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Pick's Theorem and the Todd Class of a Toric Variety

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CONTENTS

1. INTRODUCTION

It is well known that questions involving lattice points and lattice polyhedra often admit an algebro-geometric interpretation via the theory of toric varieties. For instance, Danilov [Dan] (see also [Kh]) pointed out that a formula for the Todd class of a toric variety would imply a formula for the number lattice points inside a lattice polyhedron (but did not actually show that either formula held). This paper studies these two formulas and the relation between them. An undercurrent of this work, and its original motivation, is the attempt to understand the mutual relation of these formulas to a classical result of Pick concerning the area of a lattice polygon. This relation is explained in the section below labeled *overview*.

Using combinatorial methods developed in [Mo1], I prove (Corollary 3) that there is a formula, of the type suggested by Danilov, for the number of lattice points in a polyhedron. By establishing the converse of Danilov's implication, I deduce from this that there is furthermore a formula for the Todd class of a toric variety, of the type he envisioned (Section 3.1). Following these results, which are abstract in nature, is the derivation of an explicit such formula, from the residue theorem of Baum and Bott as applied to a toric variety.

Let me now explain the latter explicit formula. A more through explanation and motivation is presented in the following section. Let $M \cong \mathbf{Z}^d$ be a lattice, and $N = M^\vee$ its dual. The convex hull in $N \otimes \mathbf{R} \cong \mathbf{R}^d$ of finitely

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many rays which pass through points of N is called a rational polyhedral cone. Denote by $G_k = G_k(N \otimes \mathbf{R})$ the grassmannian variety of k-planes in $N \otimes \mathbf{R}$. In Section 4 we construct for each k-dimensional cone σ in N, a rational function $\mu_k^{td_k}(\sigma)$ on G_{d-k+1}. This construction has the following properties:

1. (Linearity) If σ is expressed as a union $\sigma = \bigcup_i \sigma_i$ of rational cones σ_i with disjoint interiors, then

$$\mu_k^{td_k}(\sigma) = \sum_i \mu_k^{td_k}(\sigma_i).$$

2. (Number of Lattice Points) Suppose that P is a convex lattice polyhedron in M, i.e., the convex hull in $M \otimes \mathbf{R} \cong \mathbf{R}^d$ of finitely many points in M. For each face F of P we obtain a rational cone $\vartheta_F(P)^\vee$ in N consisting of all $n \in N \otimes \mathbf{R}$ which, when considered as functions on P, achieve their minimum along F. In particular, if F is a k-dimensional face, then $\vartheta_F(P)^\vee$ is a $(d-k)$-dimensional cone. Then for each k,

$$\sum_{\substack{F \subseteq P \\ \dim F = k}} \operatorname{vol}_k(F)\, \mu_{d-k}^{td_{d-k}}(\vartheta_F(P)^\vee)$$

is a constant function on G_{k+1}. Furthermore, if we denote this constant value by $\#_k(P)$ then the number of lattice points in P is

$$\#(P) = \sum_k \#_k(P).$$

For an illustration of this formula see the example in the overview below. (The $\#_k(P)$ are the coefficients of the *Ehrhart* polynomial, so that the number of lattice points in the dilation of P by a factor of n is $\sum_k \#_k(P)\, n^k$.)

3. (Danilov's Question) Let X be any toric variety (possibly singular or noncompact), and Δ its fan. For each k-dimensional cone σ in Δ, $\sigma \in \Delta(k)$, there is a codimension k orbit, whose closure determines a chow class $[V(\sigma)] \in A_{d-k}(X)$. Then the Todd classes of X are expressed as

$$Td_{d-k}(X) = \sum_{\sigma \in \Delta(k)} \mu_k^{td_k}(\sigma)[V(\sigma)],$$

where the sum on the right is evaluated after scalers are extended appropriately.

A second purpose of this paper is to draw attention to a sort of duality between the number of lattice points in lattice polyhedra, and their volumes. I do not yet understand the precise form of this duality, so that this theme is somewhat tentative. It encompasses the following result: if P is a lattice polyhedron in $N\otimes\mathbf{R}$ of dimension d at most 4, then

$$\mathrm{vol}_d(P)=\sum_F \#(F)\,\mu_d^{td_d}(\vartheta_F(P)),$$

where $\vartheta_F(P)$ is the cone subtended by P along F (whose dual cone is $\vartheta_F(P)^\vee$).

This paper is one of a series relating combinatorics and algebraic geometry. I hope that readers of one or the other viewpoint may read selectively the relevant parts without too much distraction. An overview is contained in the following section.

1.1. *Overview*

This work is an outgrowth of efforts to generalize an elementary theorem of plane geometry discovered by Pick. Pick's theorem states that the area of a convex polygon P whose vertices lie in the integer lattice $\mathbf{Z}^2$ is given by

$$\mathrm{Area}(P)=I+\frac{B}{2}-1, \tag{1}$$

where I is the number of lattice points interior to P, and B is the number on the boundary. Of course, this formula fails for higher dimensional polyhedra; for $d>2$, it is easy to construct simplices in $\mathbf{R}^d$ with arbitrarily large volume, but which meet $\mathbf{Z}^d$ in only their $d+1$ vertices.

Elementary Proof. However, another possibility is suggested by the following proof. Let us normalize angle measure $\angle$ so that the whole circle has measure 1. For any v in $\mathbf{Z}^2$, a small circle centered at v will meet P in an arc whose angle measure we denote $\angle_v(P)$. For instance, if v is interior to P, then $\angle_v(P)=1$, and if v is exterior, then $\angle_v(P)=0$. The sum of $\angle_v(P)$ over the v on the boundary of P is $(B/2-1)$, since the sum of the normalized exterior angles of a polygon is 1. Pick's theorem is therefore equivalent to

$$\mathrm{Area}(P)=\sum_{v\in\mathbf{Z}^2}\angle_v(P). \tag{2}$$

Now, let rot P be the rotation of P by 180°, and let $t_v P$ be the translation of P by the vector v. Let us moreover identify a polygon Q with its *indicator* function 1_Q on $\mathbf{R}^2$ defined

$$1_Q(x)=\begin{cases}1, & x\in Q\\ 0 & \text{otherwise.}\end{cases}$$

Under this identification we can add polygons together as functions. For instance, the sum of two polygons Q_1 and Q_2 is the function whose value is 2 on the intersection, 1 on the symmetric difference, and 0 elsewhere. Consider the sum

$$T = \sum_{r \in \mathbf{Z}^2} t_r P + \sum_{r \in \mathbf{Z}^2} t_r \operatorname{rot} P.$$

It is meaningful because all but finitely many of the terms vanish at a given point in the plane.

I claim that, outside the union of all the translates of the edges of P, T is a constant function. The trick here is that each edge of a translation $t_r P$ of P coincides with the corresponding edge of a unique translation $t_{r'}$ rot P of rot P, so that all the polygons fit together into a sort of a tessellation with overlaps. Let us determine the constant value of T in two ways. First, the integral of T over the unit square must be $\operatorname{Area}(P) + \operatorname{Area}(\operatorname{rot} P) = 2 \operatorname{Area}(P)$, since this integral is equal to the integral $\int_{\mathbf{R}^2} (P + \operatorname{rot} P)\, dA$ over the whole plane. On the other hand, the integral of T over a small circle about the origin is $(\sum_{r \in \mathbf{Z}^2} \angle_r(P) + \sum_{r \in \mathbf{Z}^2} \angle_r(\operatorname{rot} P)) = 2 \sum_{r \in \mathbf{Z}^2} \angle_r(P)$. Equating these two constants yields Pick's theorem.

Intrinsic. It is natural to conjecture here that Eq. (2) generalizes to lattice polyhedra of higher dimensions d as

$$\operatorname{vol}_d(P) = \sum_{r \in \mathbf{Z}^d} \angle_r(P),$$

but one checks that this formula fails already in dimension 3 (though it does work in dimension 1). The flaw in this generalization is that the volume of a lattice polyhedron is intrinsically defined, independent of any metric on $\mathbf{R}^d$, so the intrusion of spherical volume, which is not intrinsic, is unnatural. A closer look at the proof of Pick's theorem reveals that $\angle_r$ could be replaced by any measure μ on angles satisfying certain weak formal properties. We might therefore ask whether there exists some measure μ on angles, necessarily different from euclidean angle measure, for which the equation above holds with $\angle$ replaced by μ. Such considerations lead us to make the following definitions, which capture what is intrinsic in this problem.

Definitions

DEFINITION 1. Let $V \cong \mathbf{R}^d$ be a real vector space. The group $L(V)$ of polyhedral functions on V is defined to be the group of integer valued (discontinuous) functions on V generated by the indicator functions 1_Q of all convex polyhedra Q.

The group $L(V)$ has been used (mostly implicitly) by the school of convex geometers who study *valuations* of convex sets and polyhedra. (It also falls within a general construction of valuation groups of lattices defined by Rota.) Naturally, the field $\mathbf{R}$ may be replaced by any ordered field. More relevant for our immediate purposes is the

DEFINITION 2. The group $L(\mathbf{Z}^d)$ is the subgroup of $L(\mathbf{R}^d)$ generated by the indicator functions of lattice polyhedra.

Obviously, $L(\mathbf{Z}^d)$ does not depend on a basis, so we can also write $L(M)$ whenever $M \cong \mathbf{Z}^d$ is a free $\mathbf{Z}$-module, i.e., a lattice.

Recall that a convex polyhedral cone is the convex hull of finitely many rays emanating from the origin. Such a cone is called rational if the rays are generated by vectors with integral coordinates. As with polyhedra, we identify a polyhedral cone with its indicator function.

DEFINITION 3. Define the group of polyhedral germs $\mathcal{S}\mathcal{L}(V)$ as the group of functions generated by the set of the indicators of convex polyhedral cones. Define $\mathcal{S}\mathcal{L}(\mathbf{Z}^d)$ as the subgroup of $\mathcal{S}\mathcal{L}(\mathbf{R}^d)$ generated by the indicator functions of rational cones.

Again, it makes sense to write $\mathcal{S}\mathcal{L}(M)$ for any lattice M.

So far these definitions fail to incorporate a very important ingredient, namely translations. The device we used in the proof, of adding together all translates of a given polyhedron, is equivalent in some respects to equating a polyhedron with any translation of itself.

DEFINITION 4. Define the group $\mathcal{L}(V)$ as the quotient group of $L(V)$ by the relations

$$f = t_v f,$$

where $f \in L$, $v \in \mathbf{R}^d$, and $t_v f$ is the translation of f by v, i.e., $(t_v f)(x) = f(x - v)$. Define the group $\mathcal{L}(M)$ as the quotient group of $L(M)$ by the relations $f = t_v f$, where $f \in L$, $v \in \mathbf{Z}^d$.

Another way of saying this is that $\mathcal{L}(V)$ is the group of coinvariants of L as a V-module, or $\mathcal{L} = H_0(V, L)$. Quotients such as this of groups of polyhedra by group actions are the object of study in the theory of scissors congruence of polyhedra, an outgrowth of Hilbert's third problem.

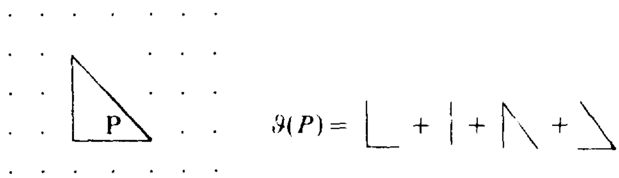
First Reformulation. Let us now recast Pick's theorem and the problem of generalizing it in terms of these definitions. A polyhedron P looks like a polyhedral cone in the neighborhood of any lattice point v. For P convex we may define this cone $\vartheta_v(P) \in \mathcal{S}\mathcal{L}$ as the cone $\mathbf{R}^+ t_{-v} P$ spanned by the translation $t_{-v}P$ of P which takes v to the origin. The expression $\angle_v(P)$ represents the angle measure of this polyhedral cone.

For each v we therefore have a *germ* homomorphism

$$\vartheta_v : L \to \mathscr{G}\mathscr{L}.$$

DEFINITION 5. The sum $\vartheta(f) = \sum_{v \in \mathbf{Z}^d} \vartheta_v(f)$ is finite, and invariant under translations, so descends to a *lattice germ* homomorphism

$$\vartheta : \mathscr{L} \to \mathscr{G}\mathscr{L}.$$



The problem can now be stated as follows:

Does there exist a factorization

$$\begin{array}{ccc} \mathscr{L} & \xrightarrow{\vartheta} & \mathscr{G}\mathscr{L} \\ & {}_{\mathrm{vol}_d}\searrow & \downarrow \mu \\ & & \mathbf{Q} \end{array} \tag{3}$$

of the volume map $\overline{\mathrm{vol}}_d$?

(Volume is normalized so that the unit lattice cube has unit volume.) Pick's theorem states that we take μ to be the euclidean angle measure for $d = 2$. A second reformulation is stated below.

Scissors Congruence. Let us now turn to the group $\mathscr{L}$, whose structure is clearly relevant. As we stated above, similar groups are considered in the theory of scissors congruence. Let us say a few words about this area.

Hilbert's third problem asks, in spirit, whether a theory of volume for polyhedra can be based upon dissections, without limiting processes. Volume is supposed to have three properties: (i) if P lies in a hyperplane, then its volume is 0; (ii) volume is invariant under rigid motions; and (iii) if P is divided into two pieces with disjoint interiors $P = Q \cup R$, $\mathrm{int}(Q \cap R) = \emptyset$, then $\mathrm{vol}(P) = \mathrm{vol}(Q) + \mathrm{vol}(R)$. A dissection theory of volume would therefore equate two polyhedra if one of them could be subdivided into pieces, and the pieces reassembled, using rigid motions, into the other. Two such polyhedra are called scissors congruent. Hilbert's third problem, as it was originally posed, asks specifically whether two polyhedra with the same volume are scissors congruent. In the nineteenth century, the answer had been shown to be yes for the plane. In higher dimensions it is no, as was shown by Dehn; scissors congruence preserves many beautiful invariants more subtle than volume. The problem has therefore been transformed and is nowadays usually taken as the problem of finding a *complete* set of invariants. For a concise account of Hilbert's third problem, see [Ca].

We are concerned here with a more manageable variation, posed by Hadwiger, which stipulates that in the reassembling, we may translate, but not rotate, the pieces. Two polyhedra are called translation scissors congruent if one of them can be subdivided into pieces, and the pieces reassembled, using only translations, into the other. It is convenient to incorporate the idea of scissors congruence into a Grothendieck group construction. We take the free abelian group generated by expressions $[P]$ for P a convex polyhedron, and mod out by the three types of relations: (i) $[P]=0$ if P lies in a hyperplane; (ii) $[P]=[t_v P]$, where $t_v P$ is the translation of P by v; and (iii) if P is divided into two pieces with disjoint interiors $P=Q\cup R$, $\text{int}(Q\cap R)=\emptyset$, then $[P]=[Q]+[R]$. Translation congruence invariants identify themselves with homomorphisms from this group into the domain of the invariant.

In this case a complete set of invariants can be given, and all of the relations that hold between these invariants (the "syzygys") are known. The invariants were discovered by Hadwiger. Their sufficiency in all dimensions was proven by Jessen and Thorup, and independently by Sah. Dupont described the relations that hold among the Hadwiger invariants.

Lattice Scissors Congruence. These results, however, were carried out over an ordered field, and what we would like to understand is the case of lattice polyhedra, which are in some sense defined over the ring Z. For this reason, a theory of scissors congruence for modules over (non-ordered) rings, which when specialized to $\mathbf{Z}^d$ gives lattice polyhedra, was worked out in [Mo1]. This was accomplished by using the observation, which is contained in [Dup], that the group of polyhedra is generated by simplices, and the subdivision relations (iii) are generated by the boundaries of degenerate $(d+1)$-simplices. The point is that simplices and their boundaries are purely algebraic constructs, not depending on any notion of convexity. This allows us to define a chain complex $\mathscr{P}_*(M)$ for any module M over any commutative domain A, and to use the methods of homological algebra to study it. The elements of $\mathscr{P}_k(M)$ are thought of as k-dimensional polyhedra in M up to translation congruence. In fact, for any ordered field K, the group $\mathscr{P}_d(K^d)$ is isomorphic to the Grothendieck group defined above. (The isomorphism is non-canonical because the elements of $\mathscr{P}_d(K^d)$ are oriented.)

The upshot of this is that both the sufficiency of the Hadwiger invariants and all the relations among them are determined by the homology of the complex $\mathscr{P}_*(M)$ which is computed in [Mo1]:

THEOREM 1. *Suppose M is free, and* char $A = 0$. *Then the homology* $H_k(\mathscr{P}_*(M)) \cong \bigwedge_A^k M$ *for* $k > 0$.

Solvability of First Reformulation. Let us now return to Pick's theorem. Specializing to the case $M \cong \mathbf{Z}^d$, the groups $\mathscr{P}_*(M)$ reflect lattice scissors congruence and are related in a precise manner to $\mathscr{L}(M)$. From the knowledge of the homology of $\mathscr{P}_*(\mathbf{Z}^d)$ it is easy to deduce that ϑ is an injection. This means that

THEOREM 2. *A lattice polyhedron P is determined up to subdivision and translation by its lattice germ $\vartheta(P)$, that is,*

$$\vartheta\colon \mathscr{L}(M) \subsetneq \mathscr{S}\mathscr{L}(M).$$

It follows that anything preserved by subdivision and translation, e.g., volume, is determined by the lattice germ:

COROLLARY 1. *A factorization* 3 *exists.*

There remains the problem of actually constructing an explicit such homomorphism μ. We will return to this problem later.

Facial Structure. Let us express the first reformulation in terms of the facial structure of P. If F is a *relatively open* face of P, denote by $\vartheta_F(P)$ the common value of $\vartheta_v(P)$ as v ranges over all points of F. Denote by $\#(Q)$ the number of lattice points contained in the lattice polyhedron Q. Since each lattice point v in P is contained in exactly one relatively open face of P,

$$\vartheta(P) = \sum_{F \subseteq P} \#(F)\, \vartheta_F(P). \tag{4}$$

Our first reformulation of Pick's theorem is therefore equivalent to

$$\mathrm{vol}_d(P) = \sum_{F \subseteq P} \#(F)\, \mu(\vartheta_F(P)).$$

It is natural to widen the question to include polyhedra of dimension k not necessarily equal to d. We can then put

$$\mathrm{vol}_k(P) = \sum_{F \subseteq P} \#(F)\, \mu_k(\vartheta_F(P)). \tag{5}$$

In this equation, P is a k-dimensional polyhedron, and μ_k is a function defined on the subgroup of $\mathcal{S}\mathcal{L}(M)$ generated by k-dimensional cones. If we denote this subgroup by $F_k^{\dim}\mathcal{S}\mathcal{L}(M)$ we can write

$$\mu_k : F_k^{\dim}\mathcal{S}\mathcal{L}(M) \to \mathbf{Q}.$$

Despite appearances, it is possible to replace P in formula (5) by an arbitrary element of the subgroup of $L(M)$ generated by k-dimensional polyhedra. For this purpose one interprets the right-hand side in terms of natural Hopf algebraic structures which are explained in [Mo2]. The existence of such μ_k reduces immediately to the $k = d$ case.

Second Reformulation. There is a second and dual reformulation of Pick's formula (1) which depends on the notion of polar dual. The polar dual of a convex polyhedral cone $C = \{x : f_1(x), ..., f_m(x) \geqslant 0\}$ in V is the cone in the dual space $V^\vee$ of V whose edges are $f_1, ..., f_m$. If σ is a rational cone in the lattice $M \cong \mathbf{Z}^d$, then $\sigma^\vee$ is a rational cone in the dual lattice $M^\vee$.

The second reformulation is

$$\#(P) = \sum_k \sum_{\substack{F \subseteq P \\ \dim F = k}} \operatorname{vol}_k(F)\, \mu_{d-k}(\vartheta_F(P)^\vee). \tag{6}$$

The subscript $d-k$ in μ_{d-k} indicates that since F has dimension k, $\vartheta_F(P)^\vee$ has dimension $d-k$. Let us write $N = M^\vee$ for the dual lattice. The function μ_k here is defined on cones in the dual space:

$$\mu_k : F_k^{\dim}\mathcal{S}\mathcal{L}(M^\vee) \to \mathbf{Q}.$$

In the two dimensional case, $\#(P) = I + B$. On the other hand, if we take euclidean angle measure for μ_k then as one easily checks that the right side of formula (6) with euclidean angle measure for μ_k is

$$\operatorname{vol}(P) + \sum_{\text{edges } e} \operatorname{vol}_1(e)\frac{1}{2} + 1 = \operatorname{vol}(P) + \frac{B}{2} + 1.$$

Thus euclidean angle measure works up to dimension 2 for this second reformulation, but again it fails in higher dimensions.

The existence of $\mu_0, ..., \mu_d$ which satisfy Eq. (6) can be established in general in a way similar to that in which the existence of a μ_k for the first formulation was established.

Relation between Formulations. I do not know in general whether the *same* μ_k can simultaneously satisfy both Eq. (5) and Eq. (6). In other words, do there exist $\mu_k : F_k^{\dim}\mathcal{S}\mathcal{L}(N) \to \mathbf{Q}$ which satisfy formula (5) *for* *lattice polyhedra P in N* and which satisfy formula (6) for P in M? I know the answer is affirmative at least for $d<5$. At any rate, the equations are not formally equivalent, and a little experimentation reveals that they make rather different demands of the μ_k. An interesting point is that if μ_k were ordinary angle measure, then two such adjoint equations would be equivalent by a theorem of McMullen. While this is suggestive, ordinary angle measure (with respect to any inner product) does not work here.

Toric Varieties. We now shift gears a bit and describe a quite different area of mathematics which is relevant.

There is a beautiful construction which associates to any convex polyhedron with integral vertices a complex algebraic variety, together with an ample line bundle, (and vice versa). The variety and line bundle are unchanged by a translation of the polyhedron, so this already suggests a connection with our problem.

The variety so constructed falls within a very special class known as toric varieties [KKMS-D]. They are intrinsically characterized as normal algebraic varieties on which an algebraic torus $T^n = (\mathbf{C}^*)^n$ acts, and into which the torus embeds equivariantly as a dense open subset. In effect, they are "equivariant partial compactifications of tori." Henceforth we denote the group of characters of T^n by $M \simeq \mathbf{Z}^d$ and its dual $M^\vee = \mathrm{Hom}_\mathbf{Z}(M, \mathbf{Z})$ by N.

A complete classification of toric varieties of a given dimension can be given in terms of the combinatorial notion of a *fan*. A fan is a collection $\Delta = \{\sigma\}$ of convex cones σ defined over $\mathbf{Z}$ (recall that such cones are the generators of $\mathscr{S}\mathscr{L}(\mathbf{Z}^d)$). We require of the fan that (i) if $\sigma \in \Delta$, then any face of σ is also in Δ, and (ii) if σ and τ are in Δ, then their intersection $\sigma \cap \tau$ is a face of each of them. The classification theorem is that there is a canonical bijection between toric varieties and fans. (The fan sits in the lattice $N \cong \mathbf{Z}^d$ of one parameter subgroups of the torus.) We denote this bijection $\Delta \mapsto X_\Delta$. We will not explain it here, except to say that the cones in the fan correspond to the orbits of the torus action, a k-dimensional cone to a codimension k orbit. This foundational work was carried out in the 1970s.

The construction of a toric variety from a polyhedron proceeds by constructing from P in $N = M^\vee$ the fan $\{\vartheta_F(P)^\vee : F \text{ a face of } P\}$ in M. Conversely, let X be a toric variety, and L an ample line bundle. We may trivialize L over the torus, thereby identifying functions with sections there. In particular, we can ask which of the characters on the torus extend to global sections of L. These characters are points in M, and in fact, they are exactly the set of lattice points interior to a certain convex polyhedron P_L.

This correspondence has been exploited by algebraic geometers as well as combinatorialists since the discovery of toric varieties, and the rule seems to be that a natural theorem in algebraic geometry gives rise to a natural theorem in combinatorics. For instance, the Alexandrov–Fenchel inequalities result from the Hodge index theorem, and important facts about the number of faces of a convex polyhedron result from the hard Lefschetz theorem.

Relevance of Toric Varieties. The relevance of toric varieties for us begins with some observations contained in [Dan] and [Kh]. For an ample line bundle L on a toric variety X, it is known that $H^i(X, L) = 0$ for $i > 0$. Danilov noticed that the number $\#(P_L) = \dim H^0(X, L)$ is therefore equal to the Euler–Poincaré characteristic $\chi(L)$, so that methods like the Riemann–Roch theorem are applicable. In particular, one derives Ehrhart's reciprocity theorem $\#(P; x) = (-1)^d \#(\text{interior}(P); -x)$ from Serre duality.

Let us assume from here on that our toric varieties are compact and smooth. This is the case if X arises from a polyhedron whose edges incident to each vertex are parallel to a subset of the coordinate axes with respect to some basis of M. We mentioned above that to each cone in the fan there corresponds an orbit $V(\sigma)$. Its closure defines a cycle in homology, and by Poincaré duality, a cohomology class $[V(\sigma)]$. It turns out that these classes $[V(\sigma)]$ generate the cohomology of X_Δ, but there are relations among them. The structure of the cohomology ring was determined by Jurkiewicz and Danilov by finding all these relations.

Danilov's Question. Since the $[V(\sigma)]$ for $\dim \sigma = k$ generate the cohomology in degree k, for any particular toric variety X_Δ, the kth Todd class can be written as a sum

$$Td^k(X_\Delta) = \sum_{\sigma \in \Delta(k)} r_\sigma [V(\sigma)]$$

for some rational numbers r_σ. Danilov poses the question of whether the r_σ can be chosen once and for all, independently of any particular Δ. He also notes that the Hirzebruch–Riemann–Roch theorem implies

$$\#(P_L) = \sum_{F_\sigma} \overline{\text{vol}}(F_\sigma) \cdot r_\sigma, \tag{7}$$

where F_σ is the face of P_L for which $\vartheta_{F_\sigma}(P_L)^\vee = \sigma$. Finally, if we extend coefficients to $\mathbf{R}$ then, in dimension 2, the r_σ may be taken to be the ordinary euclidean angle measure of σ.

Relation to Pick's Theorem. It is clear that there is a close connection between Danilov's question and the second formulation of Pick's theorem, formula (6). Let us modify it slightly to accentuate this point. We can pose the question of the r_σ as follows: does there exist a *homomorphism* $\mu_k : F_k^{\dim}\mathscr{S}\mathscr{L}(\mathbf{Z}^d) \to \mathbf{Q}$ for which

$$Td^k(X_\Delta) = \sum_{\sigma \in \Delta(k)} \mu_k(\sigma)[V(\sigma)] \tag{8}$$

for every fan Δ? It would follow that

$$\#(P) = \sum_k \sum_{\substack{F \subseteq P \\ \dim F = k}} \overline{\mathrm{vol}}_k(F)\, \mu_{d-k}(\sigma^\vee) \tag{9}$$

for any lattice polyhedron P of the special form associated with smooth toric varieties.

A natural question now is whether we can go backward from the existence of μ_k satisfying formula (6) for polyhedra to conclude the existence of μ_k satisfying Eq. (8) for toric varieties. Theorem 8 states that the two problems are *equivalent*. The proof of this depends on extending the correspondence between ample line bundles and polyhedra, to the entire K group of algebraic vector bundles. (We should point out that there exist nonprojective toric varieties, which correspond to no polyhedron, so it is hopeless to try to go backward if we do not make some extension.)

Equivariant K Theory. Recall that if a group G acts on a space X, then an equivariant vector bundle on X is a vector bundle together with an action of G on the total space, for which the projection to the base is equivariant, and the fibers are mapped to each other linearly. The isomorphism classes of equivariant vector bundles fit together into a Grothendieck group $K_G(X)$.

Now the torus T acts naturally on a toric variety X, so we have the group $K_T(X)$. If E is an equivariant vector bundle on a toric variety X, then there is a natural action of the torus on the cohomology groups $H^i(X, E)$, and these groups therefore split into sums of weight spaces: $H^i(X, E) = \bigoplus_{m \in M} H^i(X, E)_m$. Using these components we can define the weight m Euler characteristic of E as $\chi_m(E) = \sum_i (-1)^i \dim H^i(X, E)_m$. It is easy to see that χ_m passes to $K_T(X)$. Recall further that the operation of exterior power makes sense in equivariant K theory, inducing there a structure called a special λ-ring structure. In terms of the λ-ring structure certain natural ring homomorphisms $\Psi^0, \Psi^1, \ldots$ called Adams operations are defined.

DEFINITION 6. *We define a homomorphism* $x \mapsto \mathfrak{f}_T(x)$ *from* $K_T(X)$ *to* $L(M)$ *for any compact (smooth) toric variety* X *by the formula*

$$f_E(m/k) = \chi_m(\Psi^k(E)),$$

where $m \in M$, *and* k *is a positive integer.*

Now, $\text{Ɨ}_T(x)$ is well defined, and is in fact a polyhedral function, so we obtain a homomorphism

$$\text{Ɨ}_T\colon K_T(X) \to L(M).$$

Furthermore, Ɨ_T is an injection whose image can be geometrically characterized. These facts are proven in [Mo3]. By virtue of the natural map $K_T(X) \to K(X)$, which is a surjection, and the natural map $\text{Ɨ}\colon L(M) \to \mathscr{L}(M)$, we also obtain an injection from the ordinary K group $K(X)$ into $\mathscr{L}(M)$.

Volumes of Faces. When a toric variety X with ample line bundle L arises from a polyhedron P, the cones in the fan of X correspond to the faces of P. When L varies, the resulting polyhedra vary in *habit*, i.e., have parallel faces but with varying volumes. Now, the volumes of the faces do not vary arbitrarily but are constrained by certain linear relations. When we consider, more generally, all of the images of Ɨ_T we can no longer speak of faces, but it is still possible to define functions $\mathrm{Had}_F\colon L(M) \to \mathbf{Q}$ which play the same role as the volumes of faces. These functions are the Hadwiger invariants mentioned above as the separating invariants on $\mathscr{L}(M)$. Such a Hadwiger invariant is in fact an extension by linearity of the function, defined on the set of polyhedra, which assigns to a given polyhedron the volume of the face with a given orientation. Again, the different Hadwiger invariants are dependent on each other by certain linear relations. In fact, the relevant Hadwiger invariants for a given X are in bijection with the cones σ of the fan and

> *the relations holding among the Hadwiger invariants are exactly the same relations that hold among the cohomology classes* $[V(\sigma)]$.

This is a consequence of the close connection between the Hadwiger invariants of $\text{Ɨ}(x)$ and the chern character of x (see [Mo3] for details), together with the injectivity of Ɨ. One may make a comparison here. The problem of finding the linear relations holding among the volumes of the faces of a polyhedron with prescribed fan is solved by considering the cohomology of the toric variety. This is analogous to the problem of finding the linear relations among the numbers of faces of different dimensions, whose solution comes from the betti numbers of toric varieties.

Solvability of Danilov's Question. Let us return to the question of going backward from the combinatorial formula (6) to the formula (8) for the Todd class. Without going into the details, which are explained in Section 3.1, let us simply summarize:

1. The Euler–Poincaré characteristic of x is $\chi(x) = \#(\mathfrak{k}_T(x))$.

2. The Hadwiger invariants of $\mathfrak{k}_T(x)$ are connected with the chern character of x.

3. The relations on the Hadwiger invariants are the same as those on the cohomology classes $[V(\sigma)]$.

Because of these compatibilities, the Riemann–Roch situation on toric varieties, which characterizes the Todd classes, duplicates itself in the combinatorics of polyhedra. It results that functions μ_k satisfy Danilov's question if and only if they satisfy the second reformulation of Pick's formula. We have therefore shown the existence of a local formula for the Todd class:

THEOREM 3. *For each k there exists a homomorphism μ_k from the subgroup of $\mathscr{S}\mathscr{L}(M)$ generated by k-dimensional cones to $\mathbf{Q}$ satisfying*

$$Td^k(X_\Delta) = \sum_{\sigma \in \Delta(k)} \mu_k(\sigma)[V(\sigma)] \tag{10}$$

for every toric variety X_Δ.

As in the other cases, we would like to give μ_k explicit form. This is actually possible, by residue formulas of Baum and Bott, if we are willing to extend the coefficient field.

Bott Residue Theorem. The Bott residue theorem gives a formula for the characteristic numbers of a variety in terms of the local behavior of any holomorphic vector field near its zeros (assumed isolated). Now the one parameter subgroups n of the torus (elements of $N = M^\vee$) determine vector fields on any toric variety X through the action of the torus. The zeros of these vector fields occur exactly at the 0 dimensional orbits, which correspond to the cones of dimension d in the fan. We may therefore apply Bott's theorem to a toric variety X with vector field corresponding to $n \in N$ to determine the Todd number of X. We find the following:

1. The contribution from the fixed point corresponding to a cone σ depends only on σ and $n \in N$, not on the fan.

2. This contribution is linear in σ, giving rise to a homomorphism $\mathscr{S}\mathscr{L}(M) \to \mathbf{Q}$.

3. For fixed σ the contribution is a rational function of n invariant under scaler multiplication; i.e., it is a quotient of homogeneous polynomial functions of the same degree, defined on the vector space $N \otimes \mathbf{R}$.

Another way of putting this is that the local contribution from σ is a rational function on the projective space $\mathbf{P}(N \otimes \mathbf{R})$, so we obtain a homomorphism from $\mathscr{S}\mathscr{L}(M)$ to the space of rational functions on $\mathbf{P}(N \otimes \mathbf{R})$.

Now, this is not yet sufficient to solve Danilov's problem because it computes only the top Todd class Td^d (which, incidentally, is always 1 for a toric variety) while Danilov's question refers to the entire Todd class. The necessary generalization is provided by a theorem of Bott and Baum which determines characteristic classes in terms of the singularities of foliations. Now a $(d-k+1)$-dimensional subspace of $N \otimes \mathbf{R}$ determines a rank $d-k+1$ foliation on the toric variety, to which we can apply the residue theorem. Just as above, we obtain a local contribution r_σ for each cone σ of dimension k, for which the theorem states that

$$Td^k(X_\Delta) = \sum_{\sigma \in \Delta(k)} r_\sigma[V(\sigma)].$$

As before, this contribution r_σ is independent of Δ and is linear in the σ. Moreover, the $(d-k+1)$-dimensional subspaces of $N \otimes \mathbf{R}$ are parametrized by the grassmannian variety $G_{d-k+1}(N \otimes \mathbf{R})$, and the local contribution for a fixed cone σ of dimension k is a rational function on the grassmannian. Therefore, these constructions yield a homomorphism

$$\mu_k^{td_k}\colon F_k^{\dim} \mathscr{S}\mathscr{L}(N) \to R(G_{d-k+1}(N \otimes \mathbf{R})).$$

If a point of $G_{d-k+1}(N \otimes \mathbf{R})$ is chosen, then by evaluation we obtain a real valued μ_k, and these μ_k satisfy Danilov's equation, and hence the second formulation of Pick's theorem. It is more natural however to consider $\mu_k^{td_k}$ as an $R(G_{d-k+1}(N \otimes \mathbf{R}))$-valued solution. Of course in doing so we must appropriately extend scalers, passing for example to $H^*(X_\Delta) \otimes R(G_{d-k+1}(N \otimes \mathbf{R}))$.

Return to Combinatorics. At this point we can forget the geometry of toric varieties. The *Todd measures* $\mu_k^{td_k}$ constructed from Bott's formula are given explicitly in terms of the cone. We therefore obtain a purely combinatorial formula:

THEOREM 4. *For any lattice polyhedron P,*

$$\#(P) = \sum_k \sum_{\substack{F \subseteq P \\ \dim F = k}} \overline{\mathrm{vol}}_k(F)\, \mu_{d-k}^{td_{d-k}}(\vartheta_F(P)^\vee),$$

by which we mean that the right-hand side is a constant function whose value is $\#(P)$.

This formula has the advantage of naturality; it is functorial with respect to the lattice. While calculations with the $\mu_k^{td_k}$ can be cumbersome (they involve the symmetric functions), they are entirely constructive, as the fields involved are countable.

Recipe. Let us give the recipe for $\mu_d^{td_d}$ in the case of a d-dimensional cone σ in a d-dimensional lattice N. Assume first that σ has d edges, and that the primitive vectors in N on these edges constitute a basis for N. Such a cone is called a nonsingular simplicial cone. It follows that σ may be expressed as $\sigma = \{x : f_1(x), ..., f_d(x) \geqslant 0\}$ for linear functionals $f_i \in M = N^\vee$. This expression is unique if the f_i are taken to be primitive. In this case,

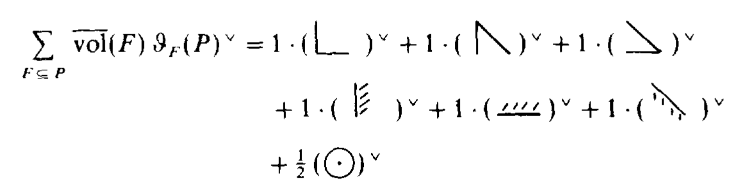
$$\mu_d^{td_d}(\sigma) = \frac{td_d(f_1, ..., f_d)}{f_1 \cdots f_d},$$

which we take to be an element of the fraction field of $\mathrm{sym}^*_{\mathbf{Q}} M$. Recall that td_d is the symmetric polynomial defined as the coefficient of t^d in the power series

$$td_t(x_1, ..., x_d) = \prod_{i=1}^{d} \frac{tx_i}{1 - e^{-tx_i}}.$$

Now we claim that any d-dimensional cone can be realized as a union $\bigcup_j \sigma_j$ of nonsingular cones with disjoint interiors, and that the expression $\sum_j \mu_d^{td_d}(\sigma_j)$ is independent of choices.

EXAMPLE. Let $P =$ be the unit triangle with vertices (0, 0), (0, 1), and (1, 0) in $M = \mathbf{Z}^2$. Now $td_2 = \frac{1}{12}(c_1^2 + c_2)$ in terms of the elementary symmetric functions c_i. Let us use the variables x and y to denote the standard basis of M. Then,



$$= \{x, y \geqslant 0\} + \{-y, (x-y) \geqslant 0\}$$
$$+ \{-x, (y-x) \geqslant 0\} + \cdots$$

so,

$$\sum_{F \subseteq P} \overline{\mathrm{vol}}(F)\, \mu_k^{td_k}(\vartheta_F(P)^\vee) = \frac{(x+y)^2+xy}{12xy} + \frac{(-y+(x-y))^2-y(x-y)}{-12y(x-y)}$$

$$+\frac{(-x+(y-x))^2-x(y-x)}{-12x(y-x)}$$

$$+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}$$

$$+\frac{1}{2}$$

$$=\frac{12xy(y-x)}{12xy(y-x)}$$

$$+2$$

$$=3$$

$$=\#(P).$$

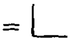
First Formulation Revisited. The inevitable question now is whether these natural functions $\mu_d^{td_d}$ work in Eq. (5). As we remarked above, this equation is not formally equivalent to formula (6), and nothing we have said so far implies that $\mu_d^{td_d}$ should work in it. The answer is that $\mu_d^{td_d}$ works for $d = 1, 2, 3$, and 4 but fails for $d = 5, 6, \ldots$.

THEOREM 5. *For any lattice polyhedron of dimension d at most four,*

$$\mathrm{vol}_d(P) = \mu_d^{td_d}(\vartheta(P)).$$

This is proven by direct calculation. Unfortunately, the same calculation that proves it for $d < 5$ also disproves it for $d \geqslant 5$. It is mysterious to me why $\mu_d^{td_d}$ works for $d < 5$, and just as mysterious why it fails at $d = 5$.

EXAMPLE. Let P be the unit triangle in N. Then, (using x and y again for the basis elements of $M = N^\vee$) we have

$\vartheta(P) =$  $+$  $+$ 

$$= \{x, y \geqslant 0\} + \{x, -(x+y) \geqslant 0\} + \{y, -(x+y) \geqslant 0\}$$

so,

$$\mu_2^{td_2}(\vartheta(P)) = \frac{(x+y)^2+xy}{12xy} + \frac{(x-(x+y))^2 - x(x+y)}{-12x(x+y)} + \frac{(y-(x+y))^2}{-12y(x+y)}$$

$$= \frac{6xy(x+y)}{12xy(x+y)}$$

$$= \frac{1}{2}$$

$$= \mathrm{vol}(P).$$

2. Lattice Polyhedra

2.1. *Definitions*

Let M denote a free $\mathbf{Z}$-module of rank d, i.e., $M \cong \mathbf{Z}^d$ is a lattice. We refer to elements of M as lattice points, and to polyhedra in $M_{\mathbf{R}} = M \otimes \mathbf{R}$, all of whose vertices lie in M as lattice polyhedra.

Definition 7. Define an abelian group $L(M)$ by taking as generators the symbols $[P]$ for all convex lattice polyhedra P, and relations

$$[P \cup Q] = [P] + [Q] - [P \cap Q] \tag{11}$$

whenever $P, Q, P \cup Q$, and $P \cap Q$ are all convex lattice polyhedra.

An alternate description of this group will also be used. Define the indicator function of a polyhedron P to be

$$1_P(x) = \begin{cases} 1, & x \in P \\ 0 & \text{otherwise.} \end{cases}$$

The indicator functions of all lattice polyhedra generate a subgroup of the group of all (discontinuous) $\mathbf{Z}$-valued functions on $M_{\mathbf{R}}$. It is proven in [Mo2] that the map sending $[P]$ to 1_P is an isomorphism from $L(M)$ onto this subgroup. For this reason, we refer to elements of $L(M)$ as polyhedral functions, and treat them, when convenient, as functions.

Definition 8. Define the group $\mathscr{L}(M)$ to be the quotient of $L(M)$ by the relations

$$[t_m P] = [P],$$

where m is any element of M, P is any lattice polyhedron, and $t_m P$ denotes the translation of P by m.

Note that we have a filtration by dimension in $\mathscr{L}$. To be explicit, let $F_k^{\dim}\mathscr{L}(M)$ be the subgroup of elements of $\mathscr{L}(M)$ generated by polyhedra of dimension at most k, and set $\mathrm{gr}_k^{\dim}\mathscr{L}(M) = F_k^{\dim}\mathscr{L}(M)/F_{k-1}^{\dim}\mathscr{L}(M)$.

By definition the group $L(M)$ is generated by lattice polyhedra. Note that there is a well defined *absolute* notion of volume in a lattice, normalized so that a fundamental parallelepiped in any k-plane has unit volume. We denote it

$$\overline{\mathrm{vol}}_k : \mathrm{gr}_k^{\dim} L(M) \to \mathbf{Q}.$$

The lattice point enumerator is the homomorphism $\#$ on $L(M)$ which returns the number of lattice points touching a given polyhedron (interior points together with points on the boundary). The map $\#$ may also be considered as a homomorphism from $\mathscr{L}$ onto $\mathbf{Z}$.

It was first proven by Ehrhart that for a polyhedron P, the integer valued function $\#(P; n) := \#(\Psi^n P)$ where $\Psi^n P$ is the dilation of P by a factor of n, is a polynomial in the positive integer variable n, whose leading coefficient is the volume of P. Let us write this polynomial as

$$\#(P, n) = \sum_k \#_k(P)\, n^k.$$

We have $\#_0(P) = 1$ and $\#_d(P) = \overline{\mathrm{vol}}_d(P)$ for any convex lattice polyhedron P.

By a rational convex cone in M we will mean the convex hull in the vector space $M \otimes \mathbf{Q}$ of finitely many rays. There is an evident notion of subdivision of a rational cone into a collection of rational cones, and hence there are scissors congruence groups of cones, analogous to the scissors congruence groups of polyhedra.

Definition 9. The group of polyhedral germs $\mathscr{S}\mathscr{L}(M)$ is the abelian group with generators $[\sigma]$, σ a rational convex cone in M, and relations

$$[\sigma \cup \tau] = [\sigma] + [\tau] - [\sigma \cap \tau]$$

whenever $\sigma \cup \tau$ is a rational convex cone.

There is a filtration by dimension on $\mathscr{S}\mathscr{L}(M)$ whose terms we denote $F_k^{\dim}\mathscr{S}\mathscr{L}(M)$ and whose associated graded groups we denote $\mathrm{gr}_k^{\dim}\mathscr{L}(M)$.

Given a lattice polyhedron P, and a point $p \in M \otimes \mathbf{Q}$, we obtain a rational cone

$$\vartheta_p(P) = \{v \in M \otimes \mathbf{Q} : p + \varepsilon v \in P \text{ for all } \varepsilon > 0 \text{ sufficiently small}\},$$

the cone subtended by P at p. It is clear that ϑ_p passes to a homomorphism from $L(M)$ to $\mathscr{S}\mathscr{L}(M)$. (We can consider $\vartheta_p(f)$ to be literally the germ of f if we consider f as a discontinuous function; see [Mo2].)

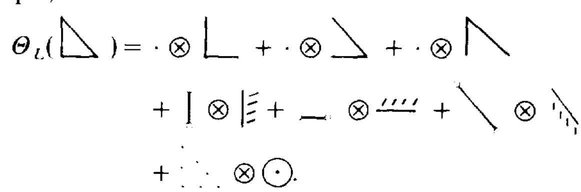
If F is an open face of P then $\vartheta_v(P)$ is the same cone for any $v \in F$. Denote this common value by $\vartheta_F(P)$.

DEFINITION 10. Define the total germ map $\Theta_L \colon L \to L \otimes \mathscr{S}\mathscr{L}$ by setting

$$\Theta_L([P]) = \sum_{F \subseteq P} F \otimes \vartheta_F(P)$$

for a convex polyhedron P, and extending linearly. The summation extends over all the open faces of P (which have been identified with elements of L).

For example,



We call Θ_L the *total germ* homomorphism. The algebra of this map is studied in [Mo2].

2.2. *Local Formulas for Volume*

We refer the reader to the first page of the overview for some of the motivation for what follows.

DEFINITION 11. Define the *lattice germ* homomorphism $\vartheta \colon L \to \mathscr{L}(M)$ as

$$\vartheta(f) = \sum_{p \in M} \vartheta_p(f).$$

This makes sense because for a given $f \in L(M)$, $\vartheta_p(f)$ will vanish for all but finitely many p. In terms of the total germ map, we have

$$\vartheta(f) = (\# \otimes \mathrm{id}) \circ \Theta(f).$$

The homomorphism ϑ is translation invariant and so it descends to a map from $\mathscr{L}(M)$ to $\mathscr{S}\mathscr{L}(M)$. In fact, it is a complete invariant for $\mathscr{L}$ as is stated by the next theorem, which we quote from [Mo1].

THEOREM 6. *The lattice germ homomorphism on $\mathscr{L}(M)$ is an injection*

$$\mathscr{L}(M) \xrightarrow{\vartheta} \mathscr{S}\mathscr{L}(M).$$

Furthermore, the dimension filtration on $\mathscr{L}(M)$ is simply that induced from $\mathscr{S}\mathscr{L}(M)$:

$$\vartheta^{-1} F_k^{\dim} \mathscr{S}\mathscr{L}(M) = F_k^{\dim} \mathscr{L}(M).$$

From this we can deduce our main objective, that volume admits a "local" expression.

COROLLARY 2. *The volume map factors through ϑ. That is, there exists a homomorphism μ from $\mathscr{S}\mathscr{L}(M)$ into $\mathbf{Q}$ such that*

$$\mu(\vartheta(f)) = \overline{\mathrm{vol}}_d(f)$$

for any element f of $\mathscr{L}$. In fact, for each $k \geqslant 0$ there exists a homomorphism $\mu_k \colon \mathrm{gr}_k^{\dim} \mathscr{S}\mathscr{L}(M) \to \mathbf{Q}$ with the property that

$$\mu_k(\vartheta(f)) = \overline{\mathrm{vol}}_k(f), \tag{12}$$

all f in $\mathrm{gr}_k^{\dim} \mathscr{L}(M)$.

Proof. The theorem identifies $\mathscr{L}$ with a subgroup of $\mathscr{S}\mathscr{L}$. Using Zorn's lemma we may extend the domain of the volume map to all of $\mathscr{S}\mathscr{L}$. (In other words, $\mathbf{Q}$ is an injective $\mathbf{Z}$-module.) Moreover, by the same identification the map $\overline{\mathrm{vol}}_k$ is defined on a subgroup of $F_k^{\dim} \mathscr{S}\mathscr{L}(M)$, and since $\vartheta^{-1} F_{k-1}^{\dim} \mathscr{S}\mathscr{L}(M) = F_{k-1}^{\dim} \mathscr{L}(M)$, we know that $\overline{\mathrm{vol}}_k$ must vanish on elements belonging to $F_{k-1}^{\dim} \mathscr{S}\mathscr{L}(M)$. By Zorn's lemma we may extend $\overline{\mathrm{vol}}_k$ to a homomorphism on all of $F_k^{\dim} \mathscr{S}\mathscr{L}(M)$ which vanishes on $F_{k-1}^{\dim} \mathscr{S}\mathscr{L}(M)$. This then defines the homomorphism μ_k.

Remark. The map ϑ extends to polyhedra with rational vertices as follows. It can be shown that if P is a lattice polyhedron, and if $\Psi^n P$ denotes the dilation of P by the factor n, then $\vartheta(\Psi^n P)$ is a polynomial in the positive integer variable n with coefficients in $\mathscr{S}\mathscr{L}(M) \otimes \mathbf{Q}$. Denote this polynomial $\vartheta(P; n)$. If P is a polyhedron with rational vertices, then for some integer n_0, $\Psi^{n_0} P$ is a lattice polyhedron. Set $\vartheta(P) = \vartheta(\Psi^{n_0} P; 1/n_0) \in \mathscr{S}\mathscr{L}(M) \otimes \mathbf{Q}$. Then for any μ_k as above,

$$(\mu_k \otimes \mathbf{Q})(\vartheta(P)) = \overline{\mathrm{vol}}_k(P).$$

Evidently, there are many maps μ which will factor volume. We do not know what additional conditions to impose in order to find a distinguished μ. One possibility is to work in a category of lattices with additional structure, such as the category of metric lattices. We can reject two natural candidates, however.

PROPOSITION 1. *Set* $M = \mathbf{Z}^d$.

(a) *For* $d > 1$, *there exists no* $SL(d, \mathbf{Z})$ *invariant* μ *factoring volume.*

(b) *Put the standard inner product on* $\mathbf{Z}^d$. *Then* $SO(d, \mathbf{Q})$ *acts on* $\mathcal{S}\mathcal{L}(\mathbf{Z}^d)$ *(though not on* $\mathcal{L}$*). For* $d > 2$, *there exists no* $SO(d, \mathbf{Q})$ *invariant* μ.

Proof. Consider the following two $\mathbf{Z}$-valued $SL(d, \mathbf{Z})$ invariant maps on $\mathcal{S}\mathcal{L}$. The Euler characteristic maps each rational convex cone (including the point at the origin) to 1, and χ^{loc} maps the point at the origin to 1 and every other rational convex cone with a vertex to 0 (see [Mo2]). Part (a) follows from the more general.

Claim. The only $SL(d, \mathbf{Z})$ invariant maps on $\mathcal{S}\mathcal{L}$ are linear combinations of the Euler characteristic and χ^{loc}.

To see this, consider the group $W = \mathcal{S}\mathcal{L}_{SL(d, \mathbf{Z})}$ of germs modulo the action of $SL(d, \mathbf{Z})$. A cone is called nonsingular if it is generated by a subset of a basis of M. It is well known that any rational convex cone may be subdivided into nonsingular cones, so nonsingular cones generate $\mathcal{S}\mathcal{L}(M)$. Clearly, any two simplicial cones of the same dimension are equivalent in W. Moreover, any simplicial cone c of dimension at least 2 may be split by a hyperplane into two such cones c_1 and c_2 intersecting in a cone of lower dimension c_3. We have $c = c_1 + c_2 - c_3$, which in W becomes $c = c_3$. Iterating, we may write any cone of positive dimension as a sum of one dimensional cones. Since any two such cones are equivalent in W, we find that W is generated by any one dimensional cone together with the zero dimensional cone at the origin. The Euler characteristic and χ^{loc} therefore generate the dual of W.

As for part (b), we construct an explicit counterexample. Let σ be the simplex in $\mathbf{Z}^3$ with vertices $(0, 0, 0)$, $(1, 1, 0)$, $(1, 0, 1)$, and $(3, -1, 1)$. It is easily checked that although σ has volume $\frac{1}{2}$, it is primitive (contains no lattice points other than its vertices). Let A be the element

$$A = \frac{1}{3}\begin{pmatrix} 2 & -2 & 1 \\ -1 & -2 & -2 \\ 2 & 1 & -2 \end{pmatrix},$$

of $SO(3, \mathbf{Q})$. We find that $A\sigma$ is again a lattice simplex with vertices $(0, 0, 0)$, $(0, -1, 1)$, $(1, -1, 0)$, and $(3, -1, 1)$. On the other hand $A\sigma$ is no longer primitive; two lattice points lie on the edge between $(0, -1, 1)$ and $(3, -1, 1)$ at $(1, -1, 1)$ and $(2, -1, 1)$. Denote by τ the dihedral angle along this edge, considered as an element of $\mathscr{S}\mathscr{L}$. Then $\vartheta(A\sigma) = A\vartheta(\sigma) + 2\tau$, so any $SO(3, \mathbf{Q})$ invariant μ would have to vanish on τ. But τ is simply the 45° dihedral angle spanned by the x-axis and the vectors $(0, 0, -1)$ and $(0, 1, -1)$. By symmetry μ must also vanish on the complementary angle and the rotations of τ about the x-axis by multiples of 90°. Since the sum of these angles differs from the entire space by two dimensional cones, and μ must take value 1 on the entire space, μ must be nonzero on these two dimensional cones. This is easily seen to be absurd.

2.3. *The Lattice Point Enumerator*

In the last section we encountered the fact that ϑ is a separating invariant for lattice polyhedra and observed that this implies the existence of a formula for the volume of a lattice polyhedron. In this section we produce dual invariants, show that these are separating, and deduce the existence of formulas for the number of lattice points in a lattice polyhedron.

Let N be the lattice dual to M, so that $M = \mathrm{Hom}_{\mathbf{Z}}(N, \mathbf{Z})$. If σ is a rational convex cone in N, then by the dual of this cone we shall mean the cone

$$\sigma^{\vee} = \{v \in N \otimes \mathbf{R} : v(\sigma) \geqslant 0\}.$$

(Note that this is contrary to the usual definition of the polar dual but agrees with the conventions of toric variety researchers.) It can be shown that $\sigma \mapsto \sigma^{\vee}$ passes to a well defined homomorphism $\mathscr{S}\mathscr{L}(N) \to \mathscr{S}\mathscr{L}(M)$ which is in fact an isomorphism.

Consider the direct sum of subquotients of $\mathscr{L}(M) \otimes \mathscr{S}\mathscr{L}(N)$,

$$B = \bigoplus_k \mathrm{gr}_k^{\dim} \mathscr{L}(M) \otimes \mathrm{gr}_{d-k}^{\dim} \mathscr{S}\mathscr{L}(N).$$

One checks that the total germ map Θ_L followed by the duality homomorphism, id $\otimes$ $^{\vee}$, maps into B in a natural way. Formally,

DEFINITION 12. Define a map $\check{\Theta}: \mathscr{L}(M) \to B$ by setting the kth component equal to

$$\check{\Theta}_k(P) = \sum_{\substack{F \subseteq P \\ \dim F = k}} F \otimes \vartheta_F(P)^{\vee}.$$

Consider the maps $\check{\vartheta}_k : \mathscr{L}(N) \to \mathbf{Q} \otimes \mathrm{gr}_{d-k}^{\dim} \mathscr{S}\mathscr{L}(M)$ for $0 \leqslant k \leqslant d$ defined by $\check{\vartheta}_k = (\overline{\mathrm{vol}}_k \otimes \mathrm{id}) \circ \check{\Theta}_k$. If P is a convex polyhedron, then

$$\check{\vartheta}_k(P) = \sum_{\substack{F \subseteq P \\ \dim F = k}} \overline{\mathrm{vol}}_k(F)\, \vartheta_F(P)^{\vee}.$$

PROPOSITION 2. *The maps $\check{\vartheta}_k$ are a set of separating invariants on $\mathscr{L}(N)$.*

Proof. It is shown in [Mo1] that the Hadwiger invariants are a set of separating invariants on $\mathscr{L}(N)$. Let us recall the definition of these invariants in the present context. For this purpose it is convenient to use the vector space $V = N \otimes \mathbf{Q}$.

By a *rigged hyperplane* I mean a hyperplane $U \subseteq V$ together with a choice of one of the two halfspaces bounded by U. A *rigged flag* of length k is a flag of subspaces $\mathbf{F} = (V = U^0 \supseteq U^1 \supseteq \cdots \supseteq U^k)$, where U^i is a rigid hyperplane in U^{i-1} for $i = 1, ..., k$. In particular, codim $U^i = i$.

There is a Hadwiger invariant associated with each rigged flag, by which I mean the following. If U^1 is a rigged hyperplane, choose a linear functional $\xi \in V^\vee$ whose kernel is U^1 and which is positive on the chosen halfspace bounded by U^1. If P is a convex polyhedron in N, set $\partial_{U^1} P = \{v \in P : \xi(v) = \min(\xi(P)) = h_P(\xi)\}$. More generally, if $\mathbf{F}$ is a rigged flag, set $\partial_{\mathbf{F}} P = \partial_{U^k} \partial_{U^{k-1}} \cdots \partial_{U^1} P$. It is easy to see that $\partial_{\mathbf{F}}$ extends by linearity to a function on L and passes to a function of $\mathscr{L}$

$$\partial_{\mathbf{F}} : \mathscr{L}(N) \to \mathscr{L}(N).$$

Let $\mathbf{F}$ be a rigged flag of length k and x an element of $\mathscr{L}(N)$. Define the Hadwiger invariant $\mathrm{Had}_{\mathbf{F}}(x)$ as

$$\mathrm{Had}_{\mathbf{F}}(P) = \mathrm{vol}_{d-k}(\partial_{\mathbf{F}}(x)).$$

In particular, if $k = 0$ then $\mathrm{Had}_{\mathbf{F}}(x) = \mathrm{vol}_d(x)$ for the unique flag of length 0. It is proven in [Mo1] that

$$\text{If } \mathrm{Had}_{\mathbf{F}}(x) = 0 \text{ for every rigged flag } \mathbf{F}, \text{ then } x = 0.$$

Our task is to show that the Hadwiger invariants factor through the maps $\check{\vartheta}_k$. In other words, we must determine $\mathrm{Had}_{\mathbf{F}}(x)$, given $\check{\vartheta}_k$.

The codimension 0 Hadwiger invariant, volume, is simply $\check{\vartheta}_d$ (after identifying $\mathrm{gr}_0^{\mathrm{dim}}\, \mathscr{S}\mathscr{L}(M)$ with $\mathbf{Z}$).

A hyperplane U in $V = N \otimes \mathbf{Q}$ determines a line $U^\perp$ in $W = M \otimes \mathbf{Q}$, while a rigged hyperplane U in V determines a ray in W, which we may consider as an element τ_U of $\mathrm{gr}_1^{\mathrm{dim}}\, \mathscr{S}\mathscr{L}(M)$. It is clear from the definitions that

$$\check{\vartheta}_1(x) = \sum_U \mathrm{Had}_U \otimes \tau_U,$$

where the sum is over all rigged hyperplanes.

Let us define another filtration (decreasing this time) on $\mathscr{S}\mathscr{L}(M)$ by the dimension of cospans. The cospan of a convex cone is the largest subspace it contains. Let $H^k_{co}\mathscr{S}\mathscr{L}(M)$ be the subgroup of elements of $\mathscr{S}\mathscr{L}(M)$ generated by convex cones whose cospans have dimension *at least* k, and set $\mathrm{gr}^k_{co}\,\mathscr{S}\mathscr{L}(M) = F^k_{co}\mathscr{S}\mathscr{L}(M)/F^{k+1}_{co}\mathscr{S}\mathscr{L}(M)$.

There is a homomorphism from $\mathscr{L}(M)$ to $\mathrm{gr}^k_{co}\,\mathscr{S}\mathscr{L}(M)$ defined by

$$\vartheta_k(P) = \sum_{\substack{F \subseteq P \\ \dim F = k}} \overline{\mathrm{vol}}_k(F)\,\vartheta_F(P).$$

Moreover, the duality map $\sigma \mapsto \sigma^\vee$ takes $\mathrm{gr}^{d-k}_{co}\,\mathscr{S}\mathscr{L}(M)$ to $\mathrm{gr}^{\dim}_k\,\mathscr{S}\mathscr{L}(N)$, and under this map ϑ_{d-k} is taken to $\check{\vartheta}_k$. It is therefore sufficient to determine the Hadwiger invariants from the ϑ_k.

Let U be a rigged hyperplane, with chosen halfspace U^+. Define as follows an operation ∂_U which takes cones in V to cones in U:

$$\partial_U(\sigma) = \begin{cases} \sigma \cap U; & \sigma \in U^+ \\ \varnothing & \text{otherwise.} \end{cases}$$

It is clear that ∂_U respects the subdivision relations in $\mathscr{S}\mathscr{L}$, so it passes to a homomorphism from $\mathscr{S}\mathscr{L}(M)$ to $\mathscr{S}\mathscr{L}(U)$. Moreover, if $\sigma \in U^+$ then its cospan must be contained in U, so ∂_U determines a homomorphism

$$\partial_U : \mathrm{gr}^k_{co}\,\mathscr{S}\mathscr{L}(M) \to \mathrm{gr}^k_{co}\,\mathscr{S}\mathscr{L}(U).$$

Let $\mathbf{F} = (V = U^0 \supseteq U^1 \supseteq \cdots \supseteq U^k)$ be a rigged flag. Set $\partial_{\mathbf{F}} P = \partial_{U^k}\partial_{U^{k-1}} \cdots \partial_{U^1} P$. Note that $\mathrm{gr}^{d-k}_{co}\,\mathscr{S}\mathscr{L}(U^k)$ is simply $\mathbf{Z}$. I claim that the general formula for $\mathrm{Had}_{\mathbf{F}}$ is

$$\mathrm{Had}_{\mathbf{F}}(x) = \partial_{\mathbf{F}}\vartheta_{d-k}(x),$$

where on the right-hand side we are identifying $\mathbf{Q} \otimes \mathrm{gr}^{d-k}_{co}\,\mathscr{S}\mathscr{L}(U^k)$ with $\mathbf{Q}$.

To see this, take x to be a convex polyhedron P. If U is a rigged hyperplane, then the cones $\vartheta_F(P)$ (F a face of P of codimension k) which are not mapped by ∂_U to 0 are exactly those for which F is contained in the face $\partial_U(P)$. If we are given a codimension k riggid flag $\mathbf{F}$, it then follows that the cones which survive $\partial_{\mathbf{F}}$ are simply the cones $\vartheta_F(P)$ for which $F \subseteq \partial_{\mathbf{F}}(P)$. Since $\partial_{\mathbf{F}}(P)$ has codimension at least k, if ϑ_F survives it must be $\partial_{\mathbf{F}}(P)$. This means that if $\check{\partial}_{\mathbf{F}}$ is applied to $\vartheta_{d-k}(P) = \sum_{F \subseteq P, \dim F = d-k} \overline{\mathrm{vol}}_{d-k}(F)\,\vartheta_F(P)$ the result will be 0 unless $\partial_{\mathbf{F}}(P)$ has codimension exactly k, in which case the result will be $\overline{\mathrm{vol}}_{d-k}(F)$. In either case it agrees with $\mathrm{Had}_{\mathbf{F}}(P)$.

COROLLARY 3. *There exist homomorphisms* $\mu_k : \mathrm{gr}^{\dim}_k\,\mathscr{S}\mathscr{L}(N) \to \mathbf{Q}$ *with the property that for all* x *in* $\mathscr{L}(M)$

$$\#_k(x) = \mu_k(\check{\vartheta}_{d-k}(x)).$$

In particular, for any lattice polyhedron P,

$$\#_k(P) = \sum_{\substack{F \subseteq P \\ \dim F = d-k}} \overline{\mathrm{vol}}_{d-k}(F)\, \mu_k(\vartheta_F(P)^\vee)$$

and

$$\#(P) = \sum_{F \subseteq P} \overline{\mathrm{vol}}_{\dim F}(F)\, \mu_{\operatorname{codim} F}(\vartheta_F(P)^\vee).$$

Proof. The maps $\check{\vartheta}_k$ are separating invariants, and $\check{\vartheta}_k$ has weight k in the sense that

$$\check{\vartheta}_k(\Psi^n P) = n^k \check{\vartheta}_k(P)$$

for a dilation $\Psi^n P$ of P. Therefore, any homomorphism on $\mathscr{L}(N)$ which has weight k must factor through $\check{\vartheta}_k$.

Remarks. 1. It is not possible to choose μ_k so that it is positive on every cone. This is simply because $\#_k(P)$ may be negative even if P is a convex polyhedron. The simplest example is the tetrahedron T with vertices at the origin, at (0, 1, 0), (1, 0, 0), and at (1, 1, 13), for which $\#_1(T) = -\frac{1}{6}$.

2. Of course, we also know that $\overline{\mathrm{vol}}_k$ factors through $\check{\vartheta}_k$ and $\#$ factors through ϑ, but these are both trivial.

3. The formal duality between Corollary 3 and Corollary 2 raises the following question. Do there exist functions $\mu_k : \mathrm{gr}_k^{\dim} \mathscr{S}\mathscr{L}(M) \to \mathbf{Q}$ so that Corollary 2 holds for polyhedra in M, and Corollary 3 holds for polyhedra in N?[1]

3. Todd Class of a Toric Variety

Notation. Recall that a toric variety (over C) is a normal algebraic variety on which a complex torus $T = (C^*)^d$ acts, and into which the torus embeds equivariantly (see [Oda]). We denote the character group of T by M, and the dual of M by N. The classification theorem constructs a bijection $\Delta \leftrightarrow X_\Delta$ between the fans Δ in N and the toric varieties X_Δ with torus T. Let $\Delta(k)$ denote the set of k-dimensional cones of Δ. To each $\sigma \in \Delta(k)$ there corresponds a codimension k orbit $V(\sigma)$ of T in X_Δ whose closure is an algebraic cycle we denote $[V(\sigma)]$.

Intersection Ring, Homology, and Cohomology. Let $A_k(X_\Delta)$ denote the group of k-cycles modulo rational equivalence. For a compact nonsingular toric variety X_Δ, the cycle class map

$$cl\colon A_*(X_\Delta) \to H_*(X_\Delta, \mathbf{Z})$$

[1] The author has recently settled this question in the affirmative.

is an isomorphism, and $A_k(X_\Delta)$ is generated by the cycles $[V(\sigma)]$ for $\sigma \in \Delta(d-k)$. The relations among these generators, and the multiplicative structure in A_*, can be elegantly expressed in terms of the fan, as was done by Jurkiewicz (projective case) and Danilov (in general).

THEOREM 7. *Let X_Δ be a compact nonsingular toric variety. The intersection ring is generated by the cycles $[V(\sigma)]$, $\sigma \in \Delta$, subject to the relations:*

(i) *For $\sigma \in \Delta(k)$ and $\tau \in \Delta(l)$,*

$$[V(\sigma)] \cdot [V(\tau)] = \begin{cases} [V(\sigma+\tau)] & \text{if } \sigma+\tau \in \Delta(k+l) \\ 0 & \text{otherwise} \end{cases}$$

(where $\sigma + \tau$ is the cone generated by σ and τ by Minkowkski sum).

(ii) *For any $m \in M$,*

$$\sum_{\varrho \in \Delta(1)} \langle m, n(\varrho) \rangle [V(\varrho)] = 0,$$

where $n(\varrho)$ is the unique primitive element of N lying in the one dimensional cone ϱ.

It is useful to bear in mind the following simple consequence. A homology class x on X_Δ may be represented as

$$x = \sum_{\sigma \in \Delta} r_\sigma [V(\sigma)].$$

The representation is not unique. A cohomology class c (with coefficients in, say, $\mathbf{Z}$) may be evaluated on the classes $[V(\sigma)]$ so gives a function on the fan

$$c\colon \Delta \to \mathbf{Z}. \tag{13}$$

Not every function on Δ represents a cohomology class.

3.1. *Existence of Local Formulas*

Suppose that X_Δ is a nonsingular compact toric variety. Then the Todd class $Td^*(X_\Delta)$ of the tangent bundle TX_Δ of X_Δ is a class in $A^*(X_\Delta) = A^*(X_\Delta) \otimes \mathbf{Q}$ defined as follows. Let $td_t(x_1, ..., x_d) = \sum_{i \geq 0} td_i(x_1, ..., x_d)\, t^i$ be the power series

$$\prod_{i=1}^{d} \frac{tx_i}{1-e^{-tx_i}}.$$

Its coefficients are symmetric polynomials in the x_i. Then the Todd class is

$$Td^*(X) = \sum_{i \geq 0} Td^i(X) = \sum_{i \geq 0} td_i(x_1, ..., x_d),$$

where the x_i are interpreted as formal roots of the chern polynomial of $TX_\Delta: c_t(TX_\Delta) = \sum_{i \geq 0} c_i(TX_\Delta)\, t^i = \prod_{i=1}^{d} (1 + x_i t)$.

We know from Theorem 7 that for any k,

$$Td^k(X_\Delta) = \sum_{\sigma \in \Delta(k)} r_\sigma [V(\sigma)], \tag{14}$$

where r_σ are rational numbers not uniquely determined. A problem, posed first by Danilov, is to decide whether the r_σ can be chosen independently of Δ. That is, can we choose a rational number r_σ for each cone σ, so that Eq. (14) holds for any fan Δ. We show below that such a choice is possible. In fact, we show that the r_σ may be chosen so that they are linear with respect to subdivisions. Incidentically, the Todd class is the only characteristic class for which one would expect linearity.

THEOREM 8. *Let $\mu_k: \mathrm{gr}_k^{\dim} \mathscr{S}\mathscr{L}(N) \to \mathbf{Q}$ be a homomorphism. The three following conditions are equivalent.*

1. *For all smooth compact toric varieties X_Δ,*

$$Td^k(X_\Delta) = \sum_{\sigma \in \Delta(k)} \mu_k(\sigma)[V(\sigma)]. \tag{15}$$

2. *For all f in $\mathscr{L}(M)$*

$$\#_k(f) = \mu_k(\check{\vartheta}_{d-k}(f)).$$

3. *For all convex lattice polyhedra*

$$\#_k(P) = \sum_{\substack{F \subseteq P \\ \dim F = d-k}} \overline{\mathrm{vol}}_{d-k}(F)\, \mu_k(\vartheta_F(P)^\vee). \tag{16}$$

Proof. This proof depends on results and constructions from [Mo3], to which we will freely refer. That paper constructs a natural injection $\mathfrak{k}: K(X_\Delta) \to \mathscr{L}(M)$ from the K group of a smooth compact toric variety to $\mathscr{L}(M)$. This injection satisfies the following properties relevant to our present purpose:

(a) $\mathfrak{k}(\Psi^n(x)) = \Psi^n \mathfrak{k}(x)$, where Ψ^n on the left is the nth Adams operation and Ψ^n on the right is dilation by a factor of n.

(b) $\#(\mathfrak{f}(x)) = \chi(x)$, where χ is the Euler Poincaré characteristic.

(c) $\check{\vartheta}_k(\mathfrak{f}(x))$ can be written uniquely in the form

$$\check{\vartheta}_k(\mathfrak{f}(x)) = \sum_{\sigma \in \Delta(d-k)} a_\sigma \otimes [\sigma].$$

The a_σ are determined by

$$a_\sigma = \int ch_{d-k}(x) \cdot [V(\sigma)],$$

where $\int : A^d(X_\Delta)_{\mathbf{Q}} \to \mathbf{Q}$ is the degree map; see [Mo3, 6].

(d) A nonsingular simplex, i.e., a simplex with volume $1/d!$, is the image under $\mathfrak{f}$ of $\mathcal{O}(1)$ on $\mathbf{P}^d$ (with some toric structure).

Conditions 2 and 3 are trivially equivalent. Assume that 1 holds. In consequence of (a) and (b) above, and the Riemann–Roch theorem, $\#_{d-k}(\mathfrak{f}(x)) = \int ch_{d-k}(x)\, Td^k(X_\Delta)$; see [Mo3, 5]. On the other hand, by (c), $ch_{d-k}(x) \cdot Td^k(X_\Delta) = ch_{d-k}(x) \cdot \sum_{\sigma \in \Delta(k)} \mu_k(\sigma)[V(\sigma)] = \sum_{\sigma \in \Delta(k)} \mu_k(\sigma)\, a_\sigma = \mu_k(\check{\vartheta}_k(\mathfrak{f}(x)))$. Thus, 1 implies 2 for elements of the image $\mathfrak{f}(K(X_\Delta))$ for a given Δ. Now by (d) above, any nonsingular simplex is such an image. But Theorem 13, which is proved in Section 4.2, states that nonsingular simplices generate $\mathscr{P}(M)$.

Conversely, suppose 2 holds. On a given toric variety X_Δ consider the cycle $c = \sum_{\sigma \in \Delta(k)} \mu_k(\sigma)[V(\sigma)]$. It follows from (c) that for any x in $K(X_\Delta)$ and any $\sigma \in \Delta(k)$, $\int c \cdot ch_k(x) = \sum_{\sigma \in \Delta(k)} \mu_k(\sigma)\, a_\sigma = \mu_k(\check{\vartheta}_k(\mathfrak{f}(x))) = \#_{d-k}(\mathfrak{f}(x))$. On the other hand, we have already seen that $\int Td^k(X_\Delta) \cdot ch_{d-k}(x) = \#_{d-k}(\mathfrak{f}(x))$. Therefore, $\int (c - Td^k(X_\Delta)) \cdot ch_{d-k}(x) = 0$ for all x. But, ch_{d-k} is surjective modulo torsion, so by Poincaré duality, $c - Td^k(X_\Delta)$ vanishes in $H^{2k}(X_\Delta, \mathbf{Q}) = A^k(X_\Delta)_{\mathbf{Q}}$.

COROLLARY 4. *The r_σ in expression* (14) *may be chosen linearly with respect to subdivisions and independently of Δ.*

Proof. This merely asserts the existence of homomorphisms μ_k as in the proposition. We know these exist by Corollary 3.

PROPOSITION 3. *Any μ_k satisfying the equivalent conditions above also automatically satisfies* (15) *for arbitrary* (*possibly singular or noncompact*) *toric varieties*:

$$Td_{d-k}(X_\Delta) = \sum_{\sigma \in \Delta(k)} \mu_k(\sigma)[V(\sigma)].$$

Proof. When the variety X_Δ is singular the Todd class $Td_k(X_\Delta) \in A_k(X_\Delta)_{\mathbf{Q}}$ is taken in the sense of [Fu]. The extension from compact smooth to arbitrary smooth toric varieties is trivial. For the extension to singular toric varieties, let $f: X_{\Delta'} \to X_\Delta$ be a resolution of singularities of X_Δ. Then by the generalized Grothendieck–Riemann–Roch theorem proved in [Fu], $Td_k(X_\Delta) = f_*(Td_k(X_{\Delta'}))$. On the other hand, the pushforward of a cycle $[V(\sigma')]$ on $X_{\Delta'}$ is

$$f_*[V(\sigma')] = \begin{cases} [V(\sigma)], & \dim(\sigma) = \dim(\sigma') \\ 0 & \text{otherwise,} \end{cases}$$

where σ is the smallest cone of Δ containing σ'. Therefore, $Td_{d-k}(X_\Delta) = f_*(\sum_{\sigma' \in \Delta'(k)} \mu_k(\sigma')[V(\sigma')]) = (\sum_{\sigma \in \Delta(k)} \mu_k(\sigma)[V(\sigma)])$, by the additivity of μ_k.
We therefore have the reformulation:

> *The Todd class can be lifted to a well defined cycle, functorial with respect to proper push forward in the category of torus embeddings of a fixed torus.*

Formula of Fulton. As an application of the existence of μ_k we prove a formula of Fulton for the Todd classes of projective space. This formula can be described as follows. Let N be the lattice of vectors in $\mathbf{Z}^{d+1}$ whose coordinates sum to 0. Let Δ be the fan in N whose k-dimensional cones are spanned by k-subsets of the vectors $(1, -1, 0, ..., 0)$, $(0, 1, -1, 0, ..., 0)$, ..., $(-1, 0, ..., 0, 1)$. One knows that $X_\Delta \simeq \mathbf{P}^d$. Endow N with the metric induced from the standard metric on $\mathbf{Z}^{d+1}$. Denote by $\angle_k(\sigma)$ the spherical angle measure of a k-dimensional cone σ with respect to this metric, normalized so that an *interior* cone, i.e., an entire k-dimensional subspace, has angle measure 1. Fulton's formula states that

$$Td^k(\mathbf{P}^d) = \sum_{\sigma \in \Delta(k)} \angle_k(\sigma)[V(\sigma)]. \tag{17}$$

Fulton has also raised the general question: given a fan Δ, when does there exist a metric on N for which such a formula (17) holds? It is known that in dimensions 1 and 2 the answer is that any metric works for any fan. In higher dimensions, one finds that for a reasonably "spread out" fan the right side of (17) tends to roughly equal $T^k(X_\Delta)$, but it is rather unusual for actual equality to hold.

DEFINITION 13. Call a fan Δ in a metric lattice N *splayed* in dimension k if formula (17) holds.

If G is a group of automorphisms of a lattice N, let us say that a cone σ of dimension k is G-commensurable with its linear span if in the group of coinvariants $(\mathrm{gr}_k^{\dim} \mathcal{PL}(N))_G$, $q\sigma = r(\operatorname{span} \sigma)$, where q and r are integers. Equivalently, $G \cdot \sigma$ is equal, as an element of $\mathrm{gr}_k^{\dim} \mathcal{PL}(N)$, to a sum of interior cones. This is the case, for instance, if the linear span of σ is the union $g_1\sigma \amalg \cdots \amalg g_l\sigma$ of non-overlapping translates of σ.

PROPOSITION 4. *Let Δ be a fan in a metric lattice N. Suppose there is a finite group G of automorphisms of N such that each $\sigma \in \Delta(k)$, is G-commensurable with its linear span. Then Δ is splayed in dimension k.*

Proof. Let μ_k satisfy the equivalent conditions of Theorem 8. Let us first make a general observation. If c is a k-dimensional interior cone, then $\mu_k(c) = 1$. To see this consider a half-open parallelepiped f in $c^\perp \cap M$. By a half-open parallelepipid in a lattice isomorphic to $\mathbf{Z}^k$ we mean a subset of $\mathbf{Q}^k$ of the form

$$\{(x_1, ..., x_k) : 0 \leqslant x_i < 1\}.$$

It is easy to see that $\#(f; n) = n^{d-k}$ so $\#_k(f) = 1$, while $\check{\vartheta}_{d-k}(f) = c$.

The dual action of G on M preserves $\#_k$. Moreover, for any lattice polyhedron P in M and any automorphism g of N, we have

$$\check{\vartheta}_k(g^\vee P) = g\check{\vartheta}_k(P),$$

so the composition of μ_k with a g again satisfies Eq. (16). Since G is finite, we can arrange, by averaging if necessary, that μ_k be G-invariant.

Let $\sigma \in \Delta(k)$, and suppose that $q\sigma$ is equal to $r(\text{span}\,\sigma)$. By the G-invariance of μ_k, $\mu_k(\sigma) = r/q$. On the other hand, G acts by isometries, so for the same reason, $\angle_k(\sigma) = r/q$. Since on the cones of $\Delta(k)$, μ_k and $\angle_k$ agree, formula (17) follows from Theorem 8.

Remarks. 1. To derive Fulton's formula, interpret the lattice described above as the root system A_d with its invariant metric. The Weyl group, which is the group of permutations of the coordinates of $\mathbf{Z}^{d+1}$, satisfies the hypothesis of the proposition, as one readily checks.

2. There are smooth toric varieties X_Δ for which $Td^k(X_\Delta)$ cannot be represented by an algebraic cycle which is a linear combination with positive coefficients of the invariant cycles $[V(\sigma)]$. This follows from Remark 1 in Section 2.3. For example, any nonsingular subdivision of the fan dual to the tetrahedron T described there gives rise to such a toric variety. For such a fan Δ, there can be no metric for which Δ is splayed.

3.2. *Bott Residue Theorem on a Toric Variety*

More satisfactory than an abstract existence statement would be explicit homomorphisms μ_k. By a straightforward application of residue formulas of Baum and Bott (see [Bo], [BB]) we construct such homomorphisms with certain desirable properties. For example, they are natural and multiplicative in a certain sense. The concession we must make to achieve these properties is that the μ_k take values not in $\mathbf{Q}$ but in extension fields of $\mathbf{Q}$ which are described below.

Let us recall the relevant statement of [BB]. Suppose M is a compact complex manifold, and ξ is a coherent subsheaf of the tangent sheaf TM which is closed under Lie brackets. The sheaf ξ will restrict to a subbundle of TM of rank, say k, on some dense open subset, so by the Frobenius theorem it will define a k (leaf-) dimensional foliation there. Denote by S the *singular set* of ξ, i.e., the set of points at which ξ is not locally free. Let $i: M - S \subsetneq M$ be the inclusion. Suppose that the image of the adjunction map $TM \to i_* i^* TM$ is the same as the image of the composite $\xi \to i_* i^* \xi \to i_* i^* TM$. Suppose further that there is a dense open subset S' of S for which the following holds:

for each point $p \in S'$, there is a neighborhood U_p of p, complex coordinates $z_1, ..., z_d$, and holomorphic functions $a_k, ..., a_d$ defined on U_p so that

1. $$S \cap U_p = \{x \in U_p : z_k(x) = \cdots = z_d(x) = 0\}$$
$$= \{x \in U_p : a_k(x) = \cdots = a_d(x) = 0\}.$$

2. If $1 \leqslant j \leqslant k-1$ and $k \leqslant i \leqslant d$ then $\partial a_i / \partial z_j$ vanishes on U_p.

3. The vector fields $\partial/\partial z_1, ..., \partial/\partial z_{k-1}$ and $\sum_{i=k}^{d} a_i \, \partial/\partial z_i$ are generating sections of ξ.

4. The matrix $A = (\partial a_i / \partial z_j)$, $k \leqslant i, j \leqslant d$, of partial derivatives is nonsingular at p.

The sheaves ξ satisfying these properties can be characterized in a more conceptual way (see [BB, p. 284]), but we will not address such points here. Note in particular that S is $(k-1)$-dimensional and that the subset of S on which ξ spans a subspace of TM of dimension smaller than $k-1$ is of codimension at least one.

Now to each irreducible $(k-1)$-dimensional component Z of S, and each degree $d-k+1$ symmetric polynomial φ, we associate a number $l(\varphi, \xi, Z)$ as follows. Let p be any point of $S' \cap Z$, and let coordinates and functions a_i be chosen as above. Then,

$$l(\varphi, \xi, Z) = \frac{\varphi(\lambda_1, ..., \lambda_{d-k+1}, 0, ..., 0)}{\lambda_1 \cdots \lambda_{d-k+1}}, \tag{18}$$

where the λ_i are the eigenvalues of the matrix A at p. The residue theorem in question then states [BB, Theorem 3].

THEOREM 9. *Let $\{Z_i\}$ be the irreducible $(k-1)$-dimensional components of S, and φ is a degree $d-k+1$ symmetric polynomial. Then under the hypotheses stated above, the chern class $\varphi(T/\xi)$ is computed as*

$$\varphi(T/\xi)=\sum_i l(\varphi, \xi, Z_i)[Z_i],$$

where $[Z_i]$ denotes the Poincaré dual of Z_i in $H^{2(d-k+1)}(M; \mathbf{C})$.

Let X_Δ be a smooth compact toric variety and choose a subalgebra of the Lie algebra of T^d. Then this theorem applies neatly to the natural subsheaf of TX_Δ generated by this subalgebra. To be precise, suppose that H is a rank k submodule of N generated by elements $n_1, ..., n_k$. Then the holomorphic vector fields $v^i(p)=(d/dt)\exp(tn_i)\cdot p$ generate a subsheaf H_X of TX which are foliations on the maximal torus but which have singularities on the orbits whose stabilizers intersect $\exp(H)$. Note that the orbits $V(\sigma)$ for which this occurs are exactly those for which the span of σ meets H nontrivially. In order that the hypotheses of the residue theorem hold, we need that there be no singularities on orbits of high dimension, i.e., H does not meet the span of any $\sigma\in\Delta(d-k)$. One checks that if this is the case, then the local condition is also automatically satisfied. Therefore, for a generic H these hypotheses are in fact met.

Since H_X is free, the residue theorem computes the characteristic classes of TX_Δ. Now let us compute the right side of (18). Since the computation is local, we restrict attention to an affine toric variety U_σ, σ a k-dimensional cone in N, with associated orbit $V(\sigma)$. Suppose then that H is a $(d-k+1)$-dimensional subalgebra generated by $n_k, ..., n_d$. The cospan of $\sigma^\vee$ is $d-k$ dimensional so we may choose a basis $\{m_1, ..., m_d\}$ of M for which $\sigma^\vee$ is generated by $m_1, m_2, ..., m_k, \pm m_{k+1}, ..., \pm m_d$. Set $z_i=e(m_i)$, the character m_i considered as a function on U_σ. We then have $V(\sigma)=\{z_1=\cdots=z_k=0\}$, as desired. In these local coordinates, the action of the torus is simply $(t_1, ..., t_d)\cdot(z_1, ..., z_d)=(t_1z_1, ..., t_dz_d)$ so the vector fields in H_X are generated by

$$\langle m_1, n_i\rangle z_1\,\partial/\partial z_1+\langle m_2, n_i\rangle z_2\,\partial/\partial z_2+\cdots+\langle m_d, n_i\rangle z_d\,\partial/\partial z_d,$$

$i=k, ..., d$. It is easy to prove that generically these generate the same subsheaf of TX as

$$\partial/\partial z_{k+1}, ..., \partial/\partial z_d$$

and

$$\sum_{i=1}^{k}\langle m_i\wedge m_{k+1}\wedge\cdots\wedge m_d, n_k\wedge\cdots\wedge n_d\rangle z_i\,\partial/\partial z_i.$$

The matrix A of partial derivatives is thus diagonal, and the number

$$l(\varphi, H_X, \overline{V(\sigma)}) = \frac{\varphi(\langle m_1 \wedge m_{k+1} \wedge \cdots \wedge m_d, n_k \wedge \cdots \wedge n_d\rangle, \ldots)}{\prod_i \langle m_i \wedge m_{k+1} \wedge \cdots \wedge m_d, n_k \wedge \cdots \wedge n_d\rangle}. \quad (19)$$

Now as it stands, the construction depends on the specific subspace H we choose. In order to make the construction natural we therefore consider the numbers $l(\varphi, H_X, \overline{V(\sigma)})$ as functions of the subspace H. That is, we consider them as functions on the grassmannian of $d-k+1$ planes in N. In fact, these functions are rational and happen to fall within a series of similarly defined functions which may be interpreted generally as meromorphic sections of the various twists $\mathcal{O}(j)$ with respect to the Plücker embedding. We will hold off writing these expressions down until Section 4, where the concrete recipes are given.

3.3. *Equivariant Cohomology*

In [AB], the residue formula was reinterpreted in terms of equivariant cohomology, at least in the case of a one dimensional foliation resulting from the action of a circle S. This new interpretation also allows the local computation of the *equivariant* characteristic "numbers," which lie naturally in the equivariant cohomology of a point, $H_S^*(pt) = H^*(BS)$.

The result of applying the ideas of [AB] to our situation is as follows. Let $n \in N$ be a generic one parameter subgroup $S \subsetneq T$ of the torus T, and let u be a corresponding generator of the character group of S. To any symmetric polynomial φ of degree k, there is a corresponding equivariant characteristic cohomology class in $H_S^k(X_\Delta)$. In ordinary cohomology one can evaluate a cohomology class over the fundamental cycle of X_Δ to yield a number. Correspondingly our equivariant class may be pushed forward to a point pt, yielding a class in $H_S^*(pt) \simeq \mathbf{C}[u]$. This equivariant characteristic "number" need not vanish even if k is distinct from d. We then have, for the equivariant characteristic number corresponding to φ, the formula

$$\sum_{\sigma \in \Delta(d)} \frac{\varphi(\langle f_1^\sigma, n\rangle, \ldots, \langle f_d^\sigma, n\rangle)\, u^k}{\prod_i \langle f_i^\sigma, n\rangle\, u^d},$$

where $f_1^\sigma, \ldots, f_d^\sigma$ are primitive generators of the dual cone $\sigma^\vee$ of σ. The right-hand side is evaluated in the fraction field of $\mathbf{C}[u]$. If $k = d$ this formula reduces to a special case of the formula of the previous section.

4. Combinatorial Consequences

In this section we distill out the combinatorics of the residues that arise in connection with the theorem of Baum and Bott. Our aim from this point onward is to sever connections with the geometry of toric varieties as much as possible.

We first define a sequence of homomorphisms on the graded pieces $\mathrm{gr}_k^{\dim} \mathcal{S}\mathcal{L}$. They will all take values in the field of rational functions in d variables. To be more precise, instead of d variables, we should say $\mathrm{sym}^* M$, the symmetric algebra (with $\mathbf{Z}$ coefficients) of the dual of N. For the sake of clarity, we first give the special case, which is easiest to state. Recall that a cone in a lattice N is called nonsingular if it can be generated by a subset of a $\mathbf{Z}$-basis of N.

Definition 14. Suppose that σ is a nonsingular d-dimensional cone in a lattice N (of rank d). Then $\sigma^\vee$ is generated by d primitive elements $f_1, ..., f_d$ of the dual lattice M, i.e., $\sigma = \{x : f_1(x), ..., f_d(x) \geqslant 0\}$. For any symmetric polynomial φ of d variables set

$$\mu_d^\varphi(\sigma) = \frac{\varphi(f_1, ..., f_d)}{f_1 \cdots f_d}.$$

This expression is meant to be a rational function lying in the fraction field $(\mathrm{sym}^* M)_0$.

The nonsingular cones generate $\mathrm{gr}_d^{\dim} \mathcal{S}\mathcal{L}$. On the other hand, there is no reason to expect a priori that these μ_d^φ will descend to well defined maps on $\mathrm{gr}_d^{\dim} \mathcal{S}\mathcal{L}$ because there are always many ways to subdivide a given cone into nonsingular cones. For most choices of φ in fact they do not descend. We will see that when φ is the Todd polynomial of degree k, $\mu_d^\varphi = \mu_d^{td_k}$ is linear with respect to subdivisions, hence gives rise to such a homomorphism on $\mathrm{gr}_d^{\dim} \mathcal{S}\mathcal{L}$. With this in view we set the total Todd measure to be

$$\mu_d^{td_t}(\sigma) = \frac{td_t(f_1, ..., f_d)}{f_1 \cdots f_d}.$$

The latter expression is a power series with rational functions as coefficients. With some caution, we can write

$$\mu_d^{td_t}(\sigma) = \prod_{i=1}^{d} \frac{1}{1 - e^{-tf_i}}.$$

For sample computations of $\mu_d^{td_d}$, see the examples in the overview.

Now for the general definition.

Definition 15. Let σ be a nonsingular k-dimensional cone in a lattice N. The cospan of $\sigma^\vee$ is a $(d-k)$-dimensional subspace $\sigma^\perp$. There are k

faces of $\sigma^\vee$ of dimension $d-k+1$, the "edges" of $\sigma^\vee$. Each "edge" is a halfspace bounded by $\sigma^\perp$. Choose generators $f_1, ..., f_k$ of the top exterior powers of the spans of these "edges." These are well defined up to a sign. Choose the generators to be compatible in the sense that the corresponding orientations on the "edges" all induce the same orientation on $\sigma^\perp$. If φ is a symmetric polynomial in k variables we set

$$\mu_k^\varphi(\sigma) = \frac{\varphi(f_1, ..., f_k)}{f_1 \cdots f_k}.$$

If the degree of φ is of parity different from k, then there is an ambiguity of sign. A way to resolve it is mentioned below. At any rate, the cases of most interest are $\deg \varphi = k$ and $k = d$, which arise from the residue formulas, and for which there is no question of sign.

In particular, we have

$$\mu_k^{td_t}(\sigma) = \frac{td_t(f_1, ..., f_k)}{f_1 \cdots f_k},$$

which is understood to be a power series whose terms are elements of the fraction field of the symmetric algebra of $\bigwedge^{d-k+1} M$.

THEOREM 10. 1. *The functions* $\mu_k^{td_n}\colon \mathrm{gr}_k^{\dim} \mathcal{S}\mathcal{L} \to (\bigwedge^{d-k+1} M)_0$ *are well defined (apart from the sign ambiguity).*

2. *The maps* $\mu_d^{td_t}$ *are multiplicative with respect to external products*

$$\mu_d^{td_t}(\sigma \times \tau) = \mu_k^{td_t}(\sigma)\, \mu_l^{td_t}(\tau),$$

where σ *and* τ *are in lattices* N_1, N_2 *of respective ranks* k *and* l, *while* $\sigma \times \tau$ *is in* $N = N_1 \oplus N_2$ *of rank* d. *If we write* M, M_1, *and* M_2 *for the duals of* N, N_1, *and* N_2, *then the product on the right is interpreted via the isomorphism* $\mathrm{sym}^* M_1 \otimes \mathrm{sym}^* M_2 = \mathrm{sym}^* M$.

3. *The maps* $\mu_k^{td_t}$ *are natural in the sense that if* $Q \in \mathrm{Aut}(N)$ *then* $\mu_k^{td_t}(Qx) = Q^*(\mu_k^{td_t}(x))$, *where on the right-hand side* Q^* *denotes the natural action of the adjoint of* Q *on the target field.*

Proof. Even prior to well definedness, the naturality property is immediate. We will also show multiplicativity before well definedness. If σ is defined by equations $f_1, ..., f_k \geq 0$ and τ by equations $g_1, ..., g_l \geq 0$ then $\sigma \times \tau$ is described by $f_1, ..., f_k, g_1, ..., g_l \geq 0$. By definition,

$$\begin{aligned}\mu_d^{td_t}(\sigma \times \tau) &= \prod_{i=1}^{k} \frac{1}{1-e^{-tf_i}} \prod_{i=1}^{l} \frac{1}{1-e^{-tg_i}} \\ &= \mu_k^{td_t}(\sigma)\, \mu_l^{td_t}(\tau).\end{aligned}$$

We show well definedness by induction on the rank d of the lattice N. Before considering the general case, we will first establish that $\mu_d^{td_i}$ is well defined. This is trivial for $d=0$ and $d=1$, there being always a unique decomposition into nonsingular cones in these dimensions. For the general case, a topological argument based on the residue formulas will be given.

Note that if the rational function $\mu_d^{td_d}$ is evaluated on a generic element n of N, then it will have a finite value equal to the local number $l(td_d, n_X, \overline{V(\sigma)})$. Bott's formula (this is the original $k=1$ case of (18) in [Bo]) then says that

$$Td^d(X_\Delta)[X_\Delta] = \sum_{\sigma \in \Delta(d)} \mu_d^{td_d}(\sigma)(n).$$

On the other hand, any toric variety is birationally equivalent to $\mathbf{P}^d$ so has arithmetic genus 1. Therefore, $\sum_{\sigma \in \Delta(d)} \mu_d^{td_d}(\sigma) = 1$ identically.

We must show that any two subdivisions of a given cone c into nonsingular cones produce the same value for $\mu_d^{td_d}(c)$. We can reduce to the case of c a nonsingular cone as follows. Take the pairwise intersections of the cones in two such subdivisions of an arbitrary cone, and further subdivide the resulting subdivision into nonsingular cones.

Suppose then that c is nonsingular. Endow N with the metric for which the primitive generators of c are an orthonormal basis of N. The dual cone $c^\vee$ is then identified with c itself, and $-c^\vee$ is the reflection of c through the origin. For each face τ of c, let $\tilde{\tau}$ be the cone

$$\tilde{\tau} = \tau + (-c^\vee \cap \tau^\perp).$$

The $\tilde{\tau}$ constitute a nonsingular complete fan.

Suppose we have a nonsingular subdivision of c. By induction, $\mu_d^{td_d}$ is well defined on all of the faces of dimension lower than d. The subdivisions of all the faces of c induce subdivisions of the cones of the complete fan $\{\tilde{\tau}\}$ described above. By the multiplicativity of $\mu_d^{td_i}$, and the inductive hypothesis, the sum of $\mu_d^{td_d}$ over this subdivision is independent of the subdivision of the face. The formula above then shows that $\mu_d^{td_d}(c)$ is well defined as $\mu_d^{td_d}(c) = 1 - \sum_{\sigma \not\subseteq c} \mu_d^{td_d}(\sigma)$.

By essentially the same argument, $\mu_d^{td_i}$ is well defined for $i \neq d$. The only difference is that now $\mu_d^{td_i}$ represents an equivariant Todd number, which vanishes. For $i < d$ this follows at once from the formula in Section 3.3. In general, we can argue as follows. If we map the circle S to itself by $z \mapsto z^q$, then the induced map on the character group is $u \mapsto qu$. According the formula in Section 3.3, the ith equivariant Todd class of X is then multiplied by q^{i-d}. Since this class is intrinsically defined, it must therefore vanish if $i \neq d$.

Now the well definedness of $\mu_k^{td_i}$ for $k \neq d$ is an easy consequence of the case $k=d$. Restrict attention to a rank k sublattice W with complement W'. Then $M=W^{\perp}\oplus(W')^{\perp}$, and we choose a primitive generator ω of $\bigwedge^{d-k} W^{\perp}$. The map

$$(W')^{\perp} \xrightarrow{\wedge\omega} \bigwedge^{d-k+1} M$$

$$x \longmapsto x\wedge\omega$$

extends to a map from $\mathrm{sym}^*(W')^{\perp}$ to $\mathrm{sym}^*(\bigwedge^{d-k+1} M)$. One sees that this map takes $\mu_k^{td_i}$ of $\sigma\in \mathrm{gr}_k^{\dim} \mathscr{SL}(W)$ to $\mu_k^{td_i}$ of σ considered as an element of $\mathrm{gr}_k^{\dim} \mathscr{SL}(N)$. Thus $\mu_k^{td_i}$ is well defined, apart from the ambiguity incurred in the choice of ω. This completes the proof.

Let us finally state the combinatorial consequences of the residue formulas. We have defined the measures $\mu_k^{td_k}$ to lie in the fraction field of $\mathrm{sym}^* \bigwedge^{d-k+1} M$, which may be considered the fraction field of coordinate ring of the projective space $\mathbf{P}(\bigwedge^{d-k+1} N)$. Actually, $\mu_k^{td_k}$ has degree 0 so it is a rational function on this projective space. Now the expression (19) from Section 3.2 is the evaluation of μ_k^{φ} on the top exterior power of the $(d-k+1)$-dimensional subspace H. Thus the local number $l(td_k, H_X, \overline{V(\sigma)})$ is a rational function of H which is obtained by restricting the rational function $\mu_k^{td_k}$ to the grassmannian $G_{d-k+1}(N)$ of $(d-k+1)$-planes in N, embedded in $\mathbf{P}(\bigwedge^{d-k+1} N)$ by the Plücker embedding. The residue theorem implies that if x is any point of $G_{d-k+1}(N)$, then $\sum_{\sigma\in\Delta(k)} \mu_k^{td_k}(\sigma)(x)[V(\sigma)]=Td^k(X_\Delta)$. Let $R(G_{d-k+1}(N))$ denote the field of rational functions on G_{d-k+1}. We have shown (see Proposition 3),

THEOREM 11. *For any toric variety,*

$$Td_{d-k}(X_\Delta)=\sum_{\sigma\in\Delta(k)} \mu_k^{td_k}(\sigma)[V(\sigma)],$$

where the sum on the right is evaluated in $A_{d-k}(X_\Delta)\otimes R(G_{d-k+1}(N))$.

Simple examples show that this expression is not generally valid in the "larger" group $A_{d-k}(X_\Delta)\otimes(\mathrm{sym}^*\bigwedge^{d-k+1} N)_0$. The ideal defining the Plücker embedding therefore comes into play; see, e.g., [GH, pp. 209–211].

More generally, any characteristic class is determined similarly by a μ_k^{φ} for some degree k symmetric polynomial φ. An equivariant characteristic "number" is equally well computed by a μ_d^{φ} via the formula in Section 3.3. The expressions μ_k^{φ} for which the degree n of φ has parity different from k involve an ambiguity. These expressions may be considered as sections of $\mathcal{O}(1)^{\otimes(n-k)}$. The ambiguity is removed by passing to the determinant line bundle of the dual of the canonical bundle. A point of this bundle determines a $(d-k+1)$-plane H together with an orientation on $M/H^{\perp}$.

Generically, the $(d-k+1)$-dimensional "edges" of the dual of a k-dimensional cone σ are naturally isomorphic to $M/H^{\perp}$, so these "edges" are provided with orientations which resolve the ambiguity in Definition 15. In any case, the μ_k^{φ} for k unequal to d or the degree of φ are a somewhat unnatural interpolation between the two geometrically meaningful cases.

COROLLARY 5. *For any element f of $\mathscr{L}$,*

$$\#_k(f) = (\overline{\mathrm{vol}}_k \otimes \mu_{d-k}^{td_{d-k}}) \circ \check{\Theta}_k(f)$$

so in particular for a convex polyhedron P

$$\#(P) = \sum_k \sum_{\substack{F \subseteq P \\ \dim F = k}} \overline{\mathrm{vol}}_k(F)\, \mu_{d-k}^{td_{d-k}}(\vartheta_F(P)^{\vee}).$$

For a worked example see the overview.

Remark. If one prefers real numbers to the function fields here, a point of the grassmannian which is generic over $\mathbf{Q}$ may be chosen, and the rational function $\mu_k^{td_k}$ evaluated there. This would define a real valued μ_k giving a local formula for the Todd class. The function fields are preferable because they are countable and functorial.

4.1. *Uniqueness*

It is natural to ask whether any symmetric polynomials other than the Todd polynomials give rise to well defined maps on $\mathrm{gr}_*^{\dim} \mathscr{S}\mathscr{L}$. Call a symmetric polynomial φ in k variables a *scissors polynomial* if the associated residue μ_k^{φ} passes to a homomorphism on $\mathrm{gr}_k^{\dim} \mathscr{S}\mathscr{L}(N)$, i.e., μ_k^{φ} is linear under subdivisions (a measure). We have the following partial converse to Theorem 10.

THEOREM 12. *Let N be a lattice of rank d and let φ be a homogeneous scissors polynomial of degree n in k variables, with $0 \leqslant n \leqslant d$. Then φ is a multiple of the Todd polynomial td_n.*

Proof. It is sufficient to assume $k = d$ as we saw in the proof of Theorem 10. Consider first the case $n = d$. It is well known that the Todd class $Td^d(M)$ is characterized among all characteristic classes by the property that it equals the generator of $H^{2d}(M; \mathbf{Z})$ whenever M is a product of projective spaces $\prod_i \mathbf{P}^{d_i}$ of dimension $\sum_i d_i$ equal to d. In particular, Td^d is the only characteristic class equaling the generator of $H^{2d}(X_\Delta; \mathbf{Z})$ for any (smooth) toric variety X_Δ. Therefore, the Todd polynomial is the only symmetric polynomial φ for which

$$\sum_{\sigma \in \Delta(d)} \mu_d^{\varphi}(\sigma) = 1$$

for every complete (nonsingular) fan Δ. Now in order for μ_d^φ to be well defined on $\mathrm{gr}_d^{\dim} \mathcal{S}\mathcal{L}$ we need that $\sum_{\sigma \in \Delta(d)} \mu_d^\varphi(\sigma)$ is independent of Δ. Hence, φ is a multiple of td_d.

We proceed by induction on d, keeping n fixed. Suppose that $\lambda = (\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_d \geqslant 0)$ is a partition of $n = \sum_i \lambda_i$. We say a monomial of degree n in $x_1, ..., x_d$ has shape λ if it is of the form $x_{i_1}^{\lambda_1} \cdots x_{i_d}^{\lambda_d}$ with $i_1, ..., i_d$ some permutation of the integers between 1 and d. Denote by p_λ the symmetric function

$$p_\lambda(x_1, ..., x_d) = \sum x_{i_1}^{\lambda_1} \cdots x_{i_d}^{\lambda_d},$$

the sum of all distinct monomials with shape λ. One knowns that as λ ranges over all partitions of n into d parts, the p_λ constitute an additive basis of the space of symmetric functions in $x_1, ..., x_d$ of degree n.

Choose a basis $y_1, ..., y_d$ of N, and denote by $x_1, ..., x_d$ the dual basis of M. Let N_1 be the span of the first s of the y_i, and N_2 the span of the remaining $t = d - s$ so that N is expressed as a direct sum $N = N_1 \oplus N_2$. If τ and σ are nonsingular simplices in N_1 and N_2, respectively, then

$$\mu_d^{p_\lambda}(\tau \times \sigma) = \sum_{\nu \cdot \rho = \lambda} \mu_s^{p_\nu}(\tau)\, \mu_t^{p_\rho}(\sigma), \tag{20}$$

where the sum is over all pairs ν, ρ of partitions (into s and t parts, respectively) whose juxtaposition is λ. The products on the right-hand side lie in the fraction field of the symmetric algebra on the dual M of N via the identification $\mathrm{sym}^* M_1 \otimes \mathrm{sym}^* M_2 = \mathrm{sym}^* M$. This is essentially the combinatorial content of a theorem of Thom about characteristic classes of direct sums of vector bundles; see [MiS, Sect. 16].

Let us apply (20) in particular when σ is a nonsingular simplex in the rank $d-1$ sublattice $x_d = 0$, and τ is the ray generated by y_d. A partition ν into one part is a nonnegative integer, and $\mu_1^{p_\nu}(\tau) = x_d^{\nu-1}$. If ρ is a partition, set

$$\iota_\nu p_\rho = \begin{cases} p_{\rho-\nu}, & \nu \text{ occurs in } \rho \\ 0 & \text{else,} \end{cases}$$

where $\rho - \nu$ is the partition obtained from ρ by deleting ν. The formula (20) takes the form

$$\mu_d^{p_\lambda}(\tau \times \sigma) = \sum_{\nu=0}^{\infty} x_d^{\nu-1} \mu_{d-1}^{\iota_\nu p_\rho}(\sigma).$$

By linearity we can extend the operations ι_ν to the entire space of symmetric polynomials. Therefore, more generally,

$$\mu_d^{\varphi}(\tau \times \sigma) = \sum_{\nu=0}^{\infty} x_d^{\nu-1} \mu_{d-1}^{\iota_\nu \varphi}(\sigma).$$

Now a subdivision of σ determines a subdivision of $\tau \times \sigma$, so if μ_d^φ respects subdivisions, then all of the terms $\mu_{d-1}^{\iota_v \varphi}$ must respect subdivisions, giving rise to homomorphisms on $\mathrm{gr}_{d-1}^{\dim} \mathscr{S}\mathscr{L}(N_2)$, in the sublattice $N_2 = x_d = 0$.

We have already considered the case $n = d$. By induction, we know that if $n \leqslant d-1$ then for each v, $\iota_v \varphi$ is a multiple of the Todd polynomial. In particular, $\iota_0 \varphi$ is a multiple of td_n. On the other hand, since $n < d$ any partition λ of n into d parts must contain at least one zero, so ι_0 is an injection. In fact, $\iota_0 \varphi(x_1, ..., x_{d-1}) = \varphi(x_1, ..., x_{d-1}, 0)$. Now it follows from definitions that $\iota_0 td_n(x_1, ..., x_d) = td_n(x_1, ..., x_{d-1})$ (as is already implicit in our notation). Therefore, φ is itself a multiple of td_n.

Remarks. 1. The assumption that $n \leqslant d$ is essential here. In dimension $d = 1$ it is trivial that any polynomial is scissors because there are no nontrivial decompositions of cones. However, the odd degree polynomials $p_3, p_5, ...$ are not multiples of Todd polynomials.

2. In dimension $d = 2$ subdivisions of cones have a rather simple structure, from which it follows that φ is a scissors polynomial exactly when

$$(x - y)\,\varphi(x, y) = x\varphi(x - y, y) - y\varphi(x, y - x).$$

The lowest degree of such a polynomial which is not a multiple of a Todd polynomial is 12, where the space of scissors polynomials is generated by td_{12} together with the polynomial $25x^{12} - 91x^{10}y^2 - 143x^6y^6 - 91x^2y^{10} + 25y^{12}$. After this the next such degree is 16, where we have the non-Todd scissors polynomial $539x^{16} - 1700x^{14}y^2 - 2431x^8y^8 - 1700x^2y^{14} + 539y^{16}$. The following two general facts about the case $d = 2$ hold:

(a) For n odd, every degree n scissors polynomial is a multiple of td_n, i.e., is a multiple of $p_{n-1,1}(x, y)$.

(b) For $n > 2$ even, all monomials in a scissors polynomial are of the form $x^{\text{even}}y^{\text{even}}$.

Since the Todd polynomials are known not to vanish it suffices for (a) to show that the space of scissors polynomials in each odd degree is one dimensional. We have already considered $n = 1$ so assume $n > 2$. The evaluation of μ_2^φ on any complete fan is independent of the fan and invariant under automorphisms of N, hence vanishes. Evaluating μ_2^φ on the fan of the Hirzebruch surface F_k, which has four rays generated by $(1, 0)$, $(0, 1)$, $(0, -1)$, and $(-1, k)$, yields

$$\frac{\varphi(x, y)}{xy} - \frac{\varphi(x, -y)}{xy} + y\frac{\varphi(-x, -kx - y)}{x(kx + y)} - y\frac{\varphi(-x, kx + y)}{x(kx + y)} = 0.$$

607/100/2-7

Writing $g(y) = \varphi(1, y) - \varphi(1, -y)$, we find that

$$(k+y)\, g(y) + (-1)^n\, g(k+y) = 0.$$

For n odd the substitution $y = 0$, $k = 1$ gives $g(0) = 0$, while $y = 1$ gives $(k+1)\, g(1) = g(k+1)$. Therefore, the space of such g is one dimensional. On the other hand, g determines the coefficients in φ of monomials of the form $x^{\text{even}} y^{\text{odd}}$, and the others are determined from these by symmetry. Therefore, we have established (a).

For (b), we substitute $k = 0$ in the equation above to obtain $2yg(y) = 0$, from which g vanishes identically. Therefore, all coefficients of monomials $x^{\text{odd}} y^{\text{odd}}$ vanish in $\varphi(x, y)$.

One is tempted to conjecture generally that if n and d have opposite parities then φ must be a multiple of the Todd polynomial.

3. Using these facts about degree 2 scissors polynomials and the method of the proof of the theorem to limit the possibilities, we find that, up to degree at least 17, the space of scissors polynomials in d variables is generated by Todd polynomials for all $d \geqslant 3$.

4. Using Danilov's factorization theorem [Dan2] on the birational geometry of toric 3-folds it can be shown that a polynomial φ in three variables is scissors exactly when

$$(x-y)\, \varphi(x, y, z) = x\varphi(x-y, y, z) - y\varphi(x, y-x, z)$$

and

$$\begin{aligned}(x-y)(x-z)(y-z)\, \varphi(x, y, z) = {} & (y-z)\, yz\varphi(x, y-x, z-x) \\ & - x(x-z)\, z\varphi(x-y, y, z-y) \\ & + xy(x-y)\, \varphi(x-z, y-z, z).\end{aligned}$$

5. The best we can hope is that if $\iota_v \varphi$ is scissors for each v, and if

$$\mu_d^{\varphi}(x_1, ..., x_d) = \sum_{i=1}^{d} \mu_d^{\varphi}(x_1 - x_i, ..., x_{i-1} - x_i, x_i, ..., x_d - x_i),$$

then φ is itself scissors. This amounts to requiring μ_d^{φ} to be invariant under equivariant blowups. It would be the case if any two complete toric varieties were connected by a sequence of equivariant blowups and blowdowns.[2]

6. By virtue of the remarks in Section 3.3, a polynomial is scissors exactly when the associated equivariant characteristic number is invariant under morphisms of toric varieties. As a corollary we therefore have that

[2] This has recently been established independently by J. Wlodarczyk and the author.

the only equivariant characteristic numbers which can be equivariant birational invariants (for varieties of all dimensions) are the Todd numbers.

4.2. *The Todd Measure and Volume*

By a *decomposition* of a lattice polyhedron P we mean a union $P = \bigcup P_i$ of lattice polyhedra P_i whose pairwise intersections have dimension at most $\dim P - 1$. We call a lattice polyhedron P *primitive* if it cannot be non-trivially decomposed. Any primitive polyhedron is a simplex, because any lattice polyhedron decomposes into lattice simplices. We define a primitive simplex to be *nonsingular* if it has volume $1/d!$. In dimension larger than 2, there are singular primitive simplices. For instance, the simplex with vertices

$$(0, ..., 0), \qquad e_1, ..., e_{d-1}, \qquad \text{and} \qquad (1, 1, ..., 1, k)$$

has volume $k/d!$. (Here, e_i is the ith standard basis vector.) In [KKMS-D] the difficult fact that any lattice polyhedron can be subdivided into nonsingular simplices after being sufficiently dilated is proven. The following theorem is a variation on this theme.

THEOREM 13. *The group $L(N)$ is generated by nonsingular simplices.*

Proof. We use a double induction on the rank d of N, and on the volume, or more properly, the integer $d! \times$ volume. The statement is trivial for $d = 0$ and also for volume $= 1/d!$. The induction hypothesis on d permits us to work modulo $F^{\dim}_{d-1} L$, in the group $\mathrm{gr}^{\dim}_d L = F^{\dim}_d L / F^{\dim}_{d-1} L$, so what we need to show is that any simplex σ in $\mathrm{gr}^{\dim}_d L(N)$ can be written as a sum of nonsingular simplices.

Let σ be a nondegenerate simplex $[v_0, ..., v_d]$ with vertices v_i. By the induction hypothesis, we may write the $(d-1)$-dimensional front face $[v_0, ..., v_{d-1}]$ of σ as a sum of nonsingular simplices. By suspending this sum with v_d, we reduce to the case of a simplex with nonsingular front face. Let us then assume $v_0 = 0$ and choose a basis $\{e_1, ..., e_d\}$ of N for which $e_1 = v_1, ..., e_{d-1} = v_{d-1}$, and $\langle e_d, v_d \rangle > 0$. Let $v_d = \sum a_i e_i$, and note in particular that $d! \operatorname{vol}(\sigma) = a_d$.

LEMMA 1. *If σ is primitive and singular then, after replacing it by its image under a suitable shear A,*

$$A = \begin{pmatrix} 1 & 0 & \cdots & 0 & c_1 \\ 0 & 1 & \cdots & 0 & c_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & c_{d-1} \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix} \in SL_d(N),$$

we can arrange that

$$-a_d < a_i < a_d \qquad \textit{for} \quad i < d$$

and

$$0 < a_1 + \cdots + a_{d-1} \leqslant a_d.$$

Proof. We have $Av_i = v_i$ for $i < d$ and $Av_d = (a_1 + c_1 a_d) e_1 + \cdots + (a_{d-1} + c_{d-1} a_d) e_{d-1} + a_d e_d$. We may first assume that the a_i for $i < d$ are all in the range $0 \leqslant a_i < a_d$, as this is a complete set of representatives mod a_d. If $a_1 = \cdots = a_{d-1} = 0$ then σ is nonsingular or nonprimitive, according as whether a_d is 1 or greater than 1. If on the other hand $a_1 + \cdots + a_{d-1} > a_d$ then we may replace positive terms a_i one by one with $a_i - a_d$ until the sum is brought into the desired range.

Returning to the proof of the theorem, assume that σ has volume greater than 1, and that the components of v_d satisfy the inequalities of the lemma. Set $v_{d+1} = v_d - a_d e_d$. Fix an orientation on N. For any collection of $d+1$ points $w_0, \ldots, w_d$ in N, we use the notation

$$(w_0, \ldots, w_d)$$

to mean $\pm [w_0, \ldots, w_d]$, the sign depending on whether the orientation determined by the w_i agrees with the fixed orientation on N. Then in $\mathrm{gr}_d^{\dim} L(N)$ we have

$$\begin{aligned} 0 &= \partial(v_0, \ldots, v_{d+1}) \\ &= (v_1, \ldots, v_{d+1}) - (v_0, v_2, \ldots, v_{d+1}) + \cdots + \mp(v_0, \ldots, v_d) \end{aligned}$$

(because the v_i are affinely dependent (see, e.g., [Mo1]). Since the final term on the right is $\mp\sigma$, we will be done if we can show that the other terms are generated by nonsingulars. Set $\omega = e_1 \wedge \cdots \wedge e_d$. As for the first term,

$$\begin{aligned} \mp d! \operatorname{vol}(v_1, \ldots, v_{d+1})\omega &= (v_1 - v_{d+1}) \wedge \cdots \wedge (v_d - v_{d+1}) \\ &= v_1 \wedge \cdots \wedge v_d \\ &\quad - \sum_{i=1}^{d} v_1 \wedge \cdots \wedge v_{i-1} \wedge v_{d+1} \wedge v_{i+1} \wedge \cdots \wedge v_d \\ &= a_d\omega - (a_1 + \cdots + a_{d-1})\omega \\ &= (a_d - a_1 - \cdots - a_{d-1})\omega. \end{aligned}$$

Since $0 \leqslant (a_d - a_1 - \cdots - a_{d-1}) < a_d = d! \operatorname{vol}(\sigma)$, this first term is handled by the induction hypothesis. The second to last term is a degenerate simplex. For the middle terms we have

$$\begin{aligned} \mp \operatorname{vol}(v_0, \ldots, \hat{v}_i, \ldots, v_{d+1})\omega &= v_1 \wedge \cdots \wedge \hat{v}_i \wedge \cdots \wedge v_{d+1} \\ &= \mp a_i a_d \omega \qquad \text{for} \quad i < d. \end{aligned}$$

On the other hand, each of these simplices contains the segment from v_d to v_{d+1} as an edge, and this segment breaks into a_d pieces. These simplices correspondingly break into a_d simplices, each of volume $\mp a_i$. Since $|a_i| < a_d = d!\ \mathrm{vol}(\sigma)$, these terms are also covered by the induction hypothesis, so we are done.

Remark. As a corollary, we obtain the result of Betke and Kneser [BK], which, in our language, states that $\mathscr{L}(M)$ modulo $\mathrm{Aut}(M) = SL_n(\mathbf{Z})$ is the free abelian group generated by the $d+1$ standard simplices, i.e., the simplices with vertices $0, e_1, ..., e_k$ for $0 \leqslant k \leqslant d$. This in turn implies Betke's theorem that a homomorphism from $\mathscr{L}(M)$ to $\mathbf{Q}$ which is invariant under $\mathrm{Aut}(M)$ is a linear combination of $\#_0, ..., \#_d$.

THEOREM 14. *For $0 \leqslant d \leqslant 4$ the functions $\mu_d^{td_d}$ factor the volume map through the lattice germ homomorphism*

$$\begin{array}{ccc} \mathscr{L}(N) & \xrightarrow{\ \vartheta\ } & \mathscr{S}\mathscr{L}(N) \\ & \underset{\mathrm{vol}}{\searrow} & \big\downarrow \mu_d^{td_d} \\ & & \mathbf{Q} \end{array}$$

(Here, N has rank d.) That is, $\mu_d^{td_d}(\vartheta(f))$ lies in $\mathbf{Q}$ for any $f \in \mathscr{L}(N)$ and

$$\overline{\mathrm{vol}}_d(f) = \mu_d^{td_d}(\vartheta(f)).$$

Proof. By Theorem 13 we may write any f in L as a virtual sum of nonsingular simplices. It therefore suffices to prove that for a nonsingular d-dimensional simplex σ, $\mu_d^{td_d}(\vartheta(\sigma)) = 1/d!$. By naturality, it is enough to check this for the standard simplex in $\mathbf{Z}^d$ whose vertices are the origin together with the d standard basis vectors. Let $x_1, ..., x_d$ be the dual basis in $(\mathbf{Z}^d)^\vee$ of the standard basis. The $d+1$ faces of this σ are determined by the $d+1$ primitive inequalities

$$x_1 \geqslant 0, ..., x_d \geqslant 0 \qquad \text{and} \qquad -(x_1 + \cdots + x_d) \geqslant 0.$$

Setting $x_{d+1} = -(x_1 + \cdots + x_d)$, we find that the lattice germ of σ is

$$\vartheta(\sigma) = \sum_{i=1}^{d+1} \sigma_i,$$

where σ_i is the simplical cone $\{x_1, ..., \hat{x}_i, ..., x_{d+1} \geqslant 0\}$. Applying to this the definition of $\mu_d^{td_d}$, we are reduced to showing the purely algebraic identity

$$\frac{1}{d!} = \sum_{i=1}^{d+1} \frac{td_d(x_1, ..., \hat{x}_i, ..., x_{d+1})}{x_1 \cdots \hat{x}_i \cdots x_{d+1}} \qquad \text{when} \quad \sum_{i=1}^{d+1} x_i = 0.$$

Equivalently,

$$\begin{aligned}\frac{1}{d!} &= \text{the zeroth degree term of} \\ &\sum_{i=1}^{d+1} \frac{td(x_1, ..., \hat{x}_i, ..., x_{d+1})}{x_1 \cdots \hat{x}_i \cdots x_{d+1}} \\ &= \sum_{i=1}^{d+1} \frac{td(x_1, ..., x_{d+1})}{x_1 \cdots x_{d+1}} (1 - e^{-x_i}) \\ &= \frac{td(x)}{c_{d+1}(x)} \sum_{i=1}^{d+1} (x_i - x_i^2/2! + \cdots) \\ &= \frac{td(x)}{c_{d+1}(x)} (s_1 - s_2/2! + \cdots),\end{aligned}$$

where $c_i(x)$ denotes the ith elementary symmetric function of the x's and where $s_i(x)$ denotes $x_1^i + \cdots + x_{d+1}^i$. Now, the c_i are algebraically independent generators of the ring of symmetric polynomials over $\mathbf{Q}$. The condition $\sum_{i=1}^{d+1} x_i = 0$ says that $c_1 = 0$, so we are really trying to prove the identity

$$c_{d+1}/d! = (d+1)\text{st degree term of } [td \cdot (s_1 - s_2/2! + \cdots)] \tag{21}$$

in $\mathbf{Q}[c_1, ..., c_d]/c_1 \cong \mathbf{Q}[c_2, ..., c_d]$. Using the well known relation (Newton's formula)

$$s_\nu - c_1 s_{\nu-1} + \cdots + (-1)^{\nu-1} c_{\nu-1} s_1 + (-1)^\nu \nu c_\nu = 0,$$

and working mod c_1, we find

$$\begin{aligned} s_1 &= c_1 = 0 \\ s_2 &= c_1 s_1 - 2c_2 = -2c_2 \\ s_3 &= \cdots = 3c_3 \\ s_4 &= \cdots = 2c_2^2 - 4c_4 \\ s_5 &= \cdots = -5c_2 c_3 + 5c_5 \\ s_6 &= \cdots = -2c_2^3 + 3c_3^2 + 6c_2 c_4 - 6c_6. \end{aligned}$$

The Todd polynomials may also be expressed in terms of the c_i, and in the first few cases these expressions are (see [Hi])

$$td_0 = 1$$
$$td_1 = c_1/2$$
$$td_2 = (c_1^2 + c_2)/12$$
$$td_3 = c_1 c_2/24$$
$$td_4 = (-c_4 + c_3 c_1 + 3c_2^2 + 4c_2 c_1^2 - c_1^4)/720$$
$$td_5 = (-c_4 c_1 + c_3 c_1^2 + 3c_2^2 c_1 - c_2 c_1^3)/1440$$
$$td_6 = (2c_6 - 2c_5 c_1 - 9c_4 c_2 - 5c_4 c_1^2 - c_3^2 + 11c_3 c_2 c_1 + 5c_3 c_1^3 + 10c_2^3 + 11c_2^2 c_1^2 - 12c_2 c_1^4 + 2c_1^6)/60{,}480$$

so mod c_1 we have

$$td_0 = 1$$
$$td_1 = 0$$
$$td_2 = c_2/12$$
$$td_3 = 0$$
$$td_4 = (-c_4 + 3c_2^2)/720$$
$$td_5 = 0$$
$$td_6 = (2c_6 - 9c_4 c_2 - c_3^2 + 10c_2^3)/60{,}480.$$

Performing the multiplication on the right side of 21 mod c_1 yields

$$\begin{aligned} td \cdot (s_1 - s_2/2! + \cdots) &= (1 + 0 + c_2/12 + \cdots)(0 + c_2 + c_3/2 - \cdots) \\ &= 0 + c_2 + c_3/2 + c_4/6 + c_5/24 \\ &\quad + (-3c_3^2 + 3c_2 c_4 + 6c_6)/720 + \cdots. \end{aligned}$$

The desired identity 21 thus holds for $d = 0, 1, 2, 3$, and 4.

Remark. 1. The last displayed term above shows that the Todd measure definitely fails for $d = 5$. In fact, because of the c_3^2 and $c_2 c_4$ terms, $\mu_5^{td_5}(\mathscr{P}(P))$ will not in general lie in $\mathbf{Q}$. The desired constant term $6c_6/720 = c_6/120$ does appear among these other terms so it may still be possible to extract the volume from the Todd measure for $d = 5$ and higher, but I do not know at present how to do this.

2. The identity 21 is reminiscent of the familiar identity

$$td_*(E)\, ch_*(\lambda_{-1}(E^\vee)) = c^{\text{top}}(E),$$

valid for a vector bundle E (or incidentally a sum of invertible elements in $\mathscr{L}$, using the natural λ-ring structure in $\mathscr{L}$ constructed in [Mo2]).

Acknowledgments

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The reader is referred in particular to [P], [KP], [B] for recent relevant work.

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