ADVANCES IN MATHEMATICS 100, 154–182 (1993)

The K Theory of a Toric Variety*

ROBERT MORELLI

Department of Mathematics, University of Chicago, 5734 University Avenue, Chicago, Illinois 60637

Contents.

1. INTRODUCTION

The discovery of toric varieties in the early 1970s began a new chapter in the interplay between combinatorics and algebraic geometry. The combinational nature of these varieties express itself in every aspect of their theory, which consists largely of a dictionary between geometric constructs (e.g., orbit structure, line bundles, cohomology) and combinatorial objects familiar from the classical theory of convex polyhedra. A window is thus opened between these two areas of mathematics, that indicates a seemingly improbable kinship. The purpose of this paper is to add a new entry to this dictionary by showing that the K-theory and equivariant K-theory of a toric variety can be naturally explained in terms of the scissors congruence theory of polyhedra. The latter theory, conversely, benefits from the importation of ideas from K-theory. This theme, and most of the combinatorial results necessary for our present purpose, are developed separately in [Mo2].

Recollection. Let us recall the well known results that are generalized here. Let T be a d-dimensional algebraic torus over $\mathbf{C}$, and let $M \approx \mathbf{Z}^d$ be its character group, imagined as the integer lattice embedded in $\mathbf{R}^d = M \otimes \mathbf{R}$. Suppose that X is a smooth, compact torus embedding of T. Let $\mathscr{E}$ be an ample line bundle on X to which the action of the torus extends, i.e., $\mathscr{E}$ is an equivariant line bundle. Then the action of the torus on the vector space $\Gamma(\mathscr{E})$ of global sections of $\mathscr{E}$ is multiplicity free. Furthermore, the weights of this representation are exactly the characters

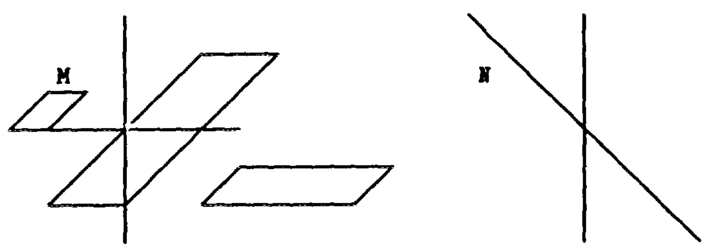
* This work was partially supported by NSF Postdoctoral Fellowship DMS 9007341.

0001-8708/93 $9.00

corresponding to the points of M which lie inside a certain convex polyhedron $P_{\mathscr{E}}$ whose vertices lie in M. The polyhedron $P_{\mathscr{E}}$ is of central importance in the theory; it determines X and $\mathscr{E}$ uniquely and captures much of their geometry in its combinatorics. The fan of X is the collection of cones in $N \otimes \mathbf{R} = (M \otimes \mathbf{R})^{\vee}$ dual to the cones along the faces of $P_{\mathscr{E}}$.

As the line bundle $\mathscr{E}$ varies, the polyhedron $P_{\mathscr{E}}$ varies in position and what we might call—to borrow a crystallographic term—its *habit*. How many faces it has, their incidences, and in which directions they face are fixed, determined by the fan of X. Only their sizes may vary. In other words, only the local geometry of the polyhedra is determined. There results a bijection between the set of all ample equivariant line bundles on X and the set of all habits of a certain polyhedral form.

Example. Below is the fan for a realization of $\mathbf{P}^1 \times \mathbf{P}^1$ as a toric variety, and $P_{\mathscr{E}}$ for three different $\mathscr{E}$'s.



Virtual Polyhedra. There is no way to associate polyhedra to non-ample (equivariant) line bundles, or to higher rank vector bundles, in a way that is compatible with the best features of the case of the ample line bundle. The main idea of this paper is to extend the construction by introducing virtuality, so that associated to a vector bundle is a formal sum of polyhedra.

By a formal sum of polyhedra I mean an element of the group $L(M)$ of functions on $\mathbf{R}^d \cong M \otimes \mathbf{R}$ generated over $\mathbf{Z}$ by the indicator functions 1_P of convex lattice polyhedra P:

$$1_P(x) = \begin{cases} 1, & x \in P \\ 0, & \text{otherwise.} \end{cases}$$

It can be shown that the only relations among the 1_P are

$$[1_{P \cup Q}] = [1_P] + [1_Q] - [1_{P \cap Q}],$$

which holds whenever P, Q, $P \cup Q$, $P \cap Q$ are all convex lattice polyhedra. The group $L(M)$ which is of independent combinatorial interest is studied in separate papers [Mo1, Mo2]. In addition to other structures, it carries a natural λ-ring structure.

Construction. The construction, which associates to an equivariant vector bundle an element of $L(M)$, is additive on exact sequences of bundles, so descends to the equivariant K-group. Denote the resulting homomorphism $\mathbf{I}_T$:

$$\mathbf{I}_T: K_T(X) \to L(M).$$

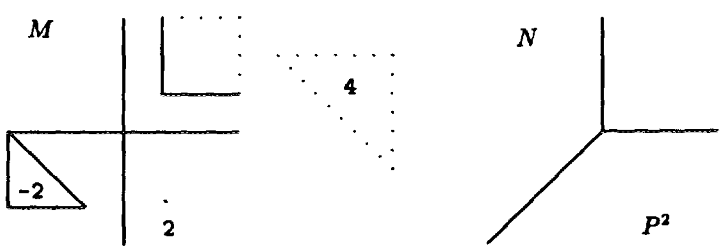
The formula for $\mathbf{I}_T(x)$, $x \in K_T(X_\Delta)$, evaluated at a point $m/k \in M_\mathbf{Q} = M \otimes \mathbf{Q}$, $m \in M$, k a positive integer, is

$$\mathbf{I}_T(x)(m/k) = \chi_m(\Psi^k(x)),$$

where χ_m is the weight m Euler characteristic and Ψ^k is the kth Adams operation (see Subsection 4.2).

Equivariant K-Group. Theorem 7 states that the homomorphism $\mathbf{I}_T$ is an injective λ-ring homomorphism. Its image in $L(M)$ is characterized by Theorem 8 in terms of a local condition on germs. (Although the functions in $L(M)$ are discontinuous, it still makes sense to speak of their germs). Specifically, a function f is in this image exactly when its germ at each point is in the linear space spanned by the duals of the cones in the fan of X. This local condition is the virtual analogue of allowing a polyhedron to vary in habit. In this way we obtain a complete combinatorial description of the equivariant K-theory of a smooth compact toric variety.

Example. The projective plane $\mathbf{P}^2$ can be realized as the toric variety with the fan at right. A typical element (or elements) of $K_T(\mathbf{P}^2)$ is depicted at left. The conventions are as follows. Numbers represent the value of $\mathbf{I}_T(x)$ in the indicated region. On an unnumbered bounded region the value is assumed to be 1. Dotted lines indicate that the value does not extend to the boundary of the region.



Picard Group. By comparison, the equivariant Picard group has the following well known description. The classical construction of Minkowski associates to a convex polyhedron in $M \otimes \mathbf{R} \cong \mathbf{R}^d$ its support function, which is a piecewise linear convex function on the dual space of $\mathbf{R}^d$. The support function of $P_{\mathscr{E}}$ is always linear on the individual cones of the fan, and is defined over $\mathbf{Z}$. It turns out that the space of *all* piecewise linear functions, which are linear on the individual cones and defined over $\mathbf{Z}$, is isomorphic to the equivariant Picard group. Virtuality manifest itself here as the passage from convex functions to arbitrary functions.

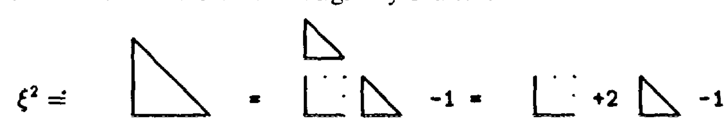
The ordinary Picard group of a toric variety is simply the equivariant group modulo M, where M is embedded in the group of piecewise linear functions as the globally linear functions. Adding a global linear function to a support function corresponds to translating the associated polyhedron by the vector defining the linear function. Therefore, to an ample line bundle without equivariant structure, one can associate a convex lattice polyhedron defined up to a translation by an element of M.

Ordinary K-Group. The relation between the ordinary and the equivariant K groups follows a pattern similar to that between the ordinary and the equivariant Picard groups. It is shown that $K_T(X)$ surjects on $K(X)$ and the map $\mathbf{I}_T$ descends to a map $\mathbf{I}$ from $K(X)$ to the group $\mathscr{L}(M)$ of coinvariants of $L(M)$ with respect to the natural action of M (Theorem 6). Recall that this latter group is defined as the quotient of $L(M)$ by the additional relations

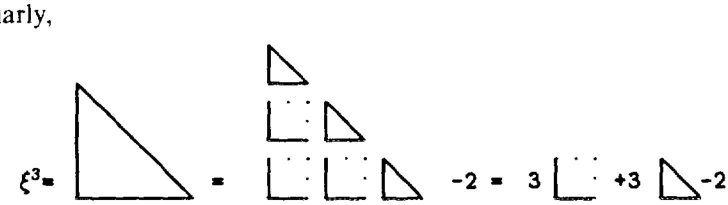
$$[1_{t_m P}] = [1_P],$$

where $m \in M$ and $t_m P$ is the translation of P by m. The map $\mathbf{I}$ is also injective, and its image is known since the image of $\mathbf{I}_T$ is known.

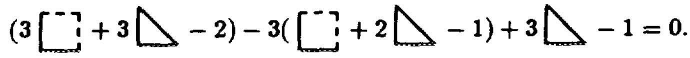
Example. One knows that the ordinary K ring of $\mathbf{P}^2$ is generated by the classes 1 and ξ represented by the trivial bundle and the line bundle $\mathscr{O}(1)$, respectively. These generators are subject to the sole relation $0 = (\xi - 1)^3 = \xi^3 - 3\xi^2 + 3\xi - 1$. In terms of polyhedra, the trivial bundle 1 corresponds to a point, while the ample line bundle $\xi = [\mathscr{O}(1)]$ corresponds to the standard triangle. The powers ξ^2 and ξ^3 correspond to the dilations of the standard triangle by 2 and 3. One has



the subtraced 1 being the point at which the two triangles overlap. Similarly,



The relation $\xi^3 - 3\xi^2 + 3\xi - 1 = 0$ is therefore represented pictorially as



The ordinary K group of a compact smooth toric variety X has been known in a weak sense via the chern character map, insofar as the intersection ring $A^*(X)$ has been computed by Jurkiewicz and Danilov. The description in terms of polyhedra is a complementary and in some ways more natural picture.

In the following section we will collect some notation and relevant basics about toric varieties. In so doing, we will introduce torus quotient embeddings, which are a trivial but, for us, very convenient generalization of torus embeddings. Our notation follows [Oda], which is a good general reference.

2. Toric Varieties—Recap and Notation

Intrinsic Definition. Let T be a d-dimensional complex algebraic torus, $T = T^d = (\mathbf{C}^*)^d$. A *torus embedding* is a normal algebraic variety X together with an action of T,

$$T \times X \to X$$

and an equivariant embedding of T as a dense open subvariety of X

$$T \hookrightarrow X.$$

If X is affine, the embedding is called affine. A variety X which admits a torus embedding is called a toric variety. By a *torus quotient embedding* we will mean a torus embedding of a quotient T/T_0 of T be an algebraic subtorus T_0.

The category of torus quotient embeddings of a given torus has equivariant algebraic maps as morphisms. More generally, if $T \hookrightarrow X$ and $T' \hookrightarrow X'$ are torus quotient embeddings of T and T', respectively, then a morphism of embeddings is a homorphism $f: T \to T'$ of algebraic tori, together with an equivariant map $F: X \to X'$, for which the diagram

$$\begin{array}{ccc} T' & \hookrightarrow & X' \\ {\scriptstyle f}\uparrow & & \uparrow{\scriptstyle F} \\ T & \hookrightarrow & X \end{array}$$

commutes.

Fans. The classification theorem identifies the category of toric varieties and torus quotient embeddings with a category of combinatorial objects called fans. We recall the basic definition pertaining to this combinatorial category.

Fix a lattice $N = \mathbf{Z}^d$. By a *rational polyhedral cone* (or simply cone) σ we mean the convex hull in $\mathbf{R}^d$ of a collection of rays defined over Z,

$$\sigma = \mathbf{R}^+ v_1 + \cdots + \mathbf{R}^+ v_m.$$

The *span* of σ is the subspace $\mathbf{R}v_1 + \cdots + \mathbf{R}v_m$ generated by σ, and the *cospan* of σ is the largest subspace contained in σ. A rational cone is called *strongly convex* if its cospan is $\{0\}$.

DEFINITION 1. A fan in N is a nonempty collection $\Delta = \{\sigma\}$ of rational cones satisfying two properties

(1) If σ is in Δ then so are all faces of σ.

(2) If σ and τ are in Δ then $\sigma \cap \tau$ is a face of each of σ and τ.

The support of Δ is the set $|\Delta| = \bigcup_{\sigma \in \Delta} \sigma \subseteq \mathbf{R}^d$. We write $\Delta(k)$ to denote the subset of k dimensional cones in Δ.

Notice that all the cones in a fan have the same cospan. I will call a fan strongly convex if all of its cones are strongly convex. This terminology departs from the literature, where strong convexity is ordinarily taken as part of the definition of fan.

The set of cones in N is a category with inclusions as morphisms. (Hence, there is at most a single morphism between two objects.) More generally, if σ is in N_1 and τ is in N_2, then define a morphism from σ to τ to be a homomorphism $f: N_1 \to N_2$ such that $f(\sigma)$ is included in τ. We can likewise define the category of fans in N: there is a unique morphism from Δ to Γ if for each $\sigma \in \Delta$ there is at least one $\tau \in \Gamma$ for which $\sigma \subseteq \tau$. It is clear now how to define morphisms between fans in different lattices. The set of all faces of a cone is a fan, and the category of cones is thereby embedded as a subcategory of the category of fans.

Classification. The set M of characters of the torus, $\{e: T \to \mathbf{C}^*\}$, is a lattice $M \approx \mathbf{Z}^d$ under pointwise multiplication. The dual group $N = M^\vee = \mathrm{Hom}_{\mathbf{Z}}(M, \mathbf{Z})$ is canonically identified with the group of one parameter subgroups of T, $\{\lambda: \mathbf{C}^* \to T\}$, under pairing $\langle e, \lambda \rangle = \deg(e \circ \lambda) \in \mathbf{Z}$. Now, $T^d = N \otimes \mathbf{C}^* = \mathrm{Spec}\, \mathbf{C}[M] = \mathrm{Spec}\, \mathbf{C}[z_1, z_1^{-1}, ..., z_d, z_d^{-1}]$, where $\mathbf{C}[M]$ is the group ring of M over $\mathbf{C}$.

THEOREM 1. *The categories of torus quotient embeddings, torus embeddings, and affine torus embeddings of T are equivalent to the categories of fans, strongly convex fans, and cones in N, respectively.*

For any rational cone $\sigma \subseteq \mathbf{R}^d = N \otimes \mathbf{R}$, there is a dual cone $\sigma^\vee \subseteq M \otimes \mathbf{R}$

$$\sigma^\vee = \{x \in M \otimes \mathbf{R} : \langle x, y \rangle \geqslant 0 \ \forall y \in \sigma\}.$$

The set $\sigma^\vee \cap M$ is a semigroup under addition, and finitely generated (Gordon's lemma), so the spectrum of the semigroup ring $\mathbf{C}[\sigma^\vee \cap M]$, $\operatorname{Spec} \mathbf{C}[\sigma^\vee \cap M]$, is an affine variety. It is in fact a torus quotient embedding—a torus embedding provided cospan $(\sigma) = \{0\}$. A fan is then essentially a gluing diagram for a torus quotient embedding.

Nonsingularity. Let c be the cospan of a fan Δ in N. Then Δ evidently determines a fan in $N/(c \cap N)$ which we call the quotient of Δ by its cospan.

DEFINITION 2. A strongly convex cone is called nonsingular if it is generated by a subset of a $\mathbf{Z}$ basis of N. A strongly convex fan is nonsingular if all its cones are nonsingular. A fan is generally called nonsingular if the quotient by its cospan is nonsingular.

Orbits. An important fact about affine toric varieties is that there is a unique minimal orbit $V(\sigma) \subseteq X_\sigma$ of dimension equal to the condimension of the span of σ. Another perspective on the fan results from this: X_Δ is the disjoint union of the orbits $V(\sigma)$, $\sigma \in \Delta$. In particular, the d dimensional cones correspond to the fixed points while the cospan of the fan, its minimal cone, corresponds to the dense orbit. If the fan is strongly convex then the 1 dimensional cones correspond to the invariant divisors. As usual, the orbits may be partially ordered by the relation "is included in the closure of."

Summarizing,

THEOREM 2. (1) *The category of fans in N is naturally equivalent to the category of torus quotient embeddings of the torus $N \otimes \mathbf{C}$.* (2) *A torus quotient embedding is nonsingular if and only if its fan is nonsingular.* (3) *A torus quotient embedding is compact if and only if the support of its fan is all $N_\mathbf{R} = N \otimes \mathbf{R}$.* (4) *There is an order-reversing bijection between the partially ordered set of cones in a given fan and the set of orbits in the corresponding torus quotient embedding.*

Fan of an Orbit Closure. The closure of an orbit $V(\sigma)$ is a torus quotient embedding of T in a natural way. We show this by simply describing its fan Δ_σ,

$$\Delta_\sigma = \{\tau + \operatorname{span}(\sigma) : \sigma \subseteq \tau \in \Delta\},$$

where $\operatorname{span}(\sigma)$ is the linear subspace spanned by σ. The cospan of this fan is $\operatorname{span}(\sigma)$. On the level of spectra, the inclusion morphism is described as follows. Denote the cone $\tau + \operatorname{span}(\sigma)$ by τ_σ. Then $\tau_\sigma^\vee \subseteq \tau^\vee$ is a face. Map $\mathbf{C}[\tau^\vee \cap M]$ to $\mathbf{C}[\tau_\sigma^\vee \cap M]$ by sending m to m if $m \in \tau_\sigma^\vee \cap M$ and to 0 otherwise.

Cohomology of Line Bundles. We quote here one final result which will play an important role:

THEOREM 3. *Suppose that X_Δ is a compact toric variety, so that $|\Delta| = N_{\mathbf{R}} = N \otimes \mathbf{R}$. Let h be a function on $N_{\mathbf{R}}$ which restricts to a linear function on each cone in Δ, and which takes integral values on N. Such a function is called Δ-linear. Then,*

(1) *There exists a T-equivariant line bundle $\mathscr{E}_h$ on X_Δ whose sections over the open set U_σ are spanned by all the characters $e(m)$ for which m satisfies $m(n) \geq h(n)$, $\forall n \in \sigma$.*

(2) *Every equivariant line bundle on X_Δ is obtained this way. Call h the* support function *of $\mathscr{E}_h$.*

(3) *The cohomology of $\mathscr{E}_h$ splits into weights spaces under the action of the torus. For a character $m \in M$, the associated weight space is canonically identified as*

$$H^k(X_\Delta, \mathscr{E}_h)_m = H^k_{Z(m,h)}(N_{\mathbf{R}}, \mathbf{C}),$$

where the right side is the local cohomology of $N_{\mathbf{R}}$ with respect to the subset $Z(m, h) = \{n \in N_{\mathbf{R}} : \langle m, n \rangle \geq h(n)\}$.

For a proof, see [Oda, p. 75].

3. K, K', AND K_T

In this and all subsequent sections all toric varieties are compact unless otherwise stated.

For an algebraic variety X, denote by $K(X)$ the grothendieck group of the category of finitely generated locally free sheaves (i.e., algebraic vector bundles) on X. This means that $K(X)$ is the abelian group generated by expressions $[E]$ for E an algebraic vector bundle, with a relation $[E] - [F] + [G] = 0$ for every exact sequence

$$0 \to E \to F \to G \to 0.$$

There is also a natural ring structure in $K(X)$ induced by $[E][F] = [E \otimes F]$. Denote by $K'(X)$ the grothendieck group of coherent sheaves on X.

The functor K is naturally contravariant, as vector bundles may be pulled back along morphisms, while K' is naturally covariant for proper morphisms f, through the formula

$$f_K[\mathscr{F}] = \sum_{n \geq 0} (-1)^n [R^n f_* \mathscr{F}], \tag{1}$$

where $R^n f_*$ are the right derived functors of the direct image functor f_*.

We will also consider the equivariant analogues of these groups. Let X be a scheme with an algebraic action of an algebraic group G, $\alpha: G \times X \to X$. Recall that an equivariant vector bundle is a vector bundle together with an action of G which is equivariant with respect to the projection map, and which maps the fibers to each other linearly. More generally, we can define an equivariant sheaf to be a sheaf $\mathscr{F}$, together with a given isomorphism of sheaves on $G \times X$, $h: \alpha^* \mathscr{F} \xrightarrow{\sim} pr_2^* \mathscr{F}$. (The map pr_2 here is projection onto the second factor.) This isomorphism is required to satisfy the associativity condition: $(pr_{23}^* h) \circ ((1 \times \alpha)^* h) = (mult \times 1)^* h$ on $G \times G \times X$. The two notions of equivariance coincide on locally free sheaves.

Denote by $K_G(X)$ the grothendieck group of equivariant algebraic vector bundles on X, and by $K'_G(X)$ the grothendieck group of coherent equivariant sheaves on X. These groups have been systematically investigated by Thomason, see [Th]. In our special case, G will be the algebraic torus T, and X will be a torus embedding X_Δ of T. The formula 1 still makes sense in the equivariant setting. We will denote the equivariant pushforward f_{K_T}.

The obvious natural map from $K(X)$ to $K'(X)$, sending a generator $[E]$ to $[\mathscr{F}]$ where $\mathscr{F}$ is the dual of the sheaf of sections of E, is called the Poincaré homomorphism. The general fact is that the Poincaré homomorphism is an isomorphism for any smooth variety (see [B2, Theorem 3.4]). The proof depends on the fact that any coherent sheaf is the quotient of a locally free sheaf. This latter fact is true for any variety which is divisorial in the sense of [B1]. Adapted to our case, the argument (of [B2]) also yields the equivariant version of this fact, from which we will conclude that the equivariant Poincaré homomorphism is an isomorphism.

PROPOSITION 1. *Let X_Δ be a d-dimensional toric variety for which the fan Δ consists of simplicial cones (cones generated by linearly independent rays). Then every coherent (respectively coherent equivariant) sheaf is the quotient of a locally free (resp. locally free equivariant) sheaf.*

Proof. Let $\sigma \in \Delta(d)$. The open set U_σ is affine and invariant and coincides with the complement of $\bigcup_{\varrho \not\subseteq \sigma} V(\varrho)$, where the union is over all one dimensional cones ϱ not contained in σ. The U_σ cover X_Δ.

Consider the divisor $D_\sigma = \sum_{\varrho \not\subset \sigma} \overline{V(\varrho)}$. In general it will not be a Cartier divisor, but some multiple of it will be. In fact, if $\mu \in \Delta(d)$ and $\{m(\varrho)\}_{\varrho \in \mu}$ is the basis of $M_{\mathbf{Q}}$ dual to the basis $\{n(\varrho)\}_{\varrho \in \mu}$ of $N_{\mathbf{Q}}$, then for some positive integer k, all the multiples $k \cdot m(\varrho)$ fall within M. For that k, kD_σ is principal on U_μ generated by $\prod_{\varrho \in \mu - \sigma} e(k \cdot m(\varrho))$. Let $\mathscr{E}_\sigma$ be the line bundle associated to some multiple of D_σ which is locally principal, and let s be the constant function 1, considered as a section of $\mathscr{E}_\sigma$. It is invariant, and vanishes on $\overline{V(\varrho)}$ if and only if $\varrho \not\subset \sigma$, so the zero section of s is exactly the complement of U_σ.

Let $\mathscr{F}$ be an (equivariant) coherent sheaf on X_Δ. Since $U_\sigma = \operatorname{spec} A_\sigma$ is affine, the space $B = \Gamma(U_\sigma, \mathscr{F})$ of sections over U_σ is finite dimensional. Choose generators $s_1, ..., s_k$ of B. For each i, there is an integer d such that $s_i \otimes s^{\otimes d}$ extends to a global section of $\mathscr{F} \otimes \mathscr{E}_\sigma^{\otimes d}$. It follows that there is some integer d_σ such that $\mathscr{F} \otimes \mathscr{E}_\sigma^{\otimes d_\sigma}$ is generated on U_σ by its global sections.

Write Γ_σ for the vector space $\Gamma(X, \mathscr{F} \otimes \mathscr{E}_\sigma^{\otimes d_\sigma})$. Then we have a sequence of sheaves

$$\mathcal{O}_X \otimes \Gamma_\sigma \to \mathscr{F} \otimes \mathscr{E}_\sigma^{\otimes d_\sigma} \to \mathscr{C}_\sigma \to 0$$

in which the cokernel $\mathscr{C}_\sigma$ is supported outside U_σ. In the equivariant case, the torus T acts algebraically on Γ_σ, so the sequence is naturally a sequence of equivariant sheaves. In any case, tensoring with the (equivariant) line bundle $\mathscr{E}_\sigma^{*\otimes d_\sigma}$ gives

$$\mathscr{E}_\sigma^{*\otimes d_\sigma} \otimes \mathcal{O}_X \otimes \Gamma_\sigma \to \mathscr{F} \to \mathscr{C}_\sigma \otimes \mathscr{E}_\sigma^{*\otimes d_\sigma} \to 0$$

from which

$$\bigoplus_\sigma \mathscr{E}_\sigma^{*\otimes d_\sigma} \otimes \mathcal{O}_X \otimes \Gamma_\sigma \to \mathscr{F} \to 0,$$

since the U_σ cover X_Δ.

PROPOSITION 2. *The Poincaré map and the equivariant Poincaré map are isomorphisms on any smooth toric variety X_Δ. Hence,*

$$K(X_\Delta) \cong K'(X_\Delta)$$

and

$$K_T(X_\Delta) \cong K'_T(X_\Delta).$$

Proof. This proposition follows from the previous one by the well known argument of [BS, Theorem 2], whose equivariant version is also valid. The idea is to apply the previous proposition repeatedly to resolve a coherent sheaf into locally free sheaves, and show that the alternating sum of the terms of this resolution is well defined, giving an inverse to the Poincaré homomorphism. This argument is presented and axiomatized in [L, Theorem 3.7].

The equivariant K-ring of a point with the trivial action of T^d is the representation ring of T^d, which is $\mathbf{Z}[M]$. Through the projection of X_Δ onto a point, $K_T(X_\Delta)$ has a canonical $\mathbf{Z}[M]$ module structure.

PROPOSITION 3. *Let X_Δ be a smooth torus quotient embedding.*

(1) *The equivariant K group $K_T(X_\Delta)$ is additively generated (over* $\mathbf{Z}$*) by the classes of equivariant line bundles.*

(2) *The natural map*

$$K'_T(X_\Delta) \to K'(X_\Delta)$$

which forgets equivariant structure is a surjection.

Proof. For part (1) we use the following fact, proved in [Th, Theorem 2.7]. For any closed invariant subset Z of a T-variety X with complement U, there is an exact sequence

$$K'_T(Z) \xrightarrow{i_*} K'_T(X) \xrightarrow{j^*} K'_T(U) \longrightarrow 0, \qquad (2)$$

where i denotes the (equivariant) closed imbedding of Z in X, and j denotes the inclusion of U.

Define a decreasing filtration of $K_T(X_\Delta)$ by setting $F^k_T K_T(X_\Delta)$ equal to the subgroup generated by elements which can be represented by equivariant sheaves whose support is contained in the closure of the union of the codimension k orbits. It follows from the exact sequence quoted above that the kth graded component of $\mathrm{gr}^*_T K_T$ is some quotient group of $\bigoplus_{\sigma \in \Delta(k)} K_T(V(\sigma))$. On the other hand, $V(\sigma)$ is just T/T_σ where T_σ is the isotropy group of the orbit, a homogeneous space for the torus. Now the equivariant K-group of such a homogeneous space is well known to be $\mathbf{Z}[M_\sigma]$, the group ring of the character group of T_σ. As a module over $\mathbf{Z}[M]$, $K_T(V(\sigma))$ is generated by the trivial bundle with trivial T action, which corresponds (mod F^{k+1}_T) to the sheaf $\mathcal{O}_{\overline{V(\sigma)}}$ with its natural action. It will therefore suffice to exhibit these bundles, on all k-dimensional orbits $V(\sigma)$, as sums of equivariant line bundles (mod F^{k+1}_T).

The locally free sheaf $\bigoplus_{\tau \subseteq \sigma,\, \dim \tau = 1} \mathcal{O}(\overline{V(\tau)})$ has an invariant section s whose component in $\mathcal{O}(\overline{V(\tau)})$ is the constant function 1. The zero section of s is $\overline{V(\sigma)}$, so we obtain the equivariant Koszul complex

$$\cdots \to \bigoplus_{\substack{\tau \subseteq \sigma \\ \dim \tau = 2}} \mathcal{O}(-\overline{V(\tau)}) \to \bigoplus_{\substack{\tau \subseteq \sigma \\ \dim \tau = 1}} \mathcal{O}(-\overline{V(\tau)}) \to \mathcal{O}_{\overline{V(\sigma)}} \to 0$$

which, as is easy to check in local coordinates, is a resolution of $\mathcal{O}_{\overline{V(\sigma)}}$ with its natural action of T.

For part (2) we use the nonequivariant version of the exact sequence quoted above:

$$K'(Z) \xrightarrow{i_*} K'(X) \xrightarrow{j^*} K'(U) \longrightarrow 0.$$

As above $K(X_\Delta)$ if filtered by setting $F^k K(X_\Delta)$ equal to the subgroup generated by elements which can be represented by sheaves whose support is contained in the closure of the union of the codimension k orbits. The kth graded component of $\mathrm{gr}^* K$ is some quotient group of $\bigoplus_{\sigma \in \Delta(k)} K(V(\sigma))$. In this case, $K(V(\sigma)) = \mathbf{Z}$. The same argument as above shows that $K(X_\Delta)$ is generated by line bundles. Since these may all be endowed with equivariant structures, we are done.

Let X_Δ be a smooth torus embedding. The ring $A^*(X_\Delta) = \bigoplus_k A^k(X_\Delta)$ of cycles modulo rational equivalence has been entirely calculated by Jurkiewicz (projective case) and Danilov (see [Dan]). Their result states that $A^k(X_\Delta)$ is generated by the classes of invariant cycles $[V(\sigma)]$, $\sigma \in \Delta(k)$, with two types of relations:

(1) for $\sigma \in \Delta(k)$ and $\tau \in \Delta(l)$,

$$[V(\sigma)] \cdot [V(\tau)] = \begin{cases} [V(\sigma + \tau)] & \text{if } \sigma + \tau \in \Delta(k+l) \\ 0 & \text{otherwise} \end{cases}$$

(where $\sigma + \tau$ is the convex hull of $\sigma \cup \tau$).

(2) For any $m \in M$,

$$\sum_{\varrho \in \Delta(1)} \langle m, n(\varrho) \rangle [V(\varrho)] = 0,$$

where $n(\varrho)$ is the unique primitive element of N lying in the one dimensional cone ϱ.

This calculation shows in particular that these groups are torsion free.

LEMMA 1. *If X_Δ is a smooth torus quotient embedding, then $K(X_\Delta)$ is free of torsion.*

Proof. It is known generally that if the group $A^*(X)$ is free of torsion then so is $K(X)$. This follows from the existence of maps $\varphi: A^*(X) \to \mathrm{gr}^* K(X)$, taking a cycle $[V]$ to the structure sheaf $\mathcal{O}_V$, and $\psi: \mathrm{gr}^* K(X) \to A^*(X)_\mathbf{Q}$, which satisfy $\psi(\varphi(x)) = i(x)$ where i is the natural inclusion of A into $A_\mathbf{Q}$. The map φ is generally a surjection. See [SGA6, Exposé IX, 4.2; Fu, Example 15.1.5 and 15.2.16] for more details.

PROPOSITION 4. *For X_Δ smooth, the quotient of $K_T(X_\Delta)$ by M, i.e., the group of coinvariants $K_T(X_\Delta)_M$, is $K(X_\Delta)$.*

Proof. With no loss of generality, we may assume that Δ is a strongly convex fan. We have already seen that $K_T(X_\Delta)$, hence $K_T(X_\Delta)_M$, surjects on $K(X_\Delta)$. It is sufficient to prove that there is an isomorphism of the associated graded groups. The map φ mentioned in the last proof is actually a ring homomorphism, which in this case is an isomorphism since $K(X_\Delta)$ is torsion free. (In this context it is convenient to bear in mind the homological description of multiplication in $K'(X_\Delta)$.) As a ring, $\mathrm{gr}^* K(X_\Delta)$ is generated by $\mathcal{O}_{\overline{V(\sigma)}}$ for $\sigma \in \Delta(1)$, and there are two kinds of relations on these generators corresponding to the two kinds of relations on the generators of $A^*(X_\Delta)$. We will show that these two kinds of relations have equivariant versions. The first relations (i) clearly hold on the nose in $\mathrm{gr}_T^* K_T(X_\Delta)$.

For the relations of the form (ii), let $m \in M$. Denote by D_0 the divisor of zeros of $e(m)$, i.e., m considered as a rational function on X_Δ. This is just $\sum_{\varrho \in \Delta(1)^+} \langle m, n(\varrho)\rangle [V(\varrho)]$ where $\Delta(1)^+$ consists of all $\varrho \in \Delta(1)$ for which $\langle m, n(\varrho)\rangle$ is positive. Similarly, denote by D_∞ the divisor of poles. Now $e(m) \cdot \mathcal{O}(-D_\infty) = \mathcal{O}(-D_0)$, while in $\mathrm{gr}_T^1 K_T(X_\Delta)$, $[\mathcal{O}(-D_0)] = 1 - \sum_{\varrho \in \Delta(1)^+} \langle m, n(\varrho)\rangle [\mathcal{O}_{\overline{V(\varrho)}}]$, and similarly for $[\mathcal{O}(-D_\infty)]$. Therefore,

$$\sum_{\varrho \in \Delta(1)^+} \langle m, n(\varrho)\rangle [\mathcal{O}_{\overline{V(\varrho)}}] + m \cdot \sum_{\varrho \in \Delta(1)^-} \langle m, n(\varrho)\rangle [\mathcal{O}_{\overline{V(\varrho)}}] = (1-m) \cdot 1.$$

Since this reduces in $K_T(X_\Delta)_M$ to the relation (ii), the map $\mathrm{gr}^* K_T(X_\Delta)_M \to \mathrm{gr}^* K(X_\Delta)$ is injective.

LEMMA 2. *Suppose X_Δ is smooth and projective. Then there exist elements $x_1, \ldots, x_n$ of $K_T(X_\Delta)$ which generate $K_T(X_\Delta)$ over $\mathbf{Z}[M]$ and whose images in $K(X_\Delta)$ constitute a basis of $K(X_\Delta)$ over $\mathbf{Z}$.*

Proof. It is known (see [Ju, Dan]) that every projective toric variety admits a filtration by closed subvarieties

$$X_0 \subseteq X_1 \subseteq \cdots \subseteq X_r = X_\Delta$$

satisfying the following properties:

(1) X_i is T-invariant

(2) the successive differences $Y_i = X_i - X_{i-1}$ are affine spaces of the form $\bigcup_{\rho_i \subseteq \tau \subseteq \sigma_i} V(\tau)$ where $\rho_i \in \Delta$ and $\sigma_i \in \Delta(d)$

(3) the cycles $[\overline{Y_i}] = [V(\rho_i)]$ constitute a basis of $A^*(X_\Delta)$.

It follows that the classes $[\mathcal{O}_{\overline{Y_i}}]$ constitute a basis of $K(X_\Delta)$. On the other hand, it is easy to see that $K_T(Y_i)$ is just the character group of the isotropy group of $V(\rho_i)$. (For this one can even use that bundles on affine space are trivial.) By the exact sequence (2) this implies that $K_T(X_i)$ is generated over $\mathbf{Z}[M]$ by $[\mathcal{O}_{\overline{Y}_i}]$ together with sheaves supported on X_{i-1}. By induction, the $[\mathcal{O}_{\overline{Y}_i}]$ generate $K_T(X_\Delta)$ over $\mathbf{Z}[M]$.

4. The Correspondence

4.1. *Polyhedral Functions*

This section contains a synopsis of the combinatorics in which the main results are stated. A fuller treatment is undertaken in [Mo2], where relevant proofs may be found

Definition. Let M denote a lattice of rank d. We will refer to elements of M as lattice points, and to polyhedra in $M_{\mathbf{R}} = M \otimes \mathbf{R}$ all of whose vertices lie in M as lattice polyhedra.

Definition 3. Define an abelian group $L(M)$ by taking as generators the symbols $[P]$ for all convex lattice polyhedra P, and relations

$$[P \cup Q] = [P] + [Q] - [P \cap Q] \tag{3}$$

whenever P, Q, $P \cup Q$, and $P \cap Q$ are all convex lattice polyhedra.

We will also denote by $L(M_Q)$ the analogous group generated by polyhedra whose vertices lie in M_Q.

By abuse, we will sometimes identify a convex polyhedron P with its image $[P]$ in $L(M)$.

An alternate description of this group is also useful. Define the indicator function of a polyhedron P to be

$$1_P(x) = \begin{cases} 1, & x \in P \\ 0, & \text{otherwise.} \end{cases}$$

The indicator functions of all lattice polyhedra generate a subgroup of the group of all (discontinuous) $\mathbf{Z}$-valued functions on M_Q. It is proven in [Mo2] that the map sending $[P]$ to 1_P is an isomorphism from $L(M)$ onto this subgroup. For this reason, we will refer to elements of $L(M)$ as polyhedral functions, and treat them, when convenient, as functions.

Euler Characteristic. There is a unique homomorphism $\chi: L(M) \to \mathbf{Z}$ which maps each convex lattice polyhedron $[P]$ to 1. If P is an arbitrary lattice polyhedron then $\chi(1_P)$ is the ordinary topological Euler characteristic of P.

Direct Image. Given a homomorphism $\pi: M \to M'$ there is induced a homomorphism $\pi_L: L(M) \to L(M')$ defined as the unique homomorphism which takes a convex polyhedron $[P]$ to $[\pi_{\mathbf{R}}(P)]$, the image of P under $\pi_{\mathbf{R}} = \pi \otimes \mathbf{R}$. This may be rephrased as $\pi_L(f)(x) = \chi(f|_{\pi^{-1}(x)})$ for any $f \in L(M)$ and $x \in M'_{\mathbf{Q}}$.

Coinvariants of $L(M)$.

DEFINITION 4. Define the group $\mathscr{L}(M)$ to be the quotient of $L(M)$ by the relations

$$[t_m P] = [P],$$

where m is any element of M, P is any lattice polyhedron, and $t_m P$ denotes the translation of P by m.

The group $\mathscr{L}$ is simply the group of coivariants of L with respect to the natural action of M, $\mathscr{L}(M) = L_M = H_0(M, L(M))$. It is similar to the group of scissors congruence classes of polyhedra studied in connection with Hilbert's third problem. In contrast to euclidean scissors congruence, where a polyhedron is considered equivalent to any subdivision, rotation, or translation of itself, we only allow subdivisions and translations. The resulting *translation scissors congruence* problem was studied and solved in low dimensions by Hadwiger who introduced what are now called Hadwiger invariants. Jessen and Thorup, and independently Sah, solved the problem in all dimensions, generalizing Hadwiger's work. The scissors problem posed by $\mathscr{L}(M)$ differes from the classical case in two respects. First, polyhedra of dimension smaller than d are not discarded. Second, the polyhedra and subdivisions are restriced to a lattice. This prolem was solved in [Mo1].

Hadwiger Invariants. Let us now define what we mean by Hadwiger invariants in the context of $\mathscr{L}(M)$. Define a *rigged hyperplane* to be a hyperplane $U \subseteq M_{\mathbf{R}}$ together with a choice of one of the two halfspaces bounded by U. A *rigged flag* of length k is a flag of subspaces $\mathbf{F} = (M_{\mathbf{R}} = U^0 \supseteq U^1 \supseteq \cdots \supseteq U^k)$, where U^i is a rigged hyperplane in U^{i-1} for $i = 1, ..., k$. In particular, codim $U^i = i$.

If U^1 is a rigged hyperplane, choose a linear functional $\xi \in N_{\mathbf{R}} = M_{\mathbf{R}}^{\vee}$ whose kernel is U^1 and which is positive on the chosen halfspace bounded by U^1. If P is a convex polyhedron, set $\partial_{U^1} P = \{v \in P: \xi(v) = \min(\xi(P))\}$. More generally, if $\mathbf{F}$ is a rigged flag, set $\partial_{\mathbf{F}} P = \partial_{U^k} \partial_{U^{k-1}} \cdots \partial_{U^1} P$. It is easy to see that $\partial_{\mathbf{F}}$ extends by linearity of a function on L

$$\partial_{\mathbf{F}}: L(M) \to L(M)$$

which also descends to $\mathscr{L}(M)$.

Notice that there is well defined *absolute* notion of volume in a lattice, normalized so that a fundamental parallelepiped in any full rank k sublattice has unit volume.

Definition 5. Let $\mathbf{F}$ be a rigged flag of length k and x an element of $L(M)$. Define the Hadwiger invariant $\mathrm{Had}_{\mathbf{F}}(x)$ as

$$\mathrm{Had}_{\mathbf{F}}(P) = \mathrm{vol}_{d-k}(\partial_{\mathbf{F}}(x)).$$

In particular, if $k=0$ then $\mathrm{Had}_{\mathbf{F}}(x) = \mathrm{vol}_d(x)$ for the unique flag of length 0.

THEOREM 4. *If $f \in \mathscr{L}(M)$ and $Had_{\mathbf{F}}(f) = 0$ for every rigged flag* $\mathbf{F}$ *in* M, *then* $f = 0$.

Lambda-Ring Structure. There is further algebraic structure in $L(M)$ and $\mathscr{L}(M)$. Most interesting for us will be the augmented special λ-ring structure. The multiplicative structures in $L(M)$ and $\mathscr{L}(M)$ are determined on generators $[P]$ and $[Q]$ by

$$[P] * [Q] = [P+Q],$$

where $P+Q$ is the Minkowski sum $P+Q = \{x+y : x \in P, y \in Q\}$. One checks that this extends by linearity to a well defined ring structure. The λ-operators are determined by the fomula

$$\lambda_t([P]) = 1 + [P]\, t.$$

The augmentation is the Euler characteristic. Despite first appearances, the λ-ring structures on L and $\mathscr{L}$ are nontrivial. Bear in mind that a convex polyhedron will correspond to a line bundle, so it should be expected to have trivial higher exterior powers.

4.2. *Definition*

Let E be an equivariant vector bundle on X_Δ. Then the torus acts on all the cohomology groups $H^i(X_\Delta, E)$, hence they decompose into weight spaces

$$H^i(X_\Delta, E) \cong \bigoplus_{m \in M} H^i(X_\Delta, E)_m.$$

The weight m component of the Euler characteristic is then defined as

$$\chi_m(E) = \sum_{i=0}^{d} (-1)^i \dim H^i(X_\Delta, E)_m.$$

Now since the long exact cohomology sequence associated to a short exact sequence of equivariant bundles is a sequence of T-modules, the weight m submodules are also an exact sequence. Hence the weight m component of the Euler characteristic passes to a well defined homomorphism from $K_T(X)$ to $\mathbf{Z}$,

$$E \mapsto \chi_m(E).$$

It makes sense therefore to take the weight m Euler charactertistic of $\Psi^i(E)$, the ith Adams operation applied to the class of E.

DEFINITION 6. Let X_Δ be a smooth compact torus embedding (or quotient embedding) of a torus T with character group M. We define the *indicator* map $\mathbf{I}_T \colon K_T(X_\Delta) \to L(M)$, by means of the formula

$$\mathbf{I}_T(x)(m/k) = \chi_m(\Psi^k(x)), \tag{4}$$

where $x \in K_T(X_\Delta)$, $m \in M$, and $k \in \mathbf{N}$.

Remarks. (1) It is not clear a priori that $\mathbf{I}_T(x)$ is well defined even as a function, since the representation m/k of an element of $M_\mathbf{Q}$ is not unique.

(2) We could use the same formula for noncompact toric varieties, but the result would not generally lie in $L(M)$ because it might not have bounded support.

(3) If X is any algebraic variety or Kaehler manifold with torus action, a map from $K_T(X)$ into an appropriate group may be defined by a similar formula. This will be taken up in a separate paper.

THEOREM 5. *Suppose that X_Δ is a smooth compact toric variety (or torus quotient embedding). Then the map $\mathbf{I}_T$ is a well defined homomorphism of augmented λ-rings.*

Proof. First we show that $\mathbf{I}_T(x)$ is well defined as a function on $M_\mathbf{Q}$. Explicitly, we need to know that $\chi_m(\Psi^k(x)) = \chi_{nm}(\Psi^{nk}(x))$. Replacing x with $\Psi^k(x)$, we are reduced to the case $k=1$, $\chi_{nm}(\Psi^n(x)) = \chi_m(x)$. Since both sides of the equation are linear, we need only show it for a set of generators, which by Proposition 3 we may take to be the classes of line bundles. Let $\mathscr{E}_h$ be the line bundle associated to the support function h on Δ, see Theorem 3. Now, $\Psi^n(\mathscr{E}_h)$ is simply the nth tensor power of $\mathscr{E}_h$, which is the line bundle associated to the support function $n \cdot h$. Referring to the explicit computation, Theorem 3(3), we have that for any $m \in M$, the sets $Z(m, h)$ and $Z(nm, nh)$ are identical, from which the desired result follows.

Futhermore, Theorem 3 gives an essentially combinatorial formula for the cohomology of $\mathscr{E}_h$ in terms of h. From this one obtains a combinatorial formula for $f_h =^{\text{def}} \mathbf{I}_T(\mathscr{E}_h)$ in terms of h. This f_h agrees with the f_h defined in [Mo2], according to Proposition 9 there. The remainder of the proof is thereby reduced to combinatorics studied in [Mo2].

It should be fairly clear from Theorem 3 that f_h lies in $L(M_{\mathbf{Q}})$, but we need to check that it actually lies in $L(M)$. This follows from [Mo2, Theorem 11], which shows that $f_h \in L(M)$.

As for compatibility with the multiplicative structure, we must show that $\mathbf{I}_T(\mathscr{E}_{h_1} \otimes E_{h_2}) = \mathbf{I}_T(\mathscr{E}_{h_1}) * \mathbf{I}_T(\mathscr{E}_{h_2})$. Now $\mathscr{E}_{h_1} \otimes \mathscr{E}_{h_2} = \mathscr{E}_{h_1+h_2}$, as one easily checks. Theorem 10 of [Mo2] implies that $f_{h_1} * f_{h_2} = f_{h_1+h_2}$, which is exactly what we need.

Finally, we must check that the λ operations and the augmentations are compatible. For this we are again reduced to the case of a line bundle $\mathscr{E}_h$. Proposition 13 of [Mo2] states that $\lambda_t(f_h) = 1 + f_h t$. This translates into $\lambda_t(\mathbf{I}_T(\mathscr{E}_h)) = \mathbf{I}_T(\lambda_t(\mathscr{E}_h))$. That $\chi(f_h) = 1$ is proved in Proposition 9 there, and this translates into $\chi(\mathbf{I}_T(\mathscr{E}_h)) = \mathrm{rk}(\mathscr{E}_h)$ where rk, the virtual rank, is the augmentation on $K_T(X_\Delta)$.

Remark. If x is the class of an ample line bundle, then $\mathbf{I}_T(x)$ is the indicator function of a convex polyhedron. This is the correspondence that underlies the well known applications of toric theory to combinatorics. The Δ-linear function h corresponding to the line bundle is the classical support function of the convex polyhedron $P = \mathbf{I}_T(x)$. For us, this support function is defined

$$h(\xi) = \inf \xi(P).$$

In order to see why this is so, note that the higher cohomology of an ample line bundle $x = \mathscr{E}_h$ on a toric variety vanishes. Therefore, $\mathbf{I}_T(x)(m) = H^0_{Z(m,h)}(N_{\mathbf{R}})$. This latter group is nonzero exactly where $Z(m, h) = N_{\mathbf{R}}$ and there it is 1. Such points m occur exactly in the intersection of all the halfspaces determined by all the one dimensional cones, $q(m) \geqslant h(n(q))$.

Let us now consider the contravariant functorial properties of $\mathbf{I}_T$. Suppose Δ is a fan in N, and Δ' is a fan in N'. Let $g: N' \to N$ be a map of lattices inducing a map of toric varieties $g_*: X_{\Delta'} \to X_\Delta$ which in turn induces g^{K_T} on the equivariant K-groups. The dual map $g^\vee: M \to M'$ induces by direct image a map on polyhedral functions $g_L^\vee: L(M) \to L(M')$.

Proposition 5. *The map $\mathbf{I}_T$ is functorial in the following sense*:

$$\mathbf{I}_T(g^{K_T}(x)) = g_L^\vee(\mathbf{I}_T(x)).$$

Proof. It follows essentially from definitions that the pullback of the line bundle $\mathscr{E}_h$ by g^{K_T} is just $\mathscr{E}_{h \circ g}$. Consequently, for any h, we have

$\mathbf{I}_T(g^{K_T}\mathscr{E}_h) = \mathbf{I}_T(\mathscr{E}_{h \circ g}) = f_{h \circ g}$. On the other hand, it is a purely combinatorial fact (Proposition 9 of [Mo2]) that $f_{h \circ g} = g_L^\vee(f_h)$, which is just $g_L(\mathbf{I}_T(\mathscr{E}_h))$. By linearity, we are done.

COROLLARY 1. *Dilation by a factor of $k \in \mathbf{N}$ in the lattice N determines a map on X_Δ. The induced (contravariant) map on $K_T(X_\Delta)$ is the Adams operation Ψ^k.*

PROPOSITION 6. *Let U be the hyperplane in M dual to a vector λ in N. Rig U by choosing the halfspace on which λ takes positive values. Let σ be the smallest cone in Δ which contains λ and let $i_\sigma : \overline{V(\sigma)} \to X_\Delta$ be the (T-equivariant) inclusion map of the closure of the corresponding orbit. Then, in the notation of Defintion 5,*

$$\mathbf{I}_T(i_\sigma^{K_T}(x)) = \partial_U \mathbf{I}_T(x)$$

for any $x \in K_T(X_\Delta)$.

Proof. By Proposition 3, it is sufficient to take for x the class of a line bundle. The proposition now reduces to Proposition 10 of [Mo2].

Much of the algebraic structure in $K_T(X_\Delta)$ is reflected in simple combinatorial terms in $L(M)$. For instance the duality involution in $K_T(X_\Delta)$ sending a bundle E to its dual bundle $E^\vee$ corresponds to the operation in $L(M)$ which takes a convex polyhedron 1_P of dimension k to $(-1)^k 1_{-\dot{P}}$, where $-\dot{P}$ is the relative interior of P reflected through the origin. The virtual rank homomorphism on $K_T(X_\Delta)$ corresponds to the Euler characteristic on $L(M)$ which takes each convex polyhedron P to 1. Beware that the virtual rank of $x \in K_T(X_\Delta)$ has no relation to the dimension of the support of $\mathbf{I}_T(x)$, which really reflects, if anything, the dimension of the variety X_Δ. It can be shown that if $\mathbf{I}_T(x)$ lies inside a *k-dimensional* affine subspace of $M_\mathbf{Q}$ then the class x is the pullback along an equivariant morphism of a class in $K_T(X_\Sigma)$ where X_Σ is some quotient embedding of T of dimension k.

4.3. *Snapper Polynomials*

It is a well known result of Snapper and Kleiman that the Euler–Poincaré characteristic, $\chi(\xi^n)$, of the powers of a line bundle ξ on a complete variety grows as a polynomial. More generally, $\chi(\xi_1^{n_1} \otimes \cdots \otimes \xi_r^{n_r})$ grows as a polynomial in the n_i. In the case of ample line bundles ξ_i on a toric variety it is known that the mixed volume of the associated convex polyhedra is equal to the coefficient of the leading term of this polynomial (see [Dan, pp. 133–135]). Because we will need it in the following section, let us fit this observation into the present framework.

First note the following slight generalization of the Snapper polynomial:

if $x_1, ..., x_r$ are classes in the K-group of a complete variety X, then $\chi(\Psi^{n_1}(x_1) \otimes \cdots \otimes \Psi^{n_r}(x_r))$ is a polynomial in the n_r. This is a direct consequence of the Hirzebruch–Riemann–Roch theorem and the fact that Ψ^k acts as multiplication by k^j in A^j (as in the case with the ordinary Snapper polynomials, see [Fu, Example 18.3.6]). For the sake of notational simplicity, take $r = 1$ and find

$$\begin{aligned}\chi(\Psi^n(x)) &= \int_X ch_*(\Psi^n(x))\, Td_*(X) \\ &= \sum_j \int_X ch_j(\Psi^n(x))\, Td_{d-j}(X) \\ &= \sum_j n^j \int_X ch_j(x)\, Td_{d-j}(X),\end{aligned} \tag{5}$$

where $\int_X$ denotes the degree homomorphism $\int_X : A_d(X)_{\mathbf{Q}} \to \mathbf{Q}$.

We will need the following simple observation. If P is a convex lattice polyhedron in $M_{\mathbf{Q}}$, denoted by $\#(P)$ the number of lattice points P touches. By linearity, $\#$ extends to a homomorphism $L(M) \to \mathbf{Z}$, given by $\#(f) = \sum_{m \in M} f(m)$. By definition, if $x \in K(X_\Delta)$, then $\chi(x) = \sum_{m \in M} \mathbf{I}_T(x)(m)$, so that

$$\chi(x) = \#(\mathbf{I}_T(x)).$$

This function $\#$ has been an object of considerable interest in combinatorics. It is studied further from the present point of view in [Mo3].

If $f \in L(M)$ and $n \in \mathbf{N}$, denote by $\Psi^n(f)$ the dilation of f by a factor of n, i.e., $\Psi^n(f)(v) = f(v/n)$ for $v \in M_{\mathbf{Q}}$. We have already seen that

$$\mathbf{I}_T(\Psi^n(x)) = \Psi^n(\mathbf{I}_T(x)).$$

It turns out that $\#(f; n) =^{\text{def}} \#(\Psi^n(f))$ is polynomial in n with leading coefficient $\mathrm{vol}_d(f)$. In fact, from what we have just seen,

$$\#(\mathbf{I}_T(x); n) = \sum_j n^j \int_{X_\Delta} ch_j(x)\, Td_{d-j}(X_\Delta).$$

In particular, the leading coefficient is $\mathrm{vol}_d(\mathbf{I}_T(x)) = \int_{X_\Delta} ch_d(x)$.

4.4. *Injectivity*

From Proposition 3, we know that for any x in $K(X_\Delta)$ there is an $\tilde{x}$ in $K_T(X_\Delta)$ mapping to x. In order to define a map $\mathbf{I}: K(X) \to \mathscr{L}$ taking x to the element represented by $\mathbf{I}_T(\tilde{x})$, we need to know that this latter element is well defined in $\mathscr{L}$.

THEOREM 6. *If X_Δ is nonsingular then the map $\mathbf{I}_T$ descends to a well defined, and in fact injective, homomorphism $\mathbf{I}: K(X) \to \mathscr{L}(M)$.*

Proof. By what we have already seen, if $\bar{\mathbf{I}}$ is well defined its image will be in $\mathscr{L}(M)$.

We saw in the last section that $\mathrm{vol}_d(\mathbf{I}_T(\tilde{x})) = \int_{X_\Delta} ch_d(\tilde{x}) = \int_{X_\Delta} ch_d(x)$. Therefore, at least the volume of $\mathbf{I}_T(\tilde{x})$ is well defined. We proceed to construct the higher Hadwiger invariants of $\mathbf{I}_T(\tilde{x})$ from x.

From a nonsingular k-dimensional simplicial cone $\sigma \in \Delta$, together with an ordering on its edges, we can construct a rigged flag of length k as follows. Choose generators $q_1, ..., q_k$ of the ordered edges of σ. The kernel of q_1 in $M_{\mathbf{R}} = N_{\mathbf{R}}^\vee$ is a rigged hyperplane U^1. Now q_2 restricts to a nonzero linear functional on U^1, hence gives a rigged hyperplane U^2 of U^1. Continuing in this fashion, we obtain a rigged flag $\mathbf{F} = (N_{\mathbf{R}} = U^0 \supseteq U^1 \supseteq \cdots \supseteq U^k)$. There are thus $k!$ different flags associated to σ.

PROPOSITION 7. *The Hadwiger invariants of $\mathbf{I}(x)$ are all determined by the chern character of x. In particular, if the flag $\mathbf{F}$ in $N_{\mathbf{R}}$ is any one of the $k!$ rigged flag constructed above from a cone σ, then*

$$\mathrm{Had}_{\mathbf{F}}(\mathbf{I}(x)) = (ch_k(x) \cdot [V(\sigma)])_d.$$

Otherwise put,

$$ch^k(x)(\sigma) = \mathrm{Had}_{\mathbf{F}}(\mathbf{I}(x)).$$

Proof. Let $\mathbf{F} = (N_{\mathbf{R}} = U^0 \supseteq U^1 \supseteq \cdots \supseteq U^k)$ be any flag. According to Proposition 6 (applied k times), the polyhedral function $\partial_{U^k} \cdots \partial_{U^1} \mathbf{I}(x)$ is the indicator of the pullback of x to the closure of one of the orbits in X_Δ. If $\mathbf{F}$ is constructed as above from a cone σ, then this orbit is the orbit $V(\sigma)$. It should be clear from Propositon 6 how to formulate which one it is in general. Now we apply the naturality of the chern character, and the observations above connecting the chern character with volume to determine the Hadwiger invariants.

It now follows from Theorem 4 that $\mathbf{I}(x)$ is well defined. We have a commutative diagram

$$\begin{array}{ccc} K_T & \xrightarrow{\mathbf{I}_T} & L \\ \downarrow & & \downarrow \\ K & \xrightarrow{\mathbf{I}} & \mathscr{L}. \end{array}$$

We now turn attention to injectivity. If x is in the kernel of $\mathbf{I}$, then the Hadwiger invariants of $\mathbf{I}(x)$ vanish. By the proposition, the chern character of x then vanishes. But the chern character is generally an isomorphism mod torsion, so x is torsion. By Lemma 1, $K(X_\Delta)$ is torsion free. Hence x is 0.

Remark. In $K'(X_\Delta) = K(X_\Delta)$ we have an increasing filtration whose kth term is the subgroup generated by coherent sheaves with support of dimension at most k. The indicator homomorphism $\mathbf{I}$ maps this filtration to the *weight* filtration F^* in $\mathscr{L}(M)$. For $k > 0$ the kth term of the weight filtration in $\mathscr{L}(M)$ is generated by k-fold products of element of the form $[P] - [Q]$, P and Q convex polyhedra.

THEOREM 7. *If X_Δ is nonsingular then $\mathbf{I}_T$ is injective, and its image lies in $L(M)$.*

Proof. We already saw that the image lies in $L(M)$ in the proof of Theorem 5. For the injectivity, we need some observations. First, we can reduce to the projective case as follows. If Δ is any fan, we can find a projective refinement Δ', so we have a proper morphism $f: X'_\Delta \to X_\Delta$. Since f is a birational morphism of normal varieties, $f_* \mathcal{O}_{X'_\Delta}$. Moreover, $R^i f_* \mathcal{O}_{X'_\Delta} = 0$ (see KKMS, Sect. 3)]. By the projection formula we then have that $f_{K_T} f^{K_T} = \mathrm{Id}$ so f_{K_T} is a split injection. By the naturality of $\mathbf{I}_T$, $\mathbf{I}_T(f^{K_T}(x)) = \mathbf{I}_T(x)$, so it is enough to show that $\mathbf{I}_T$ is injective on $X_{\Delta'}$.

Now $L(M)$ is a module over $\mathbf{Z}[M]$ by means of the natural action of M by translations, and $\mathbf{I}_T$ is clearly a homomorphism of $\mathbf{Z}[M]$-modules. We know by Lemma 2 that if X_Δ is projective we may choose generators $\xi_1, ..., \xi_l$ of $K_T(X_\Delta)$ whose images in $K(X_\Delta)$ constitute a basis. Suppose now that $x \in K_T(X_\Delta)$ is in the kernel of $\mathbf{I}_T$. Write $x = \sum_i a_i \xi_i$ with $a_i \in \mathbf{Z}[M]$. Then $\Sigma_i a_i \mathbf{I}_T(\xi_i) = \mathbf{I}_T(x) = 0$, and the $\mathbf{I}(\xi_i)$ are linearly independent as elements of $\mathscr{L}$ by Theorem 6. By the following lemma all a_i vanish.

LEMMA 3. *Let P_i be elements of $L(M)$ whose images $\bar{P}_i$ in $\mathscr{L}(M)$ are linearly independent over $\mathbf{Z}$. Then the P_i are linearly independent over $\mathbf{Z}[M]$.*

Proof. Assume that there is a relation $\sum_i a_i P_i = 0$ in L with $a_i \in \mathbf{Z}[M]$ not all zero. Then we can find a full sublattice $H \subseteq M$ of rank $d-1$ for which the images $\tilde{a}_i \in \mathbf{Z}[M/H]$ don't all vanish. Write $L(M)_H$ for the group of coinvariants of L with respect to H, and let z be a generator of M/H. Then it is clear that $z-1$ is not a zero divisor for $L(M)_H$. Now let $(z-1)^r$ be the highest power of $z-1$ which divides all $\tilde{a}_i$. Factoring, $0 = \sum_i \tilde{a}_i \tilde{P}_i = (z-1)^r \sum_i \tilde{b}_i \tilde{P}_i$ in $L(M)_H$ so $\sum_i \tilde{b}_i \tilde{P}_i = 0$. Since not all $\tilde{b}_i$ are the augmentation ideal of $\mathbf{Z}[M/H]$, reducing further to $\mathscr{L}$ gives a nontrivial relation $\sum_i \bar{b}_i \bar{P}_i = 0$.

COROLLARY 2. *If X_Δ is a smooth and projective toric variety, then there exists a $\mathbf{Z}[M]$ basis $\{\xi_1, ..., \xi_n\}$ of $K_T(X_\Delta)$ for which the images of the ξ_i constitute a $\mathbf{Z}$-basis of $K(X_\Delta)$.*

COROLLARY 3. *If the pullback of a class in $K_T(X_\Delta)$ to the closure of every one parameter subgroup of the torus vanishes, then the class itself vanishes. More generally, if $0<k<d$, and the pullback to the closure of the exponentiation of every rank k sublattice vanishes, then the class itself vanishes.*

Proof. The closure Y of a one parameter subgroup corresponds to a line l in $N_{\mathbf{Q}}$. By the naturality of the indicator map, the indicator of the pullback of x to Y is the image of $\mathbf{I}_T(x)$ under the quotient $M_{\mathbf{Q}} \to M_{\mathbf{Q}}/l^\perp$. Every such image of $\mathbf{I}_T(x)$ must therefore vanish. By the injectivity of the Radon transform for $L(M_{\mathbf{Q}})$ (see [Mo2, Theorem 5]), $\mathbf{I}_T(x)$ must itself vanish.

4.5. *Image*

The description of the image of $\mathbf{I}_T$ is intuitively simple, but requires some definitions to state.

There is an evident notion of subdivision of a rational cone into a collection of rational cones, and hence there are scissors congruence groups of cones, analogous to the scissors congruence groups of polyhedra.

DEFINITION 7. The group of *polyhedra germs* $\mathscr{S}\mathscr{L}(M)$ is the abelian group with generators $[\sigma]$, σ a rational convex cone in M, and relations

$$[\sigma \cup \tau] = [\sigma] + [\tau] - [\sigma \cap \tau]$$

whenever $\sigma \cup \tau$ is a rational convex cone.

If elements of $L(M)$ are interpreted as functions, then $\mathscr{S}\mathscr{L}(M)$ is simply the group of germs of these functions at a fixed point (the origin, say). In fact, given a lattice polyhedron P, and a point $p \in M_{\mathbf{Q}}$, we obtain a rational cone

$$\vartheta_p(P) = \{v \in M_{\mathbf{Q}} : p + \varepsilon v \in P \text{ for all } \varepsilon > 0 \text{ sufficiently small}\},$$

the cone subtended by P at p. It is clear that ϑ_p passes to a homomorphism from $L(M)$ to $\mathscr{S}\mathscr{L}(M)$. For $f \in L(M)$, $\vartheta_p(f)$ is literally the germ of f at p.

DEFINITION 8. Given a fan Δ, let $\mathscr{S}\mathscr{L}_\Delta(M)$ be the subgroup of $\mathscr{S}\mathscr{L}(M)$ generated by the duals of the cones in Δ. Define $L_\Delta(M)$ to be the subgroup of $L(M)$ consisting of those f for which $\vartheta_m(f)$ lies in $\mathscr{S}\mathscr{L}_\Delta(M)$ for each m in M. Define $\mathscr{L}_\Delta(M)$ as the image of $L_\Delta(M)$ in $\mathscr{L}(M)$.

THEOREM 8. *We have isomorphisms*

$$\mathbf{I}: K(X_\Delta) \to \mathscr{L}_\Delta(M)$$

and

$$\mathbf{I}_T: K_T(X_\Delta) \to L_\Delta(M).$$

Proof. It clearly suffices to show that the image of $\mathbf{I}_T$ is $L_\Delta(M)$. The proof depends on combinatorics developed in [Mo2] which I will only sketch here.

We know that every Δ-linear support function h corresponds to some equivariant line bundle $\mathscr{E}_h$ on X_Δ. Now any convex polyhedron P in $L_\Delta(M)$ has a support function h_P for which $\mathbf{I}_T(\mathscr{E}_{h_P}) = P$, so we would be done if we knew that such P generate $L_\Delta(M)$. But this is certainly false; for non-projective toric varieties there may not be even a single nontrivial convex P in L_Δ. However, extending the notion of support function to all of $L(M)$ allows us to follows a nearby route.

To an element of $L_\Delta(M)$ one can associate a "polysupport" function on Δ. A polysupport function is defined as a function on $N_{\mathbf{Q}}$, with values in the group ring $\mathbf{Z}[\mathbf{R}]$, which restricts on each cone of Δ to a virtual sum of linear functions, each taking M to $\mathbf{Z}$. To define this polysupport function, proceed as follows. Write f as a sum $f = \sum_k a_k[P_k]$ of convex polyhedra. (If $f \in L_\Delta(M)$, we needn't necessarily take $P_k \in L_\Delta(M)$.) Let h_k be the classical support function P_k. The polysupport function h of f is then defined $h(\xi) = \sum_k a_k[h_k(\xi)]$.

It can be shown that (1) the correspondence taking $f \in L_\Delta(M)$ to its polysupport function is an isomorphism onto the group of all Δ-linear polysupport functions; (2) if Δ is nonsingular, then every polysupport function is globally a virtual sum of Δ-linear support functions; and (3) if the polysupport function of f is a Δ-linear function h, then

$$f = \mathbf{I}_T(\mathscr{E}_h).$$

The proof of (1) is in [Mo2, Sect. 5], especially Propositions 11 and 12. Statement (2) is proven in Theorem 12 there, and statement (3) follows from Theorem 3, together with [Mo2, Proposition 9]. The three statements together obviously imply the theorem.

Here is a proof for $\mathbf{I}$ that avoids these combinatorics. First of all, it is clear that the image of $\mathbf{I}$ lies in $\mathscr{L}_\Delta(M)$. For surjectivity, choose an element x in $\mathscr{L}_\Delta$, and write x as a virtual sum of nonsingular simplices, i.e., simplices with volume $1/d!$. This is always possible because by Theorem 13 of [Mo3], the nonsingular simplices generate $\mathscr{L}(M)$. These simplices may not be in $\mathscr{L}_\Delta(M)$, but we can find a nonsingular refinement Δ' of Δ in which they all lie. As we have seen, the simplices, being convex, are all images under $\mathbf{I}$ of line bundles on $X_{\Delta'}$. We then have $x = \mathbf{I}(\xi)$ for some $\xi \in K(X_{\Delta'})$. Now the chern character of ξ can be determined in terms of the hadwiger invariants of x. Since $x \in \mathscr{L}_\Delta$, this means that $ch_*(\xi) \cdot [V(\sigma)]$ only depends on the smallest cone $\hat{\sigma}$ in Δ' containing σ. Since the image of the cycle $[V(\sigma)]$ under the natural map $X_{\Delta'} \to X_\Delta$ is $[V(\hat{\sigma})]$, this implies that the cohomology chern character $ch^*(\xi)$ is the image of a cohomology class on X_Δ. Letting $\hat{\xi}$ be a corresponding class in $K(X_\Delta)$, we find that since $\mathbf{I}(\hat{\xi})$ and x have the same hadwiger invariants, they are equal.

Remarks. (1) One would expect a similar result for X_Δ smooth but not necessarily compact, where $\mathbf{I}(x)$ is a polyhedral function which is not necessarily bounded. However (in addition to the condition on the germs of these function as above), we would expect them to be bounded from below in the directions which occur in the fan Δ. That is, we expect the support of $f \in \mathbf{I}_T(X_\Delta)$ to satisfy that $\lambda(\operatorname{supp} f)$ is bounded from below for any λ in $|\Delta| \subseteq N_{\mathbf{R}}$. (This is because $K(X_\Delta)$ is defined using coherent sheaves.) This extra condition would impose bounded support on $f \in \mathbf{I}_T(X_\Delta)$ for a complete fan even if we had not assumed it in our definition of $L(M)$.

(2) The duals $[\sigma^\vee]$ of cones σ in Δ are actually a basis of $\mathscr{SL}_\Delta(M)$. (This follows from the fact that the duality operation extends to an isomorphism $\mathscr{SL}(N) \to \mathscr{SL}(M)$.) For any element f of $L_\Delta(M)$ and any $\sigma \in \Delta$ we can therefore define f_σ as the function whose value at a point $p \in M_{\mathbf{Q}}$ is the $[\sigma^\vee]$-component of $\vartheta_p(f)$. It is easy to see that f_σ is in $L(M)$ (but not necessarily $L_\Delta(M)$). Proposition 7 says in effect that

$$\overline{\operatorname{vol}_k}(\mathbf{I}(x)_\sigma) = ch_k(x) \cdot [V(\sigma)]. \tag{6}$$

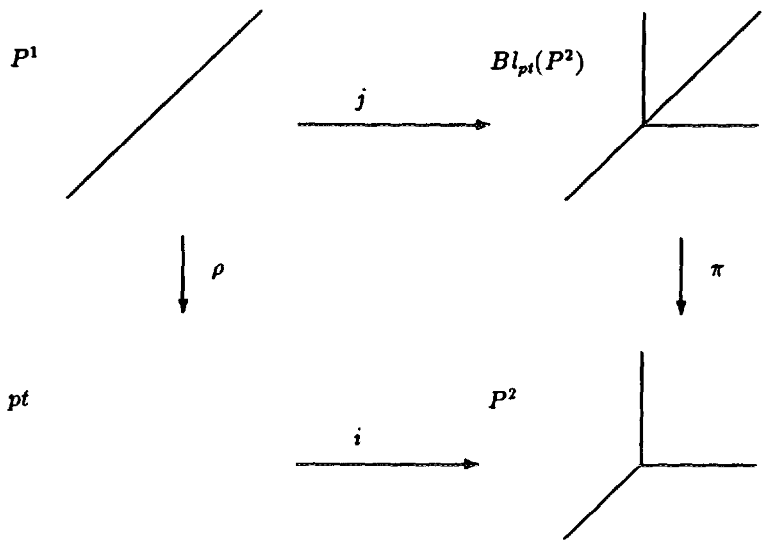
Another way of saying this is that the codimension k generalized Dehn invariant of $\mathbf{I}_T(x)$ is $ch^{d-k}(x)$, the Poincaré dual of $ch_k(x)$.

5. Example

We illustrate the ideas in this paper with a final example. Consider the blowup of the origin pt in $\mathbf{P}^2$. We have the following fiber square

$$\begin{array}{ccc} \mathbf{P}^1 & \xrightarrow{j} & Bl_{pt}\mathbf{P}^2 \\ {\scriptstyle\rho}\downarrow & & \downarrow{\scriptstyle\pi} \\ pt & \xrightarrow{i} & \mathbf{P}^2 \end{array}$$

in which all four spaces are torus quotient embeddings. The horizontal maps are equivariant regular maps, while the vertical maps are maps of torus quotient embeddings. In terms of fans this diagram is as follows.

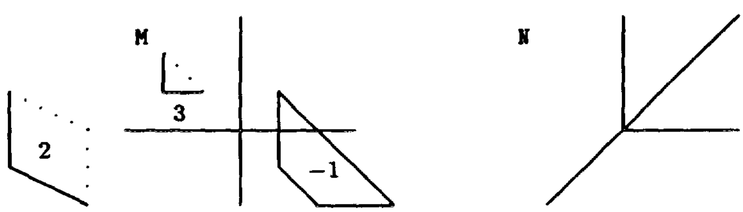


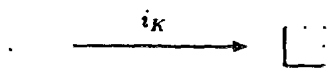
In general the K-group of a blowup is connected with the K-groups of the base space, the center, and the exceptional divisor by an exact sequence, which in the case at hand is

$$0 \longrightarrow K(pt) \longrightarrow K(\mathbf{P}^1) \oplus K(\mathbf{P}^2) \xrightarrow{j_K + \pi^K} K(Bl_{pt}\mathbf{P}^2) \longrightarrow 0.$$

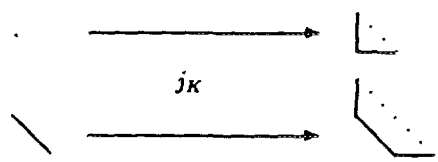
The first map in this sequence is $(-\lambda_{-1}(F) \cdot \rho^K) \oplus (i_K)$ where F is the class of the universal quotient bundle on $\mathbf{P}^1$. One computes that in $K(\mathbf{P}^1)$, $\lambda_{-1}(F) = [\mathcal{O}(1)] - 1$.

Now we know the descriptions of all these groups in terms of polyhedra. The fan of the point pt has a single cone, which is all of $N_{\mathbf{R}}$, and whose dual is the origin of $M_{\mathbf{R}}$. Therefore, $\mathbf{I}_T(K_T(pt))$ is generated by indicator functions of points. The fan of $\mathbf{P}^1$ has three cones, whose duals are the line $y = -x$ and the two rays from the origin which it contains. Therefore, $\mathbf{I}_T(K(\mathbf{P}^1))$ is generated by points and line segments parallel to $y = -x$. We already described $K(\mathbf{P}^2)$ in the second example in the Introduction. Some typical generators of $\mathbf{I}_T(K(Bl_{pt}\mathbf{P}^2))$ are depicted below.

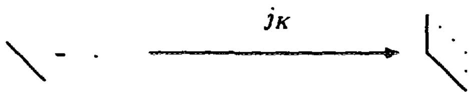




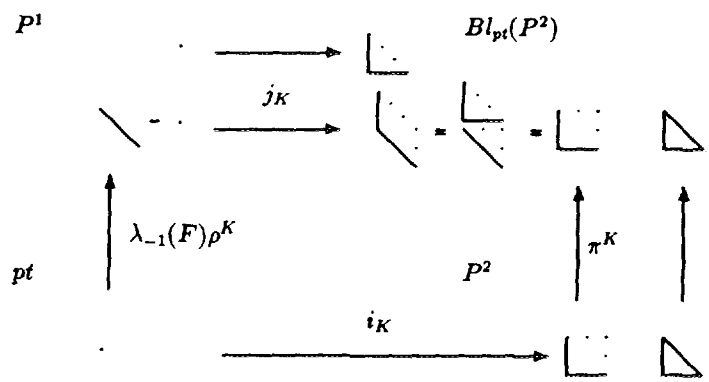
Let us now describe the maps. By naturality, the maps ρ^K and π^K do nothing. The map i_K takes a point to the half open square. The image of the map j_K on a point and a line segment is indicated as



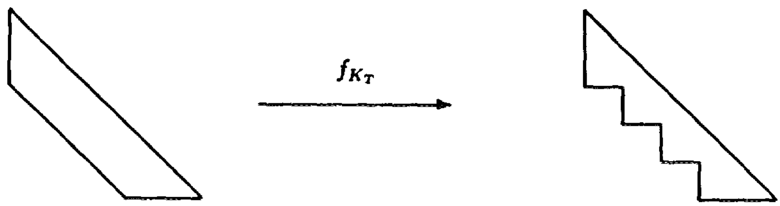
Therefore, $j_T(\lambda_{-1}(F))$ is



Assembling all of this, we have below a pictorial illustration of the exact sequence.



The map f_K is shown below. Faces parallel to $y = -x$ which face upwards are inadmissible with respect to the fan for $\mathbf{P}^2$. Such faces "buckle in" when f_K is applied. This is a typical, if simple, example of how pushforward behaves in terms of polyhedra. Incidentally, this picture illustrates the failure of Adams operations to commute with pushforward, and even hints at the appearance of bott's cannibalistic classes in the Adams Riemann–Roch formula.



In general, the Grothendieck Riemann–Roch theorem provides a description of the pushforward map for $K(X)$ which can be translated into a combinatorial description of pushforward in terms of $\mathscr{L}(M)$ (which is not geometrically intuitive). A corresponding description of pushforward for $K_T(X)$ and a consequent combinatorial description of pushforward in terms of $L(M)$ would also be interesting. At present, I don't know such a description.

References

[SGA6] P. Berthelot, A. Grothendieck, L. Illusie, *et al.*, Théorie des intersections et théorème de Riemann–Roch, *in* "Lecture Notes in Mahematics," Vol. 225, Springer-Verlag, New York/Berlin, 1971.

[B1] M. Borelli, Divisorial varieties, *Pacific J. Math.* **13**, 375–388.

[B2] M. Borelli, Some results on ampleness and divisorial schemes, *Pacific J. Math.* **23**, 217–227.

[BS] A. Borel and J.-P. Serre, Le théorème de Riemann–Roch (d'apres Grothendieck), *Bull. Soc. Math. France* **86** (1958), 97–136.

[Ca] P. Cartier, Décomposition des polyèdres; Le point sur le troisième problème de Hilbert, *in* "Séminaire Bourbaki," 37ème année, Exp. 646, 1984–1985.

[Dan] V. I. Danilov, The geometry of toric varieties, *Russian Math. Surveys* **33**, No. 2 (1978), 97–154; *Uspekhi Mat. Nauk.* **33**, No. 2 (1978), 85–134.

[FL] W. Fulton and S. Lang, "Riemann–Roch Algebra," Springer-Verlag, New York/Berlin, 1982.

[Fu] W. Fulton, "Intersection Theory," Springer-Verlag, Berlin, 1984.

[Ju] J. Jurkiewicz, Chow ring of projective non-singular torus embedding, *Colloq. Math.* **43**, No. 2 (1980), 261–270.

[KKMS] G. Kempf, F. Knudsen, D. Mumford, and B. Saint-Donat, Toroidal embeddings, I, *in* "Lecture Notes in Mathematics," Vol. 339, Springer-Verlag, New York/Berlin, 1973.

[L] S. Lang, "Algebra," 2nd ed., Addison–Wesley, Reading, MA, 1984.

[Mo1] R. Morelli, Translation scissors congruence, to appear.

[Mo2] R. Morelli, A theory of polyhedra, to appear.

[Mo3] R. Morelli, Pick's theorem and the Todd class of a toric Variety, to appear.

[Man] Y. I. Manin, Lectures on the *K*-functor in algebraic geometry, *Russian Math. Surveys* **24**, No. 5 (1969), 1–89.

[Oda] T. Oda, "Convex Bodies and Algebraic Geometry," Springer-Verlag, New York/Berlin, 1985.

[St] R. P. STANLEY, The number of faces of a simplicial convex polytope, *Adv. in Math.* **35** (1980), 236–238.

[Te] B. TEISSIER, Variétés toriques et polytopes, *in* "Séminaire Bourbaki, 1980/81," Exp. 565, pp. 71–84, Lectures Notes in Mathematics, Vol. 901, Springer, Berlin/Heidelberg/New York, 1981.

[Th] R. W. THOMASON, Algebraic K-theory of group scheme actions, *in* "Algebraic Topology and Algebraic K-Theory," Ann. of Math. Stud., Vol. 113, Princeton Univ. Press, Princeton, NJ, 1987.