in the modeling process. For this purpose, we introduce the notion of curves being in good position, that can always be assumed w.l.o.g., and we prove the main property of a set of rational curves in good position, that guarantees that the method always provides a correct solution.

1 Introduction

One of the main problems in Computer Aided Geometric Design is modeling objects (see [9]). Usually, one models the object as a collection of surfaces. However, in many cases, one wants this collection to form a composite object whose surface is smooth. This question leads to the blending problem. In fact, a blending surface is a surface that provides a smooth transition between distinct geometric features of an object.

Roughly speaking, if $V_1, \ldots, V_n$ (surfaces to be blended), and $U_1, \ldots, U_n$ (clipping surfaces) are given, the blending problem consists in finding an algebraic surface V such that $C_i = U_i \cap V_i \subset V$, and V meets each V_i at C_i with "certain" smooth conditions (G^k–continuity, see [2]). In this paper we only deal with G^1–continuity (i.e. tangent plane continuity along C_i) but the method presented can be extended to G^k–continuity. Thus, the problem of blending smoothly a collection of surfaces may be decomposed into two separate subproblems. The first one focuses on finding appropriate clipping surfaces U_i (see [8], [14]), and the second assumes that the clipping surfaces are given, and deals with the question of determining the final blending surface V (see [7], [14]). In this paper we investigate on the second problem, that we will call in the sequel the blending problem.

In addition, the blending problem can be approached from two different points of view, namely, implicitly (see [8], [15]) or parametrically (see [3],[13]). In